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Irrational Mode Locking in Quasiperiodic Systems

Alan Middleton
Syracuse University

Creighton K. Thomas
Syracuse University

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Creighton K. Thomas, A. Alan Middleton

Department of Physics, Syracuse University, Syracuse, NY 13244, USA

A model for ac-driven systems, based on the Tang-Wiesenfeld-Bak-Coppersmith-Littlewood automaton for an elastic medium, exhibits mode-locked steps with frequencies that are irrational multiples of the drive frequency, when the pinning is spatially quasiperiodic. Detailed numerical evidence is presented for the large-system-size convergence of such a mode-locked step. The irrational mode locking is stable to small thermal noise and weak disorder. Continuous time models with irrational mode locking and possible experimental realizations are discussed.

Extended non-equilibrium systems subject to spatially uniform drives exhibit a vast range of spatiotemporal behaviors, including uniform and static configurations, patterns periodic in space or time, and a great variety of spatially and temporally chaotic phenomena [1, 2]. In the cases where the local fields have a periodic attractor under a constant drive, the addition of an oscillating external drive of period τ can lead to mode locking: the local fields are translated by exactly p periods over the time $q\tau$, with fixed $p, q \in \mathbb{Z}$, for a finite range of external drive amplitudes. This rational p/q mode-locked step is evident in the invariance of a dynamical variable (a velocity or current) to changes in the strength of the external driving force and is thus seen as a plateau in the average rate of change of the fields when plotted versus a control parameter. Rational mode locking has been seen in regular Josephson junction arrays driven by microwave radiation (giant Shapiro steps) [3] and a number of translation-invariant fluid and granular systems [4]. This mode locking can also be seen in systems with quenched disorder, e.g., experiments on [5] and models of [6] charge density waves (CDWs). A natural addition to these classes of spatially uniform or periodic systems and spatially random systems is the class of systems that have fixed quasiperiodic backgrounds.

Spatially quasiperiodic systems have been extensively explored in a number of physical contexts. Quasiperiodic solids are of course one example [7]. The static and dynamic properties of mesoscopic quasiperiodic systems have also been a subject of study. In quasiperiodic wire arrays, the dependence of the normal-superconductor transition temperature on external field has cusps at irrational densities of the vortices [8, 9]. Similar complex behavior has been seen in pinning structures which consist of quasiperiodic arrays of holes or magnetic (anti)dots added to a uniform superconductor [10]. Quasiperiodic arrays of colloidal particles can also now be constructed, using holographic optical traps [11].

We present here results on temporal mode locking in ac-driven two-dimensional systems with quasiperiodic structure. In addition to the typical rational mode locking, our simulations provide clear evidence of mode-locked steps with p/q approaching an irrational value over a finite drive range in the large system limit. These steps are robust to small thermal noise and weak distributed disorder. It would be of interest to search for these anomalous steps in experiments, where an output frequency would be irrationally related to the input frequency in the limit of large samples.

For speed of computation, we carried out our initial simulations using a cellular automaton approximation of periodically driven overdamped systems. We first review its justification, based on arguments developed by Tang, Wiesenfeld, Bak, Coppersmith, and Littlewood (TWBCL) [6, 12, 13, 14]. The automaton approximates a system with scalar degrees of freedom that represent particle displacements or coarse-grained coordinates for an elastic interface, such as the phase displacements in a charge density wave, flux vortex displacements along a drive direction, or fluid-fluid interfaces without overhangs [15]. The essential features that are to be captured are displacements that can be represented by an overdamped scalar field, local elastic interactions, spatial variation (such as impurity disorder that screens elastic interactions or a periodic background) that gives a natural length scale, pinning by a potential that is periodic in the scalar field, and a periodic external force. These general physical features can be modeled by a continuous-time driven Frenkel-Kontorova (FK) model [16], where N dynamical variables x_i (the coarse-grained interface or particle displacement indexed by lattice position i) evolve according to

$$\dot{x}_i = K\Delta^2 x_i + h' \cos(kx_i - \beta_i) + F_0(t) , \qquad (1)$$

where Δ^2 is the discrete Laplacian operator, K is the elastic constant, h' is the magnitude of the pinning potential, k determines the periodicity of the pinning, $\beta_i \in [0, 2\pi)$ are pinning phases, and $F_0(t)$ is a periodic driving force. Changing variables by substituting $u_i = (kx_i - \beta_i)/2\pi$, $t \to 2\pi Kt$, $h = h'/2\pi K$, and $F(t) = F_0(t)/2\pi K$ gives

$$\dot{u}_i = \Delta^2 u_i + h \cos(2\pi u_i) + F(t) + b_i , \qquad (2)$$

with $b_i = (2\pi)^{-1}\Delta^2 \beta_i$ in the range $(-z, z)$, where z is the number of nearest neighbors ($z = 4$ for a square lattice).

Coppersmith [14] has shown that in the limit of large h and $F(t)$ of the form $F(t) = \sum_{k \in \mathbb{Z}} F\delta(t - k\tau)$ with ample relaxation time τ between pulses, an automaton model is an appropriate approximation to these driven FK dynamics. This claim is also supported by numerical work, which shows very similar mode-locking behavior for the continuous time model and the corresponding automaton [6, 17]. In this limit, the dynamical variable u_i is near a minimum of the pinning potential, $u_i \approx n_i \in \mathbb{Z}$, between drive pulses. One then defines an integer curvature variable c_i by

$$c_i = \Delta^2 n_i + \lfloor b_i + F - h \rfloor , \qquad (3)$$

where $\lfloor \rfloor$ is the floor function; this truncation of the net force approximates the selection of local minima n_i by peaks in the pinning potential. The evolution of the c_i between subsequent pulses is the TWBCL automaton. At each time step, if the curvature exceeds the critical value z, the variable n_i advances (due to the pulse and relaxation), redistributing the curvature ("grains of sand" [13]) from a site i to neighboring sites j. Defining the toppling variable $U_i(t) = 1$ at all i where $c_i(t) \geq z$ and $U_i(t) = 0$ elsewhere, the update rule for c_i is

$$c_i(t + 1) = c_i(t) + \Delta^2 U_i . \qquad (4)$$

In a given step of this automaton, the activity or toppling rate, corresponding to the velocity of the interface, is $v_t = N^{-1} \sum_i U_i(t)$. Given periodic boundary conditions (BCs), the running dynamics of the system is fixed by the initial conditions $\{c_i(0)\}$, with $n_i = 0$ in Eq. (3) [18].

In the Bak-Tang-Wiesenfeld sandpile version [13] (open BCs) and in disordered driven interfaces (reflecting BCs), the initial c_i are chosen from a Poissonian distribution. We choose the initial c_i using Eq. (3) and quasiperiodic pinning

$$b_i = 2z \operatorname{frac}\left(m\frac{P_x}{L_x} + n\frac{P_y}{L_y}\right) - z + \frac{1}{2N} \qquad (5)$$

on a 2D square lattice of size $N = L_x \times L_y$, $i = (m, n)$, $0 \leq m < L_x$, $0 \leq n < L_y$, and $\operatorname{frac}(x) = x - \lfloor x \rfloor$. The latter two terms in Eq. (5) maintain $F \rightarrow -F$ symmetry. We choose the fractions P_x/L_x and P_y/L_y to be approximants to irrational numbers ρ_x and ρ_y, respectively. The approximants are found by truncating the continued fraction representation

$$\rho \equiv (r_0, r_1, r_2, \ldots) \equiv r_0 + \cfrac{1}{r_1 + \frac{1}{r_2 + \cdots}} . \qquad (6)$$

We focus here on the case where $\rho_x = \sqrt{2} - 1 = (0, 2, 2, 2, \ldots)$ and $\rho_y = \sqrt{3} - 1 = (0, 1, 2, 1, 2, \ldots)$.

Given pinning $\{b_i(0)\}$ and drive F that fix $c_i(t = 0)$, one characterization of the dynamics is the average activity rate $v = \langle v_t \rangle$ (toppling rate or interface velocity). For any finite system there are only a finite number of
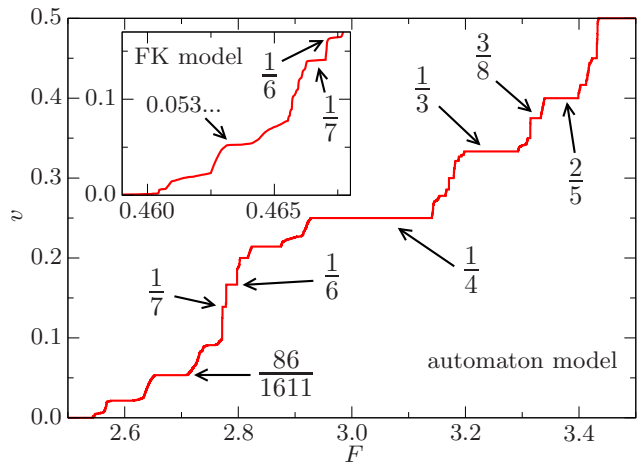

Figure 1: [color online] The activity staircase, velocity v vs. drive F, for the quasiperiodic automaton of size $L_x \times L_y = 169 \times 153$ with pinning given by the rational approximants $(70/169, 112/153)$ to $(\sqrt{2} - 1, \sqrt{3} - 1)$. Fractions with small denominators generally have the widest mode-locked steps, but the step at $86/1611$ is an anomalous step which does not fit this pattern. Inset: Results for the FK-model, Eq. (2), with $h = 10$ and the same size and quasiperiodic pinning; an anomalous step with similar $v \approx 0.053$ is seen.

configurations, so there is at least one periodic attractor (all attractors have the same v [19]). We repeat the map, Eq. (4), until a periodic orbit is found and compute $v(F) = M/(TN)$, where T is the period and $M = \sum_{t=1}^{T} \sum_i U_i(t)$ is the number of topplings in one period. We find plateaus in v, i.e. mode-locked steps (Fig. 1). Most of the wide steps seen are low-denominator rationals, but there is at least one extra "anomalous" step which has a large denominator for its width.

We find such a step, similar in position, size, and activity, for all approximants to $(\rho_x, \rho_y) = (\sqrt{2} - 1, \sqrt{3} - 1)$ with $L_x \geq 29$ and $L_y \geq 56$. The activity $v(L_x, L_y)$ on this step is not fixed at a single rational (Table I); the denominator and numerator for $v(L_x, L_y)$ grow with system size, but v appears to converge as $(L_x, L_y) \rightarrow (\infty, \infty)$ (Fig. 2). This strongly implies a limiting anomalous step at an unusual activity rate. The width of the selected step also appears to converge (Fig. 2). Scanning Table I along the $\rho_x = \sqrt{2} - 1$ approximant direction (varying L_x, fixed L_y), it can be seen that the numerators $p_{a,b}$ for $v(L_x, L_y)$ satisfy the recurrence relation

$$p_{a,b} = 2p_{a-1,b} + p_{a-2,b} , \qquad (7)$$

where a (b) is the number of terms used in the approximation of ρ_x (respectively, ρ_y). The denominators $q_{a,b}$ obey the same relation. Similarly, along the $\rho_y = \sqrt{3} - 1$ direction (varying L_y, fixed L_x) the recurrence relation that agrees with all data is

$$p_{a,b} = \begin{cases} p_{a,b-1} + p_{a,b-2} & \text{if } b \text{ is odd} \\ 2p_{a,b-1} + p_{a,b-2} & \text{if } b \text{ is even} \end{cases} , \qquad (8)$$

$\frac{P_y}{L_y}$ / $\frac{P_x}{L_x}$	$\frac{41}{56}$	$\frac{112}{153}$	$\frac{153}{209}$	$\frac{418}{571}$	$\frac{571}{780}$	$\frac{1560}{2131}$	$\frac{2131}{2911}$
$\frac{12}{29}$	$\frac{9}{167}$	$\frac{22}{405}$	$\frac{31}{572}$	$\frac{84}{1549}$	$\frac{115}{2121}$	$\frac{314}{5791}$	$\frac{429}{7912}$
$\frac{29}{70}$	$\frac{14}{266}$	$\frac{32}{603}$	$\frac{46}{869}$	$\frac{124}{2341}$	$\frac{170}{3210}$	$\frac{464}{8761}$	$\frac{634}{11971}$
$\frac{70}{169}$	$\frac{37}{699}$	$\frac{86}{1611}$	$\frac{123}{2310}$	$\frac{332}{6231}$	$\frac{455}{8541}$	$\frac{1242}{23313}$	$\frac{1697}{31854}$
$\frac{169}{408}$	$\frac{88}{1664}$	$\frac{204}{3825}$	$\frac{292}{5489}$	$\frac{788}{14803}$	$\frac{1080}{20292}$	$\frac{2948}{55387}$	$\frac{4028}{75679}$
$\frac{408}{985}$	$\frac{213}{4027}$	$\frac{494}{9261}$	$\frac{707}{13288}$	$\frac{1908}{35837}$	$\frac{2615}{49125}$	$\frac{7138}{134087}$	
$\frac{985}{2378}$	$\frac{514}{9718}$	$\frac{1192}{22347}$	$\frac{1706}{32065}$	$\frac{4604}{86477}$	$\frac{6310}{118542}$	$\frac{17224}{323561}$	
$\frac{2378}{5741}$	$\frac{1241}{23463}$	$\frac{2878}{53955}$	$\frac{4119}{77418}$				

Table I: The system-size dependence of activity (velocity) $v(L_x, L_y)$ at the anomalous step. The model is the TWBCL automaton with quasiperiodic pinning. The fractions P_x/L_x and P_y/L_y, which determine the pinning, are approximants to $\sqrt{2} - 1$ and $\sqrt{3} - 1$, respectively. To highlight the recurrence relation, Eqs. (7, 8), the fractions are not all written in lowest terms; in most cases, GCD$(p,q) = $ GCD(L_x, L_y).

with the denominators $q_{a,b}$ also obeying the same relation. Note that Eqs. (7,8) are also the recurrence relations for the numerators and denominators in, respectively, the approximants P_x/L_x and P_y/L_y. These recurrence relations suggest the temporal concatenation and synchronization of subsystems, but the precise construction of such spatiotemporal behavior is unclear to us. Jointly solving Eqs. (7,8) using four velocities as seeds, we obtain the velocity $\hat{v}$ in the $(L_x, L_y) \to (\infty, \infty)$ limit

$$\hat{v} = \frac{7903 + 379\sqrt{3} + 142\sqrt{2} - 181\sqrt{6}}{156238} , \qquad (9)$$

which is an algebraic irrational number, as our conjecture for the mode-locking value in the limit of infinite system size for this choice of (ρ_x, ρ_y).

We have found anomalous steps using other irrational pairs (ρ_x, ρ_y). For small system sizes, there are no steps that satisfy the recurrence relations, but as the system size increases, irrational candidates appear, with more than one candidate for a given (ρ_x, ρ_y). It is even possible that there are an infinite number of irrational steps with nonzero width in the infinite size limit for typical irrational pinning choices. It would be very surprising if other quasiperiodic pinning schemes (i.e., patterns for b_i) or lattices did not also give irrational steps. We have not found a way to predict, for a given driving choice, the activity of any irrational mode-locked step or the number of such steps. One-dimensional automata with nearest neighbor rules exhibit few rational steps and in the simplest case should not exhibit irrational steps [17].

Numerically more challenging versions of this model include simulations of the continuous-time FK model
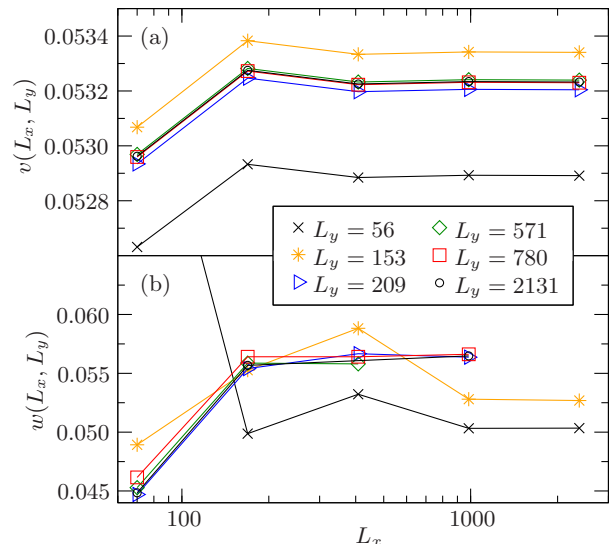

Figure 2: [color online] The size dependence of (a) activity $v(L_x, L_y)$, and (b) width, $w(L_x, L_y)$, of the anomalous step. Each curve is for fixed L_y. The convergence of v and w as a function of system size strongly suggests that there is a well defined activity $\hat{v}$ and step width $\hat{w}$ in the limit $(L_x, L_y) \to (\infty, \infty)$; we conjecture that $\hat{v} \notin \mathbb{Q}$, Eq. (9), and $\hat{w} \neq 0$.

Eq. (2). We have studied the FK model using quasiperiodic pinning variables identical to those for the automaton. We find results similar to those for the automaton, with both rational and anomalous steps converging to fixed width in a sequence of system sizes (Fig. 1, inset).

For applications to experiment, it is natural to ask about the robustness of the irrational mode-locked step. To address this question we have studied the effects of thermal noise and quenched disorder. We model thermal noise by adding random topplings to the zero-temperature parallel dynamics of the system, Eq. (4). These random topplings occur at finite rate η per time step at each site. This noise randomly increases or decreases the displacement variable n_i by one period. To implement this, we add backwards topplings, which take place when $c_i \leq -z$. The toppling variable $U_i(t)$ used in Eq. (4) is modified: in addition to setting $U_i(t) = 1$ at all i where $c_i(t) \geq z$, one sets $U_i(t) = -1$ at all i where $c_i(t) \leq -z$, with $U_i(t) = 0$ for $-z < c_i(t) < z$.

Automaton models with period $q > 2$ are unstable to thermal noise in general [20], but we find that for small thermal noise the change in the velocity is limited. Weak thermal noise, which leads to nucleation of out-of-phase regions [20] and subsequent motion of domain walls, destroys exact periodicity, but the anomalous steps can still be clearly identified (Fig. 3). Numerical simulations and analysis of similar systems, such as depinning in CDWs with weak noise, also show small corrections to $\langle v \rangle$ on most of the step and rounding near small-denominator step edges [21].
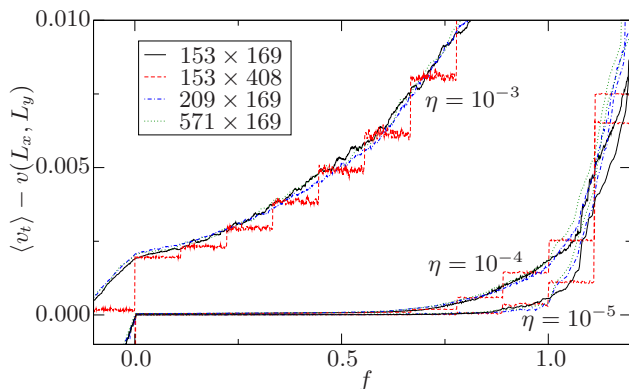
The effects of quenched disorder were included by mod-

Figure 3: [color online] The effect of thermal noise on the anomalous step for dimensionless noise values $\eta = 10^{-5}$, 10^{-4}, 10^{-3} for system sizes defined in the key. The horizontal axis is the scaled force f which is linear in F with each zero-temperature step starting at $f = 0$ and ending at $f = 1$. For easier comparison, the zero-temperature activity $v(L_x, L_y)$ has been subtracted. For each value of noise, there is similar rounding of the plateau for all system sizes.

ifying the pinning variables b_i, to model jitter in both pinning phases β_i and coupling K. Weak distributed disorder does not affect the activity $v(L_x, L_y)$ significantly, with 97% of the step unchanged for 0.1% Gaussian noise in the b_i with $(L_x, L_y) = (169, 153)$. However, irrational locking is sensitive to strong rare disorder: we find that randomizing the b_i at even one site in large systems drastically reduces the width of the anomalous step. This is to be contrasted with Poissonian initial conditions, which are maximally random, and thus have steps that are not sensitive to strong disorder, though wide large-denominator steps are not present.

Given the general applicability of the FK model and the generic features of the automaton approximation, our results suggest that irrational mode locking might be seen in a number of experimental systems. The most likely candidates are constructed mesoscopic systems, where thermal fluctuations are small enough and there is precise experimental control. As noted, quasiperiodic superconducting wire arrays and Josephson-junction arrays exhibit interesting static features as a function of applied field; dynamic ac transport experiments might well be modeled by elastically or plastically interacting vortices [22, 23] that would realize the conditions necessary for irrational mode locking. Recently [11], quasiperiodic colloidal particle arrays have been constructed using holographic optical traps. Experiments for dc driven particles in periodic arrays [24] suggest that experiments in a quasiperiodic background may be used to search for irrational mode locking in a collective system.

We have studied a broadly applicable model of dynamics in an extended system, where we have chosen a quasiperiodic spatial background. We find anomalous mode-locked steps that appear to converge to an irrational frequency relative to the drive frequency, in the limit of large system sizes. This step is stable to modest amounts of both thermal noise and quenched disorder. One avenue for comparison with experiment would be to study automaton models on quasiperiodic tilings (see [25] for the $v \to 0$ limit). There are a number of experimental systems that are closely related to this model and in which such irrational mode locking might be seen.

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