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Joseph Schechter
Syracuse University

Amir H. Fariborz
State University of New York Institute of Technology

Renata Jora
Universitat Autonoma de Barcelona

M. Naeem Shahid
Syracuse University

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Chiral Nonet Mixing in $\pi\pi$ Scattering

Amir H. Fariboz, [a] *, Renata Jora [b] †, Joseph Schechter [c] §, and M. Naeem Shahid [c] ¶
[c] *Department of Physics, Syracuse University, Syracuse, NY 13244-1130, USA,*
[a] *Department of Engineering, Sciences and Mathematics, State University of New York, Institute of Technology,Utica, NY13504-3050, USA, and*
[b] *Grup de Fisica Teorica and IFAE, Universitat Autonoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain,*

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Pion pion scattering is studied in a generalized linear sigma model which contains two scalar nonets (one of quark-antiquark type and the other of diquark-antidiquark type) and two corresponding pseudoscalar nonets. An interesting feature concerns the mixing of the four isosinglet scalar mesons which yield poles in the scattering amplitude. Some realism is introduced by enforcing exact unitarity via the K-matrix method. It is shown that a reasonable agreement with experimental data is obtained up to about 1 GeV. The poles in the unitarized scattering amplitude are studied in some detail. The lowest pole clearly represents the sigma meson (or $f_0(600)$) with a mass and decay width around 500 MeV. The second pole invites comparison with the $f_0(980)$ which has a mass around 1 GeV and decay width around 100 MeV. The third and fourth poles, resemble some of the isosinglet state in the complicated 1-2 GeV region. Some comparison is made to the situation in the usual SU(3) linear sigma model with a single scalar nonet.

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I. INTRODUCTION

Although the exact nature of the low lying scalar mesons has been a topic of intense debate, the fact that these states play important roles in our understanding of low-energy QCD seems to be shared by all. Various models have been put forward for the properties of the scalar mesons. A general discussion of the experimental situation on light scalars is given in ref. [2]. Some characteristic treatments of the last twenty years are given in refs. [3]-[35]. In particular, an important four-quark (i.e. two quarks and two antiquarks) component, as was first proposed in the MIT bag model [37], seems to explain some of their unusual properties such as the reversed mass spectrum. It has also been pointed out [21], [22], [23], [25], [26], [27]-[32], that four quark components alone are not sufficient for understanding the physical parameters of these states but seems to rquire a scenario based on an underlying mixing between quark-antiquark nonets and nonets containing two quarks as well as two anti quarks. A simple picture for scalar states below 2 GeV then seems to emerge. Amusingly, this mixing [32] automatically leads to light scalars that are dominantly of two quark- two antiquark nature and light conventional pseudoscalars that are, as expected from established phenomenology, dominantly of quark-antiquark nature.

As some of the light scalar mesons (such as the σ and the κ) are very broad their Lagrangian masses ("bare" masses) are considerably different from their pole locations in appropriate scattering amplitudes. The work of [32] investigated in detail the "bare" mass spectrum of these states, as well as their internal structure by fitting the predictions of the model for various low-energy parameters to known experimental data. This fixed all Lagrangian parameters (to the leading order). In this paper, we work with the same Lagrangian (i.e. without introducing any new parameters) and investigate the effect of unitarization on the isosinglet, zero angular momentum partial wave pi-pi scattering amplitude computed at tree order.

We will treat the pion scattering amplitude unitarization by using the K-matrix method. As the model involves two nonets of scalars, there are altogether four isosinglet scalar mesons (two from each nonet) that contribute as poles in the pion scattering amplitude. Therefore the K-matrix unitarization has to deal with all four poles at the same time resulting in a more involved version of the conventional single-pole K-matrix unitarization.

The advantages of the K-matrix approach to unitarization are that it does not introduce any new parameters and that it forces exact unitarity. It is plausible since if one starts from a pure pole in the partial wave amplitude, one ends up with a pure Breit Wigner shape. A disadvantage is that it neglects, in the simple version we use, the effects

* Email: fariboa@sunyit.edu
† Email: rjora@ifae.es
§ Email: schechte@phy.syr.edu
¶ Email: mnshahid@phy.syr.edu

of the opening of thresholds like the $K\bar{K}$ on the pi pi amplitude. This is not expected to be too serious for our initial appraisal here.

In Sec. II we give a brief review of the Lagrangian and the relevant formulas, followed by the K-matrix unitarization of the scattering amplitude in Sec. III and a summary and discussion of the results in Sec. IV.

II. BRIEF REVIEW OF THE MODEL

The model employs the 3×3 matrix chiral nonet fields:

$$M = S + i\phi, \qquad\qquad M' = S' + i\phi'. \tag{1}$$

The matrices M and M' transform in the same way under chiral SU(3) transformations but may be distinguished by their different $\mathrm{U}(1)_A$ transformation properties. M describes the "bare" quark antiquark scalar and pseudoscalar nonet fields while M' describes "bare" scalar and pseudoscalar fields containing two quarks and two antiquarks. At the symmetry level in which we are working, it is unnecessary to further specify the four quark field configuration. The four quark field may, most generally, be imagined as some linear combination of a diquark-antidiquark and a "molecule" made of two quark-antiquark "atoms".

The general Lagrangian density which defines our model is

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left(\partial_\mu M \partial_\mu M^\dagger\right) - \frac{1}{2}\mathrm{Tr}\left(\partial_\mu M' \partial_\mu M'^\dagger\right) - V_0\left(M, M'\right) - V_{SB}, \tag{2}$$

where $V_0(M, M')$ stands for a function made from $\mathrm{SU(3)_L} \times \mathrm{SU(3)_R}$ (but not necessarily $\mathrm{U(1)_A}$) invariants formed out of M and M'.

As we previously discussed [28], the leading choice of terms corresponding to eight or fewer underlying quark plus antiquark lines at each effective vertex reads:

$$\begin{aligned} V_0 = & - c_2\,\mathrm{Tr}(MM^\dagger) + c_4^a\,\mathrm{Tr}(MM^\dagger MM^\dagger) \\ & + d_2\,\mathrm{Tr}(M'M'^\dagger) + e_3^a(\epsilon_{abc}\epsilon^{def}M_d^a M_e^b M_f'^c + h.c.) \\ & + c_3\left[\gamma_1 \ln\left(\frac{\det M}{\det M^\dagger}\right) + (1-\gamma_1)\ln\frac{\mathrm{Tr}(MM'^\dagger)}{\mathrm{Tr}(M'M^\dagger)}\right]^2. \end{aligned} \tag{3}$$

All the terms except the last two (which mock up the axial anomaly) have been chosen to also possess the $\mathrm{U(1)_A}$ invariance. The symmetry breaking term which models the QCD mass term takes the form:

$$V_{SB} = -2\,\mathrm{Tr}(A\,S) \tag{4}$$

where $A = diag(A_1, A_2, A_3)$ are proportional to the three light quark masses. The model allows for two-quark condensates, $\alpha_a = \langle S_a^a \rangle$ as well as four-quark condensates $\beta_a = \langle S'^a_a \rangle$. Here we assume [**?**] isotopic spin symmetry so $A_1 = A_2$ and:

$$\alpha_1 = \alpha_2 \neq \alpha_3, \qquad\qquad \beta_1 = \beta_2 \neq \beta_3 \tag{5}$$

We also need the "minimum" conditions,

$$\left\langle \frac{\partial V_0}{\partial S} \right\rangle + \left\langle \frac{\partial V_{SB}}{\partial S} \right\rangle = 0, \qquad \left\langle \frac{\partial V_0}{\partial S'} \right\rangle = 0. \tag{6}$$

There are twelve parameters describing the Lagrangian and the vacuum. These include the six coupling constants given in Eq.(3), the two quark mass parameters, $(A_1 = A_2, A_3)$ and the four vacuum parameters $(\alpha_1 = \alpha_2, \alpha_3, \beta_1 = \beta_2, \beta_3)$.The four minimum equations reduce the number of needed input parameters to eight.

Five of these eight are supplied by the following masses together with the pion decay constant:

$$\begin{aligned} m[a_0(980)] &= 984.7 \pm 1.2\,\mathrm{MeV} \\ m[a_0(1450)] &= 1474 \pm 19\,\mathrm{MeV} \\ m[\pi(1300)] &= 1300 \pm 100\,\mathrm{MeV} \\ m_\pi &= 137\,\mathrm{MeV} \\ F_\pi &= 131\,\mathrm{MeV} \end{aligned} \tag{7}$$

Because $m[\pi(1300)]$ has such a large uncertainty, we will, as previously, examine predictions depending on the choice of this mass within its experimental range. The sixth input will be taken as the light "quark mass ratio" A_3/A_1, which will be varied over an appropriate range. The remaining two inputs will be taken from the masses of the four (mixing) isoscalar, pseudoscalar mesons. This mixing is characterized by a 4×4 matrix M_η^2. A practically convenient choice is to consider $\mathrm{Tr}M_\eta^2$ and $\det M_\eta^2$ as the inputs.

Given these inputs there are a very large number of predictions. At the level of the quadratic terms in the Lagrangian, we predict all the remaining masses and decay constants as well as the angles describing the mixing between each of (π, π'), (K, K'), (a_0, a_0'), (κ, κ') multiplets and each of the 4×4 isosinglet mixing matrices (each formally described by six angles).

In the case of the I=0 scalars there are four particles which mix with each other; the squared mass matrix then takes the form:

$$\left(X_0^2\right) = \begin{bmatrix} 4\,e_3^a\,\beta_3 - 2\,c_2 + 12\,c_4^a\,\alpha_1^2 & 4\sqrt{2}\,e_3^a\,\beta_1 & 4\,e_3^a\,\alpha_3 & 4\sqrt{2}\,e_3^a\,\alpha_1 \\ 4\sqrt{2}\,e_3^a\,\beta_1 & -2\,c_2 + 12\,c_4^a\,\alpha_3^2 & 4\sqrt{2}\,e_3^a\,\alpha_1 & 0 \\ 4\,e_3^a\,\alpha_3 & 4\sqrt{2}\,e_3^a\,\alpha_1 & 2\,d_2 & 0 \\ 4\sqrt{2}\,e_3^a\,\alpha_1 & 0 & 0 & 2\,d_2 \end{bmatrix} \tag{8}$$

For this matrix the basis states are consecutively,

$$\begin{aligned} f_a &= \frac{S_1^1 + S_2^2}{\sqrt{2}} \qquad n\bar{n}, \\ f_b &= S_3^3 \qquad s\bar{s}, \\ f_c &= \frac{S_1'^1 + S_2'^2}{\sqrt{2}} \qquad ns\bar{n}\bar{s}, \\ f_d &= S_3'^3 \qquad nn\bar{n}\bar{n}. \end{aligned} \tag{9}$$

The non-strange (n) and strange (s) quark content for each basis state has been listed at the end of each line above.

III. PION SCATTERING AMPLITUDE

Some initial discussion of the pion scattering in this model was given in refs. [28] and [29]. The tree level $\pi\pi$ scattering amplitude is:

$$A(s,t,u) = -\frac{g}{2} + \sum_i \frac{g_i^2}{m_i^2 - s}, \tag{10}$$

where the four point coupling constant is related to the "bare" four-point couplings as:

$$g = \left\langle \frac{\partial^4 V}{\partial\pi^+\,\partial\pi^-\,\partial\pi^+\,\partial\pi^-} \right\rangle = \sum_{A,B,C,D} \left\langle \frac{\partial^4 V}{\partial(\phi_1^2)_A\,\partial(\phi_2^1)_B\,\partial(\phi_1^2)_C\,\partial(\phi_2^1)_D} \right\rangle (R_\pi)_{A1}\,(R_\pi)_{B1}\,(R_\pi)_{C1}\,(R_\pi)_{D1} \tag{11}$$

where the sum is over "bare" pions and $A, B, \cdots = 1, 2$ with 1 denoting nonet M and 2 denoting nonet M'. R_π is the pion rotation matrix (given, for typical parameters in [32].

The physical scalar-pseudoscalar-pseudoscalar couplings are related to the bare couplings:

$$g_i = \left\langle \frac{\partial^3 V}{\partial f_i\,\partial\pi^+\,\partial\pi^-} \right\rangle = \sum_{M,A,B} \left\langle \frac{\partial^3 V}{\partial f_M \partial(\phi_1^2)_A\,\partial(\phi_2^1)_B} \right\rangle (L_0)_{Mi}\,(R_\pi)_{A1}\,(R_\pi)_{B1} \tag{12}$$

where A and B = 1,2 and M=1,2,3 and 4 and respectively represent the four bases in Eq. (9). L_0 is the isosinglet scalar rotation matrix.

The only non-vanishing "bare" four-point and three-point couplings are:

$$\left\langle \frac{\partial^4 V}{\partial(\phi_1^2)_1\,\partial(\phi_2^1)_1\,\partial(\phi_1^2)_1\,\partial(\phi_2^1)_1} \right\rangle = 8\,c_4^a \tag{13}$$

$$\begin{aligned} \left\langle \frac{\partial^3 V}{\partial f_a\,\partial(\phi_1^2)_1\,\partial(\phi_2^1)_1} \right\rangle &= 4\sqrt{2}\,c_4^a\,\alpha_1 \\ \left\langle \frac{\partial^3 V}{\partial f_b\,\partial(\phi_1^2)_1\,\partial(\phi_2^1)_2} \right\rangle &= \left\langle \frac{\partial^3 V}{\partial f_b\,\partial(\phi_1^2)_2\,\partial(\phi_2^1)_1} \right\rangle = \left\langle \frac{\partial^3 V}{\partial f_c\,\partial(\phi_1^2)_1\,\partial(\phi_2^1)_1} \right\rangle = 4\,e_3^a \end{aligned} \tag{14}$$

Now we project Eq. (10) to the I=J=0 partial wave amplitude. The K-matrix unitarization of this "Born" scattering amplitude ${T_0^0}^B$ defines the unitary partial wave amplitude

$$T_0^0 = \frac{{T_0^0}^B}{1 - i\,{T_0^0}^B} \tag{15}$$

wherein:

$$ {T_0^0}^B = T_\alpha + \sum_i \frac{T_\beta^i}{m_i^2 - s} \tag{16}$$

with:

$$T_\alpha = \frac{1}{64\pi}\sqrt{1 - \frac{4m_\pi^2}{s}}\left[-5g_4 + \frac{1}{p_\pi^2}\sum_i g_i^2 \ln\left(1 + \frac{4p_\pi^2}{m_i^2}\right)\right] \tag{17}$$

$$T_\beta^i = \frac{3}{32\pi}\sqrt{1 - \frac{4m_\pi^2}{s}}\, g_i^2 \tag{18}$$

$$p_\pi = \frac{1}{2}\sqrt{s - 4m_\pi^2} \tag{19}$$

IV. COMPARISON WITH EXPERIMENT

For comparison with experiment it is convenient to focus on the real part of the partial wave scattering amplitude in Eq. (15). For typical values of parameters we find the behavior illustrated in Fig. (1). The zeros which occur can be understood as follows. First, they can result from a zero of T_0^{0B}. Such a zero occurs at threshold, for example. Secondly, a zero can also result from the poles in T_0^{0B} at $s = m_i^2$ in Eq.(16) corresponding to the "bare" masses.

We compare the predictions of our model for the scattering amplitude with the corresponding experimental data up to about 1.2 GeV in Fig 2 for two values of the SU(3) symmetry breaking parameter A_3/A_1 and three choices of the only roughly known "heavy pion" mass $m[\Pi(1300)]$. One sees that, without using any new parameters, the mixing mechanism of [32] predicts the scattering amplitude in reasonable qualitative agreement with the experimental data up to around 1 GeV. This provides some support for the validity of this mixing mechanism.

For interpretation of the physical resonances it is conventional to look at the pole positions in the complex plane of the analytically continued expression for T_0^0. We examine these physical pole positions by solving for the complex roots of the denominator of the K-matrix unitarized amplitude Eq. (15):

$$\mathcal{D}(s) = 1 - i\,T_0^{0B} = 0 \tag{20}$$

with T_0^{0B} given by Eq. (16). We search for solutions, $s^{(j)} = s_r^{(j)} + i s_i^{(j)} = m_j^2 - i m_j \Gamma_j$ of this equation, where m_j and Γ_j are interpreted as the mass and decay width of the j-th physical resonance (which would hold for small Γ). A first natural attempt would be to simultaneously solve the two equations:

$$\begin{aligned} \mathrm{Re}\mathcal{D}\,(s_r, s_i) &= 0 \\ \mathrm{Im}\mathcal{D}\,(s_r, s_i) &= 0 \end{aligned} \tag{21}$$

However, this approach turns out to be rather complicated to be implement. A more efficient numerical approach is to consider the single equation involving only positive quantities:

$$\mathcal{F}\,(s_r, s_i) = |\mathrm{Re}\,(\mathcal{D}(s_r, s_i))| + |\mathrm{Im}\,(\mathcal{D}(s_r, s_i))| = 0 \tag{22}$$

A search of parameter space leads to four solutions for the pole positions [1]. As an example, for the choice of A_3/A_1 = 30 and $m[\Pi(1300)] = 1.215$ GeV, the function $\mathcal{F}$ is plotted over the complex plane around the first pole. We see a clear local minimum at which the function is zero, hence pointing to a solution of Eq. (20). Similarly, other areas of the complex plane are searched and altogether four poles are found. The results are given in Table I for $m[\Pi(1300)] =$ 1.215 GeV and two choices of $A_3/A_1 = 20$ and 30. For each choice we see that this model predicts a light and broad

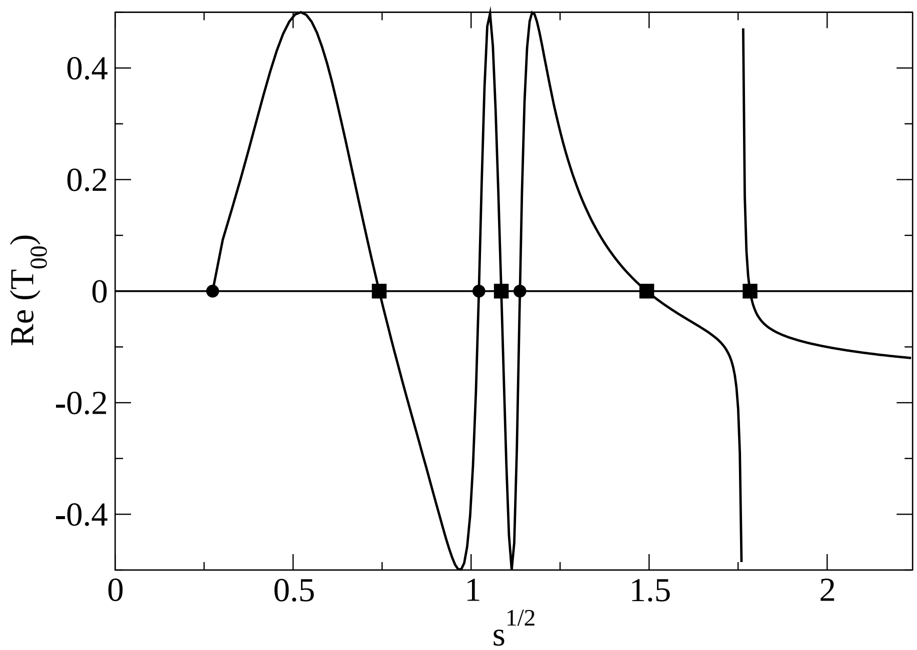


FIG. 1: The real part of the unitarized $\pi\pi$ scattering amplitude for a typical choice of parameters. The four squares correspond to the poles in T_0^{0B}.The circles correspond to locations where $T_0^{0B} = 0$.

scalar meson below 1 GeV which is a clear indication of $f_0(600)$ or σ. We see that the characteristics of the second predicted state around 1 GeV are close to those expected for $f_0(980)$. The third and the fourth predicted states should correspond to two of $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$.

We have performed the same analysis over the range of the parameter $m[\Pi(1300)] = 1.2$ - 1.4 GeV, and for two choices of $A_3/A_1 = 20$ and 30. The physical masses and the decay widths are given in Figs. 4 and 5, respectively. The effect of the unitarization can be seen in Fig. 4 where the physical masses are compared with the "bare" masses; the unitarization reduces the mass, particularly for the first and the third predicted states.

Pole	Mass (MeV)	Width (MeV)	Mass (MeV)	Width (MeV)
1	483	455	477	504
2	1012	154	1037	84
3	1082	35	1127	64
4	1663	2.1	1735	3.5

TABLE I: The physical mass and decay width of the isosinglet scalar states, with $m[\Pi(1300)] = 1.215$ GeV and with $A_3/A_1 = 20$ (the first two columns) and with $A_3/A_1 = 30$ (the last two columns).

V. CONCLUDING DISCUSSION

We studied the predictions of the real (mass) and imaginary (width) parts of the pion scattering amplitude poles representing the isoscalar scalar singlets in a chiral model containing not only the usual pseudoscalar and scalar nonets describing quark- antiquark bound states but also pseudoscalar and scalar nonets describing states with the same quantum numbers but constructed out of two quarks and two antiquarks in a general way. The physical particles correspond to mixtures of these two types.

In this model there are four scalars so the process is technically complicated. No new parameters were introduced

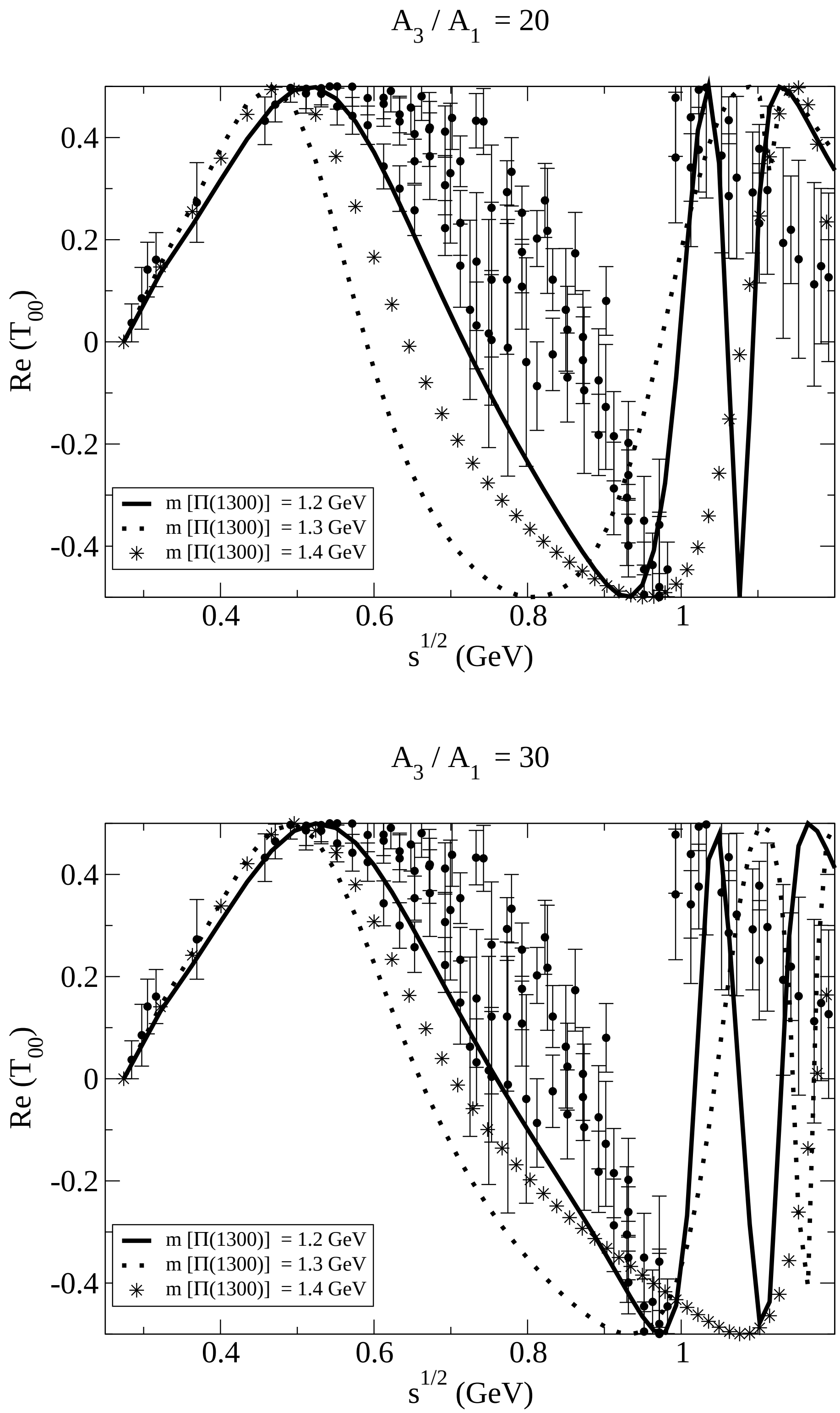


FIG. 2: Real part of unitarized scattering amplitude for two values of A_3/A_1 and three choices of $m[\Pi(1300)]$.

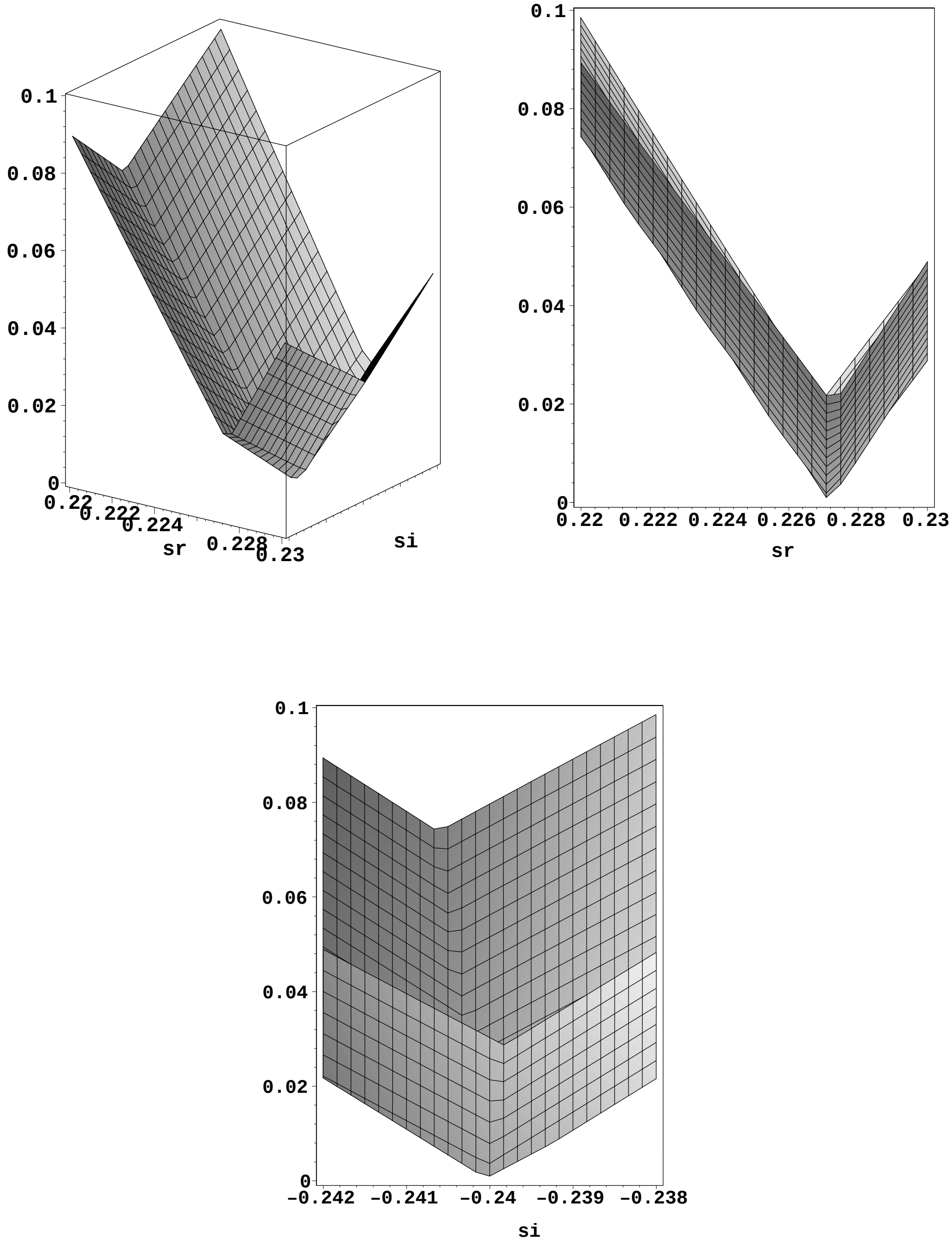
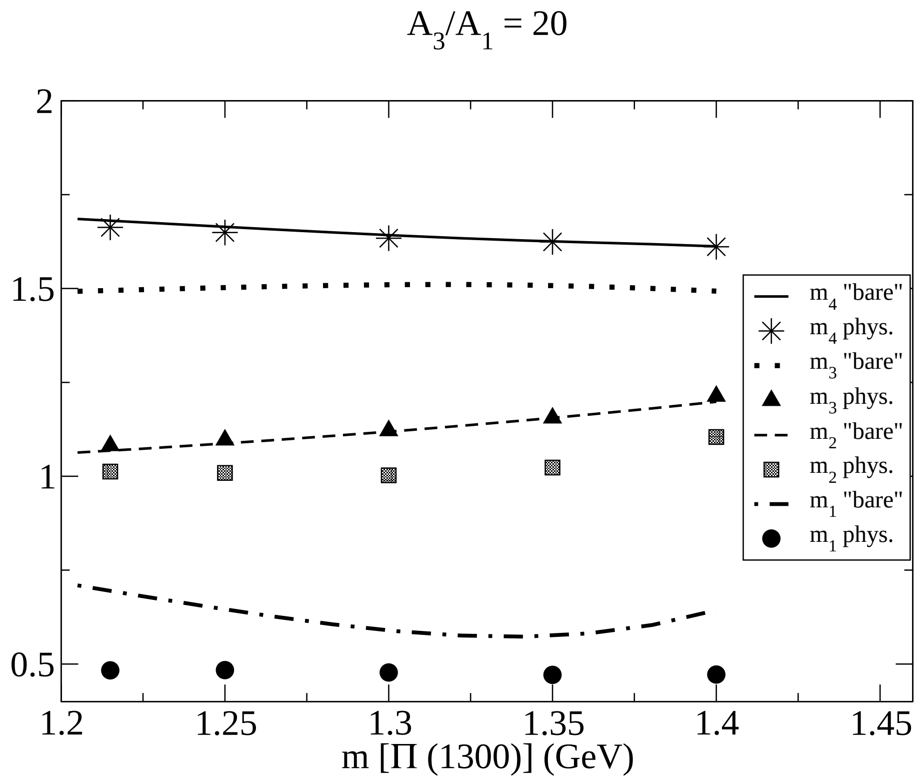


FIG. 3: The local minimum of function $\mathcal{F}(s_R, s_I)$ defined in Eq. (22) at the position of the lightest isosinglet scalar pole in the complex s plane for $m[\Pi(1300)] = 1.215$ GeV and $A_3/A_1 = 30$. Top left is the plot of function $\mathcal{F}(s_R, s_I)$ vs s_R and s_I, followed by projection of this function onto $\mathcal{F}$-s_R and onto $\mathcal{F}$-s_I planes.

here, either for the model itself or to treat the scattering. The model has been studied in a number of previous papers with the parameter determination culminating in ref. [32].

The fact that the comparison with the experimental scalar candidates, as discussed in the last section, is reasonable is in itself a non trivial conclusion. A numerical technique to facilitate this result was presented in the last section. Also the fact that the simple single channel K-matrix unitarization (using no new parameters) seems to work may be useful to point out. Presumably the results would be improved if the effect of the K - $\bar{K}$ channel were to be included. Mixing with a possible glueball state is another relevant effect.



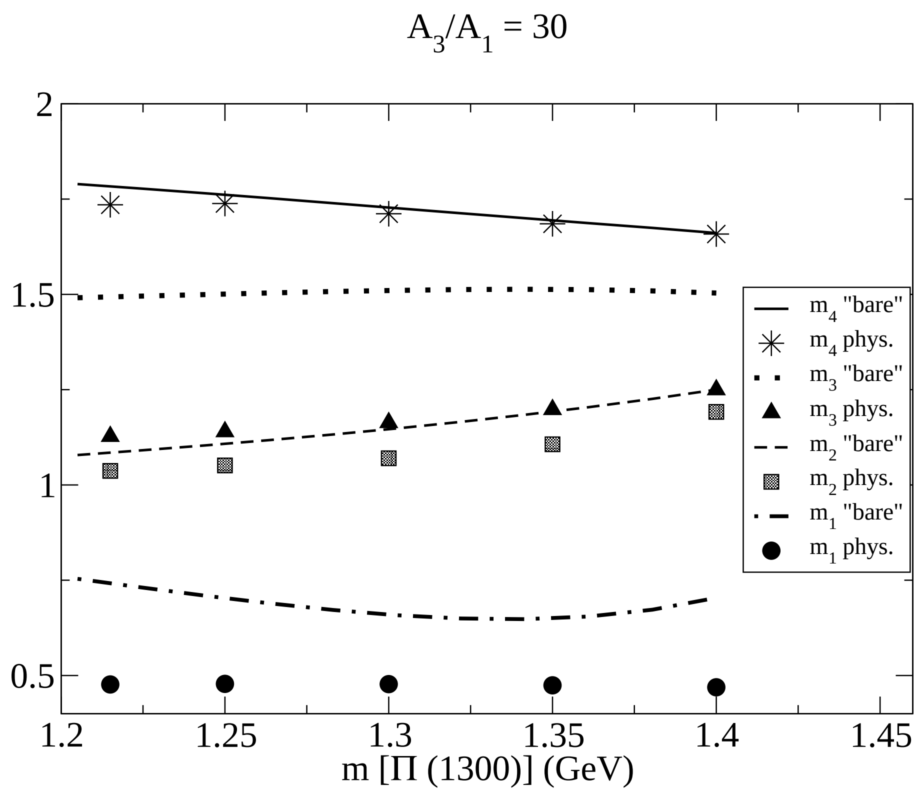


FIG. 4: Predicted physical masses are compared with the "bare" masses for two values of A_3/A_1 over the experimental range of $m[\Pi(1300)]$.

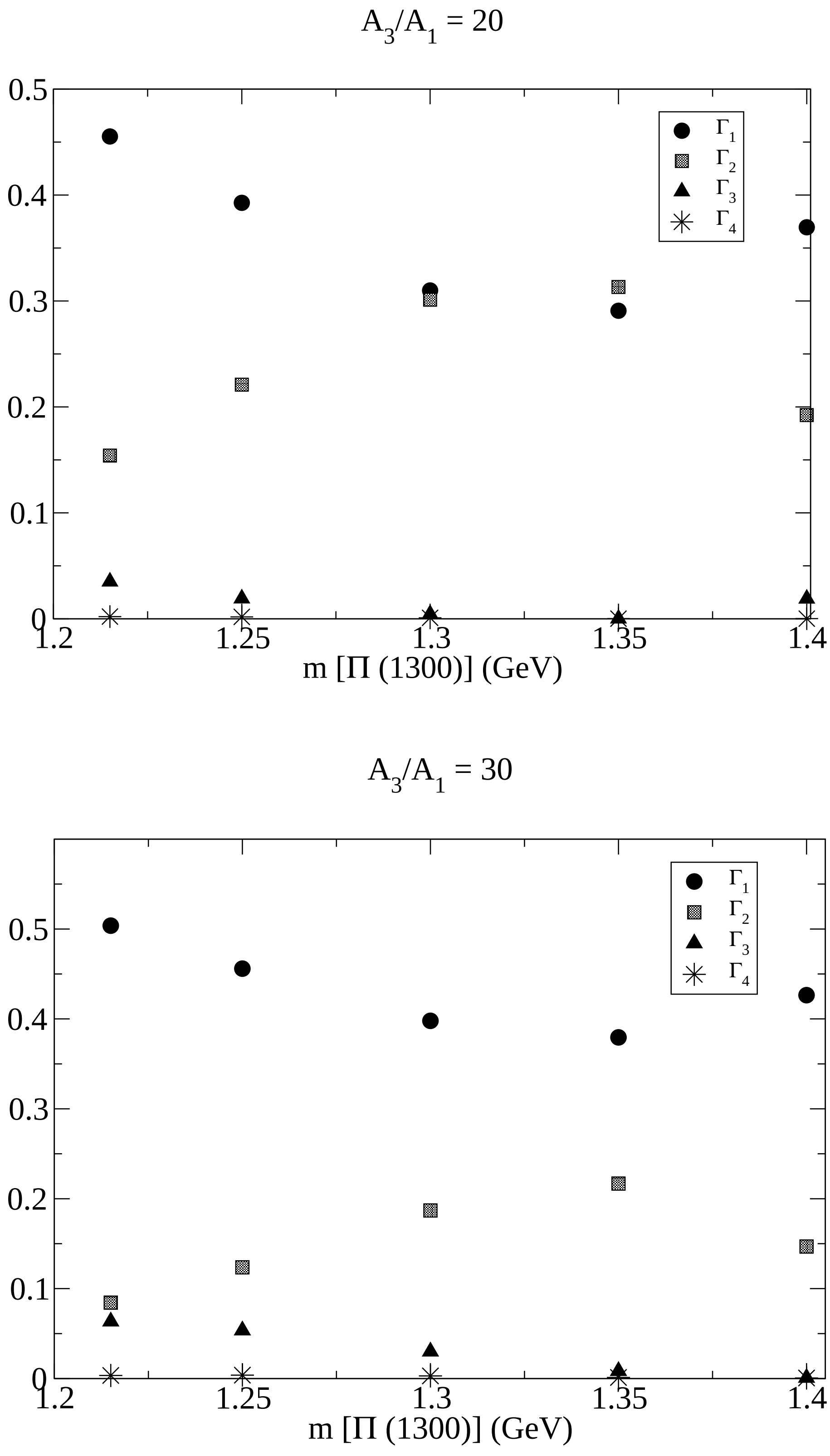


FIG. 5: Predicted decay widths for two values of A_3/A_1 over the experimental range of $m[\Pi(1300)]$.

The worst prediction seems to be the too low mass value for pole 3. We note from Fig. 4 that there is a relatively large difference between the "bare" mass and the pole mass in this case. The inclusion of the K - $\bar{K}$ threshold effects may improve this feature.

It may also be interesting to compare the predictions of pole 1 and pole 2 with those calculated in a similar manner using the single M SU(3) sigma model [22]. The agreement is quite good. However, in that model, the result was calculated using the most general form of the interaction potential involving the field matrix M; an attempt to just use the "renormalizable" terms did not give as good a result. In the present case it was not necessary to introduce any additional terms in the Lagrangian to get good results for the pi pi scattering.

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[1] We have double checked the results by developing an approximate analytical approach in which the amplitude is unitarized locally in the neighbourhood of each resonance.
[2] See p. 630 of K. Nakamura et. al.(Particle Data Group), J. Phys. G **37**, 075021 (2010).
[3] E. van Beveren, T.A. Rijken, K. Metzger, C. Dullemond, G. Rupp and J.E. Ribeiro, Z. Phys. **C30**, 615 (1986).
[4] D. Morgan and M. Pennington, Phys. Rev. **D48**, 1185 (1993).
[5] A.A. Bolokhov, A.N. Manashov, M.V. Polyakov and V.V. Vereshagin, Phys. Rev. **D48**, 3090 (1993).
[6] N.N. Achasov and G.N. Shestakov, Phys. Rev. **D49**, 5779 (1994). A summary of the recent work of the Novosibirsk group is given in N.N. Achasov, arXiv:0810.2601[hep-ph].
[7] R. Kamínski, L. Leśniak and J. P. Maillet, Phys. Rev. **D50**, 3145 (1994).
[8] F. Sannino and J. Schechter, Phys. Rev. **D52**, 96 (1995).
[9] G. Janssen, B.C. Pearce, K. Holinde and J. Speth, Phys. Rev. **D52**, 2690 (1995).
[10] R. Delbourgo and M.D. Scadron, Mod. Phys. Lett. **A10**, 251 (1995).
[11] N.A. Törnqvist and M. Roos, Phys. Rev. Lett. **76**, 1575 (1996).
[12] M. Svec, Phys. Rev. **D53**, 2343 (1996).
[13] S. Ishida, M.Y. Ishida, H. Takahashi, T. Ishida, K. Takamatsu and T Tsuru, Prog. Theor. Phys. **95**, 745 (1996).
[14] M. Harada, F. Sannino and J. Schechter, Phys. Rev. **D54**, 1991 (1996).
[15] A.V. Anisovich and A.V. Sarantsev, Phys. Lett. **B413**, 137 (1997).
[16] D. Black, A.H. Fariborz, F. Sannino and J. Schechter, Phys. Rev. **D58**, 054012 (1998).
[17] J.A. Oller, E. Oset and J.R. Pelaez, Phys. Rev. Lett. **80**, 3452 (1998).
[18] V. Elias, A.H. Fariborz, Fang Shi and T.G. Steele, Nucl. Phys. **A633**, 279 (1998).
[19] K. Igi and K. Hikasa, Phys. Rev. **D59**, 034005 (1999).
[20] D. Black, A.H. Fariborz, F. Sannino and J. Schechter, Phys. Rev. **D59**, 074026 (1999).
[21] D. Black, A. H. Fariborz and J. Schechter, Phys. Rev. D **61** 074001 (2000).
[22] See especially Sec. V of D. Black, A.H. Fariborz, S. Moussa, S. Nasri and J. Schechter, Phys. Rev. D **64**, 014031 (2001).
[23] T. Teshima, I. Kitamura and N. Morisita, J. Phys. G **28**, 1391 (2002); *ibid* **30**, 663 (2004); F. Close and N. Tornqvist, *ibid.* **28**, R249 (2002); A.H. Fariborz, Int. J. Mod. Phys. A **19**, 2095 (2004); 5417 (2004); Phys. Rev. D **74**, 054030 (2006);
[24] M. Napsuciale and S. Rodriguez, Phys. Rev. D **70**, 094043 (2004); F. Giacosa, Th. Gutsche, V.E. Lyubovitskij and A. Faessler, Phys. Lett. B **622**, 277 (2005); J. Vijande, A. Valcarce, F. Fernandez and B. Silvestre-Brac, Phys. Rev. D **72**, 034025 (2005); S. Narison, Phys. Rev. D **73**, 114024 (2006); L. Maiani, F. Piccinini, A.D. Polosa and V. Riquer, hep-ph/0604018; J.R. Pelaez, Phys. Rev. Lett. **92**, 102001 (2004); J.R. Pelaez and G. Rios, Phys. Rev. Lett. **97**, 242002 (2006); F. Giacosa, Phys. Rev. D **75**,054007 (2007).
[25] G. 't Hooft, G. Isidori, L. Maiani, A.D. Polosa and V. Riquer, arXiv:0801.2288[hep-ph].
[26] Related models for thermodynamic properties of QCD are N. Yamamoto, M. Tachibana, T. Hatsuda and G. Baym, Phys. Rev. D **76**, 074001 (2007) and A.A. Andrianov and D. Espriu, arXiv:0709.0049 [hep-ph].
[27] A.H. Fariborz, R. Jora and J. Schechter, Phys. Rev. D **72**, 034001 (2005).
[28] A.H. Fariborz, R. Jora and J. Schechter, Phys. Rev. D **76**, 014011 (2007).
[29] A.H. Fariborz, R. Jora and J. Schechter, Phys. Rev. D **77**, 034006 (2008), arXiv:0707.0843 [hep-ph]. Note that some corrections to [27] above are given in Appendix A of this reference.
[30] A.H. Fariborz, R. Jora and J. Schechter, Phys. Rev. D **76**, 114001 (2007), arXiv:0708.3402 [hep-ph].
[31] A.H. Fariborz, R. Jora and J. Schechter, Phys. Rev. D **77**, 094004 (2008).
[32] A.H. Fariborz, R. Jora and J. Schechter, Phys. Rev. D **79**, 074014 (2009).
[33] A.H. Fariborz, N.W. Park, J. Schechter and M.N. Shahid, Phys. Rev. D **80**, 113001 (2009).

[34] D. Black, A.H. Fariborz, R. Jora, N.W. Park, J. Schechter and M.N. Shahid, Mod. Phys. Lett. A **28**, 2285 (2009).
[35] A.H. Fariborz, R. Jora, J. Schechter and M.N. Shahid, Phys. Rev. D **83**, 034018 (2011).
[36] I. Caprini, G. Colangelo and H. Leutwyler, Phys. Rev. Lett. **96**, 132001 (2006).
[37] R.L. Jaffe, Phys. Rev. D **15**, 267 (1977). See also J.D. Weinstein and N. Isgur, Phys. Rev. Lett. **48**, 659 (1982).