7-19-1994

Self-Organized Criticality in Non-Conserved Systems

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(February 1, 2008)

Abstract

The origin of self-organized criticality in a model without conservation law (Olami, Feder, and Christensen, Phys. Rev. Lett. **68**, 1244 (1992)) is studied. The homogeneous system with periodic boundary condition is found to be periodic and neutrally stable. A change to open boundaries results in the invasion of the interior by a "self-organized" region. The mechanism for the self-organization is closely related to the synchronization or phase-locking of the individual elements with each other. A simplified model of marginal oscillator locking on a directed lattice is used to explain many of the features in the non-conserved model: in particular, the dependence of the avalanche-distribution exponent on the conservation parameter α is examined.

05.70.Jk,05.45.+b,91.30.-f

The phenomenon of the self-organized criticality (SOC) [1] is characterized by spontaneous and dynamical generation of scale-invariance in an extended non-equilibrium system. One of the key issues in this field is to identify the mechanism(s) of SOC. It has been conjectured that conservation laws or special symmetries are necessary [2]. Conservation laws certainly are of great importance in "sandpile" models [1,3,4], where the scale invariance can be shown to follow from a local conservation law (sand grains are conserved except at the boundaries of the pile) [5]. In this sense, the origins of long range correlations in SOC systems with conservation are well understood, though not all exponents have been calculated analytically. However, models have been constructed [6–9] that have no apparent conservation law, and yet display a power-law distribution of avalanche sizes. Of particular interest is the model proposed by Olami, Feder, and Christensen (OFC) [9–11] which, they argue, models earthquake dynamics. The OFC model is very similar to the conserved sandpile model [1], but it has a parameter which defines the degree of conservation. In this paper, we study in detail the OFC model and we find that the self-organization is due to synchronization or "phase-locking" – a mechanism very different from that in the conserved models.

In the OFC model, dynamical "height" variables h_i are defined on sites i of a square lattice. The h_i increase at unit rate until $h = 1$ at some location. The site j where $h_j = 1$ is considered to be unstable and will "topple". The rule of toppling is that when $h_j \geq 1$, then $h_j \to 0$ and $h_k \to h_k + \alpha h_j$, for all k neighboring j. The toppling on site j may cause its neighbors to become unstable ($h_k \geq 1$) and to topple. This procedure is repeated until all sites are stable ($h_i < 1$ everywhere). The magnitude of the avalanche is given by the total "energy" dissipated in the process, i.e., the total change in $\sum_i h_i$. The avalanche, which happens instantly on the time scale of driving [13], is then followed by growth. The parameter α is the measure of conservation (of h's). When $\alpha = 1/4$, the model is conserved and it is in the same universality class as for the BTW model [1,9,12]. We study here the non-conservative case $0 < \alpha < 1/4$.

We note that there is an ambiguity in this model. After some number of avalanches, it will occur that more than one site will have exactly the same height, as more than one site may topple to a height of exactly zero in a single avalanche. This can lead to two neighboring sites toppling simultaneously; the above procedure does not define the result of such events. We have modified the rules in several ways, e.g. by adding a very small amount of random noise to ensure that no two sites have the same height, and they all give the similar results.

As most avalanches are small, and do not change the h_i at many sites, we do not need to scan the whole lattice after each avalanche in order to determine the most unstable site. We use a tree structure to keep track of the highest values of h_i in order to determine avalanche trigger sites. Using this technique, we have simulated up to more than 10^{10} avalanches for single systems.

We find that the system with periodic boundary conditions quickly reaches an exactly periodic state [10], with a unique period in the slow time. It has been noted [9,11] that in the case of periodic boundary conditions, the avalanche size distribution function drops very quickly with size. In the case of our modified model, which prevents two sites from having the same value of h, all avalanches consist of the toppling of exactly one lattice site, after a brief transient time. In a periodic state, the h_i's take turns to topple, one by one. The height h decreases by one each time a site topples and increases by 4α due to the toppling of the four neighbors when they reach $h = 1$. Thus the period of all these periodic states is $1 - 4\alpha$ in the slow time variable, so that the slow "growth" is balanced exactly by the dissipation due to toppling. These periodic states are highly degenerate and neutrally stable in the sense that a typical small perturbation of the height at a single site in a periodic state is still a periodic state. They are similar to the neutrally stable periodic states in coupled oscillators [14]. There is a continuous set of periodic states in the attractor, with measure $(1 - 4\alpha)^V$ in the initial phase space, where V is the system volume.

Any inhomogeneity, such as a change in boundary conditions, destroys such simple periodic states. When the boundary conditions are open, the system can no longer have period $1-4\alpha$, as the boundary sites have 3 neighbors (we study a system that is open on one axis, with the other directions periodic). Initially, the interior sites quickly converge to a nearly periodic state and topple with period $1 - 4\alpha$, but the boundaries are aperiodic. At longer times, the aperiodic region invades the periodic interior, as shown in Fig. 1. This invasion, which destroys the periodicity in the interior and builds up long-range correlations, occurs by a mechanism similar to oscillator locking, as we describe below. The interface between the two regions is well defined on scales larger than one or two lattice constants. The invasion distance appears to have a power law dependence on time, $y(t) \sim t^{\beta(\alpha)}$ as shown in Fig. 2, with $\beta = 0.23 \pm 0.08, 0.63 \pm 0.08$ for $\alpha = 0.07, 0.15$, respectively. We see such invasion occurring even for values of $\alpha < 0.05$, though β appears to be quite small, so that the invasion is extremely slow. This suggests that the transition to non-SOC behavior claimed by Olami, Feder, and Christensen [9] may only be apparent, due to the finite time of the simulations; we note that the time for complete invasion of a 128^2 system with $\alpha = 0.07$ is greater than 10^{10} avalanches. In the limit of long times, when the invasion crosses the whole sample, the distribution $P(s;\alpha)$ of avalanches of size s is a power-law, with

$$P(s;\alpha) \sim s^{-\tau(\alpha)}. \tag{1}$$

Consistent with OFC, we find $\tau(\alpha) = 3.2 \pm 0.1, 2.3 \pm 0.1$ for $\alpha = 0.07, 0.15$, respectively. The avalanches are not uniformly distributed in the system; the typical avalanche size grows with distance from the edge.

The influence of inhomogeneity, which leads to a breaking of the $1-4\alpha$ periodic state, is quite strong. Even a single defect, where h is set to zero for all time, destroys the periodic state, and leads to a power law distribution of avalanche sizes.

The spatial distribution of length scales is apparent in Fig. 1(d), which shows an example of a configuration between avalanches in the steady state. Near the boundaries, the h_i have only short range correlations, but this correlation length grows with distance from the boundary. The *typical* avalanches in the interior, though, still involve the toppling of only one lattice site. Infrequently, an avalanche is triggered near the boundaries, which penetrates into a longer-range correlated region, giving a large avalanche. We define a toppling rate $r(y)$ as a function of distance y from the boundary, which gives the inverse of the mean time between topplings. At the boundaries, where the sites have only three neighbors, $r(y)$ is smaller than the interior, where $r(y) \to (1-4\alpha)^{-1}$ as $y \to \infty$. The toppling rate differential, defined as $\delta r(y) = (1-4\alpha)^{-1} - r(y)$ is found to behave as

$$\delta r(y) \sim y^{-\eta} \tag{2}$$

with $\eta = 3.2 \pm 0.6, 1.8 \pm 0.2$ for $\alpha = 0.07, 0.15$, respectively. Let $h_t(y)$ be the average height just before toppling, $R(y) \equiv (1-4\alpha)r(y)h_t(y)$ is then the dissipation rate and $R(y) \to 1$ as $y \to \infty$. It can be shown that in the steady state $\delta R(y) = 1 - R(y)$ behaves as $\delta R(y) = exp(-y\sqrt{\frac{1-4\alpha}{\alpha}})$. Thus the dissipation rate is rather uniform except within a boundary layer of thickness $\sqrt{\frac{\alpha}{1-4\alpha}}$. The power-law behavior of Eq. (2) must be compensated by a power-law in $\delta h_t(y) = h_t(y) - 1$:

$$\delta h_t(y) \sim y^{-\eta}. \tag{3}$$

Note that the toppling height can be larger than 1 only when the toppling is triggered by neighboring sites and hence is part of an avalanche which involves more than one site. The picture we get from Eqs. (2) and (3) is that although the toppling happens less frequently as we move towards the boundary a larger portion of it is due to multi-site avalanches.

In order to gain some insights on the build-up of long range correlations in the inhomogeneous system, we consider a system which has only two sites: h_1 and h_2. Let us first consider the homogeneous case: both h_1 and h_2 are driven with unit rate and when one of them reaches the value one it topples. The rule of toppling is that: if $h_{1(2)} \geq 1$, then $h_{2(1)} \to h_{2(1)} + \alpha h_{1(2)}$ and $h_{1(2)} \to 0$. This simple system is completely integrable. It has a continuous set of periodic states which are marginally stable. To illustrate its dynamics, we construct a Poincaré map. Denote $h_1(n)$ to be the value of h_1 right after the nth toppling of h_2. It is easy to show that [10]

$$h_1(n+1) = \begin{cases} h_1(n) & \alpha \leq h_1(n) < 1 \\ 1 + \alpha - \alpha h_1(n) & 1 \leq h_1(n) < 1/\alpha \\ \alpha^2 h_1(n) & h_1(n) \geq 1/\alpha \end{cases} \tag{4}$$

which is sketched in Fig. 3(a). We see that there is a line of marginally stable fixed points $h_1^* \in [\alpha, 1)$. These fixed points are periodic states with period $1-\alpha$: h_1 and h_2 take turns to topple and the toppling of one site will not trigger the toppling of another ($h_1^* < 1$). Now, we introduce a little inhomogeneity. We drive h_1 with rate 1, but h_2 with a slightly slower rate $(1+\epsilon)^{-1}$. The Poincaré map now reads

$$h_1(n+1) = \begin{cases} h_1(n) + \epsilon(1-\alpha) & \alpha \le h_1(n) < 1 \\ 1 + \alpha + \epsilon - \alpha(1+\epsilon)h_1(n) & 1 \le h_1(n) < 1/\alpha \\ \alpha^2 h_1(n) & h_1(n) \ge 1/\alpha \end{cases} \tag{5}$$

which is sketched in Fig. 3(b). In this case, there is only one fixed point $h_1^* = 1 + \frac{1-\alpha}{1+\alpha}\epsilon$. This fixed point is the phase-locked or synchronized state: the toppling of h_2 will trigger h_1 to topple ($h_1^* > 1$). Note that Eq. (5) is only ϵ away from Eq. (4), so the locking is rather weak and fluctuations can play a crucial role. In the OFC model with open boundary conditions, the boundaries introduce inhomogeneity. The sites at the open boundaries have only three neighbors and hence have a *slower effective growth rate*. This inhomogeneity in the effective growth rate propagates into the interior of the sample, causing phase-locking and thus long range correlation. However, the whole system is not in a synchronized state. Rather, it is only "marginally" locked so that it gives a power-law distribution of avalanche sizes.

We do not have a complete theory for the emergence of the "marginal locking" in the OFC model. However, we can abstract some of the features to construct a simpler model which also exhibits SOC. This model is defined on a directed lattice to simplify avalanches. We define dynamical variables $0 \le \phi_i < 1$ as the *phase* of the next toppling time of a site; this phase is related to the height h_i in the OFC model at a fixed time, modulo the natural period $(1-4\alpha)$. At random sites on the boundary, we initiate a toppling which changes the phase according to $\phi_i \to \phi_i' = \phi_i + \alpha$. If the phase ϕ_j at a neighboring "downhill" site $j = i + \hat{x}$ or $i + \hat{y}$ (see inset in Fig. 4) meets the locking condition $(\phi_i - \phi_j \bmod 1) < \alpha$, the site j locks onto the boundary site, with $\phi_j \to \phi_i'$. This disturbance can then continue to propagate by sites further from the boundary becoming locked, in zero time. We refer to this locking as “marginal” since neighboring phases only lock upon a *crossing* in the toppling time; the configuration is neutrally stable with respect to a continuous set of perturbations. This model differs from the OFC model most notably in the directed lattice and the instantaneous locking (even though avalanches in OFC occur in zero time, locking takes place over a time of order 1).

A snapshot of the ϕ_i in the steady state is depicted in Fig. 4(a). The domains, bounded by solid lines, are regions where the toppling times are identical. Fig. 4(b) shows a configuration after an avalanche, with the dotted lines showing the previous domain configuration. An avalanche crosses domain boundaries only when the neighboring domains have times that differ by no more than α. Numerically, it is found that the number of domains $n(S)$ of size S at a fixed time has a power law tail that is independent of alpha, $n(S) \sim S^{-\sigma}$, with $\sigma = 1.495 \pm 0.005$. Yet the avalanche distribution has an α dependent exponent, with the probability of an avalanche of size s behaving as $P(s) \sim s^{-\tau(\alpha)}$ (Fig. 5). Note that domains and avalanches are closely related, with avalanches defining domains. The exponent σ must satisfy the bound $\sigma \leq 3/2$; the fact that our numerical result saturates this inequality, to within numerical error, suggests that the domains have linear roughness, with the average width of a domain proportional to the length of the domain perpendicular to the boundary.

The relationship between τ and σ can be approximately explained. Assuming that the toppling times of neighboring domains are independent variables, the probability of an avalanche, which takes place in one domain, incorporating a given neighboring domain is just α. The incorporation of a neighboring domain increases the avalanche size to the scale of the neighboring domain, which is typically larger than the original domain. Given a distribution of domain sizes $n(S) \sim S^{-\sigma}$ and a scale independent probability $\alpha' \propto \alpha$ that a larger domain than the initial one chosen will become part of the avalanche, followed by a probability α' that an even larger domain will become part of the avalanche, etc., leads to a distribution of avalanches $P(s) \sim s^{-\tau(\alpha)}$, with $\tau(\alpha) = \sigma + \alpha'(1 - \sigma)$. This is in agreement with the linear fit of Fig. 5, with $\alpha' = (1.20 \pm 0.04)\alpha$; the fitted value of $\tau(0) = \sigma = 1.505 \pm 0.005$ agrees with the value determined by the domain distribution. The varying exponent of the avalanche distribution is then explained by the probability α' for the avalanche size to move out further into the tails of the α-independent domain size distribution.

In this paper we have examined how SOC can arise in a model without a conservation law. The OFC model with periodic boundary condition has a continuous set of neutrally stable periodic states. In general, inhomogeneity destroys these periodic states and causes phase-locking which is the building block for long range correlations. We found that an open boundary results in the invasion of the interior by a marginally locked region in which the avalanche size distribution is a power-law. A simplified model on a directed lattice has been used to demonstrate how an α-dependent avalanche size distribution exponent can arise in such non-conserved dynamical models. Finally, we note that the OFC model is similar to the coupled "integrate-and-fire" oscillators studied in the context of neural networks and biology. A close cousin is the Peskin's model for the cardiac pacemaker. [15] We found that the model in 2d with nearest neighbor coupling tends to lock into some periodic or "quasiperiodic" cluster state and that it has richer behaviors than simple synchronization.

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FIGURES

FIG. 1. Configurations of the Olami-Feder-Christensen model at various times after random initialization, demonstrating the invasion of the short-range correlated interior by the "SOC region". The density of the color corresponds to the height variables $0 \le h_i < 1$. The boundaries are periodic in the vertical direction and open in the horizontal. The lattice consists of 64^2 sites, with $\alpha = 0.07$. Times are $t =$ (a) 1.2×10^3, (b) 2.4×10^3, (c) 6.0×10^3, (d) 36.0×10^3.

FIG. 2. A plot of the invasion distance vs. time for conservation parameters $\alpha = 0.07, 0.15$. Power law fits are shown as dashed lines; symbols indicate system size ($\triangle = 64^2$, $\square = 128^2$, $\bullet = 256^2$).

FIG. 3. Return map for the two-site system: (a) homogeneous system; (b) inhomogeneous system.

FIG. 4. Domains in the simplified SOC model described in the text, for a 200^2 directed lattice. (a) Configuration with domains of identical toppling times indicated by solid lines. (b) Configuration after an avalanche, with previous domains indicated by dashed lines. The inset sketches the directed lattice.

FIG. 5. The avalanche distribution exponent $\tau(\alpha)$ for the directed model as a function of the conservation parameter α. The linear fit is discussed in the text.