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Abstract

In this paper, we study the performance limits of recovering the support of a sparse signal based on quantized noisy random projections. Although the problem of support recovery of sparse signals with real valued noisy projections with different types of projection matrices has been addressed by several authors in the recent literature, very few attempts have been made for the same problem with quantized compressive measurements. In this paper, we derive performance limits of support recovery of sparse signals when the quantized noisy corrupted compressive measurements are sent to the decoder over additive white Gaussian noise channels. The sufficient conditions which ensure the perfect recovery of sparsity pattern of a sparse signal from coarsely quantized noisy random projections are derived when the maximum likelihood decoder is used. More specifically, we find the relationships among the parameters, namely the signal dimension N, the sparsity index K, the number of noisy projections M, the number of quantization levels L, and measurement signal-to-noise ratio which ensure the asymptotic reliable recovery of the support of sparse signals when the entries of the measurement matrix are drawn from a Gaussian ensemble.

I. INTRODUCTION

Support recovery of a sparse signal is concerned with finding the locations of the non-zero elements of the sparse signal. The problem of sparsity pattern recovery arises in a wide variety of areas including source localization [1], [2], sparse approximation [3], subset selection in linear regression [4], [5], estimation of frequency band locations in cognitive radio networks [6], and signal denoising [7]. In these applications, finding the support of the signal is more important than approximating the signal itself. This problem has been addressed by many authors in the last few decades in different contexts. With the recently introduced sparse signal acquisition scheme via random projections, called *Compressed Sensing*, the problem of support recovery of sparse signals has received much attention in the context of random dictionaries.

Compressed Sensing (CS) is a new paradigm which enables the reconstruction of compressible or sparse signals with far fewer samples using a universal sampling procedure compared to that with traditional sampling methods [8]–[10]. In this framework, a small collection of linear random projections of a sparse signal contains sufficient information for signal recovery. The basic CS theory has been described in [8], [11], [12]. The authors have shown that if a signal has a sparse representation in a particular basis Ψ, then it can be recovered by a small number of projections to another basis Φ that is *incoherent* with the first. When using random projections, CS is universal in the sense that the same mechanism in acquiring measurements can be used irrespective of the sparsity level or the basis in which the signal is sparse. Development of computationally efficient and tractable algorithms to recover the sparse signals based on CS measurement scheme has been considered by many researchers [13]–[21]. The performance of the sparse signal reconstruction with *noisy* compressed measurements was also addressed by several authors in [22], [23].

A. Related work

As mentioned earlier, the problem of support recovery of sparse signals is important in many applications. Performance limits on reliable recovery of the support of sparse signals with real valued noisy corrupted compressive measurements have been addressed by several authors in recent research exploiting information theoretic tools [24]–[31]. Most of these works mainly focus on deriving sufficient and necessary conditions for the reliable support recovery, and discussing and quantifying the gap between the performance limits of existing computationally tractable practical algorithms (such as Least-Absolute Shrinkage and Selection Operator (LASSO) and Orthogonal matching pursuit (OMP)) and that can be achieved based on the optimal ML decoding algorithm if the computational constraints are removed.

Although most of the CS literature has focused on sparse signal reconstruction and/or support recovery from real valued compressive measurements, it is important to consider quantization of compressive measurements since in practice, measurements are quantized before transmission or storage. There are some recent works that have addressed the problem of recovering sparse signals based on quantized compressive measurements in different contexts [31]–[39]. In particular, the works [32], [36], [37], [39] consider the support recovery based on 1-bit quantized compressive measurements where the authors have proposed several computationally tractable algorithms to recover the sparsity pattern. In [33], [34], the effect of quantization of CS measurements is further addressed in terms of average distortion and quantization error, respectively. In [38], the design of quantizers for random measurements of sparse signals that are optimal with respect to mean-squared error of the LASSO reconstruction was discussed. However, the performance limits obtainable with quantized compressive measurements if the computational constraints are removed have not been adequately addressed in the CS literature.

B. Our contribution and the summary of main results

In this paper, we analyze the performance limits of support recovery of sparse signals based on quantized compressive measurements when the computational constraints are removed. The noisy corrupted compressive measurements are assumed to be quantized before transmitting or further processing. Our goal is to establish relationships among the parameters, namely the signal dimension N, sparsity index K, the number of compressive measurements M, the number of quantization levels L, and the measurement signal-to-noise ratio of the signal that ensure the asymptotic reliable recovery of the support of the sparse signal with quantized (noisy corrupted) compressive measurements. Our analyses are based on the assumptions that the model parameters K, N and M are large and may tend to infinity. The entries of the measurement matrix are assumed to be drawn from a Gaussian ensemble. More specifically, we derive the lower bounds on the number of quantized compressive measurements required to reliably recover the sparsity pattern of sparse signals with maximum likelihood (ML) estimator.

Although this work can be considered as an extension of the problem of support recovery of sparse signals with real valued compressive measurements in [24] for the case with quantized compressive measurements, there are some other differences in the problem formulation. In [24], the author has assumed that the sparse signal is unknown and deterministic, while in this paper the sparse signal is assumed to be random with known statistics. In [30], the performance of sparse support recovery was addressed when the non zero elements of the sparse signal are modeled as random, but those analyses are with real valued compressive measurements and for a given (deterministic) measurement matrix.

The main results in the paper can be summarized as given below.

1) In Theorem 1, a lower bound on the number of measurements required for reliable recovery of sparsity pattern with L-level quantized compressive measurements is derived when the noise power at the decoder is negligible for given values of measurement signal-to-noise ratio and threshold for the quantizer. In this case,

    a) in the regime of linear sparsity, it is sufficient to have $\Omega(N)$ quantized compressive measurements for reliable recovery of sparsity pattern with L-level quantized compressive measurements. According to the results presented in [24] it can be seen that this is the same order as with the real valued measurements. The differentiating factor which depends on the number of quantization levels, threshold values of the quantizer, the measurement signal-to-noise ratio is analyzed explicitly.

2) Theorem 2 presents a lower bound on the minimum number of measurements needed for support recovery with 1-bit quantized CS when the noise power at the decoder is negligible. In this case,

    a) the scaling with respect to K and N is similar to that obtained for L-level quantized measure-

ments as stated above.

  b) when the measurement noise power is also negligible, the minimum number of measurements required for reliable recovery of the support scales as $\Omega\left(K\log\frac{N}{K}\right)$ irrespective of the behavior of the sparsity pattern.

3) Theorem 3 presents necessary conditions for reliable support recovery based on 1-bit quantized CS with *any* classification rule. The necessary conditions, which state the minimum number of measurements required to reliably recover the sparsity pattern with any sparsity pattern recovery algorithm, derived in the paper for 1-bit quantized CS, show similar behavior as in the case of real valued CS [24]. However, the way of dependence of the measurement SNR on the minimum number of measurements is problem specific.

From the results derived in the paper, it is seen that the minimum number of measurements required for reliable recovery of sparsity pattern of sparse signals with quantized measurements scales in a similar manner (with respect to K and N) as that with real valued observations under certain asymptotic and sparsity regimes. The exact value of the lower bound differs in two cases by terms that depend on the parameters including measurement SNR, number of quantization levels, the values of quantization thresholds, etc.. We explicitly derive these terms and show how it affects the lower bound on the number of measurements required for reliable support recovery. In particular, it can be seen that the measurement SNR plays an important role in recovering the sparsity pattern with the ML decoder with coarsely quantized compressive measurements. As observed with real valued compressive measurements in [24], as the measurement SNR increases, we could numerically show that the minimum number of compressive measurements required to recover the sparsity pattern reliably with 1-bit quantized CS based on ML decoder is ultimately limited only by the sparsity index K, the signal dimension N and the statistical properties of the projection matrix.

C. Organization of the rest of the paper and notations

The rest of the paper is organized as follows. Section II presents the observation model and the problem formulation. In Section III, performance of the ML decoder with L-level quantized measurements is analyzed. The sufficient conditions that should be satisfied by the number of measurements for reliable sparsity pattern recovery with 1-bit compressive measurements with ML decoder are derived in Section III-D. The necessary conditions on the number of measurements that should be satisfied by *any* recovering algorithms are discussed in Section IV. Numerical results are presented in Section V. Concluding remarks are given in Section VI while proofs of some theorems and lemmas are given in Appendices.

Through out the rest of the paper, the expression $f(n) = \Omega(g(n))$ denotes $f(n) \geq g(n).k$ for some

positive k as $n \to \infty$. $f(n) = o(g(n))$ denotes f is dominated by g asymptotically where $|f(n)| \leq |g(n)|.\epsilon$ for all ϵ $(0 < \epsilon < 1)$ as $n \to \infty$.

II. OBSERVATION MODEL AND PROBLEM FORMULATION

A. Observation model

Consider the following $M \times 1$ real valued observation vector collected via random projections:

$$\mathbf{y} = \mathbf{\Phi s} + \mathbf{v} \tag{1}$$

where $\mathbf{\Phi}$ is the $M \times N$ $(M < N)$ measurement matrix in which the entries are assumed to be drawn from a Gaussian ensemble, $\mathbf{s} \in \mathbb{R}^N$ is the sparse signal vector of interest with only K $(K \ll N)$ non-zero elements, N is the signal dimension, $\mathbf{v}$ is the $M \times 1$ measurement noise vector which is assumed to be iid Gaussian with $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \sigma_v^2 \mathbf{I}_M)$ and $\mathbf{I}_M$ is the $M \times M$ identity matrix. For many applications, it may be required to quantize the compressive measurements before further processing or transmission. Further, if the measurements are quantized very coarsely, for example to 1-bit, the sparse signal processing is performed with access to only the sign information of the compressive measurements which reduces the sampling complexity. In the following, we assume that the measurement vector (1) is quantized element-wise into one of L levels which requires $B = \lceil \log_2 L \rceil$ bits per measurement:

$$z_i = \begin{cases} 0, & if\ \tau_0 \leq y_i < \tau_1 \\ 1, & if\ \tau_1 \leq y_i < \tau_2 \\ \quad . \\ \quad . \\ L-1, & if\ \tau_{L-1} \leq y_i < \tau_L \end{cases} \tag{2}$$

for $i = 1, 2, \cdots, M$, where $\tau_0, \tau_1, \cdots, \tau_L$ represent quantizer thresholds with $\tau_0 = -\infty$ and $\tau_L = \infty$. In many communication systems, once the observation vector is obtained, it is sent to a destination node via a communication channel for further processing. We assume that the quantized measurements are sent to the decoder over noisy communication channels. The received observation at the decoder is given by,

$$r_i = z_i + w_i,\ for\ i = 1, 2, \cdots, M \tag{3}$$

where w_i is the decoder noise which is assumed to be iid Gaussian with mean zero and variance σ_w^2. Assume that it is required to recover the support of the sparse signal $\mathbf{s}$ based on the $M \times 1$ observation vector $\mathbf{r} = [r_1, r_2, \cdots, r_M]^T$.

B. Error metric and the ML decoder

Define the support set of the signal s as $\mathcal{U} := \{i \in \{1, 2, \cdots, N\} \mid \mathbf{s}(i) \neq 0\}$ where $\mathbf{s}(i)$ is the i-th element of s for $i = 1, 2, \cdots, N$. Then we have $K = |\mathcal{U}|$ where $|.|$ denotes the cardinality of the corresponding set. In this paper, we consider the probability of error as the performance metric while recovering the support of the sparse signal s. Assuming that the support of the sparse signal is randomly and uniformly selected over $\binom{N}{K}$ possible subsets of size K, the probability of error for any decoder ϕ that maps the M-length quantized observation vector r to an estimated support $\hat{\mathcal{U}}$ is given by,

$$P_{err} = \frac{1}{N_0} \sum_{k=0}^{N_0-1} Pr(\phi(\mathbf{r}) \neq \mathcal{U}_k | \mathcal{U}_k)$$

where $N_0 = \binom{N}{K}$ and $Pr(\phi(\mathbf{r}) \neq \mathcal{U}_k | \mathcal{U}_k)$ is the probability of not selecting the support $\mathcal{U}_k$ by the decoder when the true support is $\mathcal{U}_k$.

Let s̃ denote the $K \times 1$ vector of the non-zero elements of s. In this paper, we assume that $\tilde{\mathbf{s}} \sim \mathcal{N}(\boldsymbol{\mu}, \sigma_s^2 \mathbf{I}_K)$ with $\boldsymbol{\mu} \neq [0, 0, \cdots, 0]^T$. The *maximum likelihood decoder* selects the true support of the signal s as,

$$\hat{\mathcal{U}} = \underset{\mathcal{U}_k, k=0,1,\cdots,N_0-1}{\operatorname{argmax}} p(\mathbf{r}|\mathcal{U}_k) \tag{4}$$

where $p(\mathbf{r}|\mathcal{U}_k)$ is the pdf of the observation vector r given the support $\mathcal{U}_k$ where $\mathcal{U}_k \subset \{1, 2, \cdots, N\}$ with $|\mathcal{U}_k| = K$.

III. Sufficient Conditions for Support Recovery of Sparse Signals with Quantized CS via ML Decoder

The exact analysis of the probability of error of the ML decoder (4) with the observation model (3) is difficult in general. Thus we consider an upper bound for the probability of error based on union and Chernoff bounds. Assuming the support sets are uniformly distributed, the probability of error of the ML decoder is upper bounded by,

$$
\begin{aligned}
P_{err} &= \frac{1}{N_0} \sum_{k=0}^{N_0-1} Pr(\hat{\mathcal{U}} \neq \mathcal{U}_k | \mathcal{U}_k) \\
&\leq \frac{1}{N_0} \sum_{k=0}^{N_0-1} \sum_{j=0, j \neq k}^{N_0-1} Pr(\hat{\mathcal{U}} = \mathcal{U}_k | \mathcal{U}_j) \\
&\leq \frac{1}{2N_0} \sum_{k=0}^{N_0-1} \sum_{j=0, j \neq k}^{N_0-1} \exp(-\hat{\mathcal{C}}(\alpha_0; p_k, p_j))
\end{aligned}
$$

where $\hat{\mathcal{C}}(\alpha_0; p_k, p_j) = \max_{0 \leq \alpha \leq 1} \mathcal{C}(\alpha; p_k, p_j)$ with $\mathcal{C}(\alpha; p_k, p_j)$ is the Chernoff distance between the two distributions $p(\mathbf{r}|\mathcal{U}_k)$ and $p(\mathbf{r}|\mathcal{U}_j)$ which is defined as,

$$\mathcal{C}(\alpha; p_k, p_j) = -\log\left\{\tilde{\mathcal{C}}(\alpha; p_k, p_j)\right\}. \tag{5}$$

where $\tilde{\mathcal{C}}(\alpha; p_k, p_j) = \int p(\mathbf{r}|\mathcal{U}_k)^{1-\alpha} p(\mathbf{r}|\mathcal{U}_j)^\alpha d\mathbf{r}$. For a given measurement matrix $\mathbf{\Phi}$, the joint pdf of the observation vector $\mathbf{r}$ given the support $\mathcal{U}_k$ can be approximated by the following in the high dimensional setting (derivation is given in Appendix A);

$$p(\mathbf{r}|\mathcal{U}_k) \rightarrow \prod_{i=1}^{M} \sum_{l=0}^{L-1} \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ik}^l - \lambda_{ik}^{l+1}] \tag{6}$$

where $\lambda_{ik}^l = Q\left(\dfrac{\tau_l - \sum_{j=1}^{K}\left(\tilde{\mathbf{\Phi}}_{\mathcal{U}_k}\right)_{ij}\mu_j}{\sqrt{\sigma_v^2 + \sigma_s^2 \sum_{j=1}^{K}\left(\tilde{\mathbf{\Phi}}_{\mathcal{U}_k}\right)_{ij}^2}}\right)$, $\tilde{\mathbf{\Phi}}_{\mathcal{U}_k}$ is a $M \times K$ submatrix of $\mathbf{\Phi}$ such that $\tilde{\mathbf{\Phi}}_{\mathcal{U}_k}\tilde{\mathbf{s}} = \mathbf{\Phi}\mathbf{s}$ when the

support of the signal $\mathbf{s}$ is $\mathcal{U}_k$, and $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{\frac{-t^2}{2}} dt$ is the Gaussian Q-function.

Lemma 1: With the observation model (3) at the decoder, the Chernoff distance between two pdfs $p(\mathbf{r}|\mathcal{U}_k)$ and $p(\mathbf{r}|\mathcal{U}_j)$ in (5) (when the parameters N, K, M are sufficiently large) can be approximated by,

$$\mathcal{C}(\alpha; p_k, p_j) \rightarrow -\log \prod_{i=1}^{M} g_i^\alpha(\mathcal{U}_j, \mathcal{U}_k) \tag{7}$$

where

$$g_i^\alpha(\mathcal{U}_j, \mathcal{U}_k) = \sum_{l=0}^{L-1} \int \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ik}^l - \lambda_{ik}^{l+1}] \left(\frac{\sum_{m=0}^{L-1} e^{\frac{mr_i - m^2/2}{\sigma_w^2}}[\lambda_{ij}^m - \lambda_{ij}^{m+1}]}{\sum_{n=0}^{L-1} e^{\frac{nr_i - n^2/2}{\sigma_w^2}}[\lambda_{ik}^n - \lambda_{ik}^{n+1}]}\right)^\alpha dr_i. \tag{8}$$

Proof: See Appendix B. ∎

Since it is hard to obtain an analytically tractable solution for α which maximizes the Chernoff distance, we restrict the discussion for $\alpha = \frac{1}{2}$ which yields the Bhattacharya bound on the corresponding probability of error. Then when the parameters N, K, M are sufficiently large, the probability of error of the ML decoder is upper bounded by,

$$P_{err} \leq \frac{1}{2N_0} \sum_{k=0}^{N_0-1} \sum_{j=0, j \neq k}^{N_0-1} \prod_{i=1}^{M} g_i^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k). \tag{9}$$

A. Further analysis of the bound (9) in the high-dimension with random measurement matrices

In the following, we further analyze the bound (9) in high dimension such that N, K and M are sufficiently large. We further assume that the entries of the measurement matrix $\mathbf{\Phi}$ are iid.

Proposition 1: Assume that the thresholds of the quantization scheme (2) are given by, $\tau_0 < \tau_1 < \cdots < \tau_L$, Then for given support sets $\mathcal{U}_j$ and $\mathcal{U}_k$, for $j \neq k$, $g_i^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k) \geq 0$ for $i = 1, 2, \cdots, M$.

Proof: It is noted that $g_i^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k)$ is given by,

$$g_i^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k) = \sum_{l=0}^{L-1} \int \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ik}^l - \lambda_{ik}^{l+1}] \left(\frac{\sum_{m=0}^{L-1} e^{\frac{mr_i - m^2/2}{\sigma_w^2}}[\lambda_{ij}^m - \lambda_{ij}^{m+1}]}{\sum_{n=0}^{L-1} e^{\frac{nr_i - n^2/2}{\sigma_w^2}}[\lambda_{ik}^n - \lambda_{ik}^{n+1}]} \right)^{\frac{1}{2}} dr_i \tag{10}$$

For a given support $\mathcal{U}_j$ we have,

$$\lambda_{ij}^l - \lambda_{ij}^{l+1} = Q\left(\frac{\tau_l - \sum_{r=1}^{K}\left(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_j}\right)_{ir} \mu_r}{\sigma_v^2 + \sigma_s^2 \sum_{r=1}^{K}\left(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_j}\right)_{ir}^2} \right) - Q\left(\frac{\tau_{l+1} - \sum_{r=1}^{K}\left(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_j}\right)_{ir} \mu_r}{\sigma_v^2 + \sigma_s^2 \sum_{r=1}^{K}\left(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_j}\right)_{ir}^2} \right)$$

Since $\tau_l < \tau_{l+1}$ for $l = 0, 1, \cdots, L - 1$, $Q(x)$ is a monotonically non-increasing function with x and $0 \leq Q(x) \leq 1$ for $x \in (-\infty, \infty)$, we have

$$0 \leq (\lambda_{1j}^l - \lambda_{1j}^{l+1}) \leq 1$$

Then it can be seen that the integrand in (10) is a positive function resulting in $g_i^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k)$ being non negative for given support sets $\mathcal{U}_j$ and $\mathcal{U}_k$ for $j \neq k$ and for $i = 1, 2, \cdots, M$. ∎

Using the fact that the geometric mean of non negative real numbers is always less than or equal to the arithmetic mean, we have

$$\left(\prod_{i=1}^{M} g_i^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k) \right)^{\frac{1}{M}} \leq \frac{1}{M} \sum_{i=1}^{M} g_i^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k). \tag{11}$$

Note that when the entries of the measurement matrix are iid random variables, it can be seen that the elements in the sequence $\{g_i^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k)\}_{i=1}^{M}$ are iid random variables for given support sets $\mathcal{U}_j$ and $\mathcal{U}_k$. Thus, when M is sufficiently large, from the law of large numbers, the right hand side of (11) can be approximated by the mathematical expectation of the random variable $g_1^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k)$ which we denote by

$$\bar{g}_{j,k} = \mathbb{E}\{g_1^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k)\}. \tag{12}$$

Then we have the following Lemma.

Lemma 2: When the parameters N, K, M are sufficiently large, the probability of error for the support recovery of sparse signal **s** with the observation model (3) is upper bounded by,

$$P_{err} \leq \frac{1}{2} \sum_{t=0}^{K-1} \binom{K}{t} \binom{N-K}{K-t} (\bar{g}_t)^M$$

where $\bar{g}_t = \mathbb{E}\{g_1^{\frac{1}{2}}(\mathcal{U}_j, \mathcal{U}_k)| \ (|\mathcal{U}_j \cap \mathcal{U}_k| = t)\}.$

Proof: With $\bar{g}_{j,k}$ as given in (12), the probability of error in (9) can be upper bounded by,

$$P_{err} \leq \frac{1}{2N_0} \sum_{k=0}^{N_0-1} \sum_{j=0, j\neq k}^{N_0-1} (\bar{g}_{j,k})^M. \tag{13}$$

When the entries of the measurement matrix are iid random variables, the value of $\bar{g}_{j,k}$ is the same for any j, k as far as the number of overlapping elements of the sets $\mathcal{U}_j$ and $\mathcal{U}_k$ is the same for given support sets $\mathcal{U}_j$ and $\mathcal{U}_k$ for $j \neq k$. Note that for a given support set $\mathcal{U}_k$ with $|\mathcal{U}_k| = K$, there are $\binom{K}{t}\binom{N-K}{K-t}$ support sets which have t overlapping elements with any other support set $\mathcal{U}_j$ with $|\mathcal{U}_j| = K$ for $k \neq j$ and $t = 0, 1, 2, \cdots, K-1$. Thus, the upper bound for the probability of error in (13) reduces to,

$$\begin{aligned} P_{err} &\leq \frac{1}{2N_0} \sum_{k=0}^{N_0-1} \sum_{t=0}^{K-1} \binom{K}{t}\binom{N-K}{K-t} (\bar{g}_t)^M \\ &= \frac{1}{2} \sum_{t=0}^{K-1} \binom{K}{t}\binom{N-K}{K-t} (\bar{g}_t)^M \end{aligned}$$

∎

It is noted that the value $\bar{g}_t$ depends on many parameters including the sparsity index K, signal dimension N, statistical properties of the measurement matrix $\boldsymbol{\Phi}$, the measurement SNR, the number of quantization levels L, and the values of quantization thresholds. Computation of the quantity $\bar{g}_t$ for a given measurement matrix is quite complex in general. In the following, we consider several important special cases where a tractable analysis can be obtained.

B. Performance of the ML decoder with L-level quantized CS when the average SNR at the decoder is good ($\sigma_w^2 \to 0$)

We consider the case where the noise power at the decoder σ_w^2 is negligible such that $\sigma_w^2 \to 0$.

Lemma 3: With the observation model (3) at the decoder, the Bhattacharya distance $\mathcal{C}(\frac{1}{2}; p_k, p_j)$ in (7) reduces to

$$\mathcal{C}\left(\frac{1}{2}; p_k, p_j\right) \to -\log\left\{\prod_{i=1}^{M} \sum_{l=0}^{L-1} (\lambda_{ij}^l - \lambda_{ij}^{l+1})^{1/2}(\lambda_{ik}^l - \lambda_{ik}^{l+1})^{1/2}\right\}.$$

as $\sigma_w^2 \to 0$ when the parameters N, K, M are sufficiently large. Then the probability of error of the ML decoder is upper bounded by,

$$P_{err} \leq \frac{1}{2} \sum_{t=0}^{K-1} \binom{K}{t}\binom{N-K}{K-t} (\bar{a}_{t,L})^M$$

where $\bar{a}_{t,L} = \sum_{l=0}^{L-1} \mathbb{E}\{\tilde{a}_t^l\}$ *and* $\mathbb{E}\{\tilde{a}_t^l\} = \mathbb{E}\{(\lambda_{1j}^l - \lambda_{1j}^{l+1})^{\frac{1}{2}}(\lambda_{1k}^l - \lambda_{1k}^{l+1})^{\frac{1}{2}}| \ (|\mathcal{U}_j \cap \mathcal{U}_k| = t)\}.$

Proof: See Appendix C. ∎

The sufficient conditions for reliable recovery of the sparsity pattern with L-level quantized observations (2) when $\sigma_w^2 \to 0$ are stated in the following theorem:

Theorem 1: In the high dimensional setting (such that the parameters N, K, M are sufficiently large), for L-level quantized compressive measurements with given values of thresholds $\tau_0, \tau_1, \cdots, \tau_L$, the sufficient condition to have a vanishing probability of error asymptotically when using the ML decoder for support recovery of the sparse signal $\mathbf{s}$ is:

$$M \geq \max\left\{ f_0(N, K, \gamma, L), f_1(N, K, \gamma, L), \cdots, f_{K-1}(N, K, \gamma, L) \right\} \tag{14}$$

where

$$f_t(N, K, \gamma, L) = \frac{1}{\log \frac{1}{\bar{a}_{t,L}}} \left[(K - t)\left(2 + \log \frac{K}{K-t} + \log \frac{(N-K)}{K-t} \right) + \log \frac{1}{2} \right]$$

for $t = 0, 1, \cdots, K - 1$.

Proof: See Appendix D. ∎

(14) explicitly shows how the minimum number of measurements required for sparsity pattern recovery scales with the parameters N and K with L-level quantized quantized compressive measurements when the decoder noise power is negligible. It is also noted that this minimum number of measurements depends on a term related to several parameters including number of quantization levels L, values of quantization thresholds, measurement SNR and statistical properties of the measurement matrix $\mathbf{\Phi}$.

In the following, we further analyze the lower bound (14) in linear sparsity regime for given values of the parameters, the measurement SNR γ, sparsity index K, number of quantization levels L, and values of quantization thresholds, and compare the results with some related existing work. Following a direct information theoretic approach the author in [24] showed that the sufficient condition for sparsity pattern recovery with real valued observations (given that the non-zero elements of sparse signal are unknown and deterministic) is $M > C \max\left\{ K \log \frac{N}{K}, \frac{1}{\mathcal{P}_{\min}^2} \log(N - K) \right\}$ where $\mathcal{P}_{\min}$ is the minimum value of the unknown signal and C is a constant. With a similar problem formulation, the authors in [40] have shown that $M = \max\left\{ \Omega(K \log K), \Omega\left(\frac{K}{\log K} \log N \right) \right\}$ measurements are required for sparsity pattern recovery in which the analyses are based on the analogy of channel capacity in additive white Gaussian noise-multiple access channel (AWGN-MAC) channels. Considering the quantization error as to perturbations to the measurements, the authors in [31] have shown that the number of measurements required for reliable sparsity pattern recovery is given by $M > \frac{K}{SNR}[(1 - \epsilon) \log(N - K) - 1]$ for some $\epsilon > 0$.

Linear sparsity regime: Consider the linear sparsity regime where K varies linearly with N. Let $\max\limits_{0 \leq t \leq K-1} \{f_t(N, K, \gamma, L)\}$ occur at $t = t_0$. If $\frac{t_0}{K} = \beta$ for $\beta \in (0, 1)$, we have,

$$f_{t_0}(N, K, \gamma, L) = \frac{1}{\log \frac{1}{\bar{a}_{t_0, L}(\gamma, K)}} \left[C_1 K \left(\log \frac{N-K}{K} + C_2 \right) + C_3 \right] \tag{15}$$

where $C_1 = 1 - \beta$, $C_2 = 2 \left(1 + \log \frac{1}{1-\beta} \right)$, and $C_3 = \log \frac{1}{2}$. Note that the right hand side of (15) is dominated by the term $K \log \frac{N-K}{K}$. Thus in the linear sparsity regime, the number of measurements required for reliably sparsity pattern recovery scales as $\Omega \left(K \log \frac{N}{K} \right)$ for given values of SNR, L and quantization thresholds. This is the same scaling as observed in the real valued case in [24], [28] in the linear sparsity regime. Also, the authors in [40] have shown that their results on the sufficient conditions in the linear sparsity regime (i.e. $\Omega(K \log N)$ measurements) are inferior to that in [24] as well as to our results. From (14), it can be seen that the effect of the quantization as considered in (2) is related to the number of measurements required for reliable sparsity pattern via the term $\bar{a}_{t, L}$. In fact $\bar{a}_{t, L}$ depends on many factors including the measurement SNR γ, number of quantization levels L, and values of quantization thresholds. We provide a precise bound for M in (14) clearly demonstrating the impact of these factors and the scaling with respect to K and N on the number of measurements required for sparsity pattern recovery, in contrast to the results obtained in [24] and [40] for the real valued compressive measurements. Further, the scaling $\Omega(K \log(N-K))$ grows much more rapidly compared to $\Omega \left(K \log \frac{N}{K} \right)$ in the linear sparsity regime. Thus the scaling obtained in this paper with respect to K and N shows a lower order growth compared to that obtained in [31] for the sparsity pattern recovery in the linear sparsity regime.

The author in [24] has shown that the practical computationally tractable algorithms in the literature (such as LASSO) for reliable sparsity recovery in the linear sparsity regime require many more measurements compared to $\Omega(N)$ that is obtained with real valued observations. Thus, it is an interesting open problem to develop efficient algorithms to reach the performance bounds achievable by the optimal decoder when the sparsity pattern recovery is to be done based on the quantized version of the real valued compressive measurements.

As mentioned before, the effect of quantization is related to the minimum number of measurements via the term $\bar{a}_{t, L}$. Thus, it is difficult to characterize the behavior of the bound on M in (14) with the number of quantization levels L directly since the term $\bar{a}_{t, L}$ for given t is dependent on the thresholds used in the quantization scheme (2) and the values of these thresholds vary as L increases. In the numerical results section, we show the impact of number of quantization levels and the values of the thresholds on the minimum number of measurements required for reliable sparsity pattern recovery. It is of interest to

consider the design of the thresholds used in the quantization scheme (2) such that the lower bound on M decreases as L increases. This issue will be addressed in a future work.

C. Performance of the ML decoder with uniform quantizer when $\sigma_w^2 \to 0$

Now we consider the simple but important special case; the uniform quantizer in which the quantizer levels are equally spaced. With the bin width Δ, the finite level uniform quantizer is given by,

$$
z_i = \begin{cases}
-L_0\Delta, & if \ y_i < \frac{-(2L_0-1)\Delta}{2} \\
\quad . \\
-\Delta, & if \ -\frac{3\Delta}{2} \le y_i < \frac{-\Delta}{2} \\
0, & if \ -\frac{\Delta}{2} \le y_i < \frac{\Delta}{2} \\
\Delta, & if \ -\frac{\Delta}{2} \le y_i < \frac{3\Delta}{2} \\
\quad . \\
L_0\Delta, & if \ y_i > \frac{(2L_0-1)\Delta}{2}
\end{cases}
\tag{16}
$$

for $i = 1, 2, \cdots, M$ where it has $L = 2L_0 + 1$ quantizer levels.

Following a similar approach as in Section III-B, it can be shown that $\tilde{\mathcal{C}}\left(\frac{1}{2}; p_k, p_j\right)$ is given by (in the high dimensional setting),

$$
\tilde{\mathcal{C}}\left(\frac{1}{2}; p_k, p_j\right) \to \prod_{i=1}^{M} \left((1 - \xi_{ij}^{-L_0})^{\frac{1}{2}}(1 - \xi_{ik}^{-L_0})^{\frac{1}{2}} + \sum_{l=-(L_0)}^{L_0-2} (\xi_{ij}^l - \xi_{ij}^{l+1})^{1/2}(\xi_{ik}^l - \xi_{ik}^{l+1})^{1/2} + (\xi_{ij}^{L_0-1})^{\frac{1}{2}}(\xi_{ik}^{L_0-1})^{\frac{1}{2}} \right) .
$$

as $\sigma_w^2 \to 0$ where $\xi_{ij}^l = Q\left(\frac{\frac{(2l+1)\Delta}{2} - \sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}\mu_j}{\sqrt{\sigma_v^2 + \sigma_s^2 \sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}^2}} \right)$ for $l = -L_0, \cdots, L_0 - 1$, and the probability of error of the ML decoder is upper bounded by,

$$
P_{err} \ \le \ \frac{1}{2} \sum_{t=0}^{K-1} \binom{K}{t}\binom{N-K}{K-t} \left(\bar{b}_{t,\Delta}\right)^M
\tag{17}
$$

where $\bar{b}_{t,\Delta} = \mathbb{E}\{\tilde{b}_t^{-L_0}\} + \mathbb{E}\{\tilde{b}_t^{L_0}\} + \sum_{l=-(L_0-1)}^{L_0-1} \mathbb{E}\{\tilde{b}_t^l\}$, $\mathbb{E}\{\tilde{b}_t^l\} = \mathbb{E}\{(\xi_{1j}^l - \xi_{1j}^{l+1})^{\frac{1}{2}}(\xi_{1k}^l - \xi_{1k}^{l+1})^{\frac{1}{2}}|\ (|\mathcal{U}_j \cap \mathcal{U}_k| = t)\}$, $\mathbb{E}\{\tilde{b}_t^{-L_0}\} = \mathbb{E}\{(\xi_{1j}^{-L_0})^{\frac{1}{2}}(\xi_{1k}^{-L_0})^{\frac{1}{2}}|\ (|\mathcal{U}_j \cap \mathcal{U}_k| = t)\}$, and $\mathbb{E}\{\tilde{b}_t^{L_0}\} = \mathbb{E}\{(\xi_{1j}^{L_0})^{\frac{1}{2}}(\xi_{1k}^{L_0})^{\frac{1}{2}}|\ (|\mathcal{U}_j \cap \mathcal{U}_k| = t)\}$. Following a similar procedure as in subsection III-B, the minimum number of measurements for asymptotic reliable recovery of sparse signals can be obtained and details are avoided here for brevity.

It is noted that when $L_0 \to \infty$, the uniform quantizer has infinite number of levels with the bin width Δ and the number of bits required is infinite. From (17), it can be seen that the quantizer parameter Δ affects the probability of error via the term $\bar{b}_{t,\Delta}$. It can be verified that each term $\bar{b}_{t,\Delta}$ is not a monotonic (increasing or decreasing) function of Δ. However, it is computationally difficult to find the optimal value

of the bin width Δ analytically which results the minimum probability of error in (17). We provide some numerical results in Section V to observe the dependence of the bin width Δ on the minimum number of measurements required for asymptotic reliable recovery of the support of sparse signals.

D. Performance of the ML decoder of the support Recovery of Sparse Signals with 1-Bit Quantized CS when $\sigma_w^2 \to 0$

Next we consider another important special case where the measurements are quantized into only two levels such that $L = 2$, i.e., 1-bit quantization. One example under this special case is to use only the sign information of the compressive measurements to recover the support of the sparse signal [32], [36]. Sparse signal processing with 1-bit quantized CS is attractive since 1-bit CS techniques are robust under different kinds of non-linearties applied to measurements and have less sampling complexities at the hardware level because the quantizer takes the form of a comparator [36], [41]. More specifically let,

$$
z_i = \begin{cases} 1, & if \ y_i \geq 0 \\ -1, & otherwise \end{cases}
\tag{18}
$$

for $i = 1, 2, \cdots, M$. The following theorem (a sketch of the proof can be found in Appendix E) states the sufficient conditions on the number of measurements to achieve a reliable recovery of the support of the sparse signal with 1-bit quantized compressive measurements.

Theorem 2: In the high dimensional setting, for 1- bit quantized compressive measurements, the probability of error of the support recovery of the sparse signal s is upper bounded by,

$$
P_{err} \leq \frac{1}{2} \sum_{t=0}^{K-1} \binom{K}{t} \binom{N-K}{K-t} (\bar{a}_{t,2}(\gamma, K))^M,
$$

and the sufficient condition to have a vanishing probability of error asymptotically when using the ML decoder for sparse support recovery is:

$$
M \geq \max \left\{ \tilde{f}_0(N, K, \gamma), \tilde{f}_1(N, K, \gamma), \cdots, \tilde{f}_{K-1}(N, K, \gamma) \right\}
\tag{19}
$$

where

$$
\tilde{f}_t(N, K, \gamma) = \frac{1}{\log \frac{1}{\bar{a}_{t,2}(\gamma, K)}} \left[(K - t) \left(2 + \log \frac{K}{K - t} + \log \frac{(N - K)}{K - t} \right) + \log \frac{1}{2} \right]
\tag{20}
$$

for $t = 0, 1, \cdots, K - 1$ where $\bar{a}_{t,2}(\gamma, K) = \mathbb{E}\{\tilde{a}_t^0\} + \mathbb{E}\{\tilde{a}_t^1\}$ where

$$
\mathbb{E}\{\tilde{a}_t^0\} = \mathbb{E}\{(1 - \lambda_j)^{\frac{1}{2}}(1 - \lambda_k)^{\frac{1}{2}} | \ (|\mathcal{U}_j \cap \mathcal{U}_k| = t)\}
$$

and

$$\mathbb{E}\{\tilde{a}_t^1\} = \mathbb{E}\{(\lambda_j)^{\frac{1}{2}}(\lambda_k)^{\frac{1}{2}} | \ (|\mathcal{U}_j \cap \mathcal{U}_k| = t)\}$$

with

$$\lambda_k = Q\left(\frac{-\sum_{i=1}^K \left(\tilde{\mathbf{\Phi}}_{\mathcal{U}_k}\right)_{1i} \mu_i}{\sigma_v^2 + \sigma_s^2 \sum_{i=1}^K \left(\tilde{\mathbf{\Phi}}_{\mathcal{U}_k}\right)_{1i}^2} \right).$$

Remark 1: With the definition of λ_k and assuming finite σ_s^2 and σ_v^2, it can be easily shown that $0 < \bar{a}_{t,2}(\gamma, K) < 1$ resulting in the fact that $\log \frac{1}{\bar{a}_{t,2}(\gamma,K)}$ is always positive. Further if we assume that $K \leq \frac{N}{2}$, the lower bound on the number of measurements M in (19) is always positive.

The lower bound in (19) explicitly shows the minimum number of measurements required to recover the sparsity pattern of the sparse signal with only sign information of the compressive measurement vector (1) with the ML decoder. It is further noted that, as mentioned earlier, the value of $\bar{a}_{t,2}(\gamma, K)$ also depends on the sparsity index K and the measurement SNR γ. Illustration of the dependence of M in (19) on both γ and K is shown in the numerical results section.

Computation of $\mathbb{E}\{\tilde{a}_t^0\}$ and $\mathbb{E}\{\tilde{a}_t^1\}$: Let $u_k = \sum_{i=1}^K \left(\tilde{\mathbf{\Phi}}_{\mathcal{U}_k}\right)_{1i} \mu_i$. When the entries of the projection matrix $\mathbf{\Phi}$ are iid Gaussian with mean zero and variance $\frac{1}{N}$, it can be shown that u_k is a Gaussian random variable with mean zero and the variance $\frac{1}{N}||\boldsymbol{\mu}||_2^2$. Then we have,

$$\begin{aligned}
\mathbb{E}\{\tilde{a}_t^0\} &= \mathbb{E}\{(1 - \lambda_j)^{\frac{1}{2}}(1 - \lambda_k)^{\frac{1}{2}} | \ |\mathcal{U}_j \cap \mathcal{U}_k| = t\} \\
&= \int \int (1 - \lambda_j(u_j))^{\frac{1}{2}}(1 - \lambda_k(u_k))^{\frac{1}{2}} f_{U_k U_j}(u_k, u_j) du_k du_j
\end{aligned}$$

which can be found by a 2-fold integration, where we write $\lambda_j(u_j) = Q\left(\frac{-u_j}{\sigma_v^2 + \frac{K}{N}\sigma_s^2}\right)$.

Proposition 2: In the high dimensional setting, given that $|\mathcal{U}_j \cap \mathcal{U}_k| = t$, the joint pdf of (u_k, u_j), $f_{U_k U_j}(u_k, u_j)$ tends to a bi-variate Gaussian with mean $\mathbf{0}$ and the covariance matrix Σ_t where

$$\Sigma_t = \frac{\boldsymbol{\mu}^T \boldsymbol{\mu}}{N} \begin{bmatrix} 1 & \rho_t \\ \rho_t & 1 \end{bmatrix}$$

where $\rho_0 = 0$, $\rho_t = \frac{\sum_{i=1}^t \tilde{\mu}_i}{\boldsymbol{\mu}^T \boldsymbol{\mu}}$ for $t = 1, \cdots, K-1$, $\tilde{\boldsymbol{\mu}}_t = \{\mu_m \mu_n; if \left(\tilde{\mathbf{\Phi}}_{\mathcal{U}_k}\right)_{1m} = \left(\tilde{\mathbf{\Phi}}_{\mathcal{U}_j}\right)_{1n} \ for \ m, n = 1, 2, \cdots, K\}$ for $|\mathcal{U}_j \cap \mathcal{U}_k| = t$ and $\tilde{\boldsymbol{\mu}}_t = [\tilde{\mu}_1, \cdots, \tilde{\mu}_t]^T$.

- In the case where the vector corresponding to non zero elements of the sparse signal, $\tilde{\mathbf{s}}$ (with any sparsity model) is characterized as a first order Gaussian such that $\sigma_s^2 = 0$ we have the following results. When the measurement quality is good such that $\sigma_v^2 \to 0$, it can be shown that (see Appendix

F),

$$\mathbb{E}\{\tilde{a}_t^0\} = \mathbb{E}\{\tilde{a}_t^1\} = \frac{1}{4} + \frac{arcsin(\rho_t)}{2\pi} \tag{21}$$

for $t = 0, 1, \cdots, K-1$ and thus $\bar{a}_{t,2} = \frac{1}{2} + \frac{arcsin(\rho_t)}{\pi}$. Further, if we assume $\boldsymbol{\mu} = \mu_0[1 \ 1 \ \cdots 1]^T$, we have $\rho_t = \frac{t}{K}$ and $\max_{0 \le t \le K-1} \bar{a}_{t,2} = \frac{1}{2} + \frac{arcsin(\frac{K-1}{K})}{\pi} = a_K$ where $0 < a_K < 1$. Then the upper bound on the probability of error (30) with 1-bit quantization can be written as,

$$
\begin{aligned}
P_{err} &\le \frac{1}{2} \sum_{t=0}^{K-1} \binom{K}{t} \binom{N-K}{K-t} \left(\max_{1 \le t \le K} \bar{a}_{t,2} \right)^M \\
&= \frac{1}{2} (a_K)^M \sum_{t=0}^{K-1} \binom{K}{t} \binom{N-K}{K-t}
\end{aligned}
\tag{22}
$$

Using *Chu-Vandermonde* identity (which says that $\sum_{j=0}^{k} \binom{m}{j} \binom{n-m}{k-j} = \binom{n}{k}$ for any complex values m and n and non-negative integer k), the bound in (22) reduces to,

$$
\begin{aligned}
P_{err} &\le \frac{1}{2} (a_K)^M \left[\sum_{t=0}^{K} \binom{K}{t} \binom{N-K}{K-t} - 1 \right] \\
&= \frac{1}{2} (a_K)^M \left[\binom{N}{K} - 1 \right] < \frac{1}{2} (a_K)^M \binom{N}{K}
\end{aligned}
$$

Thus, to have a vanishing probability of error it is required that,

$$M \ge C_K K \log \frac{N}{K},$$

where $C_K = \frac{1}{\log \frac{1}{a_K}}$ only depends on K. This says that, when the measurement noise power is negligible such that $\sigma_v^2 \to 0$, it is sufficient to have $\Omega(K \log \frac{N}{K})$ measurements to recover the sparsity pattern with only the noisy corrupted sign information of the real valued compressive measurement vector (3) with the ML decoder. In [24], the author has shown that when the minimum value of the sparse signal, $\mathcal{P}_{\min} \to \infty$ (the sparse signal is modeled as deterministic and unknown), the sufficient condition to achieve a reliable sparse support recovery by the ML decoder with real valued compressive measurements is given by $M \ge CK \log \frac{N}{K}$ for some constant C. Our results show that, even with coarsely quantized compressive measurements, it is sufficient to have $\Omega(K \log \frac{N}{K})$ measurements (with a different constant which depends on K) when the measurement quality is good irrespective of the nature of the sparsity pattern.

IV. NECESSARY CONDITIONS FOR SPARSITY PATTERN RECOVERY WITH 1-BIT QUANTIZED CS

In this section, we derive the necessary conditions that should be satisfied by any recovery algorithm for support recovery of sparse signals with 1-bit quantized CS. With the observation model in (3), Fano's Lemma states that the probability of error of the support recovery of the sparse signal s is lower bounded by [24], [42],

$$
P_e \geq 1 - \frac{1}{\log(N_0 - 1)} \left(\frac{1}{N_0^2} \sum_{j,k} \mathcal{D}(p(\mathbf{r}|\mathcal{U}_j)||p(\mathbf{r}|\mathcal{U}_k)) + \log 2 \right)
$$

where $\mathcal{D}(p(\mathbf{r}|\mathcal{U}_j)||p(\mathbf{r}|\mathcal{U}_k))$ is the *Kullback-Leibler* distance between two pdfs $p(\mathbf{r}|\mathcal{U}_j)$ and $p(\mathbf{r}|\mathcal{U}_k)$.

Proposition 3: In the high dimensional setting, the Kullback-Leibler distance between two pdfs $p(\mathbf{r}|\mathcal{U}_j)$ and $p(\mathbf{r}|\mathcal{U}_k)$ with 1-bit quantized CS is upper bounded by,

$$
\mathcal{D}(p(\mathbf{r}|\mathcal{U}_j)||p(\mathbf{r}|\mathcal{U}_k)) \leq \frac{4}{\sigma_w^2} \sum_{i=1}^{M} [\lambda_{ij}(1 - \lambda_{ik}) + \lambda_{ik}(1 - \lambda_{ij})].
$$

Proof: The proof follows from the assumption that r_i's for $i = 1, 2, \cdots, M$ are uncorrelated when the parameters N, K, M are sufficiently large given the measurement matrix $\boldsymbol{\Phi}$ and using the convexity bound for the KL distance between two Gaussian mixture pdfs [43]. ■

Let $\Xi = \frac{1}{N_0^2} \sum_{j,k} \mathcal{D}(p(\mathbf{r}|\mathcal{U}_j)||p(\mathbf{r}|\mathcal{U}_k))$. Then we have,

$$
\Xi \leq \frac{4}{\sigma_w^2 N_0^2} \sum_{j=0}^{N_0-1} \sum_{k=0, k\neq j}^{N_0-1} \sum_{i=1}^{M} [\lambda_{ij}(1 - \lambda_{ik}) + \lambda_{ik}(1 - \lambda_{ij})] \tag{23}
$$

Following a similar argument as in Subsection III-A, when the entries of the measurement matrix are iid random variables, we can approximate the quantities $\frac{1}{M} \sum_{i=1}^{M} \lambda_{ij}$ and $\frac{1}{M} \sum_{i=1}^{M} \lambda_{ij}\lambda_{ik}$ by the mathematical expectation of the random variables λ_{1j} and $\lambda_{1j}\lambda_{1k}$ respectively, for given support sets $\mathcal{U}_j$ and $\mathcal{U}_k$ when the number of measurements M is sufficiently large. Letting $\bar{\lambda}_j = \mathbb{E}\{\lambda_{1j}\}$ and $\hat{\lambda}_{jk} = \mathbb{E}\{\lambda_{1j}\lambda_{1k}\}$, (23) reduces to,

$$
\Xi \leq \frac{4M}{\sigma_w^2 N_0^2} \sum_{j=0}^{N_0-1} \sum_{k=0, k\neq j}^{N_0-1} (\bar{\lambda}_j + \bar{\lambda}_k - 2\hat{\lambda}_{jk})
$$

Again, since the entries of the measurement matrix are iid, we have that $\bar{\lambda}_j$ is the same for all $\mathcal{U}_j$ with $|\mathcal{U}_j| = K$ for $j = 0, 1, \cdots, N_0 - 1$. Similar to the discussion in the proof of Lemma 2, $\hat{\lambda}_{jk}$ is the same as far as the number of overlapping elements of two support sets $\mathcal{U}_j$ and $\mathcal{U}_k$ is the same for $j \neq k$. Thus we get the following bound for Ξ.

$$
\Xi \leq \frac{4M}{\sigma_w^2} \left(2\bar{\lambda} - \frac{2}{N_0} \sum_{t=0}^{K-1} \binom{K}{t} \binom{N-K}{K-t} \hat{\lambda}_t \right)
$$

where we have let $\bar{\lambda} = \bar{\lambda}_j$ for $j = 0, 1, \cdots, N_0 - 1$ and $\hat{\lambda}_t = \mathbb{E}\{\lambda_{1j}\lambda_{1k} | (|\mathcal{U}_j \cap \mathcal{U}_k| = t)\}$. Letting $\hat{\lambda}_{min} = \min\limits_{0 \leq t \leq K}\{\hat{\lambda}_t\}$, we have,

$$
\begin{aligned}
\Xi &\leq \frac{8M}{\sigma_w^2}\left(\bar{\lambda} - \frac{1}{N_0}\sum_{t=0}^{K-1}\binom{K}{t}\binom{N-K}{K-t}\hat{\lambda}_{min}\right) \\
&= \frac{8M}{\sigma_w^2}\left(\bar{\lambda} - \hat{\lambda}_{min}\left(1 - \frac{1}{N_0}\right)\right) \\
&\approx \frac{8M}{\sigma_w^2}\left(\bar{\lambda} - \hat{\lambda}_{min}\right)
\end{aligned}
$$

Thus we conclude that the probability of error is lower bounded by,

$$
P_e \geq 1 - \frac{8M}{\sigma_w^2 \log(\binom{N}{K} - 1)}(\bar{\lambda} - \hat{\lambda}_{min}) - \epsilon_0
$$

where $\epsilon_0 = \frac{\log 2}{\log(\binom{N}{K}-1)}$. Thus when $M \leq \frac{(1-\epsilon_0)\sigma_w^2}{8(\bar{\lambda}-\hat{\lambda}_{min})}\log\binom{N}{K}$, the probability of error of the support recovery of sparse signal with 1-bit CS with any recovery rule is bounded away from zero. Subsequently, we have the following Theorem.

Theorem 3: When the parameters N, K and M are sufficiently large, the support recovery of sparse signal s *with 1-bit CS with any recovery algorithm is impossible if,*

$$
M < \frac{(1-\epsilon_0)\sigma_w^2}{8(\bar{\lambda}(\gamma, K) - \hat{\lambda}_{min}(\gamma, K))}K\log\frac{N}{K} \tag{24}
$$

where we write $\bar{\lambda}(\gamma, K)$ ($\hat{\lambda}_{min}(\gamma, K)$) to denote $\bar{\lambda}$ ($\hat{\lambda}_{min}$) since the term $\bar{\lambda}$ ($\hat{\lambda}_{min}$) depends on the parameters γ and K.

It should be noted that, with the real valued compressive measurements, the author in [24] has shown that the asymptotic reliable recovery of the support of a sparse signal is impossible if $M < \tilde{C}K\log\frac{N}{K}$ where the value of $\tilde{C}$ depends on the sparsity index K and the minimum value of the unknown deterministic signal vector of interest and this dependence is explicitly derived. From (24), a similar behavior is observed on the necessary conditions for sparsity pattern recovery with coarsely quantized compressive measurements in our problem set up in which we model the non-zero elements of the sparse signal as random. However, again the way of dependence of the measurement SNR and the sparsity index K on the terms $\bar{\lambda}(\gamma, K)$ and $\hat{\lambda}_{min}(\gamma, K)$ in (24) determines the exact value of M and to directly evaluate this dependence analytically with 1-bit quantized CS is difficult.

V. NUMERICAL RESULTS

In this section, we provide some numerical results to illustrate the performance of the support recovery of sparse signals with quantized compressive measurements.

From the theoretical analysis performed in Section III, it was noted that the ability to estimate the sparsity pattern of the sparse signal based on the ML decoder with quantized compressive measurements has a considerable dependence on a term related to the measurement SNR (and it depends on the sparsity index K as well), number of quantization levels and quantization thresholds, in addition to the parameters M, N and K. Since it is difficult to see a direct dependence of the measurement SNR on the minimum number of measurements required to reliably recover the support of the sparse signal analytically, in this section, we numerically demonstrate how the measurement SNR and the sparsity index affect the reliable recovery of the support of the sparse signal with 1-bit quantized compressive measurements with the ML decoder.

In Fig. 1, we show the dependence of the minimum number of measurements required to reliably recover the support of the sparse signal on the measurement SNR with 1-bit CS with the ML decoder as stated in Theorem 2 for given values of K. In Fig. 2, we let $N = 768$. In the following figures, the measurement SNR is defined as $\gamma = \frac{\boldsymbol{\mu}^T \boldsymbol{\mu} + K\sigma_s^2}{N\sigma_v^2}$. To vary the value of SNR for given K and N the value of $\boldsymbol{\mu}$ is varied keeping σ_v^2 and σ_s^2 at fixed values. From Fig. 1, it can be seen that, for given values of K and N, the minimum number of measurements required to recover the sparsity pattern with 1-bit CS reduces and converges to a certain value as the average measurement SNR increases and this value is different for different values of K. This observation demonstrates that when the measurement SNR increases beyond a certain value for a given sparsity level of the sparse signal, by reducing the number of compressive measurements further will not allow the reliable recovery of the sparsity pattern of the sparse signal with 1-bit quantized CS for given values of K and N. Alternatively, this observation implies that the minimum number of measurements required for reliable recovery of the support of the sparse signal with 1-bit CS, M, scales only with K and N as the measurement SNR increases. Although computationally difficult, finding this scaling analytically will be an interesting problem to be considered. We further consider this issue in the next figure. Fig. 1 further illustrates that a small fraction of compressive measurements (compared to the original signal dimension) is sufficient for reliable recovery of the support of the sparse signal with 1-bit quantized CS at moderate values of average measurement SNR when the signal becomes more sparse.

In Theorem 2, the minimum number of measurements required for reliable sparsity pattern recovery with the ML decoder depends on the measurement SNR via the term $\bar{a}_{t,2}(\gamma, K)$ for given t. With 1-bit quantized CS, we numerically show the behavior of the term $\bar{a}_{t,2}(\gamma, K)$ as the measurement SNR increases for given t for $t = 0, 1, \cdots, K - 1$. In Fig. 2, we plot $\bar{a}_{t,2}(\gamma, K)$ Vs the measurement SNR for a given value of K. In Fig. 2, we have let $N = 768$ and $K = 20$. As can be seen from Fig. 2, the term related to the measurement SNR in the lower bound in (19), converges to a constant value as the SNR increases
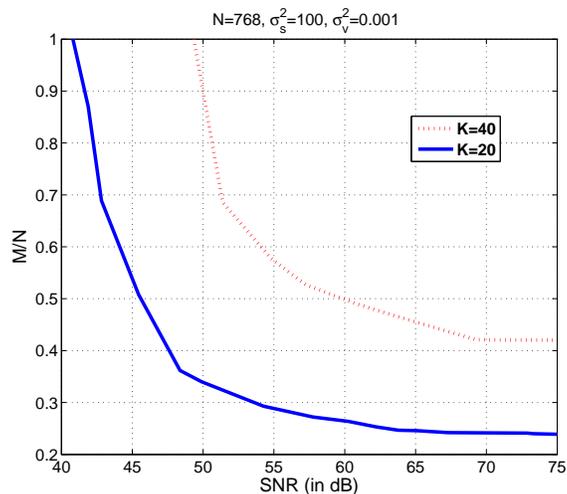
Fig. 1. Lower bound on the minimum number of measurements Vs. measurement signal-to-noise ratio with 1-bit quantized CS; solid line-$K = 20$, dotted line-$K = 40$, $N = 768$
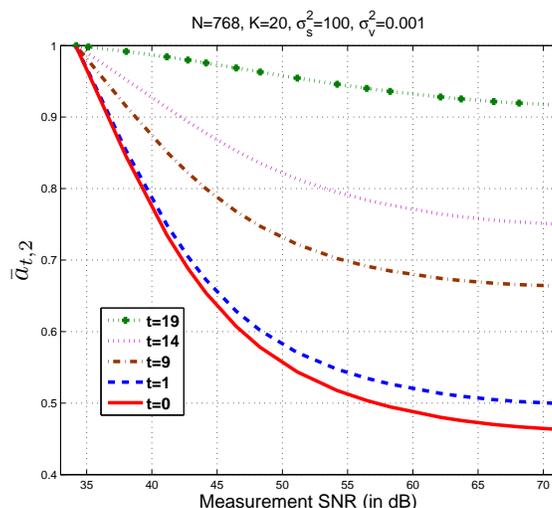

Fig. 2. Behavior of $\bar{a}_{t,2}$ in Theorem 2 as the measurement SNR increases; $K = 20$, $N = 768$

which is determined only by the number of overlapping elements (t) in any two support sets for a given value of K. In other words, it is understood that this converged value for a given t is determined only by the statistical properties of the random measurement matrix as the measurement SNR increases for a given value of sparsity index. This impact of $\bar{a}_{t,2}(\gamma, K)$ on the minimum number of measurements required to reliably recover the sparsity pattern as the measurement SNR increases was illustrated in Fig. 1.

We further discuss the behavior of the minimum number of measurements required for reliable recovery of the support of sparse signals with 1-bit quantized CS as the measurement SNR increases, based on the results observed from Fig. 2. From the Theorem 2, we have

$$M \geq \max_{0 \leq t \leq K-1} \left\{ \tilde{f}_t(N, K, \gamma) \right\}$$
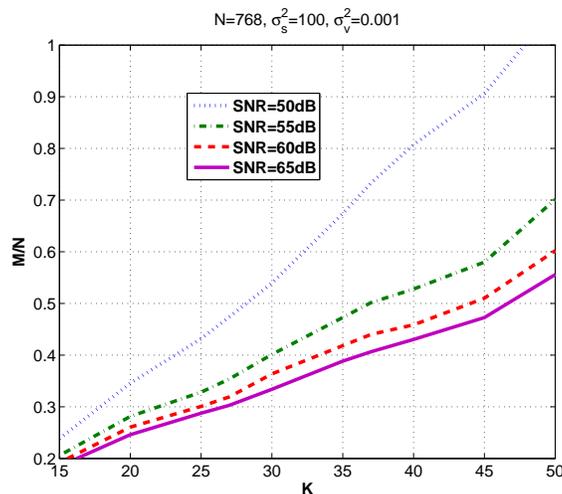
Fig. 3. Lower bound on the minimum number of measurements Vs. the sparsity index K with 1-bit quantized CS

where $\tilde{f}_t(N, K, \gamma)$ is as given by (20). When the measurement SNR $\gamma \to \infty$, we observe from Fig. 2 that $\bar{a}_{t,2}(K, \gamma) \to \hat{a}_t(K)$ where the term $\hat{a}_t(K)$ only depends on K for a given t. Then the term $\tilde{f}_t(N, K, \gamma)$ in (20) can be written as,

$$\tilde{f}_t(N, K, \gamma) = \frac{1}{\log \frac{1}{\hat{a}_t(K)}} \left[(K - t) \left(2 + \log \frac{K}{K - t} + \log \frac{(N - K)}{K - t} \right) + \log \frac{1}{2} \right]$$

When $\gamma \to \infty$, let maximum of $\tilde{f}_t(N, K)$ occur at $t = \eta_K K$, where $0 \le \eta_K \le 1$ (in fact it can be shown that $\frac{1}{2} \le \eta_K \le 1$ as K increases). Then we have,

$$
\begin{aligned}
M &\ge \frac{1}{\log \frac{1}{\hat{a}_{\eta_K}(K)}} \left[(1 - \eta_K) K \left(\log \frac{(N - K)}{K} + 2 \log \frac{1}{1 - \eta_K} + 2 \right) + \log \frac{1}{2} \right] \\
&\ge \hat{C}_K K \log \frac{N}{K}
\end{aligned}
$$

where the term $\hat{C}_K$ depends only on K. Thus, it is interesting to see that the minimum number of measurements required for reliable recovery of the support of the sparse signals with 1-bit quantized CS based on the ML decoder scales with the parameters N and K in a similar manner compared to that with real valued observations (in [24], [27]) when the measurement quality is good ($\gamma \to \infty$). This observation was analytically verified at the end of Section III-D for the special case where $\sigma_s^2 = 0$.

It is noted that the term $\bar{a}_{t,2}(\gamma, K)$ in Theorem 2 depends on the sparsity index K in addition to the measurement SNR. The dependence of the minimum number of measurements required for sparsity pattern recovery on the sparsity index K with 1-bit quantized CS is shown in Fig. 3 for different values of average measurement SNR. As can be seen from Fig. 3, it is possible to recover the sparsity pattern with 1-bit quantized CS reliably at moderate SNR values as signal becomes more sparse with the ML
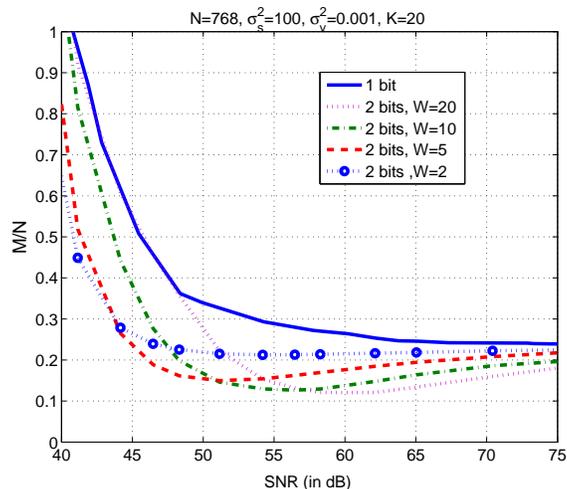
Fig. 4. Lower bound on the minimum number of measurements Vs. measurement signal-to-noise ratio with different levels of quantization: $K = 20$, $N = 768$

decoder. However, as the signal becomes less sparse, the minimum number of compressive measurements required to recover the sparsity pattern with 1-bit CS increases rapidly for moderate SNR values. Figures 1 and 3 clearly illustrate the dependence of the term $\bar{a}_{t,2}(\gamma, K)$ (which is a function of both K and γ) on the minimum number of compressive measurements required with the ML decoder for reliable sparsity pattern recovery.

In Fig. 4, we illustrate the impact of the number of quantization levels and the values of quantization thresholds of the quantization scheme in (2) on the minimum number of measurements required for reliable sparsity pattern recovery. We consider 2-bit quantizer with different values of quantization thresholds and compare the results with 1-bit quantizer. In Fig. 4 we let $\tau_1 = \infty$, $\tau_2 = W$, $\tau_3 = 0$, $\tau_4 = -W$ and $\tau_5 = \infty$ and the value of W is varied. With 2-bit quantization, it can be seen that the minimum number of measurements required for reliable sparsity pattern recovery highly depends on the selection of the quantization thresholds. For a given value of SNR, a particular set of threshold values provides the best performance in terms of the minimum number of measurements required for sparsity pattern recovery and this set of threshold values varies for different SNR values. By proper selection of quantization thresholds with 2-bit quantizer, the performance of the sparsity pattern recovery can be improved in a large margin compared to that with the 1-bit quantization. It is also noted that, when the SNR increases the performance gap with 1-bit and 2-bit quantization is not significant irrespective of the selection of the quantization thresholds.

Next, we investigate the performance of the support recovery of sparse signals based on uniformly quantized compressive measurements. As mentioned in subsection III-C, the minimum number of measurements required for asymptotic reliable recovery of sparse signals with uniformly quantized compressive
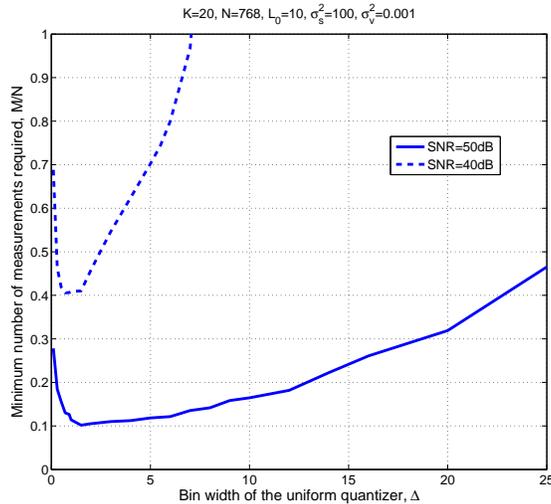
Fig. 5. Lower bound on the minimum number of measurements Vs. the bin width Δ of the uniform quantizer

is given by (14) where $\bar{a}_{t,L}$ in (14) is replaced by $\bar{b}_{t,\Delta}$. In Fig. 5, we show how the minimum number of measurements depends on the bin width Δ of the uniform quantizer for given values of measurement SNR. In Fig. 5, we let $L_0 = 10$ such that $L = 21$ quantization levels. From Fig. 5, it can be seen that the lower bound on the number of measurements required for reliable support recovery tends to its minimum at a particular value of Δ. Further, it can be seen that when the bin width Δ deviates slightly from its optimal value, the lower bound on the number of measurements required for reliable support recovery increases to a large value especially when the measurement SNR is low.

VI. CONCLUSIONS

In this paper, we have determined the performance bounds for the support recovery of sparse signals with quantized compressive measurements. The entries of the compressive measurement matrix are assumed to be drawn from a Gaussian ensemble and the noisy corrupted compressive measurements are assumed to be quantized into L quantization levels. The sufficient conditions to achieve reliable recovery of the sparsity pattern with the ML decoder with quantized compressive measurements were derived. It was explicitly shown how the measurement quality, the sparsity index, the original signal dimension and the statistical properties of the measurement matrix depend on the minimum number of measurements needed to reliably recover the sparsity pattern of the sparse signal with coarsely quantized CS. Further, when the measurement noise power is negligible, it was shown that it is sufficient to have $C_K K \log \frac{N}{K}$ number of measurements, where C_K is a constant which only depends on K, to reliably recover the sparse support with 1-bit compressive measurements. We also analyzed the necessary conditions that should be satisfied by the number of compressive measurements for reliable recovery of the sparsity pattern with 1-bit quantized compressive measurements with any support recovery algorithm.

There are several directions that the work can be extended. In this paper, we derived the sufficient conditions to achieve sparse support recovery with L level quantized compressive measurements for a given set of quantization threshold values. The design of quantization thresholds so as to minimize the probability of error or other performance criteria is worth considering as a future work. As observed with real valued compressive measurements in [24], [27], there is an obvious gap in the performance achieved by optimal ML decoder and the existing practical algorithms in the literature for sparse support recovery with quantized compressive measurements. Thus, it is an open problem to develop computationally tractable algorithms for sparse support recovery with coarsely quantized compressive measurements which have performance guarantees closer to that is achieved by the optimal decoder.

APPENDIX A

Derivation of (6): For a given support set $\mathcal{U}_k$, we have

$$
\begin{aligned}
\mathbb{E}\{y_i|\mathcal{U}_k\} &= \mathbb{E}\left\{\left(\sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}\tilde{s}_j + v_i\right)\right\} \\
&= \sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}\mu_j
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\{y_i y_t|\mathcal{U}_k\} &= \mathbb{E}\left\{\left(\sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}\tilde{s}_j + v_i\right)\left(\sum_{l=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{tl}\tilde{s}_l + v_t\right)\right\} \\
&= \sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{tj}\mathbb{E}\{\tilde{s}_j^2\} + \sum_{j\neq l}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{tl}\mathbb{E}\{\tilde{s}_j\tilde{s}_l\} + \mathbb{E}\{v_i v_t\}
\end{aligned}
$$

$$
\mathbb{E}\{y_i y_t|\mathcal{U}_k\} = \begin{cases} \sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}^2\mathbb{E}\{\tilde{s}_j^2\} + \sum_{j\neq l}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{il}\mathbb{E}\{\tilde{s}_j\tilde{s}_l\} + \sigma_v^2 & \text{if } i = t \\ \sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{tj}\mathbb{E}\{\tilde{s}_j^2\} + \sum_{j\neq l}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{tl}\mathbb{E}\{\tilde{s}_j\tilde{s}_l\} & \text{if } i \neq t \end{cases}
$$

for $i, t = 1, 2, \cdots, M$. Thus

$$
\begin{aligned}
Var(y_i|\mathcal{U}_k) &= \sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}^2\mathbb{E}\{\tilde{s}_j^2\} + \sum_{j\neq l}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{il}\mathbb{E}\{\tilde{s}_j\tilde{s}_l\} + \sigma_v^2 - \left(\sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}\mu_j\right)^2 \\
&= \sigma_s^2\sum_{j=1}^{K}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij}^2 + \sigma_v^2
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{cov}(y_i, y_t | \mathcal{U}_k) &= \mathbb{E}\{y_i y_t | \mathcal{U}_k\} - \mathbb{E}\{y_i | \mathcal{U}_k\}\mathbb{E}\{y_t | \mathcal{U}_k\} \\
&= \sum_{j=1}^{K} (\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij} (\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{tj} \mathbb{E}\{\tilde{s}_j^2\} + \sum_{j \neq l} (\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij} (\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{tl} \mathbb{E}\{\tilde{s}_j \tilde{s}_l\} \\
&\quad - \sum_{j=1}^{K} (\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij} \mu_j \sum_{l=1}^{K} (\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{tl} \mu_l \\
&= \sigma_s^2 \sum_{j=1}^{K} (\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{ij} (\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k})_{tj}
\end{aligned}
$$

for $i, t = 1, 2, \cdots, M$. This gives $\mathbf{y} | \mathcal{U}_k \sim \mathcal{N}(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k} \boldsymbol{\mu}, \sigma_s^2 \tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k} \tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k}^T + \sigma_v^2 \mathbf{I}_M)$. When the sparsity index K is large enough and the entries of the measurement matrix $\boldsymbol{\Phi}$ are drawn from a Gaussian ensemble with mean zero and variance $\frac{1}{N}$, invoking law of large numbers, we may approximate,

$$
\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k} \tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k}^T \to \frac{K}{N} \mathbf{I}_M
$$

resulting $\mathrm{cov}(y_i, y_t) \to 0$ for $i \neq t$. Then we have,

$$
\begin{aligned}
p(r_i | \mathcal{U}_k) &= \sum_l Pr(r_i | z_i = l) Pr(z_i = l) \\
&\to \sum_{l=0}^{L-1} \mathcal{N}(l, \sigma_w^2) Pr(\tau_l \leq y_i < \tau_{l+1}) \\
&= \sum_{l=0}^{L-1} \mathcal{N}(l, \sigma_w^2) \left[Q\left(\frac{\tau_l - \sum_{j=1}^{K} \left(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k}\right)_{ij} \mu_j}{\sqrt{\sigma_v^2 + \sigma_s^2 \sum_{j=1}^{K} \left(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k}\right)_{ij}^2}} \right) - \left(\frac{\tau_{l+1} - \sum_{j=1}^{K} \left(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k}\right)_{ij} \mu_j}{\sqrt{\sigma_v^2 + \sigma_s^2 \sum_{j=1}^{K} \left(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k}\right)_{ij}^2}} \right) \right]
\end{aligned}
$$

(6) is obtained since $p(\mathbf{r} | \mathcal{U}_k) \to \prod_{i=1}^{M} p(r_i | \mathcal{U}_k)$ in the high dimensional setting and defining $\lambda_{ik}^l = Q\left(\frac{\tau_l - \sum_{j=1}^{K} \left(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k}\right)_{ij} \mu_j}{\sqrt{\sigma_v^2 + \sigma_s^2 \sum_{j=1}^{K} \left(\tilde{\boldsymbol{\Phi}}_{\mathcal{U}_k}\right)_{ij}^2}} \right)$.

APPENDIX B

Proof of Lemma 1: The Chernoff distance between pdfs p_k and p_j, $\mathcal{C}(\alpha; p_k, p_j)$, can be written as,

$$
\mathcal{C}(\alpha; p_k, p_j) = -\log\left\{ \tilde{\mathcal{C}}(\alpha; p_k, p_j) \right\}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{C}}(\alpha; p_k, p_j) &= \int p(\mathbf{r}|\mathcal{U}_k)^{1-\alpha} p(\mathbf{r}|\mathcal{U}_j)^{\alpha} d\mathbf{r} \\
&\rightarrow \int \left[\prod_{i=1}^{M} \sum_{l=0}^{L-1} \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ik}^l - \lambda_{ik}^{l+1}] \right]^{1-\alpha} \left[\prod_{i=1}^{M} \sum_{l=0}^{L-1} \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ij}^l - \lambda_{ij}^{l+1}] \right]^{\alpha} d\mathbf{r}
\end{aligned}
$$

as the parameters (N, K, M) are large enough with $\lambda_{ik}^l = Q\left(\dfrac{\tau_l - \sum_{j=1}^{K} \left(\tilde{\Phi}_{\mathcal{U}_k} \right)_{ij} \mu_j}{\sqrt{\sigma_v^2 + \sigma_s^2 \sum_{j=1}^{K} \left(\tilde{\Phi}_{\mathcal{U}_k} \right)_{ij}^2}} \right)$. It can be shown that,

$$
\begin{aligned}
\frac{p(\mathbf{r}|\mathcal{U}_j)}{p(\mathbf{r}|\mathcal{U}_k)} &\rightarrow \frac{\prod_{i=1}^{M} \sum_{l=0}^{L-1} \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ij}^l - \lambda_{ij}^{l+1}]}{\prod_{i=1}^{M} \sum_{l=0}^{L-1} \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ik}^l - \lambda_{ik}^{l+1}]} \\
&= \frac{\prod_{i=1}^{M} \sum_{l=0}^{L-1} e^{\frac{(lr_i - \frac{l^2}{2})}{\sigma_w^2}}[\lambda_{ij}^l - \lambda_{ij}^{l+1}]}{\prod_{i=1}^{M} \sum_{l=0}^{L-1} e^{\frac{(lr_i - \frac{l^2}{2})}{\sigma_w^2}}[\lambda_{ik}^l - \lambda_{ik}^{l+1}]}
\end{aligned}
$$

Then we have,

$$
\begin{aligned}
\tilde{\mathcal{C}}(\alpha; p_k, p_j) &\rightarrow \int \left(\frac{\prod_{i=1}^{M} \sum_{l=0}^{L-1} e^{\frac{(lr_i - \frac{l^2}{2})}{\sigma_w^2}}[\lambda_{ij}^l - \lambda_{ij}^{l+1}]}{\prod_{i=1}^{M} \sum_{l=0}^{L-1} e^{\frac{(lr_i - \frac{l^2}{2})}{\sigma_w^2}}[\lambda_{ik}^l - \lambda_{ik}^{l+1}]} \right)^{\alpha} \prod_{i=1}^{M} \sum_{l=0}^{L-1} \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ik}^l - \lambda_{ik}^{l+1}] d\mathbf{r} \\
&= \prod_{i=1}^{M} \int \left(\frac{\sum_{l=0}^{L-1} e^{\frac{(lr_i - \frac{l^2}{2})}{\sigma_w^2}}[\lambda_{ij}^l - \lambda_{ij}^{l+1}]}{\sum_{l=0}^{L-1} e^{\frac{(lr_i - \frac{l^2}{2})}{\sigma_w^2}}[\lambda_{ik}^l - \lambda_{ik}^{l+1}]} \right)^{\alpha} \sum_{l=0}^{L-1} \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ik}^l - \lambda_{ik}^{l+1}] dr_i \\
&= \prod_{i=1}^{M} \sum_{m=0}^{L-1} \int \left(\frac{\sum_{l=0}^{L-1} e^{\frac{(lr_i - \frac{l^2}{2})}{\sigma_w^2}}[\lambda_{ij}^l - \lambda_{ij}^{l+1}]}{\sum_{n=0}^{L-1} e^{\frac{(nr_i - \frac{n^2}{2})}{\sigma_w^2}}[\lambda_{ik}^n - \lambda_{ik}^{n+1}]} \right)^{\alpha} \mathcal{N}(r_i; m, \sigma_w^2)[\lambda_{ik}^m - \lambda_{ik}^{m+1}] dr_i \quad (25)
\end{aligned}
$$

Letting $g_i^\alpha(\mathcal{U}_j, \mathcal{U}_k)$ as in (8), we get (7).

APPENDIX C

Proof of Lemma 3: We consider the case where $\sigma_w^2 \rightarrow 0$. Defining sets, $\mathcal{S}_0 = \{i; r_i \leq \frac{1}{2}\}$, $\mathcal{S}_1 = \{i; \frac{1}{2} < r_i \leq \frac{3}{2}\}$, $\mathcal{S}_2 = \{i; \frac{3}{2} < r_i \leq \frac{5}{2}\}$, $\cdots$, $\mathcal{S}_l = \{i; \frac{(2l-1)}{2} < r_i \leq \frac{(2l+1)}{2}\}$, $\cdots$ $\mathcal{S}_{L-1} = \{i; r_i > \frac{(2L-3)}{2}\}$ and following a similar procedure as in [44], it can be shown that,

$$
\frac{p(\mathbf{r}|\mathcal{U}_j)}{p(\mathbf{r}|\mathcal{U}_k)} \rightarrow \prod_{i \in \mathcal{S}_0} \frac{\lambda_{ij}^0 - \lambda_{ij}^1}{\lambda_{ik}^0 - \lambda_{ik}^1} \prod_{i \in \mathcal{S}_1} \frac{\lambda_{ij}^1 - \lambda_{ij}^2}{\lambda_{ik}^1 - \lambda_{ik}^2} \cdots \prod_{i \in \mathcal{S}_{L-1}} \frac{\lambda_{ij}^{L-1} - \lambda_{ij}^L}{\lambda_{ik}^{L-1} - \lambda_{ik}^L}
$$

as $\sigma_w^2 \to 0$ and N, K, M are sufficiently large. Then $\tilde{\mathcal{C}}(\alpha; p_k, p_j)$ in (25) becomes,

$$\tilde{\mathcal{C}}(\alpha; p_k, p_j) \to \int \prod_{i \in \mathcal{S}_0} \left(\frac{\lambda_{ij}^0 - \lambda_{ij}^1}{\lambda_{ik}^0 - \lambda_{ik}^1} \right)^\alpha \prod_{i \in \mathcal{S}_1} \left(\frac{\lambda_{ij}^1 - \lambda_{ij}^2}{\lambda_{ik}^1 - \lambda_{ik}^2} \right)^\alpha \cdots$$

$$\prod_{i \in \mathcal{S}_{L-1}} \left(\frac{\lambda_{ij}^{L-1} - \lambda_{ij}^L}{\lambda_{ik}^{L-1} - \lambda_{ik}^L} \right)^\alpha \prod_{i=1}^M \sum_{l=0}^{L-1} \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ik}^l - \lambda_{ik}^{l+1}] d\mathbf{r}$$

$$= \prod_{i=1}^M \sum_{l=0}^{L-1} \int \left[\prod_{i \in \mathcal{S}_0} \left(\frac{\lambda_{ij}^0 - \lambda_{ij}^1}{\lambda_{ik}^0 - \lambda_{ik}^1} \right)^\alpha \prod_{i \in \mathcal{S}_1} \left(\frac{\lambda_{ij}^1 - \lambda_{ij}^2}{\lambda_{ik}^1 - \lambda_{ik}^2} \right)^\alpha \cdots \prod_{i \in \mathcal{S}_{L-1}} \left(\frac{\lambda_{ij}^{L-1} - \lambda_{ij}^L}{\lambda_{ik}^{L-1} - \lambda_{ik}^L} \right)^\alpha \mathcal{N}(r_i; l, \sigma_w^2)[\lambda_{ik}^l - \lambda_{ik}^{l+1}] \right] dr_i$$

$$= \prod_{i=1}^M \left\{ \left(\frac{\lambda_{ij}^0 - \lambda_{ij}^1}{\lambda_{ik}^0 - \lambda_{ik}^1} \right)^\alpha \sum_{l=0}^{L-1} \left[1 - Q\left(\frac{1/2 - l}{\sigma_w} \right) \right] (\lambda_{ik}^l - \lambda_{ik}^{l+1}) \right.$$

$$+ \sum_{m=1}^{L-2} \left(\frac{\lambda_{ij}^m - \lambda_{ij}^{m+1}}{\lambda_{ik}^m - \lambda_{ik}^{m+1}} \right)^\alpha \sum_{l=0}^{L-1} \left[Q\left(\frac{(2m-1)/2 - l}{\sigma_w} \right) - Q\left(\frac{(2m+1)/2 - l}{\sigma_w} \right) \right] (\lambda_{ik}^l - \lambda_{ik}^{l+1})$$

$$\left. + \left(\frac{\lambda_{ij}^{L-1} - \lambda_{ij}^L}{\lambda_{ik}^{L-1} - \lambda_{ik}^L} \right)^\alpha \sum_{l=0}^{L-1} \left[Q\left(\frac{(2L-3)/2 - l}{\sigma_w} \right) \right] (\lambda_{ik}^l - \lambda_{ik}^{l+1}) \right\}$$

which reduces to,

$$\tilde{\mathcal{C}}(\alpha; p_k, p_j) \to \prod_{i=1}^M \sum_{l=0}^{L-1} (\lambda_{ij}^l - \lambda_{ij}^{l+1})^\alpha (\lambda_{ik}^l - \lambda_{ik}^{l+1})^{1-\alpha}$$

as $\sigma_w^2 \to 0$ where we use the fact that $Q(x) \to 0$ as $x \to +\infty$ and $Q(x) \to 1$ as $x \to -\infty$.

Then the probability of error of the ML decoder is upper bounded by,

$$P_{err} \le \frac{1}{2N_0} \sum_{k=0}^{N_0-1} \sum_{j=0, j \neq k}^{N_0-1} \prod_{i=1}^M \sum_{l=0}^{L-1} (\lambda_{ij}^l - \lambda_{ij}^{l+1})^{\frac{1}{2}} (\lambda_{ik}^l - \lambda_{ik}^{l+1})^{\frac{1}{2}} \tag{26}$$

Using the fact that the geometric mean of a sequence of non negative real numbers is always less than or equal to corresponding arithmetic mean we have

$$\left(\prod_{i=1}^M \sum_{l=0}^{L-1} (\lambda_{ij}^l - \lambda_{ij}^{l+1})^{\frac{1}{2}} (\lambda_{ik}^l - \lambda_{ik}^{l+1})^{\frac{1}{2}} \right)^{\frac{1}{M}} \le \frac{1}{M} \sum_{i=1}^M \sum_{l=0}^{L-1} (\lambda_{ij}^l - \lambda_{ij}^{l+1})^{\frac{1}{2}} (\lambda_{ik}^l - \lambda_{ik}^{l+1})^{\frac{1}{2}}$$

$$= \sum_{l=0}^{L-1} \frac{1}{M} \sum_{i=1}^M (\lambda_{ij}^l - \lambda_{ij}^{l+1})^{\frac{1}{2}} (\lambda_{ik}^l - \lambda_{ik}^{l+1})^{\frac{1}{2}}$$

Following the similar argument in the proof of Lemma 2, it can be shown that the probability of error is

upper bounded by,

$$P_{err} \leq \frac{1}{2} \sum_{t=0}^{K-1} \binom{K}{t} \binom{N-K}{K-t} \left(\sum_{l=0}^{L-1} \mathbb{E}\{\tilde{a}_t^l\} \right)^M \tag{27}$$

where $\mathbb{E}\{\tilde{a}_t^l\} = \mathbb{E}\{(\lambda_{1j}^l - \lambda_{1j}^{l+1})^{\frac{1}{2}} (\lambda_{1k}^l - \lambda_{1k}^{l+1})^{\frac{1}{2}} | (|\mathcal{U}_j \cap \mathcal{U}_k| = t)\}$. Letting $\bar{a}_{t,L} = \sum_{l=0}^{L-1} \mathbb{E}\{\tilde{a}_t^l\}$, (27) reduces to,

$$P_{err} \leq \frac{1}{2} \sum_{t=0}^{K-1} \binom{K}{t} \binom{N-K}{K-t} (\bar{a}_{t,L})^M .$$

APPENDIX D

Proof of Theorem 1: As given by Lemma 3, the probability of error for sparse support recovery with L-level quantized compressive measurements is upper bounded by,

$$P_{err} \leq \frac{1}{2} \sum_{t=0}^{K-1} \binom{K}{t} \binom{N-K}{K-t} (\bar{a}_{t,L})^M . \tag{28}$$

The goal is to find the conditions under which the bound on the probability of error in (28) tends to zero. Let l_t be the logarithm of the t-th term in (28) which is given by,

$$
\begin{aligned}
l_t &= \log \binom{K}{t} + \log \binom{N-K}{K-t} + \log \frac{1}{2} - M \log \frac{1}{\bar{a}_{t,L}} \\
&= \log \binom{K}{K-t} + \log \binom{N-K}{K-t} + \log \frac{1}{2} - M \log \frac{1}{\bar{a}_{t,L}}
\end{aligned}
\tag{29}
$$

Based on the fact that the Binomial coefficient $\binom{N}{K}$ is upper bounded by $\left(\frac{N.e}{K} \right)^K$, we have,

$$l_t \leq (K-t) \left(2 + \log \frac{K}{K-t} + \log \frac{(N-K)}{K-t} \right) + \log \frac{1}{2} - M \log \frac{1}{\bar{a}_{t,L}}$$

For this term to be asymptotically negative, it is required that,

$$M \geq \frac{1}{\log \frac{1}{\bar{a}_{t,L}}} \left[(K-t) \left(2 + \log \frac{K}{K-t} + \log \frac{(N-K)}{K-t} \right) + \log \frac{1}{2} \right].$$

APPENDIX E

Proof of Theorem 2: With 1-bit quantized compressive measurements (18), it can be shown that the probability of the ML decoder is upper bounded by,

$$P_{err} \leq \frac{1}{2} \sum_{t=0}^{K-1} \binom{K}{t} \binom{N-K}{K-t} \left[(\mathbb{E}\{\tilde{a}_t^0\}) + (\mathbb{E}\{\tilde{a}_t^1\}) \right]^M$$

where $\mathbb{E}\{\tilde{a}_t^0\}$ and $\mathbb{E}\{\tilde{a}_t^1\}$ are as defined below.

$$\mathbb{E}\{\tilde{a}_t^0\} = \mathbb{E}\{(1-\lambda_j)^{\frac{1}{2}}(1-\lambda_k)^{\frac{1}{2}}|\ |\mathcal{U}_j\cap\mathcal{U}_k|=t\}$$

and

$$\mathbb{E}\{\tilde{a}_t^1\} = \mathbb{E}\{(\lambda_j)^{\frac{1}{2}}(\lambda_k)^{\frac{1}{2}}|\ |\mathcal{U}_j\cap\mathcal{U}_k|=t\}$$

with $\lambda_k = Q\left(\frac{-\sum_{i=1}^{K}(\tilde{\Phi}u_k)_{1i}\mu_i}{\sigma_v^2+\sigma_s^2\sum_{i=1}^{K}(\tilde{\Phi}u_k)_{1i}^2}\right)$. Letting $\bar{a}_t = \mathbb{E}\{\tilde{a}_t^0\} + \mathbb{E}\{\tilde{a}_t^1\}$, the upper bound for the probability of error of the 1-bit quantizer (18) reduces to,

$$P_{err} \leq \frac{1}{2}\sum_{t=0}^{K-1}\binom{K}{t}\binom{N-K}{K-t}\bar{a}_t^M. \tag{30}$$

Following a similar procedure as in Theorem 1, it can be shown that the required condition for the t-th term in (30) tend to zero is given by, For this term to be asymptotically negative, it is required that,

$$M \geq \frac{1}{\log\frac{1}{\bar{a}_t}}\left[(K-t)\left(2+\log\frac{K}{K-t}+\log\frac{(N-K)}{K-t}\right)+\log\frac{1}{2}\right].$$

APPENDIX F

Derivation of (21): When $\tilde{s}$ is a first order Gaussian, $\lambda_k = Q\left(\frac{-u_k}{\sigma_v^2}\right)$ with $u_k \sim \mathcal{N}(0, \frac{1}{N}||\boldsymbol{\mu}||_2^2)$. When the measurement noise power is negligible such that $\sigma_v^2 \to 0$, we have,

$$\lambda_k = \begin{cases} 1 & if\ u_k > 0 \\ 0 & if\ u_k < 0 \end{cases}$$

Then,

$$\begin{aligned} \mathbb{E}\{\tilde{a}_t^0\} &= \mathbb{E}\{(1-\lambda_j)^{\frac{1}{2}}(1-\lambda_k)^{\frac{1}{2}}|\ |\mathcal{U}_j\cap\mathcal{U}_k|=t\} \\ &= 1.Pr(u_k<0 \& u_j<0|\ |\mathcal{U}_j\cap\mathcal{U}_k|=t) \\ &= \int_{-\infty}^{0}\int_{\infty}^{0}f_{U_kU_j}(u_k,u_j)du_kdu_j \end{aligned} \tag{31}$$

where $f_{U_kU_j}(u_k,u_j)$ is bi-variate Gaussian with mean zero and the covariance matrix Σ_t. The quadrant probability of the bivariate Gaussian density in (31) is given by [45],

$$\mathbb{E}\{\tilde{a}_t^0\} = \frac{1}{4} + \frac{arcsin(\rho_t)}{2\pi}$$

Following a similar procedure, it can be shown that,

$$
\begin{aligned}
\mathbb{E}\{\tilde{a}_t^1\} &= \mathbb{E}\{(\lambda_j)^{\frac{1}{2}}(\lambda_k)^{\frac{1}{2}}|\ |\mathcal{U}_j \cap \mathcal{U}_k| = t\} \\
&= 1.Pr(u_k > 0 \& u_j > 0|\ |\mathcal{U}_j \cap \mathcal{U}_k| = t) \\
&= \frac{1}{4} + \frac{arcsin(\rho_t)}{2\pi}.
\end{aligned}
$$

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