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Augmented Lagrangian approach to design of structured optimal state feedback gains

Fu Lin, Makan Fardad, and Mihailo R. Jovanović

Abstract—**We consider the design of optimal state feedback gains subject to structural constraints on the distributed controllers. These constraints are in the form of sparsity requirements for the feedback matrix, implying that each controller has access to information from only a limited number of subsystems. The minimizer of this constrained optimal control problem is sought using the augmented Lagrangian method. Notably, this approach does not require a stabilizing structured gain to initialize the optimization algorithm. Motivated by the structure of the necessary conditions for optimality of the augmented Lagrangian, we develop an alternating descent method to determine the structured optimal gain. We also utilize the sensitivity interpretation of the Lagrange multiplier to identify favorable communication architectures for structured optimal design. Examples are provided to illustrate the effectiveness of the developed method.**

Index Terms—**Augmented Lagrangian, optimal distributed design, sparse matrices, structured feedback gains.**

I. INTRODUCTION

The design of distributed controllers for interconnected systems has received considerable attention in recent years [1]–[13]. For linear spatially invariant plants, it was shown in [1] that optimal controllers are themselves spatially invariant. Furthermore, for optimal distributed problems with quadratic performance indices the dependence of a controller on information coming from other parts of the system decays exponentially as one moves away from that controller [1]. These developments motivate the search for inherently *localized* controllers that communicate only to a subset of other controllers.

The main focus of this work is to search for an optimal distributed controller that is a static gain with *a priori* assigned structural constraints. The localized architectural requirements are formulated using matrix sparsity constraints. For example, for banded feedback gains, which are non-zero only on the main diagonal and a relatively small number of sub-diagonals, each controller uses information only from a limited number of neighboring subsystems. We search for structured controllers that minimize the H_2 norm and find a set of *coupled* algebraic matrix equations that characterize necessary conditions for the optimality.

The unstructured output feedback problem has been studied extensively since the original work of Levine and Athans [14]. Many computational methods have been proposed and, in general, they fall into two categories: (i) the general-purpose minimization methods, which include Newton's method [15] and quasi-Newton method [16]; and (ii) the special-purpose iterative methods [17], [18]. The advent of linear matrix inequality (LMI) has sparked renewed interest in fixed-order output feedback design [19]–[21]. Recently, nonsmooth optimization methods have been successfully employed for the design

of the fixed-order H_∞ and H_2 controllers [22], [23]. HIFOO, a Matlab package for fixed-order controller design, provides an effective means for solving many benchmark problems [24], [25].

In this note we employ the augmented Lagrangian method to design structured optimal state feedback gains. This approach does not require knowledge of a stabilizing structured gain to initialize the algorithm. A sequence of unstructured problems is instead minimized and the resulting minimizers converge to the optimal structured gain. We note that the augmented Lagrangian method was previously used to design decentralized dynamic controllers [26] and fixed-order H_∞ controllers [27], [28]. In contrast to these papers, we utilize structure of the necessary conditions for optimality of the augmented Lagrangian to develop an alternating descent method to determine the structured optimal gain. Furthermore, we use sensitivity interpretation of the Lagrange multiplier to identify favorable architectures for performance improvement.

Our presentation is organized as follows. In Section II, we formulate the structured optimal state feedback problem, introduce the augmented Lagrangian approach, and demonstrate how sensitivity interpretation of Lagrange multiplier can be utilized to identify favorable sparsity patterns for performance improvement. In Section III, we develop an alternating descent method for the minimization of the augmented Lagrangian. In Section IV, we illustrate the effectiveness of the proposed approach via two examples. We summarize our developments and comment on future directions in Section V.

II. PROBLEM FORMULATION AND AUGMENTED LAGRANGIAN METHOD

Let a linear time-invariant system be given by its state-space representation

$$
\begin{aligned}
\dot{x} &= A\,x + B_1\,d + B_2\,u, \\
z &= \begin{bmatrix} Q^{1/2}x \\ R^{1/2}u \end{bmatrix},
\end{aligned}
\tag{1}
$$

where x is the state vector, d is the disturbance, u is the control input, and z is the performance output. All matrices are of appropriate dimensions, and $Q^{1/2}$ and $R^{1/2}$ denote the square-roots of the state and control performance weights. We consider the *structured* state feedback design problem

$$
u = -Fx,
$$

where matrix F is subject to structural constraints that dictate the zero entries of F. For a mass-spring system in Fig. 1, if the controller acting on the ith mass has access to displacement and velocity of the ith mass and displacements of the two neighboring masses, then the feedback gain can be partitioned into $F = [\, F_p \quad F_v \,]$ where F_p is a tridiagonal matrix and F_v is a diagonal matrix.

For systems defined on general graphs the feedback matrix sparsity patterns can be more complex. Let the subspace $\mathcal{S}$ encapsulate these structural constraints and let us assume that there exists a non-empty set of stabilizing F that belongs to $\mathcal{S}$. Upon closing the loop, we have

$$
\begin{aligned}
\dot{x} &= (A - B_2 F)\,x + B_1\,d, \\
z &= \begin{bmatrix} Q^{1/2} \\ -R^{1/2}F \end{bmatrix} x, \quad F \in \mathcal{S}.
\end{aligned}
$$

Our objective is to find $F \in \mathcal{S}$ that minimizes the H_2 norm of the transfer function from d to z. This structured optimal control problem can be formulated as

$$
\begin{aligned}
\text{minimize} \quad & J(F) = \operatorname{trace}\Big(B_1^T \int_0^\infty \mathrm{e}^{(A-B_2F)^T t} \times \\
& \qquad (Q + F^T RF)\,\mathrm{e}^{(A-B_2F)t}\,\mathrm{d}t\, B_1 \Big) \\
\text{subject to} \quad & F \in \mathcal{S}.
\end{aligned}
\tag{2}
$$

Fig. 1: Mass-spring system on a line.

For stabilizing F, the integral in (2) is bounded and it can be evaluated by solving the Lyapunov equation

$$(A - B_2F)^T P + P(A - B_2F) = -\left(Q + F^T RF\right), \quad (3)$$

thereby yielding $J(F) = \text{trace}\left(B_1^T P(F)B_1\right)$.

The closed-loop H_2 norm of a stabilizable and detectable system increases to infinity as the least stable eigenvalue of $A_{cl} := A - B_2F$ goes towards the imaginary axis. For marginally stable and unstable systems, we define the H_2 norm to be infinity, which is consistent with the integral in the definition of the H_2 norm (2). Furthermore, $J(F)$ is a smooth function of F, since it is a product of the exponential and polynomial functions of the feedback gain. Therefore, the closed-loop H_2 norm is a smooth function that increases to infinity as one approaches the boundary of the set of stabilizing feedback gains. However, in general, the H_2 norm of the closed-loop system is not convex in the feedback gain [29], that is, $J(F)$ is not a convex function of F. Moreover, the set of all stabilizing feedback gains is not a convex set. On the other hand, $\mathcal{S}$ defines a linear subspace and thus $F \in \mathcal{S}$ is a linear constraint on the feedback gain.

If a stabilizing $F \in \mathcal{S}$ is known, descent algorithms can be employed to determine a local minimum of (2). However, finding a structured stabilizing gain is in general a challenging problem. To alleviate this difficulty, we employ the augmented Lagrangian method in Section II-A. We then provide the sensitivity interpretation of the Lagrange multiplier in Section II-B and introduce an alternating method for the minimization of the augmented Lagrangian in Section III.

A. Augmented Lagrangian method

The augmented Lagrangian method minimizes a sequence of *unstructured* problems such that the minimizers of the unstructured problems converge to the minimizer of (2). Therefore, the augmented Lagrangian method does not require a stabilizing *structured* feedback gain to initialize the optimization algorithm.

We begin by providing an algebraic characterization of the structural constraint $F \in \mathcal{S}$. Let $I_\mathcal{S}$ be the *structural identity* of the subspace $\mathcal{S}$ with its ijth entry defined as

$$[I_\mathcal{S}]_{ij} = \begin{cases} 1, & \text{if } F_{ij} \text{ is a free variable;} \\ 0, & \text{if } F_{ij} = 0 \text{ is required.} \end{cases}$$

If $I_\mathcal{S}^c := \mathbf{1} - I_\mathcal{S}$ denotes the structural identity of the *complementary* subspace $\mathcal{S}^c$, where $\mathbf{1}$ is the matrix with all its entries equal to one, then

$$F \in \mathcal{S} \Leftrightarrow F \circ I_\mathcal{S} = F \Leftrightarrow F \circ I_\mathcal{S}^c = 0,$$

where $\circ$ denotes the entry-wise multiplication of matrices. Therefore, the structured H_2 optimal control problem (2) can be rewritten as

$$\begin{aligned} \text{minimize} \quad & J(F) = \text{trace}\left(B_1^T P(F)B_1\right) \\ \text{subject to} \quad & F \circ I_\mathcal{S}^c = 0, \end{aligned} \quad \text{(SH2)}$$

where $P(F)$ is the solution of (3).

The Lagrangian function for (SH2) is given by

$$\mathcal{L}(F, V) = J(F) + \text{trace}\left(V^T(F \circ I_\mathcal{S}^c)\right).$$

From Lagrange duality theory [30]–[32], it follows that there exists a unique Lagrange multiplier $V^\star \in \mathcal{S}^c$ such that the minimizer of $\mathcal{L}(F, V^\star)$ with respect to F is a local minimum of (SH2). The Lagrange dual approach minimizes $\mathcal{L}(F, V)$ with respect to *unstructured* F for fixed V (the estimate of $V^\star$), and then updates V such that it converges to the Lagrange multiplier $V^\star$. Consequently, as V converges to $V^\star$, the minimizer of $\mathcal{L}(F, V)$ with respect to F converges to the minimizer of (SH2). This Lagrange dual approach is most powerful for convex problems [32]; for nonconvex problems, it relies on local convexity assumptions [31] that may not be satisfied in problem (SH2).

In what follows, a quadratic term is introduced to locally convexify the Lagrangian [30], [31] yielding the augmented Lagrangian for (SH2)

$$\mathcal{L}_c(F, V) = J(F) + \text{trace}\left(V^T(F \circ I_\mathcal{S}^c)\right) + (c/2)\|F \circ I_\mathcal{S}^c\|^2,$$

where the penalty weight c is a positive scalar and $\|\cdot\|$ is the Frobenius norm. Starting with an initial estimate of the Lagrange multiplier, e.g., $V^0 = 0$, the augmented Lagrangian method iterates between minimizing $\mathcal{L}_c(F, V^i)$ with respect to unstructured F (for fixed V^i) and updating

$$V^{i+1} = V^i + c(F^i \circ I_\mathcal{S}^c),$$

where F^i is the minimizer of $\mathcal{L}_c(F, V^i)$. Note that, by construction, V^i belongs to the complementary subspace $\mathcal{S}^c$, that is,

$$V^i \circ I_\mathcal{S}^c = V^i.$$

It can be shown [30], [31] that the sequence $\{V^i\}$ converges to the Lagrange multiplier $V^\star$, and consequently, the sequence of the minimizers $\{F^i\}$ converges to the structured optimal feedback gain $F^\star$.

Augmented Lagrangian method for (SH2):
Let $V^0 = 0$ and $c^0 > 0$, **for** $i = 0, 1, \ldots,$ **do**

(1) for fixed V^i, minimize $\mathcal{L}_c(F, V^i)$ with respect to unstructured F (see Section III);

(2) update $V^{i+1} = V^i + c^i(F^i \circ I_\mathcal{S}^c)$;

(3) update $c^{i+1} = \gamma c^i$ with $\gamma > 1$;

until: the stopping criterion $\|F^i \circ I_\mathcal{S}^c\| < \epsilon$ is reached.

The convergence rate of the augmented Lagrangian method depends heavily on the penalty weight c. In general, large c results in fast convergence rate. However, large values of c may introduce computational difficulty in minimizing the augmented Lagrangian. This is because the condition number of the Hessian matrix $\nabla^2 \mathcal{L}_c(F, V)$ becomes larger as c increases. It is thus recommended [30] to increase the penalty weight gradually until it reaches a certain threshold value τ. Our numerical experiments suggest that $c^0 \in [1, 5]$, $\gamma \in [3, 10]$, and $\tau \in [10^4, 10^6]$ work well in practice. Additional guidelines for choosing these parameters can be found in [30, Section 4.2].

B. Sensitivity interpretation

It is a standard fact that the Lagrange multiplier provides useful information about the sensitivity of the optimal value with respect to the perturbations of the constraints [30]–[32]. In particular, for the structured design problem, the Lagrange multiplier indicates how sensitive the optimal H_2 norm is with respect to the change of the structural constraints. We use this sensitivity interpretation to identify favorable sparsity patterns for improving H_2 performance.

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product of matrices $\langle M_1, M_2 \rangle = \text{trace}(M_1^T M_2)$. It is readily verified that $\|F \circ I_\mathcal{S}^c\|^2 = \langle F \circ I_\mathcal{S}^c, F \circ I_\mathcal{S}^c \rangle = \langle F \circ I_\mathcal{S}^c, F \rangle$ and $\langle V, F \circ I_\mathcal{S}^c \rangle = \langle V \circ I_\mathcal{S}^c, F \rangle = \langle V, F \rangle$ where we used the fact that $V \circ I_\mathcal{S}^c = V$. Thus the augmented

Lagrangian can be rewritten as

$$\mathcal{L}_c(F,V) \;=\; J(F) \;+\; \langle V, F \rangle \;+\; (c/2)\,\langle F \circ I_{\mathcal{S}}^c, F \rangle,$$

and its gradient with respect to F is given by

$$\nabla \mathcal{L}_c(F,V) \;=\; \nabla J(F) \;+\; V \;+\; c\,(F \circ I_{\mathcal{S}}^c).$$

Since the minimizer $F^\star$ of $\mathcal{L}_c(F, V^\star)$ satisfies $\nabla \mathcal{L}_c(F^\star, V^\star) = 0$ and $F^\star \circ I_{\mathcal{S}}^c = 0$, we have

$$\nabla J(F^\star) \;+\; V^\star \;=\; 0.$$

Let the structural constraints $\{F_{ij} = 0, \ (i,j) \in \mathcal{S}^c\}$ be relaxed to $\{|F_{ij}| \le w, \ (i,j) \in \mathcal{S}^c\}$ with $w > 0$, and let $\hat{F}$ be the minimizer of

$$\begin{aligned} \text{minimize} \quad & J(F) \\ \text{subject to} \quad & \{|F_{ij}| \le w, \ (i,j) \in \mathcal{S}^c\}. \end{aligned} \tag{RH2}$$

Since the constraint set in (RH2) contains the constraint set in (SH2), $J(\hat{F})$ is smaller than or equal to $J(F^\star)$,

$$J(\hat{F}) \;:=\; J(F^\star + \tilde{F}^\star) \;\le\; J(F^\star), \tag{4}$$

where $\tilde{F}^\star$ denotes the difference between $\hat{F}$ and $F^\star$. Now, the Taylor series expansion of $J(F^\star + \tilde{F}^\star)$ around $F^\star$ in conjunction with (4) yields

$$\begin{aligned} J(F^\star) - J(F^\star + \tilde{F}^\star) &= -\langle \nabla J(F^\star), \tilde{F}^\star \rangle + O(\|\tilde{F}^\star\|^2) \\ &= \langle V^\star, \tilde{F}^\star \rangle + O(\|\tilde{F}^\star\|^2) \\ &\ge 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} \langle V^\star, \tilde{F}^\star \rangle &\le \sum_{i,j} |V_{ij}^\star||\tilde{F}_{ij}^\star| \\ &= \sum_{(i,j)\in\mathcal{S}} |V_{ij}^\star||\tilde{F}_{ij}^\star| + \sum_{(i,j)\in\mathcal{S}^c} |V_{ij}^\star||\tilde{F}_{ij}^\star| \\ &\le w \sum_{(i,j)\in\mathcal{S}^c} |V_{ij}^\star|, \end{aligned}$$

where we have used the fact that $V_{ij}^\star = 0$ for $(i,j) \in \mathcal{S}$ and $|\tilde{F}_{ij}^\star| \le w$ for $(i,j) \in \mathcal{S}^c$. Thus, up to the first order in $\tilde{F}^\star$, we have

$$J(F^\star) \;-\; J(F^\star + \tilde{F}^\star) \;\le\; w \sum_{(i,j)\in\mathcal{S}^c} |V_{ij}^\star|.$$

Note that larger $|V_{ij}^\star|$ indicates larger decrease in the H_2 norm if the corresponding constraint $F_{ij} = 0$ is relaxed. This sensitivity interpretation can be utilized to identify favorable controller architectures; see Section IV-B for an illustrative example.

III. ALTERNATING METHOD FOR MINIMIZATION OF AUGMENTED LAGRANGIAN

In this section, we develop an alternating iterative method for minimization of the augmented Lagrangian. This method is motivated by the structure encountered in the necessary conditions for optimality (NC-L), (NC-P), and (NC-F) given below. We note that Newton's method, which is well-suited for dealing with ill-conditioning in $\mathcal{L}_c$ for large values of c [30], can also be employed to minimize the augmented Lagrangian.

Using standard techniques [14], [16], we obtain the expression for the gradient of $\mathcal{L}_c(F)$[1]

$$\begin{aligned} \nabla \mathcal{L}_c(F) &= \nabla J(F) + V + c\,(F \circ I_{\mathcal{S}}^c) \\ &= 2(RF - B_2^T P)L + V + c\,(F \circ I_{\mathcal{S}}^c). \end{aligned}$$

[1]Since V is fixed in minimizing $\mathcal{L}_c(F, V)$, we will use $\mathcal{L}_c(F)$ to denote the augmented Lagrangian.

Here, L and P are the controllability and observability Gramians of the closed-loop system,

$$(A - B_2 F)L + L(A - B_2 F)^T = -B_1 B_1^T, \tag{NC-L}$$
$$(A - B_2 F)^T P + P(A - B_2 F) = -(Q + F^T RF), \tag{NC-P}$$

and the necessary condition for optimality of $\mathcal{L}_c(F)$ is given by

$$2(RF - B_2^T P)L + V + c\,(F \circ I_{\mathcal{S}}^c) = 0. \tag{NC-F}$$

Solving the system of equations (NC-L), (NC-P), and (NC-F) is a non-trivial task. In the absence of structural constraints, setting $\nabla J(F) = 2(RF - B_2^T P)L = 0$ yields the optimal unstructured feedback gain

$$F_c \;=\; R^{-1} B_2^T P,$$

where the pair $(A - B_2 F, B_1)$ is assumed to be controllable and therefore L is invertible. Here, P is the positive definite solution of the algebraic Riccati equation obtained by substituting F_c in (NC-P)

$$A^T P + PA + Q - PB_2 R^{-1} B_2^T P = 0.$$

Starting with $F = F_c$, we can solve Lyapunov equations (NC-L) and (NC-P), and then solve (NC-F) to obtain a new feedback gain $\bar{F}$. We can thus alternate between solving (NC-L), (NC-P) and solving (NC-F).

In Proposition 1, we show that the difference between two consecutive steps $\bar{F} - F$ is a descent direction of $\mathcal{L}_c(F)$. Therefore, we can employ the Armijo rule to choose the step-size s in $F + s(\bar{F} - F)$ such that the alternating method converges to a stationary point of $\mathcal{L}_c(F)$. By virtue of the fact that the augmented Lagrangian $\mathcal{L}_c(F)$ is locally convex [30], [31], the stationary point indeed provides a local minimum of $\mathcal{L}_c(F)$. We then update V and c in the augmented Lagrangian (see Section II-A for details), and use the minimizer of $\mathcal{L}_c(F)$ to initialize another round of the alternating descent iterations. As V converges to $V^\star$, the minimizer of $\mathcal{L}_c(F)$ converges to $F^\star$. Therefore, the augmented Lagrangian method traces a solution path (parameterized by V and c) between the unstructured optimal gain F_c and the structured optimal gain $F^\star$. Here, we assume that F_c is contained in a connected set of stabilizing feedback gains that has a non-empty intersection with the subspace $\mathcal{S}$.

We summarize this approach in the following algorithm.

Alternating method to minimize augmented Lagrangian $\mathcal{L}_c(F, V^i)$

For $V^0 = 0$, start with the optimal unstructured feedback gain F_c;
For V^i with $i \ge 1$, start with the minimizer of $\mathcal{L}_c(F, V^{i-1})$;
for $k = 0, 1, \ldots,$ **do**

(1) solve Lyapunov equations (NC-L) and (NC-P) with $F = F_k$ to obtain L_k and P_k;

(2) solve linear equation (NC-F) with $L = L_k$ and $P = P_k$ to obtain $\bar{F}_k$;

(3) update $F_{k+1} = F_k + s_k(\bar{F}_k - F_k)$ where s_k is determined by Armijo rule;

until: The stopping criterion $\|\nabla \mathcal{L}_c(F_k)\| < \epsilon$ is reached.

Armijo rule [30, Section 1.2] for step-size s_k:
Let $s_k = 1$, repeat $s_k = \beta s_k$
until

$$\mathcal{L}_c(F_k + s_k(\bar{F}_k - F_k)) < \mathcal{L}_c(F_k) + \alpha\, s_k\, \langle \nabla \mathcal{L}_c(F_k), \bar{F}_k - F_k \rangle,$$

where $\alpha, \beta \in (0,1)$, e.g., $\alpha = 0.3$ and $\beta = 0.5$.

The descent property of $\bar{F}_k - F_k$ established in Proposition 1, continuity of $\bar{F}_k$ with respect to F_k, and the step-size selection using the Armijo rule guarantee the convergence of the alternating method [30]. Furthermore, for F_k sufficiently close to the local minimum we have established the linear convergence rate of the

alternating method; due to page limitations these convergence rate results will be reported elsewhere.

We conclude this section by establishing the descent property of the difference between two consecutive steps in the alternating method, $\bar{F} - F$.

Proposition 1: The difference between two consecutive steps, $\tilde{F} := \bar{F} - F$, is a descent direction of the augmented Lagrangian, $\langle \nabla \mathcal{L}_c(F), \tilde{F} \rangle < 0$. Moreover, $\langle \nabla \mathcal{L}_c(F), \tilde{F} \rangle = 0$ if and only if F is a stationary point of $\mathcal{L}_c(F)$, that is, $\nabla \mathcal{L}_c(F) = 0$.

Proof: Substituting $\bar{F} = F + \tilde{F}$ in (NC-F) yields

$$2R\tilde{F}L + c(\tilde{F} \circ I_{\mathcal{S}}^c) + \nabla \mathcal{L}_c(F) = 0. \tag{5}$$

Since R and L are positive definite matrices, we have

$$\langle \nabla \mathcal{L}_c(F), \tilde{F} \rangle = -2\langle R\tilde{F}L, \tilde{F} \rangle - c\langle \tilde{F} \circ I_{\mathcal{S}}^c, \tilde{F} \circ I_{\mathcal{S}}^c \rangle \leq 0. \tag{6}$$

We next show that the equality is achieved if and only if F is a stationary point, that is,

$$\langle \nabla \mathcal{L}_c(F), \tilde{F} \rangle = 0 \quad \Leftrightarrow \quad \nabla \mathcal{L}_c(F) = 0.$$

The necessity is immediate and the sufficiency follows from two facts: (i) equality in (6) implies $\tilde{F} = 0$, that is,

$$-2\langle R\tilde{F}L, \tilde{F} \rangle - c\langle \tilde{F} \circ I_{\mathcal{S}}^c, \tilde{F} \circ I_{\mathcal{S}}^c \rangle = 0 \Rightarrow \tilde{F} = 0,$$

and (ii) setting $\tilde{F} = 0$ in (5) yields $\nabla \mathcal{L}_c(F) = 0$. This completes the proof.

Remark: If R is a diagonal matrix, we can write the jth row of (5) as

$$\tilde{F}_j \big(2R_{jj}L + c\,\mathrm{diag}\,\{I_{\mathcal{S}j}^c\}\big) + \nabla \mathcal{L}_c(F)_j = 0,$$

where $(\cdot)_j$ denotes the jth row of a matrix and $\mathrm{diag}\,\{I_{\mathcal{S}j}^c\}$ is a diagonal matrix with $I_{\mathcal{S}j}^c$ on its main diagonal. Therefore each row of $\tilde{F}$ can be computed independently.

IV. EXAMPLES

We next demonstrate the utility of the augmented Lagrangian approach in the design of optimal structured controllers. The mass-spring system in Section IV-A illustrates the efficiency of the augmented Lagrangian method, and the vehicle formation example in Section IV-B illustrates the effectiveness of the Lagrange multiplier in identifying favorable controller architectures for improving H_2 performance.

A. Mass-spring system

We consider a mass-spring system with unit masses and unit spring constants shown in Fig. 1. If restoring forces are considered as linear functions of displacements, the state-space representation of this system is given by (1) with

$$A = \begin{bmatrix} O & I \\ T & O \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} O \\ I \end{bmatrix},$$

where I and O are $n \times n$ identity and zero matrices, and T is an $n \times n$ symmetric Toeplitz matrix with the first row given by $\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^n$. The state and control weights are assigned to be $Q = I$ and $R = 10I$.

We consider a situation in which the control applied to the ith mass has access to displacement and velocity of the ith mass, and displacements of p neighboring masses on the left and p neighboring masses on the right. Thus, $I_{\mathcal{S}} = \begin{bmatrix} S_p & I \end{bmatrix}$ where S_p is a banded matrix with ones on p upper and p lower sub-diagonals. For $n = 100$ masses with $p = 0, 1, 2, 3$, the results are summarized in Table I. Here, the stopping criterion for the augmented Lagrangian method is $\|F \circ I_{\mathcal{S}}^c\| < 10^{-6}$, and the stopping criterion for the alternating method is $\|\nabla \mathcal{L}_c(F)\| < 10^{-3}$.

TABLE I: Mass-spring system with $n = 100$ masses: p determines the spatial spread of the distributed controller, ALT# is the number of alternating steps.

p	ALT#	$J(F^\star)$	$J(F_c \circ I_{\mathcal{S}})$
0	92	499.9	546.5
1	83	491.2	497.2
2	71	488.0	489.6
3	70	486.8	487.6

We note that as the spatial spread p of the distributed controller increases (i) the improvement of $J(F^\star)$ becomes less significant; and (ii) $J(F_c \circ I_{\mathcal{S}}) \approx J(F^\star)$, i.e., near optimal performance can be achieved by the truncated optimal unstructured controller $F_c \circ I_{\mathcal{S}}$. These observations are consistent with the spatially decaying property of the optimal unstructured controller on the information from neighboring subsystems [1], [10].

B. Formation of vehicles

We consider a formation of nine vehicles in a plane. The control objective is to keep constant distances between neighboring vehicles. Modeling these independently actuated vehicles as double-integrators, in both horizontal and vertical directions, yields the state-space representation (1) with

$$A = \mathrm{diag}\{A_i\}, \quad B_1 = \mathrm{diag}\{B_{1i}\}, \quad B_2 = \mathrm{diag}\{B_{2i}\},$$

$$A_i = \begin{bmatrix} O_2 & I_2 \\ O_2 & O_2 \end{bmatrix}, \quad B_{1i} = B_{2i} = \begin{bmatrix} O_2 \\ I_2 \end{bmatrix}, \quad i = 1, \ldots, 9,$$

where I_2 and O_2 are 2×2 identity and zero matrices. The control weight R is set to identity, and the state weight Q is obtained by penalizing both the absolute and the relative position errors

$$x^T Q x = \sum_{i=1}^{9} \left(p_{1i}^2 + p_{2i}^2 + 10 \sum_{j \in \mathcal{N}_i} \big((p_{1i} - p_{1j})^2 + (p_{2i} - p_{2j})^2\big) \right),$$

where p_{1i} and p_{2i} are the absolute position errors of the ith vehicle in the horizontal and vertical directions, respectively, and set $\mathcal{N}_i$ determines neighbors of the ith vehicle.

The decentralized control architecture with no communication between vehicles specifies the block diagonal structure $\mathcal{S}_d$; see Fig. 2a. We solve (SH2) for $F \in \mathcal{S}_d$ and obtain the Lagrange multiplier $V^\star \in \mathcal{S}_d^c$; see Fig. 2b. Let

$$V_{ij}^\star \text{ be in group } \begin{cases} \text{small,} & \text{if } 0 < |V_{ij}^\star| \leq 0.5\,V_M, \\ \text{large,} & \text{if } |V_{ij}^\star| > 0.5\,V_M, \end{cases} \tag{7}$$

where V_M is the maximum absolute value of the entries of $V^\star$. We solve (SH2) for $F \in \mathcal{S}_s$ or $F \in \mathcal{S}_l$, where $\mathcal{S}_s$ and $\mathcal{S}_l$ are the subspaces obtained from removing the constraints $\{F_{ij} = 0\}$ corresponding to $\{V_{ij}^\star\}$ in groups small and large, respectively. We also consider the performance of the optimal controller in the unstructured subspace $\mathcal{S}_u$ with no constraints on F.

Table II shows the influence of the number of optimization variables on the performance improvement. Note that $\mathcal{S}_l$ has the largest improvement *per variable* among all three structures $\mathcal{S}_s$, $\mathcal{S}_l$, and $\mathcal{S}_u$. As illustrated in Fig. 3, $\mathcal{S}_l$ determines a *localized* communication architecture in which each vehicle communicates only with its neighbors. Therefore, the Lagrange multiplier $V^\star$ identifies distributed controller with nearest neighbor interactions as the favorable controller architecture. This is in agreement with [10] where it was shown that optimal unstructured controllers for systems on general graphs possess spatially decaying property; similar result was proved earlier for spatially invariant systems [1].
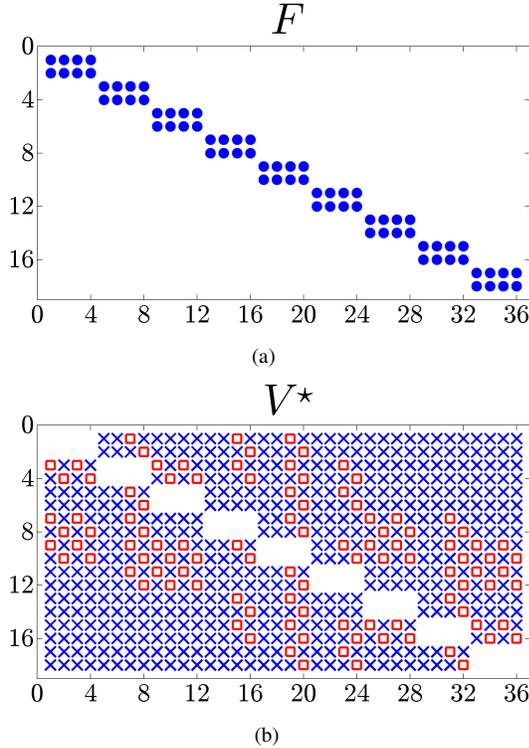
Fig. 2: (a) Block diagonal feedback gain F where each block signifies that the two control inputs acting on each vehicle only have access to the four states of that vehicle; (b) Lagrange multiplier $V^\star$ with entries separated into groups small ($\times$) and large ($\square$) according to (7).
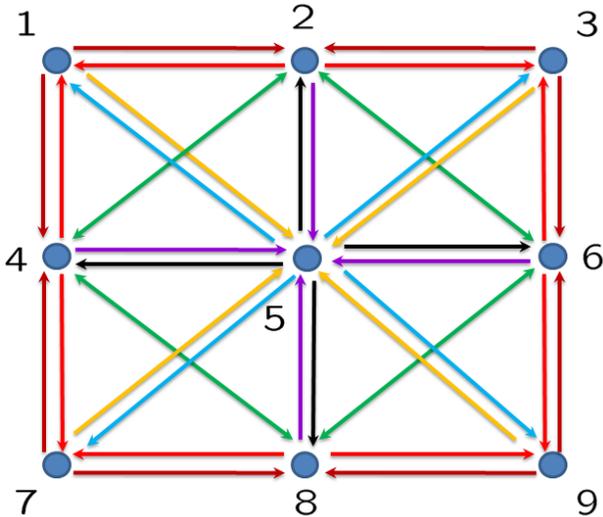

Fig. 3: Localized controller architecture in which each vehicle communicates only with its neighbors. The arrow directed from node i to node j indicates that node i is sending information to node j. Priority order of communication channels is determined by the absolute values of $V^\star_{ij}$, ranging from the highest to the lowest: brown, red, orange, green, blue, purple, and black.

TABLE II: Performance improvement, $\kappa = (J^\star_d - J^\star)/J^\star_d$, relative to the optimal H_2 norm $J^\star_d = 65.4154$ with decentralized structure $\mathcal{S}_d$. Here, ρ is the number of extra variables in $\mathcal{S}_s$, $\mathcal{S}_l$, and $\mathcal{S}_u$ compared with $\mathcal{S}_d$, and κ/ρ is the performance improvement per variable.

	$J^\star$	κ	ρ	κ/ρ
$\mathcal{S}_s$	64.1408	1.95%	472	0.0041%
$\mathcal{S}_l$	64.2112	1.84%	104	0.0177%
$\mathcal{S}_u$	62.1183	5.04%	576	0.0088%

V. CONCLUDING REMARKS

In this note, we consider the design of structured optimal state feedback gains for interconnected systems. We employ the augmented Lagrangian method and utilize the sensitivity interpretation of the Lagrange multiplier to identify favorable communication architectures for structured optimal design. The necessary conditions for optimality of the augmented Lagrangian are given by coupled matrix equations. Motivated by the structure of these equations, we develop an alternating descent method for obtaining the optimal feedback gain. The proposed approach does not require a stabilizing structured controller to initialize the iterative procedure and its utility is illustrated by two examples.

Although we focus on structural equality constraints, we note that it is also possible to incorporate inequality constraints, e.g., $|F_{ij}| \le w_{ij}$, in the augmented Lagrangian method [30], [31]. This extension is expected to be useful in applications where controller saturations or limited communication budgets are incorporated in the design.

In our ongoing efforts, we are applying the tools developed here to the control of vehicular formations [33], [34], and to the design of consensus-type algorithms for optimal performance over general connected networks. These problems have received considerable attention in recent years but a systematic procedure for the design of optimal localized controllers is yet to be developed. The algorithms developed here will also be useful in analyzing the scaling of different performance measures with respect to the network size [34]. Such analysis will provide insight into the fundamental limitations of the performance achievable using localized control strategies with relative information exchange in systems with arbitrary communication topologies.

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