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Abstract

This paper studies the asymptotic properties of within groups *k*-class estimators in a panel data model with weak instruments. Weak instruments are characterized by the coefficients of the instruments in the reduced form equation shrinking to zero at a rate proportional to $\sqrt{n}T^{\delta}$; where n is the dimension of the cross-section and T is the dimension of the time series. Joint limits as $(n,T)\to\infty$ show that this within group *k*-class estimator is consistent if $0\le\delta\le ½$ and inconsistent if $½\le\delta\le\infty$.

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Key Words: Weak Instrument; Panel Data; fixed effects; Pitman drift local-to-zero

On the Estimation and Testing of Fixed Effects Panel Data Models with Weak Instruments

Badi H. Baltagi[*], Chihwa Kao[†], Long Liu[‡]

August 8, 2012

Abstract

This paper studies the asymptotic properties of within groups k-class estimators in a panel data model with weak instruments. Weak instruments are characterized by the coefficients of the instruments in the reduced form equation shrinking to zero at a rate proportional to $\sqrt{n}T^{\delta}$, where n is the dimension of the cross-section and T is the dimension of the time series. Joint limits as $(n,T) \to \infty$ show that this within group k-class estimator is consistent if $0 \le \delta < \frac{1}{2}$ and inconsistent if $\frac{1}{2} \le \delta < \infty$.

Key Words: *Weak Instrument; Panel Data; fixed effects; Pitman drift local-to-zero.*

1 Introduction

This paper contributes to the literature on weak instrumental variable (IV) for panel data models with fixed effects. The problem of weak instruments have attracted considerable attention in recent years, see Stock, Wright and Yogo (2002) for an excellent survey. Weak instruments are characterized by the coefficients of the instruments in the reduced form equation shrinking to zero at a rate proportional to the square root of the sample size. In case of weak instruments, the usual asymptotic normal approximations of the 2SLS estimator can be quite poor, even if the number of observations is large. Staiger and Stock (1997) use we*ak-instrument asymptotics* to show that the 2SLS estimator is inconsistent (i.e., converges to a random variable) and has a nonstandard limiting distribution. This is a serious problem as inference, test of hypotheses and confidence intervals in the case of weak-instruments becomes unreliable and misleading.

Bai and Ng (2010) show that for panel data models in which all regressors are endogenous but share exogenous common factors, valid instruments can be constructed from the endogenous regressors that are themselves invalid instruments in a conventional sense. This requires both dimensions of the panel n and T

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to be large. More recently, Cai, Fang and Li (2012) argue that there may be benefits to using panel data when the available instruments are cross-sectionally weak. They consider the within-group 2SLS (W2SLS) estimator in a panel context where the degree of weakness of the instruments depends upon the number of cross-sectional observations n only. For large n, and fixed T, they show that the bias of W2SLS is of order $1/T$ as $n \to \infty$. They argue that leting the degree of weakness of the instruments depend on n only is an "analytical device" and with T fixed, "it is natural to relate the degree of weakness to n only". However, from Staiger and Stock (1997), the degree of weakness of the instruments depends upon the total number of observations nT and how n and T tend to infinity is crucial for the asymptotics of weak instruments in panel data.[1] This paper extends the results presented in Cai, Fang and Li (2012) to the case where the weak instruments are modeled as "Pitman drift" local-to-zero sense, and the degree of weakness of the instruments is allowed to depend upon both n and T, but with different impact. To be specific, we let the degree of weakness of the instruments depend upon $\sqrt{n}T^{\delta}$ where $\delta \geq 0$. When $\delta = 0$, it reduces to the weak instrument case in Cai, Fang and Li (2012). When $\delta = 1/2$, it reduces to the weak instrument case in Staiger and Stock (1997). The basic argument is that with enough time periods observed, panel data may provide enough information to yield consistent estimation. In fact, it is well known that for cross-sectional data, when the concentration parameter stays constant as the sample size grows, the signal of the model is too weak compared to the corresponding noise. Hence the model is weakly identified, and 2SLS converges to a random variable. However, in the panel data set-up, if the time series dimension is large, the weak signal can be strengthened by the repeating regression across the time series dimension. This argument is similar in spirit to the argument of establishing consistency for the panel spurious regression, see for example Phillips and Moon (1999) and Kao (1999).

Cai, Fang and Li (2012) also considered the case where the degree of weakness of the instruments depends upon n^{δ}, where $\delta \geq 0$. For a fixed T, when $0 < \delta < \frac{1}{2}$, the correlation between the instruments and endogenous variables converges to zero more slowly than the square root of the sample size, as $n \to \infty$. This corresponds to the *nearly weak instruments* case of Hahn and Kuersteiner (2002) and Hahn, Hausman and Kuersteiner (2004). For $\delta = 1/2$, this is the *weak instruments* case, and for $\delta > \frac{1}{2}$, this is the *nearly non-identified* case because the correlation converges to zero faster than the square root of the sample size, as $n \to \infty$. For cross-section or time-series models, Hahn and Kuersteiner (2002) showed that 2SLS for the *nearly weak instruments* case is consistent and its limiting distribution is normal. However, for the *weak instruments* case as well as the *nearly non-identified* case, 2SLS is inconsistent and its limiting distribution is not normal. Cai, Fang and Li (2012) similarly showed that for panel data models with fixed T, the bias

[1] How n and T tend to infinity was emphasized by Phillips and Moon (1999) for panel unit root testing.

of W2SLS estimator with *weak* or *nearly non-identified* instruments is of order $1/T$ as $n \to \infty$. They argue that as $T \to \infty$, W2SLS is consistent and asymptotically normal. They also consider a mixed case where some instrumental variables are weak and others are nearly weak and show that as $n \to \infty$, with T fixed, the W2SLS estimator of the weak instruments is biased of order $1/T$, while the W2SLS estimator of the nearly weak instruments is consistent. We generalize the Cai, Fang and Li (2012) panel data results by studying the asymptotic properties of the general within-group k-class estimator, which includes W2SLS and within-group LIML as special cases. We allow the degree of weakness of the instruments to depend upon $\sqrt{n}T^{\delta}$ where $\delta \geq 0$. We study the asymptotics using joint limits in n and T, rather than fixing T and letting $n \to \infty$. We show that for the simple case of one right hand side endogenous variable and no included exogenous variables, W2SLS is consistent if $0 \leq \delta < \frac{1}{2}$ and inconsistent if $\frac{1}{2} \leq \delta < \infty$. Next, we generalize these results to the within group k-class estimator with included exogenous regressors applied to fixed effects panel data. We show using joint limits that this within group k-class estimator is consistent if $0 \leq \delta < \frac{1}{2}$ and inconsistent if $\frac{1}{2} \leq \delta < \infty$. We characterize these conditions for three special cases of the within group k-class estimator including W2SLS, within group LIML, and within group bias-adjusted 2SLS. We also generalize the test for weak instruments proposed by Cragg and Donald (1993) and Stock and Yogo (2005) to the case of fixed effects panel data as well as test of hypothesis that is robust to weak instruments in the fixed effects panel data set-up. We study the asymptotic properties of these tests as both $(n, T) \to \infty$.[2]

The rest of the paper is organized as follows. Section 2 introduces the fixed effects panel data model with weak instruments. Section 3 discusses the within group k-class estimator. Section 4 generalizes the test for weak instruments proposed by Cragg and Donald (1993) and Stock and Yogo (2005) to the case of fixed effects panel data. Section 5 considers the problem of hypothesis testing whose size is robust to weak instruments in the fixed effects panel data set-up. Section 6 provides Monte Carlo results, while Section 7 concludes. All the proofs are relegated to the appendix. All the limits in the paper are taken as $(n, T) \to \infty$ jointly, except when otherwise noted.

2 Model and Assumptions

Consider the following panel IV regression model with endogenous regressors

$$y_t = Y_t\beta + X_t\gamma + \mu + u_t \tag{1}$$

[2]Cai, Fang and Li (2012) also consider some asymptotically pivotal tests in the case of fixed effects panel data and study their asymptotic properties for fixed T and $n \to \infty$.

and

$$Y_t = Z_t \Pi + X_t \Gamma + \alpha + V_t \tag{2}$$

for $t = 1, 2, \cdots, T$, where y_t is a $n \times 1$ vector and Y_t is a $n \times L$ matrix of endogenous variables, X_t is a $n \times K_1$ matrix of K_1 exogenous regressors, Z_t is a $n \times K_2$ matrix of K_2 instruments, and β, γ, Π, and Γ are unknown parameters. μ and α denote the individual effects which are of dimensions $n \times 1$ and $n \times L$ respectively. The remainder disturbances $(u_t, V_t')'$ are of dimensions $n \times 1$ and $n \times L$ respectively. These disturbances $(u_t, V_t)'$ are assumed to be i.i.d. $N(0, \Sigma)$ across $t = 1, 2, \cdots, T$, with the elements of Σ denoted by σ_{uu}, Σ_{Vu}, and Σ_{VV}. Let $Z^* = [X, Z]$, $Y^* = [y, Y]$ and let $\Phi = EZ_{it}^{*\prime} Z_{it}^*$, partitioned so that $EX_{it}X_{it}' = \Phi_{XX}$, $EX_{it}Z_{it}' = \Phi_{XZ}$, and $EZ_{it}Z_{it}' = \Phi_{ZZ}$. It is assumed throughout that $EZ_{it}^*(u_{it}, V_{it}') = 0$ for all i and t. This i.i.d. assumption for the errors can be relaxed to allow for weak dependence across the time series and cross-section dimensions at the expense of more complicated notation. This will be taken up in a future extension of this paper. Equation (1) is the structural equation and β is the parameter of interest. The reduced-form equation (2) relates the endogenous regressors to the instruments. In matrix form, equations (1) and (2) can be rewritten as

$$y = Y\beta + X\gamma + \mu \otimes \iota_T + u \tag{3}$$

and

$$Y = Z\Pi + X\Gamma + \alpha \otimes \iota_T + V \tag{4}$$

where $y = (y_1', y_2', \cdots, y_T')'$ is a $nT \times 1$ vector, ι_T is a vector of ones of dimension T, and Y, X, Z, u, and V are similarly defined.

To wipe out the individual effects, we premultiply equations (3) and (4) by the within transformation $Q = I_n \otimes E_T$, where $E_T = I_T - \bar{J}_T$, $\bar{J}_T = J_T/T$; J_T is a matrix of ones of dimension T and I_n is an identity matrix of dimension n. This yields

$$\tilde{y} = \tilde{Y}\beta + \tilde{X}\gamma + \tilde{u} \tag{5}$$

and

$$\tilde{Y} = \tilde{Z}\Pi + \tilde{X}\Gamma + \tilde{V} \tag{6}$$

where $\tilde{y} = Qy$, and $\tilde{Y}$, $\tilde{X}$, $\tilde{Z}$, $\tilde{u}$, and $\tilde{V}$ are similarly defined. This wipes out possible correlation between these individual effects and the regressors. It also wipes out time-invariant variables that may cause omission bias if not included in the model. We model weak instruments by focussing on Π being local to zero which is analogous to the local-to-unity panel unit root literature as in Moon et al. (2007).

Assumption 1 *Let $\Pi = \frac{C}{\sqrt{n}T^{\delta}}$, where C is a $K_2 \times L$ constant matrix and $\delta \geq 0$.*

Assumption 1 controls the relative magnitude of the instrument strength, as measured by δ. When $\delta = 1/2$, it is the standard weak instrument case introduced by Staiger and Stock (1997). When $\delta = 0$, it reduces to the weak instrument case in Cai, Fang and Li (2012).

Following Staiger and Stock (1997), we assume:

Assumption 2 *The following joint limits hold, as* $(n, T) \to \infty$

1. $\left(\frac{1}{Tn}\sum_{t=1}^{T} u_t' u_t, \frac{1}{Tn}\sum_{t=1}^{T} V_t' u_t, \frac{1}{Tn}\sum_{t=1}^{T} V_t' V_t\right) \xrightarrow{p} (\sigma_{uu}, \Sigma_{Vu}, \Sigma_{VV})$;
2. $\frac{1}{Tn}\sum_{t=1}^{T}\left(\tilde{X}_t, \tilde{Z}_t\right)'\left(\tilde{X}_t, \tilde{Z}_t\right) \xrightarrow{p} \Phi \equiv \begin{pmatrix} \Phi_{XX} & \Phi_{XZ} \\ \Phi_{ZX} & \Phi_{ZZ} \end{pmatrix}$;
3. $\left(\frac{1}{\sqrt{Tn}}\sum_{t=1}^{T} X_t' u_t, \frac{1}{\sqrt{Tn}}\sum_{t=1}^{T} Z_t' u_t, \frac{1}{\sqrt{Tn}}\sum_{t=1}^{T} X_t' V_t, \frac{1}{\sqrt{Tn}}\sum_{t=1}^{T} Z_t' V_t\right) \xrightarrow{d} (\Psi_{Xu}, \Psi_{Zu}, \Psi_{XV}, \Psi_{ZV})$, *where* $\Psi = \left(\Psi_{Xu}', \Psi_{Zu}', vec(\Psi_{XV})', vec(\Psi_{ZV})'\right)'$ *is distributed* $N(0, \Sigma \otimes \Phi)$.

Notice that Assumption 2 implies that $\left(\frac{1}{Tn}\sum_{t=1}^{T} \tilde{u}_t'\tilde{u}_t, \frac{1}{Tn}\sum_{t=1}^{T} \tilde{V}_t'\tilde{u}_t, \frac{1}{Tn}\sum_{t=1}^{T} \tilde{V}_t'\tilde{V}_t\right) \xrightarrow{p} (\sigma_{uu}, \Sigma_{Vu}, \Sigma_{VV})$ and $\left(\frac{1}{\sqrt{Tn}}\sum_{t=1}^{T} \tilde{X}_t'\tilde{u}_t, \frac{1}{\sqrt{Tn}}\sum_{t=1}^{T} \tilde{Z}_t'\tilde{u}_t, \frac{1}{\sqrt{Tn}}\sum_{t=1}^{T} \tilde{X}_t'\tilde{V}_t, \frac{1}{\sqrt{Tn}}\sum_{t=1}^{T} \tilde{Z}_t'\tilde{V}_t\right) \xrightarrow{d} (\Psi_{Xu}, \Psi_{Zu}, \Psi_{XV}, \Psi_{ZV})$ since $\frac{1}{T}\sum_{t=1}^{T} u_t \xrightarrow{p} 0$ and $\frac{1}{T}\sum_{t=1}^{T} V_t \xrightarrow{p} 0$. Following Staiger and Stock (1997), we define

$$\lambda = \Omega^{1/2} C \Sigma_{VV}^{-1/2}, \tag{7}$$

where $\Omega = \Phi_{ZZ} - \Phi_{ZX}\Phi_{XX}^{-1}\Phi_{XZ}$. Also define

$$z_u = \Omega^{-1/2\prime}\left(\Psi_{Zu} - \Phi_{ZX}\Phi_{XX}^{-1}\Psi_{Xu}\right)\sigma_{uu}^{-1/2}, \tag{8}$$

and

$$z_V = \Omega^{-1/2\prime}\left(\Psi_{ZV} - \Phi_{ZX}\Phi_{XX}^{-1}\Psi_{XV}\right)\Sigma_{VV}^{-1/2}. \tag{9}$$

The random variable $\left[z_u', vec(z_V)'\right]'$ is distributed $N\left(0, \bar{\Sigma} \otimes I_{K_2}\right)$, where

$$\bar{\Sigma} = \begin{pmatrix} 1 & \rho' \\ \rho & I_L \end{pmatrix} \tag{10}$$

with $\rho = \Sigma_{VV}^{-1/2\prime}\Sigma_{Vu}\sigma_{uu}^{-1/2}$ and I_L is an identity matrix of dimension L.

3 Estimation

Most of the Theorems in this section are developed for the within-group k-class estimator. However, we start by deriving the asymptotic properties of W2SLS for the simple case of one right hand side endogenous regressor and no included exogenous regressors.

3.1 A special case: 2SLS when $L = 1$ and $K_1 = 0$

Let $P_{\tilde{Z}} = \tilde{Z}\left(\tilde{Z}'\tilde{Z}\right)^{-1}\tilde{Z}'$ be the projection matrix on the space spanned by the columns of $\tilde{Z}$. The W2SLS estimator is defined as

$$\widehat{\beta}_{W2SLS} = \frac{\tilde{Y}'P_{\tilde{Z}}\tilde{y}}{\tilde{Y}'P_{\tilde{Z}}\tilde{Y}}.$$

Theorem 1 *As $(n,T) \to \infty$, for $0 \leq \delta < \frac{1}{2}$*

$$T^{\frac{1}{2}-\delta}\left(\widehat{\beta}_{W2SLS} - \beta\right) \xrightarrow{d} N\left(0, \sigma_{uu}\left(C'\Phi_{ZZ}C\right)^{-1}\right)$$

and for $\frac{1}{2} \leq \delta < \infty$

$$\widehat{\beta}_{W2SLS} - \beta = O_p(1).$$

The results in Theorem 1 imply that $\widehat{\beta}_{W2SLS}$ is consistent only if $0 \leq \delta < \frac{1}{2}$ and inconsistent if $\frac{1}{2} \leq \delta < \infty$. The strength of the instruments is measured by the following concentration matrix $\Lambda_{Tn} = \Sigma_{VV}^{-1/2\prime}\Pi'\tilde{Z}'\tilde{Z}\Pi\Sigma_{VV}^{-1/2}$. Using Assumptions (1) and (2), we have

$$\begin{aligned}
\Lambda_{Tn} &= \frac{1}{T^{2\delta-1}}\Sigma_{VV}^{-1/2\prime}C'\frac{\tilde{Z}'\tilde{Z}}{Tn}C\Sigma_{VV}^{-1/2} \\
&= \frac{1}{T^{2\delta-1}}\Sigma_{VV}^{-1/2\prime}C'\Phi_{ZZ}C\Sigma_{VV}^{-1/2} + o_p(1) \\
&= O_p\left(T^{1-2\delta}\right).
\end{aligned}$$

Note that $T^{1-2\delta}$ can be interpreted as the rate at which Λ_{Tn} grows as T increases. Clearly, for the consistency of W2SLS, one needs $\Lambda_{Tn} \to \infty$ as $T^{1-2\delta} \to \infty$ which holds if $0 \leq \delta < \frac{1}{2}$. We also note from Theorem 1 that the limiting distribution near the point of non-identification, i.e., $\delta = \frac{1}{2}$, is discontinuous.[3]

3.2 Within-group k-class Panel Data Estimators

We now generalize the results to the within-group k-class estimator with included regressors.[4] Let $P_{\tilde{X}} = \tilde{X}\left(\tilde{X}'\tilde{X}\right)^{-1}\tilde{X}'$ be the projection matrix on the space spanned by the columns of $\tilde{X}$ and $M_{\tilde{X}} = I - P_{\tilde{X}}$. Premultiplying equations (3) and (4) by $M_{\tilde{X}}$, we get

$$\tilde{y}^{\perp} = \tilde{Y}^{\perp}\beta + \tilde{u}^{\perp}$$

and

$$\tilde{Y}^{\perp} = \tilde{Z}^{\perp}\Pi + V^{\perp}$$

[3] This is the Hahn and Kuersteiner (2002) result for a cross-sectional IV regression.

[4] See Stock, Wright and Yogo (2002) for an important summary of the advantages and disadvantages of k-class estimators.

where the superscript "$\perp$" denotes the residuals from the projection on $\tilde{X}$, such as $\tilde{y}^{\perp} = M_{\tilde{X}}\tilde{y}$, $\tilde{Z}^{\perp} = M_{\tilde{X}}\tilde{Z}$, and $\tilde{Y}^{\perp} = M_{\tilde{X}}\tilde{Y}$. The within-group k-class estimator of β is given by

$$\widehat{\beta}(k) = \left[\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp}\right]^{-1}\left[\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{y}^{\perp}\right]$$

for some choice of k. Note that the W2SLS estimator is a special case of the within-group k-class estimator when $k = 1$. Theorem 2 derives the asymptotic properties of this within-group k-class panel data estimator.

Theorem 2 *Under Assumptions 1 and 2. As $(n, T) \to \infty$ we have*

1. *For $0 \leq \delta < \frac{1}{2}$, joint with $\kappa_{Tn} = T^{1/2+\delta}n(k-1) \xrightarrow{d} \kappa$,*

$$T^{1/2-\delta}\left(\widehat{\beta}(k) - \beta\right) + \theta_{Tn} \xrightarrow{d} N\left(0, \sigma_{uu}\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}\right)$$

   where $\theta_{Tn} = \left[\frac{1}{T^{1-2\delta}}\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp}\right]^{-1}\kappa_{Tn}\Sigma_{Vu}$.

2. *For $\delta = \frac{1}{2}$, joint with $\kappa_{Tn} = Tn(k-1) \xrightarrow{d} \kappa$,*

$$\widehat{\beta}(k) - \beta \xrightarrow{d} \sigma_{uu}^{1/2}\Sigma_{VV}^{-1/2}\Delta_1(\kappa)$$

   where $\Delta_1(\kappa) = \left[(\lambda + z_V)'(\lambda + z_V) - \kappa I_{Tn}\right]^{-1}\left[(\lambda + z_V)' z_u - \kappa\rho\right]$.

3. *For $\frac{1}{2} < \delta < \infty$, joint with $\kappa_{Tn} = Tn(k-1) \xrightarrow{d} \kappa$,*

$$\widehat{\beta}(k) - \beta \xrightarrow{d} \sigma_{uu}^{1/2}\Sigma_{VV}^{-1/2}\Delta_2(\kappa)$$

   where $\Delta_2(\kappa) = [z_V' z_V - \kappa I_{Tn}]^{-1}[z_V' z_u - \kappa\rho]$.

Similar to the results of Theorem 1 for $\widehat{\beta}_{W2SLS}$, Theorem 2 shows that $\widehat{\beta}(k)$ is consistent if $0 \leq \delta < \frac{1}{2}$ and inconsistent if $\frac{1}{2} \leq \delta < \infty$. Similarly, using Assumptions (1), (2) and Lemma 2, the strength of the instruments is measured by the following concentration matrix:

$$\begin{aligned}
\Lambda_{Tn} &= \Sigma_{VV}^{-1/2\prime}\Pi'\tilde{Z}^{\perp\prime}\tilde{Z}^{\perp}\Pi\Sigma_{VV}^{-1/2} \\
&= \frac{1}{T^{2\delta-1}}\Sigma_{VV}^{-1/2\prime}C'\frac{\tilde{Z}^{\perp\prime}\tilde{Z}^{\perp}}{Tn}C\Sigma_{VV}^{-1/2} \\
&= \frac{1}{T^{2\delta-1}}\Sigma_{VV}^{-1/2\prime}C'\Omega C\Sigma_{VV}^{-1/2} + o_p(1) \\
&= O_p\left(T^{1-2\delta}\right).
\end{aligned}$$

Note that $T^{1-2\delta}$ can be interpreted as the rate at which Λ_{Tn} grows as T increases. Clearly, for consistency of the within-group k-class estimator, one needs $\Lambda_{Tn} \to \infty$ as $T^{1-2\delta} \to \infty$ which holds if $0 \leq \delta < \frac{1}{2}$.

For the W2SLS estimator with $k = 1$,it follows that $T^{1/2+\delta} n\left(k-1\right) = 0$ and $Tn\left(k-1\right) = 0$. Therefore, the W2SLS estimator satisfies the conditions of κ_{Tn} for the three cases considered in Theorem 2.

The within-group k-class estimator also includes the within-group bias-adjusted 2SLS (B2SLS) described in Donald and Newey (2001) for the cross-section or time-series regression case. This is a special case of the k-class estimator with $k = nT/\left(nT - K_2 + 2\right)$. Rothenberg (1984) showed that B2SLS is unbiased to the second order for the fixed-instrument, normal error model. For this special case, $T^{1/2+\delta} n\left(k-1\right) = \left(K_2 - 2\right)/T^{1/2-\delta} = o_p\left(1\right)$ and $Tn\left(k-1\right) = K_2 - 2 = O_p\left(1\right)$. Hence, the within-group B2SLS estimator satisfies the conditions of κ_{Tn} for the three cases considered in Theorem 2.

For the within-group LIML estimator in panel data, we obtain the following results:

Theorem 3 *Under Assumptions 1 and 2, with $\bar{\Sigma}$ is defined in equation (10), we have*

1. *For $0 \leq \delta < \frac{1}{2}$, $T^{2\delta} n\left(\hat{k}_{LIML} - 1\right) \xrightarrow{p} 0$.*
2. *For $\delta = \frac{1}{2}$, $Tn\left(\hat{k}_{LIML} - 1\right) \xrightarrow{d} \kappa^*_{LIML}$, where κ^*_{LIML} is the smallest root of the determinantal equation, $\left|\Xi_2 - \kappa\bar{\Sigma}\right| = 0$, where $\Xi_2 = \begin{pmatrix} z_u' z_u & z_u'\left(\lambda + z_V\right) \\ \left(\lambda + z_V\right)' z_u & \left(\lambda + z_V\right)'\left(\lambda + z_V\right) \end{pmatrix}$.*
3. *For $\frac{1}{2} < \delta < \infty$, $Tn\left(\hat{k}_{LIML} - 1\right) \xrightarrow{d} \kappa^*_{LIML}$, where κ^*_{LIML} is the smallest root of the determinantal equation, $\left|\Xi_3 - \kappa\bar{\Sigma}\right| = 0$, where $\Xi_3 = \begin{pmatrix} z_u' z_u & z_u' z_V \\ z_V' z_u & z_V' z_V \end{pmatrix}$.*

3.3 Wald Test Under Weak Identification

Next, we consider testing the q linear restrictions $R\beta = r$, where R is $q \times L$. The standard formula for the Wald statistic, based on the within-group k-class estimator, is given by

$$W\left(k\right) = \left[R\hat{\beta}\left(k\right) - r\right]' \left\{\hat{\sigma}_{uu}\left(k\right) R\left[\tilde{Y}^{\perp\prime}\left(I - kM_{\tilde{Z}^{\perp}}\right)\tilde{Y}^{\perp}\right]^{-1} R'\right\}^{-1} \left[R\hat{\beta}\left(k\right) - r\right]$$

where $\hat{\sigma}_{uu}\left(k\right) = \widehat{\tilde{u}}\left(k\right)' \widehat{\tilde{u}}\left(k\right)/\left(Tn - K_1 - L\right)$, and $\widehat{\tilde{u}}\left(k\right) = \tilde{y} - \tilde{Y}\hat{\beta}\left(k\right) - \tilde{X}\hat{\gamma}\left(k\right) = \tilde{y}^{\perp} - \tilde{Y}^{\perp}\hat{\beta}\left(k\right)$.

Theorem 4 *Under Assumptions 1 and 2. As $\left(n, T\right) \to \infty$ we have*

1. *For $0 \leq \delta < \frac{1}{2}$, joint with $\kappa_{Tn} = T^{1/2+\delta} n\left(k-1\right) \xrightarrow{d} \kappa$,*

$$W\left(k\right) \xrightarrow{d} \chi^2\left(q, \Lambda\right)$$

a noncentral chi-squared distribution with q degrees of freedom and noncentrality parameter $\Lambda = \theta' R'\left[\sigma_{uu} R\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1} R'\right]^{-1} R\theta$, where $\theta = \kappa\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}\Sigma_{Vu}$.

2. *For $\delta = \frac{1}{2}$, joint with $\kappa_{Tn} = Tn\,(k-1) \xrightarrow{d} \kappa$,*

$$W(k) \xrightarrow{d} \Delta_1'(\kappa)\,\Sigma_{VV}^{-1/2}R'\left\{S(\Delta_1(\kappa))\,R\left\{\Sigma_{VV}^{1/2\prime}\left[(\lambda+z_V)'(\lambda+z_V)-\kappa I_{Tn}\right]\Sigma_{VV}^{1/2}\right\}^{-1}R'\right\}^{-1}R\Sigma_{VV}^{-1/2}\Delta_1(\kappa)\,.$$

3. *For $\frac{1}{2} < \delta < \infty$, joint with $\kappa_{Tn} = Tn\,(k-1) \xrightarrow{d} \kappa$,*

$$W(k) \xrightarrow{d} \Delta_2'(\kappa)\,\Sigma_{VV}^{-1/2}R'\left\{S(\Delta_2(\kappa))\,R\left\{\Sigma_{VV}^{1/2\prime}\left[z_V'z_V-\kappa I_{Tn}\right]\Sigma_{VV}^{1/2}\right\}^{-1}R'\right\}^{-1}R\Sigma_{VV}^{-1/2}\Delta_2(\kappa)\,.$$

Note that for $0 \leq \delta < \frac{1}{2}$, if $\kappa = 0$, then $\theta = 0$ and $\Lambda = 0$. Hence $W(k) \xrightarrow{d} \chi^2(q)$ a central chi-squared distribution with q degrees of freedom.

4 Testing for Weak Instruments

Following Stock and Yogo (2005), we focus in this section on testing the null hypothesis that the set of instruments is weak against the alternative that they are strong. In this case, the instruments are defined to be strong if W2SLS inference is reliable for any linear combination of the coefficients. From the results in Theorems 2 and 4, weak instruments can produce biased IV estimators and test of hypotheses with large size distortions, e.g., when $\frac{1}{2} \leq \delta < \infty$. The Stock and Yogo (2005) test is based on the partial identification test statistic proposed by Cragg and Donald (1993). For our case, this statistic is $g_{\min}$, the smallest eigenvalue of the matrix analog of the F statistic from the first stage regression of W2SLS, i.e.,

$$g_{\min} = \text{mineval} G_{Tn}$$

where

$$G_{Tn} = \frac{\widehat{\Sigma}_{VV}^{-1/2\prime}\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{Y}^{\perp}\widehat{\Sigma}_{VV}^{-1/2}}{K_2}$$

with $\hat{\Sigma}_{VV} = \tilde{Y}'M_{\tilde{Z}}\tilde{Y}/(Tn-K_1-K_2)$. A small $g_{\min}$ indicates that the instruments are weak, see Stock and Yogo (2005). Let $W(K,\Omega,\Upsilon)$ denote the Wishart distribution with K denoting the degrees of freedom, Ω denoting the covariance matrix, and Υ denoting the noncentrality matrix, we have the following result:

Theorem 5 *Under Assumptions 1 and 2, we have*

1. *For $0 \leq \delta < \frac{1}{2}$, $\frac{1}{T^{1-2\delta}}K_2G_{Tn} \xrightarrow{p} \lambda'\lambda$.*
2. *For $\delta = \frac{1}{2}$, $K_2G_{Tn} \xrightarrow{d} (\lambda+z_V)'(\lambda+z_V) \backsim W(K_2, I_L, \lambda'\lambda)$.*
3. *For $\frac{1}{2} < \delta < \infty$, $K_2G_{Tn} \xrightarrow{d} z_V'z_V \backsim W(K_2, I_L, 0)$.*

Note that for $0 \leq \delta < \frac{1}{2}$, $G_{Tn} \cong T^{1-2\delta}\lambda'\lambda/K_2 \to \infty$. For $\delta = \frac{1}{2}$, $E\left((\lambda + z_V)'(\lambda + z_V)\right) = K_2 I_L + \lambda'\lambda$, hence $G_{Tn} \cong I_L + \lambda'\lambda/K_2$. For $\frac{1}{2} < \delta < \infty$, $E\left(z_V' z_V\right) = K_2 I_L$, hence $G_{Tn} \cong I_L$. Therefore, as pointed out in Stock, Wright and Yogo (2002), $tr\left(G_{Tn}\right)/L$ can be thought of as a measure of the strength of the instruments. It is clear that $g_{\min} \to \infty$ if $0 \leq \delta < \frac{1}{2}$, $g_{\min} \xrightarrow{d} \text{mineval}\frac{(\lambda+z_V)'(\lambda+z_V)}{K_2}$ if $\delta = \frac{1}{2}$, and $g_{\min} \xrightarrow{d} \text{mineval}\frac{z_V' z_V}{K_2}$ if $\frac{1}{2} < \delta < \infty$. Next we discuss how to use $g_{\min}$ to detect the presence of weak instruments.

When $\delta = \frac{1}{2}$, $K_2 G_{Tn} \xrightarrow{d} (\lambda + z_V)'(\lambda + z_V)$ which has a noncentral Wishart distribution with noncentrality matrix $\lambda'\lambda$. This noncentrality matrix is the limit of the concentration matrix

$$\Lambda_{Tn} = \Sigma_{VV}^{-1/2\prime}\Pi'\tilde{Z}'\tilde{Z}\Pi\Sigma_{VV}^{-1/2} \xrightarrow{p} \lambda'\lambda.$$

On the other hand, when $0 \leq \delta < \frac{1}{2}$, $K_2 G_{Tn} \to \infty$, because $\Lambda_{Tn} \longrightarrow \infty$. Also note that $z_V' z_V$ has a Wishart distribution (i.e., $\lambda'\lambda = 0$). This corresponds to

$$\Lambda_{Tn} \xrightarrow{p} 0$$

when $\frac{1}{2} < \delta < \infty$. Let $\delta_{\min}$ be the smallest eigenvalue of $\lambda'\lambda$. Following Stock and Yogo (2005), we propose using the conservative critical value x which satisfies the relationship[5]

$$P\left(g_{\min} \leq x\right) \leq P\left(\chi^2\left(K_2, \delta_{\min}\right) \geq \nu x\right)$$

where $\chi^2\left(\nu, \delta_{\min}\right)$ denotes the noncentral chi-squared random variable with ν degrees of freedom and noncentrality parameter $\delta_{\min}$. Stock and Yogo (2005) focus on the worst-behaved linear combination and it is in this sense that this test is conservative. We refer the reader to their tables for critical values.

5 Robust Inference with Weak Instruments

The above results indicate that for $\delta \geq \frac{1}{2}$, the within-group k-class estimator is inconsistent. In this section, we discuss hypothesis testing whose size is robust to the weak instruments in the panel data set-up. Following the survey by Stock, Wright and Yogo (2002), we will discuss the AR test of Anderson and Rubin (1949), the Lagrange multiplier (LM) test of Kleibergen (2002) and Moreira (2009), and the conditional likelihood ratio (CLR) test of Moreira (2003) but applied to the *fixed effects* panel data model. For simplicity, we only consider the case of one right hand side endogenous variable, i.e., $L = 1$.[6]

[5]Stock and Yogo (2005) observe that the limiting distribution of $g_{\min}$ will depend upon all of the eigenvalues of $\lambda'\lambda$.

[6]It is important to note that Cai, Fang and Li (2012) also considered the Anderson and Rubin (1949), the Kleibergen (2002), and the Moreira (2003) conditional likelihood ratio and studied their asymptotic properties for fixed T and $n \to \infty$.

For convenience, we assume that $\tilde{X}_t$ and $\tilde{Z}_t$ are non-stochastic such that $\tilde{Z}_t'\tilde{X}_t = 0$. The reduced form equations corresponding to the structural equations (5) and (6) are as follows:

$$\tilde{y} = \tilde{Z}\Pi\beta + \tilde{X}\gamma^* + \tilde{u}^*$$

and

$$\tilde{Y} = \tilde{Z}\Pi + \tilde{X}\Gamma + \tilde{V}$$

where $\gamma^* = \gamma + \Gamma\beta$ and $\tilde{u}^* = \tilde{u} + \tilde{V}\beta$. The reduced-form errors are assumed to be homoskedastic with covariance matrix

$$\Sigma^* = \begin{bmatrix} \sigma_{uu}^* & \Sigma_{Vu}^{*\prime} \\ \Sigma_{Vu}^* & \Sigma_{VV} \end{bmatrix}.$$

The concentration parameter can be rewritten as $\Sigma_{VV}^{-1/2\prime}\Pi'\tilde{Z}'\tilde{Z}\Pi\Sigma_{VV}^{-1/2}$.

Consider the null hypothesis

$$H_0 : \beta = \beta_0.$$

Define

$$\mathbb{S} = \frac{\left(\tilde{Z}'\tilde{Z}\right)^{-1/2}\tilde{Z}'\tilde{Y}^*b_0}{\left(b_0'\Sigma^{*-1}b_0\right)^{1/2}}$$

and

$$\mathbb{T} = \frac{\left(\tilde{Z}'\tilde{Z}\right)^{-1/2}\tilde{Z}'\tilde{Y}^*\Sigma^{*-1}a_0}{\left(a_0'\Sigma^{*-1}a_0\right)^{1/2}}$$

where $b_0 = [1, -\beta_0]'$, $a_0 = [\beta_0, 1]'$, and $\tilde{Y}^* = \left[\tilde{y}, \tilde{Y}\right]$. Define

$$\Theta = \begin{bmatrix} \mathbb{S}' \\ \mathbb{T}' \end{bmatrix} [\mathbb{S}, \mathbb{T}] = \begin{bmatrix} \mathbb{S}'\mathbb{S} & \mathbb{T}'\mathbb{S} \\ \mathbb{S}'\mathbb{T} & \mathbb{T}'\mathbb{T} \end{bmatrix} = \begin{bmatrix} \Theta_{\mathbb{S}} & \Theta_{\mathbb{ST}} \\ \Theta_{\mathbb{ST}} & \Theta_{\mathbb{T}} \end{bmatrix}.$$

Three test statistics that are functions of Θ are the LM statistics of Kleibergen (2002) and Moreira (2009), the Anderson and Rubin (1949) statistic (AR), and the Moreira (2003) conditional likelihood ratio statistic (CLR). We now define the AR, LM, and CLR test statistics as follows:

$$AR = \frac{\Theta_{\mathbb{S}}}{K_2},$$

$$LM = \frac{\Theta_{\mathbb{ST}}^2}{\Theta_{\mathbb{T}}},$$

and

$$CLR = \frac{1}{2}\left(\Theta_{\mathbb{S}} - \Theta_{\mathbb{T}} + \sqrt{\left(\Theta_{\mathbb{S}} - \Theta_{\mathbb{T}}\right)^2 + 4\Theta_{\mathbb{ST}}^2}\right).$$

see Andrews and Stock (2006a, b). Define

$$\mu_\pi = \left(\tilde{Z}'\tilde{Z}\right)^{1/2}\Pi,$$

$$c_\beta = \frac{\beta-\beta_0}{\left(b_0'\Sigma^{*-1}b_0\right)^{1/2}},$$

and

$$d_\beta = \frac{a'\Sigma^{*-1}a_0}{\left(a_0'\Sigma^{*-1}a_0\right)^{1/2}}.$$

Note that $\tilde{Y}^*$ can be written as

$$\tilde{Y}^* = \tilde{Z}\Pi a + \tilde{X}\eta + \left[\tilde{u}^*, \tilde{V}\right]$$

where $a = (\beta, 1)'$, and $\eta = [\gamma^*, \Gamma]'$. That is, $\tilde{Y}$ is multivariate normal with mean matrix $\tilde{Z}\Pi a + \tilde{X}\eta$. Then $\mathbb{S}$ is $K_2 \times 1$ multivariate normal with mean

$$\begin{aligned}
E[\mathbb{S}] &= E\left[\frac{\left(\tilde{Z}'\tilde{Z}\right)^{-1/2}\tilde{Z}'\tilde{Y}b_0}{\left(b_0'\Sigma^{*-1}b_0\right)^{1/2}}\right] \\
&= \frac{\left(\tilde{Z}'\tilde{Z}\right)^{-1/2}\tilde{Z}'\left(\tilde{Z}\Pi a + \tilde{X}\eta\right)b_0}{\left(b_0'\Sigma^{*-1}b_0\right)^{1/2}} \\
&= \frac{\left(\tilde{Z}'\tilde{Z}\right)^{1/2}\Pi\left(\beta-\beta_0\right)}{\left(b_0'\Sigma^{*-1}b_0\right)^{1/2}} = c_\beta\mu_\pi
\end{aligned}$$

and

$$Var[\mathbb{S}] = I_{K_2}$$

using

$$\tilde{Z}'\tilde{X} = 0$$

and

$$a'b_0 = (\beta, 1)\begin{pmatrix} 1 \\ -\beta_0 \end{pmatrix} = \beta - \beta_0.$$

Similarly, $\mathbb{T}$ is $K_2 \times 1$ multivariate normal with mean

$$E[\mathbb{T}] = d_\beta\mu_\pi$$

and variance

$$Var[\mathbb{T}] = I_{K_2}.$$

It is also easy to show that $\mathbb{S}$ and $\mathbb{T}$ are independent using $b_0^{'}a_0 = 0$. Under the null, $H_0 : \beta = \beta_0$

$$\mathbb{S} \sim N\left(0, I_{K_2}\right)$$

which does not depend on Π since $c_\beta = 0$. However, the distribution of $\mathbb{T}$ depends on Π under the null.

Assume $\frac{\tilde{Z}^{'}\tilde{Z}}{Tn} \to D_Z$. The asymptotic distributions of $\mathbb{S}$ and $\mathbb{T}$ are given in the following theorem:

Theorem 6 *Suppose Assumptions 1 and 2, hold. We have*

$$\mathbb{S} - c_\beta \mu_\pi \xrightarrow{d} \mathbb{S}^* \sim N\left(0, I_{K_2}\right)$$

and

$$\mathbb{T} - d_\beta \mu_\pi \xrightarrow{d} \mathbb{T}^* \sim N\left(0, I_{K_2}\right)$$

where $\mathbb{S}$ and $\mathbb{T}$ are independent with

1. *If $0 \leq \delta < \frac{1}{2}$, $\mu_\pi \to \infty$,*

2. *If $\delta = \frac{1}{2}$, $\mu_\pi = O(1)$, and*

3. *If $\frac{1}{2} < \delta < \infty$, $\mu_\pi = o(1)$.*

Hence, under the null,

$$\mathbb{S} \xrightarrow{d} N\left(0, I_{K_2}\right)$$

and

$$AR = \frac{\Theta_{\mathbb{S}}}{K_2} \xrightarrow{d} \frac{\chi^2_{K_2}}{K_2}$$

for all values of δ. Note that

$$LM = \frac{\Theta^2_{\mathbb{ST}}}{\Theta_{\mathbb{T}}} = \frac{\left(\mathbb{S}^{'}\mathbb{T}\right)^2}{\mathbb{T}^{'}\mathbb{T}} = \mathbb{S}^{'}P_{\mathbb{T}}\mathbb{S}$$

where $P_{\mathbb{T}} = \mathbb{T}\left(\mathbb{T}^{'}\mathbb{T}\right)^{-1}\mathbb{T}^{'}$ is a symmetric idempotent matrix with $rank(P_{\mathbb{T}}) = K_2$. By Proposition B.3.1 in Lütkepohl (2005), we have

$$LM \xrightarrow{d} \chi^2_{K_2}$$

for all values of δ. Because Σ^* is unknown, it must be replaced by a consistent estimator, $\hat{\Sigma}^*$. Critical values for the CLR statistic can be found in Andrews et al. (2006).

CLR, LM and AR tests have good size properties under all values of δ, i.e., strong and weak IVs. However, they may have different power properties. Deriving the asymptotic power envelops and power upper bounds is an interesting question which we leave for future research.

6 Monte Carlo Simulation

In this section, we report some Monte Carlo results that examine the finite sample properties of the cross-section 2SLS and the panel data W2SLS estimator when $L = 1$. Following Staiger and Stock (1997), instruments Z_{1t}, Z_{2t}, Z_{3t}, and Z_{4t} are assumed to be standard normal variables and X_t is the constant 1; the errors $(u_t, V_t)'$ are generated from an i.i.d. bivariate normal distribution with $\Sigma = \begin{pmatrix} 1 & 0.99 \\ 0.99 & 1 \end{pmatrix}$. The true value of β, γ, and Γ is set as 0, 1, and 1 respectively. We set $C = 0.5$. Individual fixed effects μ and α are generated from independent standard normal distributions. To summarize, the data generating process (DGP) is given by

$$y_t = 1 + \mu + u_t \tag{11}$$

and

$$Y_t = 1 + \frac{0.5}{\sqrt{n T^{\delta}}} \left(Z_{1t} + Z_{2t} + Z_{3t} + Z_{4t} \right) + \alpha + V_t \tag{12}$$

for $t = 1, 2, \cdots, T$. The cross-section sample size n takes the values $(50, 100)$, while the time-series sample size T takes the values $(1, 10, 20, 50, 100)$. δ takes the values $(0, 0.2, 0.5, 0.8)$ for the panel data case, i.e., for $T > 1$ and $\delta = 0$ for the cross-sectional case, i.e., $T = 1$. For each experiment, we perform $1,000$ replications. For each replication we estimate the model using W2SLS and LIML estimators of β. Table 1 reports the root mean squared error (RMSE) of these estimators for various values of n, T and δ. Following Kelejian and Prucha (1999), RMSE is defined as $\left[bias^2 + (IQR/1.35)^2 \right]^{1/2}$, where $bias$ is the difference between the median and the true parameter value and IQR is the interquantile range. That is $IQR = c_1 - c_2$, where c_1 and c_2 are the 0.75 and 0.25 quantiles respectively. As explained in Kelejian and Prucha (1999), these characteristics are closely related to the standard measures of bias and root mean squared error (RMSE) but, unlike these measures, are assured to exist. We can see that LIML has a smaller bias than the W2SLS estimator, however, W2SLS has a smaller IQR and RMSE than the LIML estimator. Figure 1 shows the density function of W2SLS estimator for $n = 100$. As we can see in the graph, when $\delta = 0$ or 0.2, the distribution tends to center at zero as T increases. when $\delta = 0.5$ or 0.8, the distribution does not change much as T increases. Table 2 reports the size of the t-test for $\beta = 0$. Results from table 2 confirm that the t-tests using the W2SLS and LIML estimators are not robust with respect to weak instruments. Table 3 reports the results of the AR, LM and CLR tests. Table 3 indicates that the robust tests are indeed, robust to the weak instruments in this panel data design.

7 Conclusion

Following the extensive literature on weak instruments surveyed by Stock, Wright and Yogo (2002), this paper proposes k-class IV estimators and test statistics and studies their behaviour when the available instruments are weak in a fixed effects panel data model. It is important to note that Cai, Fang and Li (2012) studied this fixed effects panel data model, but they let the degree of weakness of the instruments depend upon n^{δ}, where $\delta \geq 0$, and studied the asymptotic properties of W2SLS and pivotal statistics for fixed T and $n \to \infty$. In contrast, our study let the degree of weakness of the instruments depend upon $\sqrt{n}T^{\delta}$ and studies the asymptotic properties of k-class IV estimators and pivotal test statistics as both $(n, T) \to \infty$. Both papers argue that there are benefits to panel data in reducing the bias of W2SLS and k-class IV estimators in case of weak instruments. Monte Carlo results confirm these asymptotic results.

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Table 1: Bias, IQR and RMSE of W2SLS and LIML Estimators

			Bias		IQR		RMSE	
n	T	δ	W2SLS	LIML	W2SLS	LIML	W2SLS	LIML
50	1	0	0.473	0.447	0.647	1.692	0.673	1.331
50	10	0	0.259	-0.014	0.293	0.485	0.338	0.359
50	10	0.2	0.490	0.010	0.299	0.779	0.537	0.577
50	10	0.5	0.798	0.329	0.262	0.913	0.821	0.752
50	10	0.8	0.933	0.735	0.194	1.057	0.944	1.074
50	20	0	0.142	0.004	0.225	0.297	0.219	0.220
50	20	0.2	0.367	0.012	0.275	0.581	0.419	0.430
50	20	0.5	0.769	0.300	0.263	0.822	0.793	0.679
50	20	0.8	0.943	0.776	0.180	0.825	0.952	0.988
50	50	0	0.059	0.010	0.173	0.189	0.142	0.140
50	50	0.2	0.253	0.021	0.275	0.416	0.325	0.309
50	50	0.5	0.758	0.299	0.260	0.795	0.782	0.661
50	50	0.8	0.954	0.858	0.147	0.693	0.960	1.000
50	100	0	0.037	0.008	0.121	0.132	0.097	0.098
50	100	0.2	0.179	0.020	0.231	0.332	0.248	0.246
50	100	0.5	0.758	0.327	0.242	0.796	0.779	0.674
50	100	0.8	0.969	0.928	0.131	0.502	0.974	1.000
100	1	0	0.447	0.375	0.621	1.593	0.641	1.238
100	10	0	0.246	-0.027	0.287	0.488	0.325	0.362
100	10	0.2	0.477	0.004	0.292	0.777	0.523	0.576
100	10	0.5	0.784	0.303	0.287	0.887	0.813	0.724
100	10	0.8	0.929	0.723	0.214	1.156	0.943	1.121
100	20	0	0.139	0.007	0.241	0.324	0.227	0.240
100	20	0.2	0.361	0.014	0.271	0.605	0.413	0.449
100	20	0.5	0.767	0.317	0.271	0.936	0.793	0.762
100	20	0.8	0.941	0.798	0.183	0.885	0.951	1.033
100	50	0	0.052	-0.005	0.161	0.186	0.130	0.138
100	50	0.2	0.224	-0.010	0.267	0.430	0.299	0.319
100	50	0.5	0.755	0.266	0.260	0.915	0.780	0.728
100	50	0.8	0.964	0.857	0.157	0.684	0.971	0.995
100	100	0	0.031	0.000	0.126	0.135	0.098	0.100
100	100	0.2	0.170	-0.001	0.237	0.349	0.244	0.258
100	100	0.5	0.760	0.303	0.246	0.953	0.782	0.768
100	100	0.8	0.973	0.916	0.142	0.511	0.978	0.991

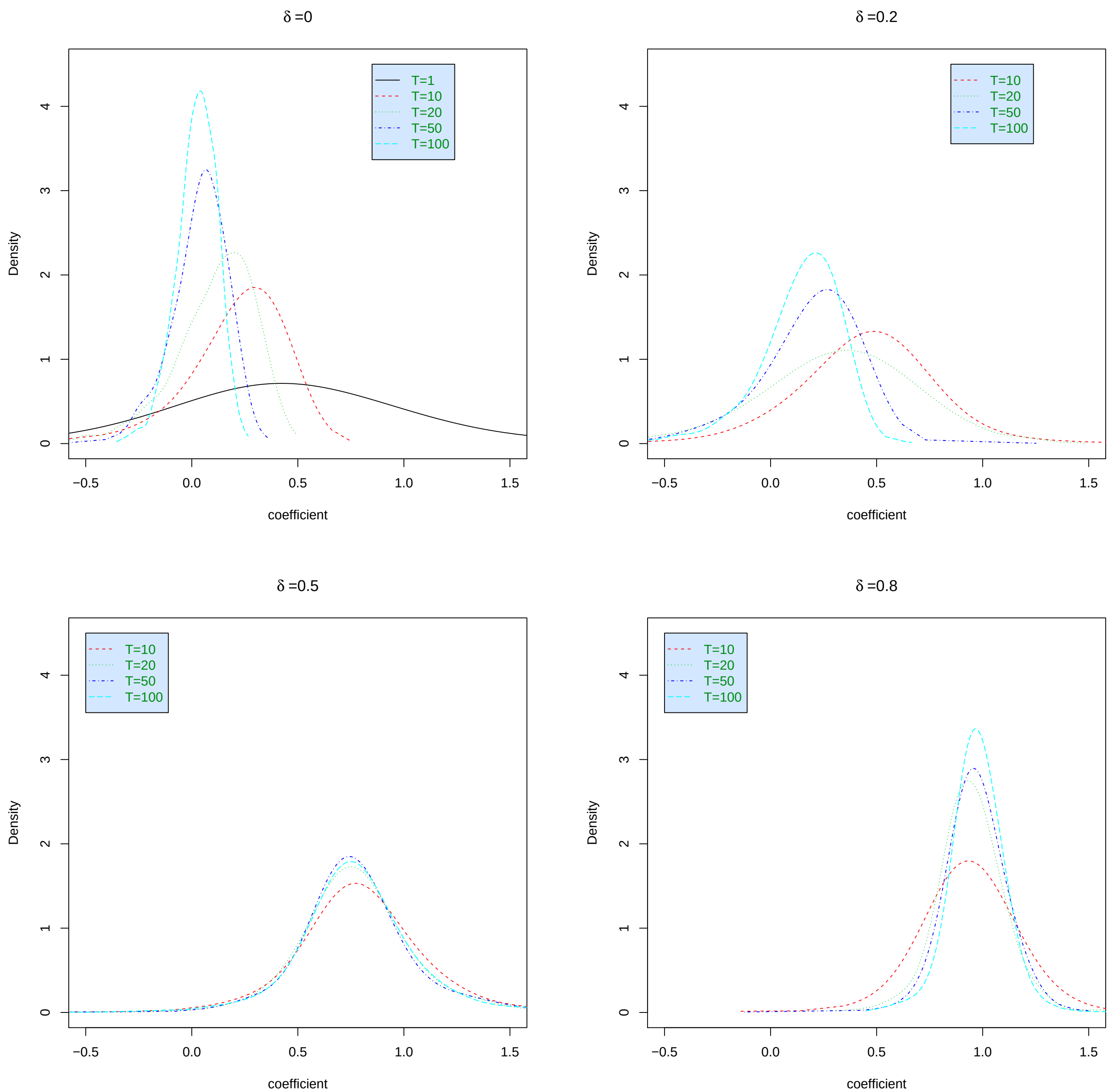
Table 2: Size of t-test

n	T	δ	W2SLS	LIML
50	1	0	0.153	0.098
50	10	0	0.347	0.091
50	10	0.2	0.587	0.127
50	10	0.5	0.851	0.208
50	10	0.8	0.946	0.393
50	20	0	0.191	0.075
50	20	0.2	0.461	0.108
50	20	0.5	0.862	0.213
50	20	0.8	0.972	0.473
50	50	0	0.118	0.062
50	50	0.2	0.331	0.087
50	50	0.5	0.835	0.214
50	50	0.8	0.967	0.534
50	100	0	0.098	0.055
50	100	0.2	0.252	0.091
50	100	0.5	0.862	0.201
50	100	0.8	0.988	0.634
100	1	0	0.132	0.080
100	10	0	0.309	0.084
100	10	0.2	0.565	0.115
100	10	0.5	0.850	0.188
100	10	0.8	0.935	0.384
100	20	0	0.211	0.081
100	20	0.2	0.444	0.116
100	20	0.5	0.847	0.215
100	20	0.8	0.963	0.468
100	50	0	0.102	0.049
100	50	0.2	0.289	0.085
100	50	0.5	0.849	0.161
100	50	0.8	0.974	0.527
100	100	0	0.080	0.054
100	100	0.2	0.232	0.077
100	100	0.5	0.879	0.198
100	100	0.8	0.974	0.609

Table 3: Size of Robust Tests

n	T	δ	AR	LM	CLR
50	1	0	0.065	0.065	0.069
50	10	0	0.055	0.057	0.056
50	10	0.2	0.055	0.055	0.055
50	10	0.5	0.055	0.055	0.051
50	10	0.8	0.055	0.050	0.053
50	20	0	0.050	0.048	0.046
50	20	0.2	0.050	0.050	0.052
50	20	0.5	0.050	0.058	0.059
50	20	0.8	0.050	0.062	0.057
50	50	0	0.058	0.053	0.053
50	50	0.2	0.058	0.053	0.052
50	50	0.5	0.058	0.057	0.057
50	50	0.8	0.058	0.056	0.056
50	100	0	0.050	0.042	0.042
50	100	0.2	0.050	0.042	0.042
50	100	0.5	0.050	0.045	0.047
50	100	0.8	0.050	0.054	0.048
100	1	0	0.054	0.050	0.050
100	10	0	0.040	0.053	0.053
100	10	0.2	0.040	0.053	0.054
100	10	0.5	0.040	0.048	0.047
100	10	0.8	0.040	0.046	0.046
100	20	0	0.050	0.045	0.045
100	20	0.2	0.050	0.046	0.046
100	20	0.5	0.050	0.042	0.041
100	20	0.8	0.050	0.044	0.043
100	50	0	0.043	0.047	0.047
100	50	0.2	0.043	0.048	0.048
100	50	0.5	0.043	0.049	0.050
100	50	0.8	0.043	0.041	0.054
100	100	0	0.044	0.056	0.056
100	100	0.2	0.044	0.058	0.058
100	100	0.5	0.044	0.064	0.061
100	100	0.8	0.044	0.057	0.055

Figure 1: Density of the W2SLS Estimator (n=100)



Appendix

To prove Theorem 1, we need the following Lemma.

Lemma 1 *As* $(n,T)\to\infty$, *for* $0\le\delta<\frac{1}{2}$

$$\frac{1}{T^{1-\delta}n^{\frac{1}{2}}}\tilde{Z}'\tilde{Y}=\Phi_{ZZ}C+O_p\left(\frac{1}{T^{\frac{1}{2}-\delta}}\right),$$

for $\delta=\frac{1}{2}$

$$\frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{Y}\xrightarrow{d}\Phi_{ZZ}C+\Psi_{ZV},$$

and for $\frac{1}{2}<\delta<\infty$

$$\frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{Y}\xrightarrow{d}\Psi_{ZV}.$$

Proof. First we note that for $0\le\delta\le\frac{1}{2}$

$$\begin{aligned}
\frac{1}{T^{1-\delta}n^{\frac{1}{2}}}\tilde{Z}'\tilde{Y} &= \frac{1}{T^{1-\delta}n}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{Y}_t\\
&= \frac{1}{T^{1-\delta}n^{\frac{1}{2}}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{Z}_t\Pi+\frac{1}{T^{1-\delta}n^{\frac{1}{2}}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{V}_t\\
&= \frac{1}{Tn}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{Z}_t\left(T^{\delta}n^{\frac{1}{2}}\Pi\right)+\frac{1}{T^{1-\delta}n^{\frac{1}{2}}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{V}_t\\
&= \frac{1}{Tn}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{Z}_tC+\frac{1}{T^{\frac{1}{2}-\delta}}\frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{V}_t\\
&= \Phi_{ZZ}C+O_p\left(\frac{1}{T^{\frac{1}{2}-\delta}}\right)
\end{aligned}$$

as $(n,T)\to\infty$ since $\frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{V}_t=O_p(1)$.

For $\delta=\frac{1}{2}$

$$\frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{Y}=\frac{1}{Tn}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{Z}_t\left(\sqrt{Tn}\Pi\right)+\frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{V}_t\xrightarrow{d}\Phi_{ZZ}C+\Psi_{ZV}.$$

Finally, for $\frac{1}{2}<\delta<\infty$

$$\begin{aligned}
\frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{Y} &= \frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{Y}_t\\
&= \frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{Z}_t\Pi+\frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{V}_t\\
&= \left(\frac{1}{Tn}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{Z}_t\right)\left(\frac{C}{T^{\delta-1/2}}\right)+\frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{V}_t\xrightarrow{d}\Psi_{ZV}+O_p\left(\frac{1}{T^{\delta-1/2}}\right)
\end{aligned}$$

using $\frac{1}{Tn}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{Z}_t=O_p(1)$ and $\frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\tilde{Z}_t'\tilde{V}_t\xrightarrow{d}\Psi_{ZV}$.This proves the lemma. ■

A Proof of Theorem 1

Proof. Consider

$$\widehat{\beta}_{W2SLS}-\beta=\frac{\tilde{Y}'P_{\tilde{Z}}\tilde{u}}{\tilde{Y}'P_{\tilde{Z}}\tilde{Y}}=\frac{\tilde{Y}'\tilde{Z}\left(\tilde{Z}\tilde{Z}\right)^{-1}\tilde{Z}'\tilde{u}}{\tilde{Y}'\tilde{Z}\left(\tilde{Z}\tilde{Z}\right)^{-1}\tilde{Z}'\tilde{Y}}.$$

For $0<\delta<\frac{1}{2}$,

$$T^{\frac{1}{2}-\delta}\left(\widehat{\beta}_{W2SLS}-\beta\right)=\frac{\left(\frac{1}{T^{1-\delta}\sqrt{n}}\tilde{Y}'\tilde{Z}\right)\left(\frac{1}{nT}\tilde{Z}\tilde{Z}\right)^{-1}\left(\frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{u}\right)}{\left(\frac{1}{T^{1-\delta}\sqrt{n}}\tilde{Y}'\tilde{Z}\right)\left(\frac{1}{nT}\tilde{Z}\tilde{Z}\right)^{-1}\left(\frac{1}{T^{1-\delta}\sqrt{n}}\tilde{Z}'\tilde{Y}\right)}\xrightarrow{d}\frac{\Psi_{Zu}}{\Phi_{ZZ}C}=N\left(0,\sigma_{uu}\left(C'\Phi_{ZZ}C\right)^{-1}\right),$$

for $\delta=\frac{1}{2}$,

$$\widehat{\beta}_{W2SLS}-\beta=\frac{\left(\frac{1}{\sqrt{Tn}}\tilde{Y}'\tilde{Z}\right)\left(\frac{1}{nT}\tilde{Z}\tilde{Z}\right)^{-1}\left(\frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{u}\right)}{\left(\frac{1}{\sqrt{Tn}}\tilde{Y}'\tilde{Z}\right)\left(\frac{1}{nT}\tilde{Z}\tilde{Z}\right)^{-1}\left(\frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{Y}\right)}=\frac{\left(\Phi_{ZZ}C+\Psi_{ZV}\right)'\Phi_{ZZ}^{-1}\Psi_{Zu}}{\left(\Phi_{ZZ}C+\Psi_{ZV}\right)'\Phi_{ZZ}^{-1}\left(\Phi_{ZZ}C+\Psi_{ZV}\right)}=O_p\left(1\right),$$

and for $\frac{1}{2}<\delta<\infty$,

$$\widehat{\beta}_{W2SLS}-\beta=\frac{\tilde{Y}'P_{\tilde{Z}}\tilde{u}}{\tilde{Y}'P_{\tilde{Z}}\tilde{Y}}=\frac{\left(\frac{1}{\sqrt{Tn}}\tilde{Y}'\tilde{Z}\right)\left(\frac{1}{nT}\tilde{Z}\tilde{Z}\right)^{-1}\left(\frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{u}\right)}{\left(\frac{1}{\sqrt{Tn}}\tilde{Y}'\tilde{Z}\right)\left(\frac{1}{nT}\tilde{Z}\tilde{Z}\right)^{-1}\left(\frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{Y}\right)}=\frac{\Psi_{ZV}'\Phi_{ZZ}^{-1}\Psi_{Zu}}{\Psi_{ZV}'\Phi_{ZZ}^{-1}\Psi_{ZV}}=O_p\left(1\right).$$

This proves the theorem. ■

To prove Theorem 2, we need the following Lemma.

Lemma 2 *Under Assumptions 1 and 2, as $(n,T)\to\infty$,*

1. $\frac{1}{Tn}\tilde{u}^{\perp\prime}\tilde{u}^{\perp}\xrightarrow{p}\sigma_{uu}$, $\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{u}^{\perp}\xrightarrow{p}\Sigma_{Vu}$, $\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{Y}^{\perp}\xrightarrow{p}\Sigma_{VV}$, *and* $\frac{1}{Tn}\tilde{Z}^{\perp\prime}\tilde{Z}^{\perp}\xrightarrow{p}\Phi_{ZZ}-\Phi_{Z\tilde{X}}\Phi_{\tilde{X}\tilde{X}}^{-1}\Phi_{\tilde{X}Z}=\Omega$.

2. $\frac{1}{\sqrt{Tn}}\tilde{Z}^{\perp\prime}\tilde{u}^{\perp}\xrightarrow{d}\Psi_{Zu}-\Phi_{Z\tilde{X}}\Phi_{\tilde{X}\tilde{X}}^{-1}\Psi_{\tilde{X}u}$, $P_{\tilde{Z}^{\perp}}^{1/2}\tilde{u}^{\perp}\xrightarrow{d}\sigma_{uu}^{1/2}z_u$, *and* $\tilde{u}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{u}^{\perp}\xrightarrow{d}\sigma_{uu}z_u'z_u$.

3. *For* $0\leq\delta\leq\frac{1}{2}$, $\frac{1}{T^{1-\delta}n^{\frac{1}{2}}}\tilde{Z}^{\perp\prime}\tilde{Y}^{\perp}\xrightarrow{p}\Omega C$, $\frac{1}{T^{1/2-\delta}}P_{\tilde{Z}^{\perp}}^{1/2}\tilde{Y}^{\perp}\xrightarrow{p}\lambda\Sigma_{VV}^{1/2}$, $\frac{1}{T^{1/2-\delta}}\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{u}^{\perp}\xrightarrow{d}\sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\lambda'z_u$, *and* $\frac{1}{T^{1-2\delta}}\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{Y}^{\perp}\xrightarrow{p}\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}$; *For* $\delta=\frac{1}{2}$ $\frac{1}{\sqrt{Tn}}\tilde{Z}^{\perp\prime}\tilde{Y}^{\perp}\xrightarrow{d}\Omega C+\Omega^{1/2}z_V\Sigma_{VV}^{1/2}$, $P_{\tilde{Z}^{\perp}}^{1/2}\tilde{Y}^{\perp}\xrightarrow{d}(\lambda+z_V)\Sigma_{VV}^{1/2}$, $\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{u}^{\perp}\xrightarrow{d}\sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\left(\lambda+z_V\right)'z_u$, *and* $\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{Y}^{\perp}\xrightarrow{d}\Sigma_{VV}^{1/2\prime}\left(\lambda+z_V\right)'\left(\lambda+z_V\right)\Sigma_{VV}^{1/2}$; *For* $\frac{1}{2}<\delta<\infty$, $\frac{1}{\sqrt{Tn}}\tilde{Z}^{\perp\prime}\tilde{Y}^{\perp}\xrightarrow{d}\Omega^{1/2}z_V\Sigma_{VV}^{1/2}$, $P_{\tilde{Z}^{\perp}}^{1/2}\tilde{Y}^{\perp}\xrightarrow{d}z_V\Sigma_{VV}^{1/2}$, $\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{u}^{\perp}\xrightarrow{d}\sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}z_V'z_u$, *and* $\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{Y}^{\perp}\xrightarrow{d}\Sigma_{VV}^{1/2\prime}z_V'z_V\Sigma_{VV}^{1/2}$.

Proof. Consider part (1). First we note that

$$\begin{aligned}
\frac{1}{Tn}\tilde{u}^{\perp\prime}\tilde{u}^{\perp} &= \frac{1}{Tn}\tilde{u}' M_{\tilde{X}}\tilde{u} \\
&= \frac{1}{Tn}\tilde{u}'\tilde{u} - \left(\frac{1}{Tn}\tilde{u}'\tilde{X}\right)\left(\frac{1}{Tn}\tilde{X}'\tilde{X}\right)^{-1}\left(\frac{1}{Tn}\tilde{X}'\tilde{u}\right) \xrightarrow{p} \sigma_{uu}
\end{aligned}$$

using $\frac{1}{Tn}\tilde{u}'\tilde{u} \xrightarrow{p} \sigma_{uu}$, $\frac{1}{Tn}\tilde{X}'\tilde{X} = O_p(1)$, and $\frac{1}{Tn}\tilde{X}'\tilde{u} = o_p(1)$. Next,

$$\begin{aligned}
\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{u}^{\perp} &= \frac{1}{Tn}\tilde{Y}' M_{\tilde{X}}\tilde{u} \\
&= \frac{1}{Tn}\Pi'\tilde{Z}' M_{\tilde{X}}\tilde{u} + \frac{1}{Tn}\tilde{V}' M_{\tilde{X}}\tilde{u} \\
&= \Pi'\left[\frac{1}{Tn}\tilde{Z}'\tilde{u} - \left(\frac{1}{Tn}\tilde{Z}'\tilde{X}\right)\left(\frac{1}{Tn}\tilde{X}'\tilde{X}\right)^{-1}\left(\frac{1}{Tn}\tilde{X}'\tilde{u}\right)\right] \\
&\quad + \left[\frac{1}{Tn}\tilde{V}'\tilde{u} - \left(\frac{1}{Tn}\tilde{V}'\tilde{X}\right)\left(\frac{1}{Tn}\tilde{X}'\tilde{X}\right)^{-1}\left(\frac{1}{Tn}\tilde{X}'\tilde{u}\right)\right] \xrightarrow{p} \Sigma_{Vu}
\end{aligned}$$

using $\Pi = o(1)$, $\frac{1}{Tn}\tilde{Z}'\tilde{u} = o_p(1)$, $\frac{1}{Tn}\tilde{Z}'\tilde{X} = O_p(1)$, $\frac{1}{Tn}\tilde{X}'\tilde{X} = O_p(1)$, $\frac{1}{Tn}\tilde{X}'\tilde{u} = o_p(1)$, $\frac{1}{Tn}\tilde{V}'\tilde{X} = o_p(1)$, and $\frac{1}{Tn}\tilde{V}'\tilde{u} \xrightarrow{p} \Sigma_{Vu}$. Similarly,

$$\begin{aligned}
\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{Y}^{\perp} &= \frac{1}{Tn}\tilde{Y}' M_{\tilde{X}}\tilde{Y} \\
&= \frac{1}{Tn}\Pi'\tilde{Z}' M_{\tilde{X}}\tilde{Z}\Pi + \frac{1}{Tn}\tilde{V}' M_{\tilde{X}}\tilde{Z}\Pi + \frac{1}{Tn}\Pi'\tilde{Z}' M_{\tilde{X}}\tilde{V} + \frac{1}{Tn}\tilde{V}' M_{\tilde{X}}\tilde{V} \\
&= \left[\frac{1}{Tn}\tilde{V}'\tilde{u} - \left(\frac{1}{Tn}\tilde{V}'\tilde{X}\right)\left(\frac{1}{Tn}\tilde{X}'\tilde{X}\right)^{-1}\left(\frac{1}{Tn}\tilde{X}'\tilde{V}\right)\right] + o_p(1) \xrightarrow{p} \Sigma_{VV}
\end{aligned}$$

since $\Pi = o(1)$, $\frac{1}{Tn}\tilde{X}'\tilde{X} \xrightarrow{p} \Phi_{\tilde{X}\tilde{X}}$, $\frac{1}{Tn}\tilde{V}'\tilde{X} \xrightarrow{p} 0$, and $\frac{1}{Tn}\tilde{V}'\tilde{u} \xrightarrow{p} \Sigma_{Vu}$. Finally,

$$\frac{1}{Tn}\tilde{Z}^{\perp\prime}\tilde{Z}^{\perp} = \frac{1}{Tn}\tilde{Z}' M_{\tilde{X}}\tilde{Z} = \frac{1}{Tn}\tilde{Z}'\tilde{Z} - \left(\frac{1}{Tn}\tilde{Z}'\tilde{X}\right)\left(\frac{1}{Tn}\tilde{X}'\tilde{X}\right)^{-1}\left(\frac{1}{Tn}\tilde{X}'\tilde{Z}\right) \xrightarrow{p} \Phi_{ZZ} - \Phi_{Z\tilde{X}}\Phi_{\tilde{X}\tilde{X}}^{-1}\Phi_{\tilde{X}Z} = \Omega$$

as $(n, T) \to \infty$ proving part (1).

Consider (2). Note that

$$\begin{aligned}
\frac{1}{\sqrt{Tn}}\tilde{Z}^{\perp\prime}\tilde{u}^{\perp} &= \frac{1}{\sqrt{Tn}}\tilde{Z}' M_{\tilde{X}}\tilde{u} \\
&= \frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{u} - \left(\frac{1}{Tn}\tilde{Z}'\tilde{X}\right)\left(\frac{1}{Tn}\tilde{X}'\tilde{X}\right)^{-1}\left(\frac{1}{\sqrt{Tn}}\tilde{X}'\tilde{u}\right) \xrightarrow{d} \Psi_{Zu} - \Phi_{Z\tilde{X}}\Phi_{\tilde{X}\tilde{X}}^{-1}\Psi_{\tilde{X}u}
\end{aligned}$$

since $\frac{1}{\sqrt{Tn}}\tilde{X}'\tilde{u} \xrightarrow{d} \Psi_{\tilde{X}u}$ and $\frac{1}{\sqrt{Tn}}\tilde{Z}'\tilde{u} \xrightarrow{d} \Psi_{Zu}$. Therefore,

$$P_{\tilde{Z}^{\perp}}^{1/2}\tilde{u}^{\perp} = \left(\frac{1}{Tn}\tilde{Z}^{\perp\prime}\tilde{Z}^{\perp}\right)^{-1/2}\left(\frac{1}{\sqrt{Tn}}\tilde{Z}^{\perp\prime}\tilde{u}^{\perp}\right) \xrightarrow{d} \Omega^{-1/2}\left(\Psi_{Zu} - \Phi_{Z\tilde{X}}\Phi_{\tilde{X}\tilde{X}}^{-1}\Psi_{\tilde{X}u}\right) = \sigma_{uu}^{1/2} z_u$$

and

$$\tilde{u}^{\perp\prime} P_{\tilde{Z}^{\perp}} \tilde{u}^{\perp} = \left(\tilde{u}^{\perp\prime} P_{\tilde{Z}^{\perp}}^{1/2}\right)\left(P_{\tilde{Z}^{\perp}}^{1/2} \tilde{u}^{\perp}\right) \xrightarrow{d} \sigma_{uu} z_u' z_u.$$

Consider (3). For $0 \leq \delta < \frac{1}{2}$, we have

$$\begin{aligned}
\frac{1}{T^{1-\delta} n^{\frac{1}{2}}} \tilde{Z}^{\perp\prime} \tilde{Y}^{\perp} &= \frac{1}{T^{1-\delta} n^{\frac{1}{2}}} \tilde{Z}' M_{\tilde{X}} \tilde{Z} \Pi + \frac{1}{T^{1-\delta} n^{\frac{1}{2}}} \tilde{Z}' M_{\tilde{X}} \tilde{V} \\
&= \left[\left(\frac{1}{Tn} \tilde{Z}' \tilde{Z}\right) - \left(\frac{1}{Tn} \tilde{Z}' \tilde{X}\right)\left(\frac{1}{Tn} \tilde{X}' \tilde{X}\right)^{-1}\left(\frac{1}{Tn} \tilde{X}' \tilde{Z}\right)\right]\left(T^{\delta} n^{\frac{1}{2}} \Pi\right) \\
&\quad + \frac{1}{T^{\frac{1}{2}-\delta}}\left[\left(\frac{1}{\sqrt{Tn}} \tilde{Z}' \tilde{V}\right) - \left(\frac{1}{Tn} \tilde{Z}' \tilde{X}\right)\left(\frac{1}{Tn} \tilde{X}' \tilde{X}\right)^{-1}\left(\frac{1}{\sqrt{Tn}} \tilde{X}' \tilde{V}\right)\right] = \Omega C + O_p\left(\frac{1}{T^{\frac{1}{2}-\delta}}\right)
\end{aligned}$$

with $\Omega = \Phi_{ZZ} - \Phi_{Z\tilde{X}} \Phi_{\tilde{X}\tilde{X}}^{-1} \Phi_{\tilde{X}Z}$ because $\frac{1}{\sqrt{Tn}} \tilde{Z}' \tilde{V} \xrightarrow{d} \Psi_{ZV}$, $\frac{1}{Tn} \tilde{Z}' \tilde{X} \xrightarrow{p} \Phi_{Z\tilde{X}}$, $\frac{1}{Tn} \tilde{X}' \tilde{X} \xrightarrow{p} \Phi_{\tilde{X}\tilde{X}}$, and $\frac{1}{\sqrt{Tn}} \tilde{X}' \tilde{V} \xrightarrow{d} \Psi_{\tilde{X}V}$. Together with Lemmas (1) and (2), we obtain

$$\frac{1}{T^{\frac{1}{2}-\delta}} P_{\tilde{Z}^{\perp}}^{1/2} \tilde{Y}^{\perp} = \left(\frac{1}{Tn} \tilde{Z}^{\perp\prime} \tilde{Z}^{\perp}\right)^{-1/2}\left(\frac{1}{T^{1-\delta} n^{\frac{1}{2}}} \tilde{Z}^{\perp\prime} \tilde{Y}^{\perp}\right) \xrightarrow{p} \Omega^{-1/2} \Omega C = \Omega^{1/2} C = \lambda \Sigma_{VV}^{1/2},$$

$$\frac{1}{\sqrt{Tn}} \tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}} \tilde{u}^{\perp} = \left(\frac{1}{\sqrt{Tn}} \tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}}^{1/2}\right)\left(P_{\tilde{Z}^{\perp}}^{1/2} \tilde{u}^{\perp}\right) \xrightarrow{d} \sigma_{uu}^{1/2} \Sigma_{VV}^{1/2\prime} \lambda' z_u,$$

and

$$\frac{1}{T^{1-2\delta}} \tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}} \tilde{Y}^{\perp} = \left(\frac{1}{T^{1/2-\delta}} \tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}}^{1/2}\right)\left(\frac{1}{T^{1/2-\delta}} P_{\tilde{Z}^{\perp}}^{1/2} \tilde{Y}^{\perp}\right) \xrightarrow{p} \Sigma_{VV}^{1/2\prime} \lambda' \lambda \Sigma_{VV}^{1/2}.$$

Next for $\delta = \frac{1}{2}$, we have

$$\begin{aligned}
\frac{1}{\sqrt{Tn}} \tilde{Z}^{\perp\prime} \tilde{Y}^{\perp} &= \frac{1}{\sqrt{Tn}} \tilde{Z}' M_{\tilde{X}} \tilde{Z} \Pi + \frac{1}{\sqrt{Tn}} \tilde{Z}' M_{\tilde{X}} \tilde{V} \\
&= \left[\left(\frac{1}{Tn} \tilde{Z}' \tilde{Z}\right) - \left(\frac{1}{Tn} \tilde{Z}' \tilde{X}\right)\left(\frac{1}{Tn} \tilde{X}' \tilde{X}\right)^{-1}\left(\frac{1}{Tn} \tilde{X}' \tilde{Z}\right)\right]\left(\sqrt{Tn} \Pi\right) \\
&\quad + \left[\left(\frac{1}{\sqrt{Tn}} \tilde{Z}' \tilde{V}\right) - \left(\frac{1}{Tn} \tilde{Z}' \tilde{X}\right)\left(\frac{1}{Tn} \tilde{X}' \tilde{X}\right)^{-1}\left(\frac{1}{\sqrt{Tn}} \tilde{X}' \tilde{V}\right)\right] \\
&\quad \xrightarrow{d} \left(\Phi_{ZZ} - \Phi_{Z\tilde{X}} \Phi_{\tilde{X}\tilde{X}}^{-1} \Phi_{\tilde{X}Z}\right) C + \left(\Psi_{ZV} - \Phi_{Z\tilde{X}} \Phi_{\tilde{X}\tilde{X}}^{-1} \Psi_{\tilde{X}V}\right) = \Omega C + \Omega^{1/2} z_V \Sigma_{VV}^{1/2}
\end{aligned}$$

by the definition of Ω and z_V. Together with Lemmas (1) and (2), we have

$$P_{\tilde{Z}^{\perp}}^{1/2} \tilde{Y}^{\perp} = \left(\frac{1}{Tn} \tilde{Z}^{\perp\prime} \tilde{Z}^{\perp}\right)^{-1/2}\left(\frac{1}{\sqrt{Tn}} \tilde{Z}^{\perp\prime} \tilde{Y}^{\perp}\right) \xrightarrow{d} \Omega^{-1/2}\left[\Omega C + \Omega^{1/2} z_V \Sigma_{VV}^{1/2}\right] = \Omega^{1/2} C + z_V \Sigma_{VV}^{1/2} = \left(\lambda + z_V\right) \Sigma_{VV}^{1/2},$$

$$\tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}} \tilde{u}^{\perp} = \left(\tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}}^{1/2}\right)\left(P_{\tilde{Z}^{\perp}}^{1/2} \tilde{u}^{\perp}\right) \xrightarrow{d} \sigma_{uu}^{1/2} \Sigma_{VV}^{1/2\prime} \left(\lambda + z_V\right)' z_u,$$

and

$$\tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}} \tilde{Y}^{\perp} = \left(\tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}}^{1/2}\right)\left(P_{\tilde{Z}^{\perp}}^{1/2} \tilde{Y}^{\perp}\right) \xrightarrow{d} \Sigma_{VV}^{1/2\prime} \left(\lambda + z_V\right)' \left(\lambda + z_V\right) \Sigma_{VV}^{1/2}.$$

Finally for $\frac{1}{2} < \delta < \infty$, we have

$$\begin{aligned}
\frac{1}{\sqrt{Tn}} \tilde{Z}^{\perp\prime} \tilde{Y}^{\perp} &= \frac{1}{\sqrt{Tn}} \tilde{Z}' M_{\tilde{X}} \tilde{Z} \Pi + \frac{1}{\sqrt{Tn}} \tilde{Z}' M_{\tilde{X}} \tilde{V} \\
&= \frac{1}{T^{\delta-1/2}} \left[\left(\frac{1}{Tn} \tilde{Z}' \tilde{Z}\right) - \left(\frac{1}{Tn} \tilde{Z}' \tilde{X}\right)\left(\frac{1}{Tn} \tilde{X}' \tilde{X}\right)^{-1}\left(\frac{1}{Tn} \tilde{X}' \tilde{Z}\right)\right] (\sqrt{n} T^{\delta} \Pi) \\
&\quad + \left[\left(\frac{1}{\sqrt{Tn}} \tilde{Z}' \tilde{V}\right) - \left(\frac{1}{Tn} \tilde{Z}' \tilde{X}\right)\left(\frac{1}{Tn} \tilde{X}' \tilde{X}\right)^{-1}\left(\frac{1}{\sqrt{Tn}} \tilde{X}' \tilde{V}\right)\right] \\
&= O_p\left(\frac{1}{T^{\delta-1/2}}\right) + \left(\Psi_{ZV} - \Phi_{Z\tilde{X}} \Phi_{\tilde{X}\tilde{X}}^{-1} \Psi_{\tilde{X}V}\right) = \Omega^{1/2} z_V \Sigma_{VV}^{1/2} + O_p\left(\frac{1}{T^{\delta-1/2}}\right).
\end{aligned}$$

Using Lemmas (1) and (2), we have

$$P_{\tilde{Z}^{\perp}}^{1/2} \tilde{Y}^{\perp} = \left(\frac{1}{Tn} \tilde{Z}^{\perp\prime} \tilde{Z}^{\perp}\right)^{-1/2}\left(\frac{1}{\sqrt{Tn}} \tilde{Z}^{\perp\prime} \tilde{Y}^{\perp}\right) \xrightarrow{d} \Omega^{-1/2} \Omega^{1/2} z_V \Sigma_{VV}^{1/2} = z_V \Sigma_{VV}^{1/2},$$

$$\tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}} \tilde{u}^{\perp} = \left(\tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}}^{1/2}\right)\left(P_{\tilde{Z}^{\perp}}^{1/2} \tilde{u}^{\perp}\right) \xrightarrow{d} \sigma_{uu}^{1/2} \Sigma_{VV}^{1/2} z_V{}' z_u,$$

and

$$\tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}} \tilde{Y}^{\perp} = \left(\tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}}^{1/2}\right)\left(P_{\tilde{Z}^{\perp}}^{1/2} \tilde{Y}^{\perp}\right) \xrightarrow{d} \Sigma_{VV}^{1/2} z_V' z_V \Sigma_{VV}^{1/2}.$$

This proves (3). ■

B Proof of Theorem 2

Proof. We write

$$\widehat{\beta}(k) = \beta + \left[\tilde{Y}^{\perp\prime} (I - k M_{\tilde{Z}^{\perp}}) \tilde{Y}^{\perp}\right]^{-1}\left[\tilde{Y}^{\perp\prime} (I - k M_{\tilde{Z}^{\perp}}) \tilde{u}^{\perp}\right].$$

First for $0 \leq \delta < \frac{1}{2}$, let $\kappa_{Tn} = T^{1/2+\delta} n (k-1)$. Using $k = \left(1 + \frac{\kappa_{Tn}}{T^{1/2+\delta} n}\right)$ and $M_{\tilde{Z}^{\perp}} = I - P_{\tilde{Z}^{\perp}}$, by Lemmas 2.1 and 2.3, we have

$$\begin{aligned}
\frac{1}{T^{1-2\delta}} \tilde{Y}^{\perp\prime} (I - k M_{\tilde{Z}^{\perp}}) \tilde{Y}^{\perp} &= \frac{1}{T^{1-2\delta}} \tilde{Y}^{\perp\prime}\left[I - \left(1 + \frac{\kappa_{Tn}}{T^{1/2+\delta} n}\right)(I - P_{\tilde{Z}^{\perp}})\right] \tilde{Y}^{\perp} \\
&= \left(1 + \frac{\kappa_{Tn}}{\sqrt{Tn}}\right)\left(\frac{1}{T^{1-2\delta}} \tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}} \tilde{Y}^{\perp}\right) - \frac{\kappa_{Tn}}{T^{1/2-\delta}}\left(\frac{1}{Tn} \tilde{Y}^{\perp\prime} \tilde{Y}^{\perp}\right) \xrightarrow{p} \Sigma_{VV}^{1/2\prime} \lambda' \lambda \Sigma_{VV}^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{T^{1/2-\delta}} \tilde{Y}^{\perp\prime} (I - k M_{\tilde{Z}^{\perp}}) \tilde{u}^{\perp} &= \frac{1}{T^{1/2-\delta}} \tilde{Y}^{\perp\prime}\left[I - \left(1 + \frac{\kappa_{Tn}}{T^{1/2+\delta} n}\right)(I - P_{\tilde{Z}^{\perp}})\right] \tilde{u}^{\perp} \\
&= \left(1 + \frac{\kappa_{Tn}}{\sqrt{Tn}}\right)\left(\frac{1}{T^{1/2-\delta}} \tilde{Y}^{\perp\prime} P_{\tilde{Z}^{\perp}} \tilde{u}^{\perp}\right) - \kappa_{Tn}\left(\frac{1}{Tn} \tilde{Y}^{\perp\prime} \tilde{u}^{\perp}\right) \xrightarrow{d} \sigma_{uu}^{1/2} \Sigma_{VV}^{1/2\prime} \lambda' z_u - \kappa \Sigma_{Vu}
\end{aligned}$$

uniformly in κ. Therefore,

$$
\begin{aligned}
& T^{1/2-\delta}\left(\widehat{\beta}\left(k\right)-\beta\right)+\left[\frac{1}{T^{1-2\delta}}\tilde{Y}^{\perp\prime}\left(I-kM_{\tilde{Z}^{\perp}}\right)\tilde{Y}^{\perp}\right]^{-1}\kappa_{Tn}\Sigma_{Vu} \\
= & \left[\frac{1}{T^{1-2\delta}}\tilde{Y}^{\perp\prime}\left(I-kM_{\tilde{Z}^{\perp}}\right)\tilde{Y}^{\perp}\right]^{-1}\left[\frac{1}{T^{1/2-\delta}}\tilde{Y}^{\perp\prime}\left(I-kM_{\tilde{Z}^{\perp}}\right)\tilde{u}^{\perp}-\kappa_{Tn}\left(\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{u}^{\perp}\right)+\kappa_{Tn}\Sigma_{Vu}\right] \\
& \xrightarrow{d}\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}\left(\sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\lambda' z_u\right)\sim N\left(0,\sigma_{uu}\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}\right)
\end{aligned}
$$

as $(n,T)\to\infty$ joint with $\kappa_{Tn}\xrightarrow{d}\kappa$.

Consider $\delta=\frac{1}{2}$ and let $\kappa_{Tn}=Tn\left(k-1\right)$. Using $k=\left(1+\frac{\kappa_{Tn}}{Tn}\right)$ and $M_{\tilde{Z}^{\perp}}=I-P_{\tilde{Z}^{\perp}}$, by Lemmas 2.1 and 2.3, we have

$$
\begin{aligned}
\tilde{Y}^{\perp\prime}\left(I-kM_{\tilde{Z}^{\perp}}\right)\tilde{Y}^{\perp} &= \tilde{Y}^{\perp\prime}\left[I-\left(1+\frac{\kappa_{Tn}}{Tn}\right)\left(I-P_{\tilde{Z}^{\perp}}\right)\right]\tilde{Y}^{\perp} \\
&= \left(1+\frac{\kappa_{Tn}}{Tn}\right)\left(\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{Y}^{\perp}\right)-\kappa_{Tn}\left(\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{Y}^{\perp}\right) \\
&\xrightarrow{d}\Sigma_{VV}^{1/2\prime}\left(\lambda+z_V\right)'\left(\lambda+z_V\right)\Sigma_{VV}^{1/2}-\kappa\Sigma_{VV}=\Sigma_{VV}^{1/2\prime}\left[\left(\lambda+z_V\right)'\left(\lambda+z_V\right)-\kappa I_{Tn}\right]\Sigma_{VV}^{1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{Y}^{\perp\prime}\left(I-kM_{\tilde{Z}^{\perp}}\right)\tilde{u}^{\perp} &= \tilde{Y}^{\perp\prime}\left[I-\left(1+\frac{\kappa_{Tn}}{T^{1/2+\delta}n}\right)\left(I-P_{\tilde{Z}^{\perp}}\right)\right]\tilde{u}^{\perp} \\
&= \left(1+\frac{\kappa_{Tn}}{\sqrt{T}n}\right)\left(\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{u}^{\perp}\right)-\kappa_{Tn}\left(\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{u}^{\perp}\right) \\
&\xrightarrow{d}\sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\left(\lambda+z_V\right)'z_u-\kappa\Sigma_{Vu}=\sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\left[\left(\lambda+z_V\right)'z_u-\kappa\rho\right]
\end{aligned}
$$

uniformly in κ. Therefore,

$$
\widehat{\beta}\left(k\right)-\beta\xrightarrow{d}\left\{\Sigma_{VV}^{1/2\prime}\left[\left(\lambda+z_V\right)'\left(\lambda+z_V\right)-\kappa I_{Tn}\right]\Sigma_{VV}^{1/2}\right\}^{-1}\left\{\sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\left[\left(\lambda+z_V\right)'z_u-\kappa\rho\right]\right\}=\sigma_{uu}^{1/2}\Sigma_{VV}^{-1/2}\Delta_1\left(\kappa\right),
$$

as $(n,T)\to\infty$ joint with $\kappa_{Tn}\xrightarrow{d}\kappa$, where $\Delta_1\kappa=\left[\left(\lambda+z_V\right)'\left(\lambda+z_V\right)-\kappa I_{Tn}\right]^{-1}\left[\left(\lambda+z_V\right)'z_u-\kappa\rho\right]$.

Consider $\frac{1}{2}<\delta<\infty$ and let $\kappa_{Tn}=Tn\left(k-1\right)$. Using $k=\left(1+\frac{\kappa_{Tn}}{Tn}\right)$ and $M_{\tilde{Z}^{\perp}}=I-P_{\tilde{Z}^{\perp}}$, by Lemmas 2.1 and 2.3, we have

$$
\begin{aligned}
\tilde{Y}^{\perp\prime}\left(I-kM_{\tilde{Z}^{\perp}}\right)\tilde{Y}^{\perp} &= \tilde{Y}^{\perp\prime}\left[I-\left(1+\frac{\kappa_{Tn}}{Tn}\right)\left(I-P_{\tilde{Z}^{\perp}}\right)\right]\tilde{Y}^{\perp} \\
&= \left(1+\frac{\kappa_{Tn}}{Tn}\right)\left(\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{Y}^{\perp}\right)-\kappa_{Tn}\left(\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{Y}^{\perp}\right) \\
&\xrightarrow{d}\Sigma_{VV}^{1/2\prime}z_V'z_V\Sigma_{VV}^{1/2}-\kappa\Sigma_{VV} \\
&= \Sigma_{VV}^{1/2\prime}\left[z_V'z_V-\kappa I_{Tn}\right]\Sigma_{VV}^{1/2}
\end{aligned}
$$

and

$$\begin{aligned}
\tilde{Y}^{\perp\prime}(I-kM_{\tilde{Z}^{\perp}})\,\tilde{u}^{\perp} &= \tilde{Y}^{\perp\prime}\left[I-\left(1+\frac{\kappa_{Tn}}{Tn}\right)(I-P_{\tilde{Z}^{\perp}})\right]\tilde{u}^{\perp} \\
&= \left(1+\frac{\kappa_{Tn}}{Tn}\right)\left(\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{\perp}}\tilde{u}^{\perp}\right)-\kappa_{Tn}\left(\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{u}^{\perp}\right) \\
&\xrightarrow{d} \sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}z_V' z_u-\kappa\Sigma_{Vu} \\
&= \sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\left[z_V' z_u-\kappa\rho\right].
\end{aligned}$$

Therefore,

$$\widehat{\beta}(k)-\beta\xrightarrow{d}\left\{\Sigma_{VV}^{1/2\prime}\left[z_V' z_V-\kappa I_{Tn}\right]\Sigma_{VV}^{1/2}\right\}^{-1}\left\{\sigma_{uu}^{1/2}\Sigma_{VV}^{1/2\prime}\left[z_V' z_u-\kappa\rho\right]\right\}=\sigma_{uu}^{1/2}\Sigma_{VV}^{-1/2}\Delta_2(\kappa),$$

as $(n,T)\to\infty$ joint with $\kappa_{Tn}\xrightarrow{d}\kappa$, where $\Delta_2(\kappa)=\left[z_V' z_V-\kappa I_{Tn}\right]^{-1}\left[z_V' z_u-\kappa\rho\right]$. ■

C Proof of Theorem 3

Proof. Let us denote J, partitioned conformably with $\tilde{Y}^{*\perp}$, to be $J_{11}=I_T$, $J_{21}=-\beta\otimes\iota_T$, $J_{11}=0\otimes\iota_T'$ and $J_{11}=I_{Tn}$. Because $\tilde{y}^{\perp}=\tilde{Y}^{\perp}\beta+\tilde{u}^{\perp}$, hence $\tilde{Y}^{*\perp}J=\left[\tilde{u}^{\perp},\tilde{Y}^{\perp}\right]$. By Lemma 2.1, we have $\frac{1}{Tn}J'\tilde{Y}^{*\perp\prime}\tilde{Y}^{*\perp}J=\begin{pmatrix}\frac{1}{Tn}\tilde{u}^{\perp\prime}\tilde{u}^{\perp} & \frac{1}{Tn}\tilde{u}^{\perp\prime}\tilde{Y}^{\perp}\\ \frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{u}^{\perp} & \frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{Y}^{\perp}\end{pmatrix}\xrightarrow{p}\begin{pmatrix}\sigma_{uu} & \Sigma_{uV}\\ \Sigma_{Vu} & \Sigma_{VV}\end{pmatrix}=\Upsilon'\bar{\Sigma}\Upsilon$ where $\Upsilon=diag\left(\sigma_{uu}^{1/2},\Sigma_{VV}^{1/2}\right)$ and $\bar{\Sigma}$ is defined in Equation (10). If $\frac{1}{Tn}J'\tilde{Y}^{*\perp\prime}P_{\tilde{Z}^{*\perp}}\tilde{Y}^{*\perp}J=o_p(1)$ which will be shown below, then we have $\frac{1}{Tn}J'\tilde{Y}^{*\perp\prime}M_{\tilde{Z}^{*\perp}}\tilde{Y}^{*\perp}J=\frac{1}{Tn}J'\tilde{Y}^{*\perp\prime}\tilde{Y}^{*\perp}J-\frac{1}{Tn}J'\tilde{Y}^{*\perp\prime}P_{\tilde{Z}^{*\perp}}\tilde{Y}^{*\perp}J\xrightarrow{p}\Upsilon'\bar{\Sigma}\Upsilon$.

For $0\leq\delta<\frac{1}{2}$, note that for any nonsingular $(n+1)T\times(n+1)T$ matrix J, the roots of $\left|\tilde{Y}^{*\perp\prime}\left[I_{Tn}-\hat{k}_{LIML}M_{\tilde{Z}^{*\perp}}\right]\tilde{Y}^{*\perp}\right|=0$ are the same as the roots of

$$\begin{aligned}
\left|\frac{1}{T^{1-2\delta}}J'\tilde{Y}^{*\perp\prime}\left[I_{Tn}-\hat{k}_{LIML}M_{\tilde{Z}^{*\perp}}\right]\tilde{Y}^{*\perp}J\right| &= \left|\frac{1}{T^{1-2\delta}}J'\tilde{Y}^{*\perp\prime}\left[P_{\tilde{Z}^{*\perp}}-\left(\hat{k}_{LIML}-1\right)M_{\tilde{Z}^{*\perp}}\right]\tilde{Y}^{*\perp}J\right| \\
&= \left|\left(\frac{1}{T^{1-2\delta}}J'\tilde{Y}^{*\perp\prime}P_{\tilde{Z}^{*\perp}}\tilde{Y}^{*\perp}J\right)-\hat{\kappa}_{LIML}\left(\frac{1}{Tn}J'\tilde{Y}^{*\perp\prime}M_{\tilde{Z}^{*\perp}}\tilde{Y}^{*\perp}J\right)\right|=0,
\end{aligned}$$

where $\hat{\kappa}_{LIML}=T^{2\delta}n\left(\hat{k}_{LIML}-1\right)$. By Lemma 2.2 and 2.3, we have

$$\begin{aligned}
\frac{1}{T^{1-2\delta}}J'\tilde{Y}^{*\perp\prime}P_{\tilde{Z}^{*\perp}}\tilde{Y}^{*\perp}J &= \begin{pmatrix}\frac{1}{T^{1-2\delta}}\left(\tilde{u}^{\perp\prime}P_{\tilde{Z}^{*\perp}}\tilde{u}^{\perp}\right) & \frac{1}{T^{1/2-\delta}}\left(\frac{1}{T^{1/2-\delta}}\tilde{u}^{\perp\prime}P_{\tilde{Z}^{*\perp}}\tilde{Y}^{\perp}\right)\\ \frac{1}{T^{1/2-\delta}}\left(\frac{1}{T^{1/2-\delta}}\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{*\perp}}\tilde{u}^{\perp}\right) & \left(\frac{1}{T^{1-2\delta}}\tilde{Y}^{\perp\prime}P_{\tilde{Z}^{*\perp}}\tilde{Y}^{\perp}\right)\end{pmatrix} \\
&\xrightarrow{d}\begin{pmatrix}0 & 0\\ 0 & \Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\end{pmatrix} \\
&= \Upsilon'\Xi_1\Upsilon,
\end{aligned}$$

where $\Xi_1 = \begin{pmatrix} 0 & 0 \\ 0 & \lambda'\lambda \end{pmatrix}$. The solutions to $\left|\left(\frac{1}{T^{1-2\delta}} J'\tilde{Y}^{*\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J\right) - \hat{\kappa}_{LIML}\left(\frac{1}{Tn} J'\tilde{Y}^{*\perp\prime} M_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J\right)\right| = 0$ therefore converge to those of $\left|\Xi_1 - \kappa\bar{\Sigma}\right| = 0$, among them the smallest root is zero. Thus $\hat{\kappa}_{LIML} = T^{2\delta} n \left(\hat{k}_{LIML} - 1\right) \xrightarrow{p} 0$.

For $\delta = \frac{1}{2}$, the roots of $\left|\tilde{Y}^{*\perp\prime}\left[I_{Tn} - \hat{k}_{LIML} M_{\tilde{Z}^{*\perp}}\right]\tilde{Y}^{*\perp}\right| = 0$ are the same as the roots of

$$\left|J'\tilde{Y}^{*\perp\prime}\left[P_{\tilde{Z}^{*\perp}} - \left(\hat{k}_{LIML} - 1\right) M_{\tilde{Z}^{*\perp}}\right]\tilde{Y}^{*\perp} J\right| = \left|\left(J'\tilde{Y}^{*\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J\right) - \hat{\kappa}_{LIML}\left(\frac{1}{Tn} J'\tilde{Y}^{*\perp\prime} M_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J\right)\right| = 0,$$

where $\hat{\kappa}_{LIML} = Tn\left(\hat{k}_{LIML} - 1\right)$. By Lemma 2.2 and 2.3, we have

$$\begin{aligned} J'\tilde{Y}^{*\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J &= \begin{pmatrix} \tilde{u}^{\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{u}^{\perp} & \tilde{u}^{\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{\perp} \\ \tilde{Y}^{\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{u}^{\perp} & \tilde{Y}^{\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{\perp} \end{pmatrix} \\ &\xrightarrow{d} \begin{pmatrix} \sigma_{uu} z_u' z_u & \sigma_{uu}^{1/2} z_u' \left(\lambda + z_V\right) \Sigma_{VV}^{1/2} \\ \sigma_{uu}^{1/2} \Sigma_{VV}^{1/2\prime} \left(\lambda + z_V\right)' z_u & \Sigma_{VV}^{1/2\prime} \left(\lambda + z_V\right)' \left(\lambda + z_V\right) \Sigma_{VV}^{1/2} \end{pmatrix} \\ &= \Upsilon' \Xi_2 \Upsilon \end{aligned}$$

where $\Xi_2 = \begin{pmatrix} z_u' z_u & z_u' \left(\lambda + z_V\right) \\ \left(\lambda + z_V\right)' z_u & \left(\lambda + z_V\right)' \left(\lambda + z_V\right) \end{pmatrix}$. The solutions to $\left|\left(J'\tilde{Y}^{*\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J\right) - \hat{\kappa}_{LIML}\left(\frac{1}{Tn} J'\tilde{Y}^{*\perp\prime} M_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J\right)\right| = 0$ therefore converge to those of $\left|\Xi_2 - \kappa\bar{\Sigma}\right| = 0$. Thus $\hat{\kappa}_{LIML} = Tn\left(\hat{k}_{LIML} - 1\right) \xrightarrow{d} \kappa_{LIML}^*$, where κ_{LIML}^* is the smallest root of $\left|\Xi_2 - \kappa\bar{\Sigma}\right| = 0$.

For $\frac{1}{2} < \delta < \infty$, the roots of $\left|\tilde{Y}^{*\perp\prime}\left[I_{Tn} - \hat{k}_{LIML} M_{\tilde{Z}^{*\perp}}\right]\tilde{Y}^{*\perp}\right| = 0$ are the same as the roots of

$$\left|J'\tilde{Y}^{*\perp\prime}\left[P_{\tilde{Z}^{*\perp}} - \left(\hat{k}_{LIML} - 1\right) M_{\tilde{Z}^{*\perp}}\right]\tilde{Y}^{*\perp} J\right| = \left|\left(J'\tilde{Y}^{*\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J\right) - \hat{\kappa}_{LIML}\left(\frac{1}{Tn} J'\tilde{Y}^{*\perp\prime} M_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J\right)\right| = 0,$$

where $\hat{\kappa}_{LIML} = Tn\left(\hat{k}_{LIML} - 1\right)$. By Lemma 2.2 and 2.3, we have

$$\begin{aligned} J'\tilde{Y}^{*\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J &= \begin{pmatrix} \tilde{u}^{\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{u}^{\perp} & \tilde{u}^{\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{\perp} \\ \tilde{Y}^{\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{u}^{\perp} & \tilde{Y}^{\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{\perp} \end{pmatrix} \\ &\xrightarrow{d} \begin{pmatrix} \sigma_{uu} z_u' z_u & \sigma_{uu}^{1/2} z_u' z_V \Sigma_{VV}^{1/2} \\ \sigma_{uu}^{1/2} \Sigma_{VV}^{1/2\prime} z_V' z_u & \Sigma_{VV}^{1/2\prime} z_V' z_V \Sigma_{VV}^{1/2} \end{pmatrix} \\ &= \Upsilon' \Xi_3 \Upsilon \end{aligned}$$

where $\Xi_3 = \begin{pmatrix} z_u' z_u & z_u' z_V \\ z_V' z_u & z_V' z_V \end{pmatrix}$. The solutions to $\left|\left(J'\tilde{Y}^{*\perp\prime} P_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J\right) - \hat{\kappa}_{LIML}\left(\frac{1}{Tn} J'\tilde{Y}^{*\perp\prime} M_{\tilde{Z}^{*\perp}} \tilde{Y}^{*\perp} J\right)\right| = 0$ therefore converge to those of $\left|\Xi_3 - \kappa\bar{\Sigma}\right| = 0$. Thus $\hat{\kappa}_{LIML} = Tn\left(\hat{k}_{LIML} - 1\right) \xrightarrow{d} \kappa_{LIML}^*$, where κ_{LIML}^* is the smallest root of $\left|\Xi_3 - \kappa\bar{\Sigma}\right| = 0$.

■

D Proof of Theorem 4

Proof. Using $R\hat{\beta}(k) - r = R\left[\hat{\beta}(k) - \beta\right]$, we have

$$
\begin{aligned}
W(k) &= \left[R\hat{\beta}(k) - r\right]' \left\{\hat{\sigma}_{uu}(k) R \left[\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp}\right]^{-1} R'\right\}^{-1} \left[R\hat{\beta}(k) - r\right] \\
&= \left\{\left[\hat{\beta}(k) - \beta\right]' R'\right\} \left\{\hat{\sigma}_{uu}(k) R \left[\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp}\right]^{-1} R'\right\}^{-1} \left\{R\left[\hat{\beta}(k) - \beta\right]\right\}.
\end{aligned}
$$

and note that $\widehat{\tilde{u}}(k) = \tilde{y}^{\perp} - \tilde{Y}^{\perp}\hat{\beta}(k) = \tilde{u}^{\perp} - \tilde{Y}^{\perp}\left[\hat{\beta}(k) - \beta\right]$, so

$$
\begin{aligned}
\hat{\sigma}_{uu}(k) &= \frac{1}{Tn - K_1 - L}\widehat{\tilde{u}}(k)'\widehat{\tilde{u}}(k) \\
&= \frac{1}{Tn - K_1 - L}\left\{\tilde{u}^{\perp} - \tilde{Y}^{\perp}\left[\hat{\beta}(k) - \beta\right]\right\}'\left\{\tilde{u}^{\perp} - \tilde{Y}^{\perp}\left[\hat{\beta}(k) - \beta\right]\right\} \\
&= \frac{Tn}{Tn - K_1 - L}\left\{\left(\frac{1}{Tn}\tilde{u}^{\perp\prime}\tilde{u}^{\perp}\right) - 2\left[\hat{\beta}(k) - \beta\right]'\left(\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{u}^{\perp}\right) + \left[\hat{\beta}(k) - \beta\right]'\left(\frac{1}{Tn}\tilde{Y}^{\perp\prime}\tilde{Y}^{\perp}\right)\left[\hat{\beta}(k) - \beta\right]\right\}.
\end{aligned}
$$

For $0 \leq \delta < \frac{1}{2}$, Theorem 2 implies $\hat{\beta}(k) - \beta \xrightarrow{p} 0$. By Lemma 2.1,

$$
\hat{\sigma}_{uu}(k) \xrightarrow{p} \sigma_{uu}
$$

as $(n, T) \to \infty$. Because $\theta_{Tn} = \left[\frac{1}{T^{1-2\delta}}\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp}\right]^{-1}\kappa_{Tn}\Sigma_{Vu} \xrightarrow{p} \kappa\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}\Sigma_{Vu} = \theta$, then

$$
RT^{1/2-\delta}\left(\widehat{\beta}(k) - \beta\right) \sim N\left(R\theta, \sigma_{uu}R\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}R'\right)
$$

Recall that $\frac{1}{T^{1-2\delta}}\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp} \xrightarrow{p} \Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}$ by the proof of Theorem 2. By Proposition B.7 in Lütkepohl (2005), we have

$$
\begin{aligned}
W(k) &= \left\{T^{1/2-\delta}\left[\hat{\beta}(k) - \beta\right]'R'\right\}\left\{\hat{\sigma}_{uu}(k) R\left[\frac{1}{T^{1-2\delta}}\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp}\right]^{-1}R'\right\}^{-1}\left\{RT^{1/2-\delta}\left[\hat{\beta}(k) - \beta\right]\right\} \\
&= \left\{T^{1/2-\delta}\left[\hat{\beta}(k) - \beta\right]'R'\right\}\left\{\sigma_{uu}R\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}R'\right\}^{-1}\left\{RT^{1/2-\delta}\left[\hat{\beta}(k) - \beta\right]\right\} + o_p(1) \\
&\xrightarrow{d} \chi^2_*(\Phi, \Lambda),
\end{aligned}
$$

which is a noncentral chi-squared distribution with Φ degrees of freedom and noncentrality parameter $\Lambda = \theta' R'\left[\sigma_{uu}R\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)^{-1}R'\right]^{-1}R\theta$. Note that if $\kappa = 0$, $\theta = 0$ and $\Lambda = 0$, hence $W(k) \xrightarrow{d} \chi^2(\Phi)$, a central chi-squared distribution with Φ degrees of freedom.

For $\delta = \frac{1}{2}$, by Lemma 2.1,

$$\hat{\sigma}_{uu}(k) \xrightarrow{d} \sigma_{uu} - 2\sigma_{uu}^{1/2}\left[(\lambda+z_V)' z_u - \kappa\rho\right]' \left[(\lambda+z_V)'(\lambda+z_V) - \kappa I_{Tn}\right]^{-1} \Sigma_{VV}^{-1/2}\Sigma_{Vu}$$
$$+\sigma_{uu}\left[(\lambda+z_V)' z_u - \kappa\rho\right]' \left[(\lambda+z_V)'(\lambda+z_V) - \kappa I_{Tn}\right]^{-1} \left[(\lambda+z_V)'(\lambda+z_V) - \kappa I_{Tn}\right]^{-1} \left[(\lambda+z_V)' z_u - \kappa\rho\right]$$
$$= \sigma_{uu} S(\Delta_1(\kappa))$$

where $S(b) = 1 - 2\rho' b + b'b$ and $\Delta_1(k) = \left[(\lambda+z_V)'(\lambda+z_V) - \kappa I_{Tn}\right]^{-1}\left[(\lambda+z_V)' z_u - \kappa\rho\right]$.Therefore,

$$\begin{aligned} W(k) &= \left\{\left[\hat{\beta}(k) - \beta\right]' R'\right\}\left\{\hat{\sigma}_{uu}(k) R\left[\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp}\right]^{-1} R'\right\}^{-1}\left\{R\left[\hat{\beta}(k) - \beta\right]\right\} \\ &\xrightarrow{d} \left[\sigma_{uu}^{1/2}\Sigma_{VV}^{-1/2}\Delta_1((\kappa))\right]' R'\left\{\sigma_{uu} S(\Delta_1((\kappa))) R\left\{\Sigma_{VV}^{1/2\prime}\left[(\lambda+z_V)'(\lambda+z_V) - \kappa I_{Tn}\right]\Sigma_{VV}^{1/2}\right\}^{-1} R'\right\}^{-1} R\left[\sigma_{uu}^{1/2}\Sigma_{VV}^{-1/}\right. \\ &= \Delta_1'((\kappa))\Sigma_{VV}^{-1/2} R'\left\{S(\Delta_1((\kappa))) R\left\{\Sigma_{VV}^{1/2\prime}\left[(\lambda+z_V)'(\lambda+z_V) - \kappa I_{Tn}\right]\Sigma_{VV}^{1/2}\right\}^{-1} R'\right\}^{-1} R\Sigma_{VV}^{-1/2}\Delta_1((\kappa)). \end{aligned}$$

as $(n,T) \to \infty$ because $\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp} \xrightarrow{d} \Sigma_{VV}^{1/2\prime}\left[(\lambda+z_V)'(\lambda+z_V) - \kappa I_{Tn}\right]\Sigma_{VV}^{1/2}$.

For $0 \le \delta < \frac{1}{2}$, by Lemma 2.1,

$$\begin{aligned} \hat{\sigma}_{uu}(k) &\xrightarrow{d} \sigma_{uu} - 2\sigma_{uu}^{1/2}\left[z_V' z_u - \kappa\rho\right]'\left[z_V' z_V - \kappa I_{Tn}\right]^{-1}\Sigma_{VV}^{-1/2}\Sigma_{Vu} \\ &\quad +\sigma_{uu}\left[z_V' z_u - \kappa\rho\right]'\left[z_V' z_V - \kappa I_{Tn}\right]^{-1}\left[z_V' z_V - \kappa I_{Tn}\right]^{-1}\left[z_V' z_u - \kappa\rho\right] = \sigma_{uu} S(\Delta_2((\kappa))) \end{aligned}$$

where $S(b) = 1 - 2\rho' b + b'b$ and $\Delta_2((\kappa)) = \left[z_V' z_V - \kappa I_{Tn}\right]^{-1}\left[z_V' z_u - \kappa\rho\right]$.Therefore,

$$\begin{aligned} W(k) &= \left\{\left[\hat{\beta}(k) - \beta\right]' R'\right\}\left\{\hat{\sigma}_{uu}(k) R\left[\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp}\right]^{-1} R'\right\}^{-1}\left\{R\left[\hat{\beta}(k) - \beta\right]\right\} \\ &\xrightarrow{d} \left[\sigma_{uu}^{1/2}\Sigma_{VV}^{-1/2}\Delta_2((\kappa))\right]' R'\left\{\sigma_{uu} S(\Delta_2((\kappa))) R\left\{\Sigma_{VV}^{1/2\prime}\left[z_V' z_V - \kappa I_{Tn}\right]\Sigma_{VV}^{1/2}\right\}^{-1} R'\right\}^{-1} R\left[\sigma_{uu}^{1/2}\Sigma_{VV}^{-1/2}\Delta_2((\kappa))\right] \\ &= \Delta_2'((\kappa))\Sigma_{VV}^{-1/2} R'\left\{S(\Delta_2((\kappa))) R\left\{\Sigma_{VV}^{1/2\prime}\left[z_V' z_V - \kappa I_{Tn}\right]\Sigma_{VV}^{1/2}\right\}^{-1} R'\right\}^{-1} R\Sigma_{VV}^{-1/2}\Delta_2((\kappa)). \end{aligned}$$

as $(n,T) \to \infty$ because $\tilde{Y}^{\perp\prime}(I - kM_{\tilde{Z}^{\perp}})\tilde{Y}^{\perp} \xrightarrow{d} \Sigma_{VV}^{1/2\prime}\left[z_V' z_V - \kappa I_{Tn}\right]\Sigma_{VV}^{1/2}$. ■

E Proof of Theorem 5

Proof. Note that $M_{\tilde{Z}^*}\tilde{Y} = M_{\tilde{Z}^*}\tilde{V}$, we have

$$\begin{aligned} \hat{\Sigma}_{VV} &= \frac{1}{Tn - K_1 - K_2}\tilde{V}' M_{\tilde{Z}^*}\tilde{V} \\ &= \frac{Tn}{Tn - K_1 - K_2}\left[\left(\frac{1}{Tn}\tilde{V}'\tilde{V}\right) - \frac{1}{Tn}\left(\frac{1}{\sqrt{Tn}}\tilde{V}'\tilde{Z}^*\right)\left(\frac{1}{Tn}\tilde{Z}^{*\prime}\tilde{Z}^*\right)^{-1}\left(\frac{1}{\sqrt{Tn}}\tilde{Z}^{*\prime}\tilde{V}\right)\right] \xrightarrow{p} \Sigma_{VV} \end{aligned}$$

because $\frac{1}{\sqrt{Tn}}\tilde{V}'\tilde{Z}^* = \left(\frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\tilde{V}_t'\tilde{X}_t, \frac{1}{\sqrt{Tn}}\sum_{t=1}^{T}\tilde{V}_t'\tilde{Z}_t\right) \xrightarrow{d} \left(\Psi_{\tilde{X}V}', \Psi_{ZV}'\right)$ and $\frac{1}{Tn}\sum_{t=1}^{T}\tilde{Z}_t^{*\prime}\tilde{Z}_t^* \xrightarrow{p} \Phi$.

For $0 \le \delta < \frac{1}{2}$, by Lemma 2.3,

$$\frac{K_2}{T^{1-2\delta}}G_{Tn} = \hat{\Sigma}_{VV}^{-1/2\prime}\left(\frac{1}{T^{1-2\delta}}\tilde{Y}^{\perp\prime}P_{\tilde{Z}^\perp}\tilde{Y}^\perp\right)\hat{\Sigma}_{VV}^{-1/2} \xrightarrow{p} \Sigma_{VV}^{-1/2\prime}\left(\Sigma_{VV}^{1/2\prime}\lambda'\lambda\Sigma_{VV}^{1/2}\right)\Sigma_{VV}^{-1/2} = \lambda'\lambda.$$

For $\delta = \frac{1}{2}$, by Lemma 2.3,

$$K_2G_{Tn} = \hat{\Sigma}_{VV}^{-1/2\prime}\left(\tilde{Y}^{\perp\prime}P_{\tilde{Z}^\perp}\tilde{Y}^\perp\right)\hat{\Sigma}_{VV}^{-1/2} \xrightarrow{d} \Sigma_{VV}^{-1/2\prime}\left[\Sigma_{VV}^{1/2\prime}\left(\lambda+z_V\right)'\left(\lambda+z_V\right)\Sigma_{VV}^{1/2}\right]\Sigma_{VV}^{-1/2} = \left(\lambda+z_V\right)'\left(\lambda+z_V\right),$$

which is a noncentral Wishart distribution with K_2 degrees of freedom and noncentrality parameter $\lambda'\lambda$.

For $\frac{1}{2} < \delta < \infty$, by Lemma 2.3,

$$K_2G_{Tn} = \hat{\Sigma}_{VV}^{-1/2\prime}\left(\tilde{Y}^{\perp\prime}P_{\tilde{Z}^\perp}\tilde{Y}^\perp\right)\hat{\Sigma}_{VV}^{-1/2} \xrightarrow{d} \Sigma_{VV}^{-1/2\prime}\left(\Sigma_{VV}^{1/2\prime}z_V'z_V\Sigma_{VV}^{1/2}\right)\Sigma_{VV}^{-1/2} = z_V'z_V.$$

which is a Wishart distribution with K_2 degrees of freedom. ■

F Proof of Theorem 6

Proof. Consider (1). When $0 \le \delta < \frac{1}{2}$, we note that

$$\begin{aligned}\mu_\pi &= \left(\tilde{Z}'\tilde{Z}\right)^{1/2}\pi = \left(\frac{\tilde{Z}'\tilde{Z}}{Tn}\right)^{1/2}(Tn)^{1/2}\frac{C}{\sqrt{n}T^\delta} \\ &= \left(\frac{\tilde{Z}'\tilde{Z}}{Tn}\right)^{1/2}T^{1/2}\frac{C}{T^\delta} = O\left(T^{\frac{1}{2}-\delta}\right).\end{aligned}$$

It follows that

$$S - c_\beta\mu_\pi \xrightarrow{d} N\left(0, I_k\right)$$

and

$$T - d_\beta\mu_\beta \xrightarrow{d} N\left(0, I_k\right).$$

Consider (2). When $\delta = \frac{1}{2}$, we have

$$\mu_\pi \to D_ZC = O_p\left(1\right).$$

Then

$$S \xrightarrow{d} N\left(c_\beta D_ZC, I_k\right)$$

and

$$T \xrightarrow{d} N\left(d_\beta D_ZC, I_k\right).$$

Consider (3). When $\frac{1}{2} < \delta < \infty$,

$$\mu_\pi = o_p(1).$$

Then

$$S \xrightarrow{d} N(0, I_k)$$

and

$$T \xrightarrow{d} N(0, I_k).$$

This proves the theorem. ■