9-1992

Parallel Cellular Automata: A Model Program for Computational Science

Per Brinch Hansen
Syracuse University, School of Computer and Information Science, pbh@top.cis.syr.edu

SU-CIS-92-18

Parallel Cellular Automata: A Model Program for Computational Science

Per Brinch Hansen

September 1992

School of Computer and Information Science
Syracuse University
Suite 4-116, Center for Science and Technology
Syracuse, NY 13244-4100

Parallel Cellular Automata: A Model Program for Computational Science[1]

PER BRINCH HANSEN

Syracuse University, Syracuse, New York 13244

September 1992

We develop a model program for parallel execution of cellular automata on a multicomputer. The model program is then adapted for simulation of forest fires and numerical solution of Laplace's equation for stationary heat flow. The performance of the parallel program is analyzed and measured on a Computing Surface configured as a matrix of transputers with distributed memory.

Categories and Subject Descriptors: D.1.3 [**Programming Techniques**]: Concurrent programming—*distributed programming*

General Terms: Algorithms

Additional Key Words and Phrases: Cellular automata, Multicomputers, Programming methodology

CONTENTS

INTRODUCTION

This is one of several papers that explore the benefits of developing *model programs for computational science* [Brinch Hansen 1990, 1991a, 1991b, 1992a]. The theme of this paper is *parallel cellular automata.*

A cellular automaton is a discrete model of a system that varies in space and time. The discrete space is an array of identical cells, each representing a local state. As time advances in discrete steps, the system evolves according to universal laws. Every time the clock ticks, the cells update their states simultaneously. The next state of a cell depends only on the current state of the cell and its nearest neighbors.

In 1950 John von Neumann and Stan Ulam introduced cellular automata to study self-reproducing systems [von Neumann 1966, Ulam 1986]. John Conway's game of *Life* is undoubtedly the most widely known cellular automaton [Gardner 1970, 1971, Berlekamp et al. 1982]. Another well-known automaton simulates the life cycles of sharks and fish on the imaginary planet *Wa-Tor* [Dewdney 1984]. The numerous applications include forest infestation [Hoppenstadt 1978], fluid flow [Frisch et al. 1986], earthquakes [Bak and Tang 1989], forest fires [Bak and Chen 1990], and sandpile avalanches [Hwa and Kardar 1989].

Cellular automata can simulate continuous physical systems described by *partial differential equations.* The numerical solution of, say, Laplace's equation by grid relaxation is really a discrete simulation of heat flow performed by a cellular automaton.

Cellular automata are ideally suited for parallel computing. Our goal is to explore *programming methodology for multicomputers.* We will illustrate this theme by developing a model program for parallel execution of cellular automata on a multicomputer with a square matrix of processor nodes. We will then show how easy it is to adapt the model program for two different applications: (1) simulation of a *forest fire*, and (2) numerical solution of *Laplace's equation* for stationary heat flow. On a *Computing Surface* with transputer nodes, the parallel efficiency of the model program is close to one.

1. CELLULAR AUTOMATA

A *cellular automaton* is an array of parallel processes, known as *cells.* Every cell has a discrete *state.* At discrete moments in *time*, the cells update their states *simultaneously.* The state *transition* of a cell depends only on its previous state and the states of the *adjacent* cells.

We will program a *two-dimensional* cellular automaton with *fixed boundary states* (Fig. 1).

−	+	+	+	+	+	+	−
+	?	?	?	?	?	?	+
+	?	?	?	?	?	?	+
+	?	?	?	?	?	?	+
+	?	?	?	?	?	?	+
+	?	?	?	?	?	?	+
+	?	?	?	?	?	?	+
−	+	+	+	+	+	+	−

Fig. 1 A cellular automaton

The automaton is a square matrix with three kinds of cells:

1. *Interior cells*, marked "?", may change their states dynamically.

2. *Boundary cells*, marked "+", have fixed states.

3. *Corner cells*, marked "−", are not used.

Figure 2 shows an interior cell and the four neighbors that may influence its state. These five cells are labeled c (central), n (north), s (south), e (east), and w (west).

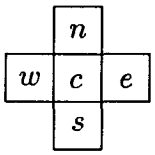

Fig. 2 Adjacent cells

The cellular automaton will be programmed in Pascal, extended with statements for parallel execution and message communication.

The execution of k statements $S_1, S_2, \ldots, S_k$ as *parallel processes* is denoted

$$\textbf{parbegin } S_1|S_2|\ldots|S_k \textbf{ end}$$

The parallel execution continues until every one of the k processes has terminated.

The *parallel for* statement

$$\textbf{parfor } i := 1 \textbf{ to } k \textbf{ do } S(i)$$

is equivalent to

$$\textbf{parbegin } S(1)|S(2)|\ldots|S(k) \textbf{ end}$$

We assume that parallel processes communicate through *synchronous channels* only. The *input* and *output* of a value x through a channel c are denoted

```
c?x        c!x
```

A cellular automaton is a set of parallel communicating cells. If we ignore boundary cells and communication details, a two-dimensional automaton is defined as follows:

```
parfor i := 1 to n do
  parfor j := 1 to n do
    cell(i, j)
```

After initializing its own state, every interior cell goes through a fixed number of state transitions before outputting its final state:

```
initialize own state;
for k := 1 to steps do
  begin
    exchange states with
      adjacent elements;
    update own state
  end;
output own state
```

The challenge is to transform this fine-grained parallel model into an efficient program for a *multicomputer* with distributed memory.

2. INITIAL STATES

Consider a cellular automaton with 36 interior cells and 24 boundary cells. In a sequential computer, the combined state of the automaton can be represented by an 8×8 matrix, called a *grid* (Fig. 3). For reasons that will be explained later, the grid elements are indicated by 0's and 1's.

–	1	0	1	0	1	0	–
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
–	0	1	0	1	0	1	–

Fig. 3 A square grid

Figure 4 shows the *initial values* of the elements. The boundary elements have fixed values u_1, u_2, u_3, and u_4. Every interior element has the same initial value u_5.

	u_1	
u_4	u_5	u_3
	u_2	

Fig. 4 Initial values

In general, a grid u has $n \times n$ interior elements and $4n$ boundary elements:

```
const n = ...;
type state = (...);
  row = array [0..n+1] of state;
  grid = array [0..n+1] of row;
var u: grid;
```

Since the possible *states* of every cell vary from one application to another, we deliberately leave them unspecified. The grid dimension n and the initial states u_1, u_2, u_3, u_4, and u_5 are also application dependent.

On a sequential computer, the grid is initialized as follows:

```
for i := 0 to n + 1 do
  for j := 0 to n + 1 do
    u[i,j] := initial(i, j)
```

Algorithm 1 defines the *initial* value of element $u[i, j]$. The values of the corner elements are arbitrary (and irrelevant).

```
function initial(i, j: integer)
  : state;
begin
  if i = 0 then
    initial := u1
  else if i = n + 1 then
    initial := u2
  else if j = n + 1 then
    initial := u3
  else if j = 0 then
    initial := u4
  else
    initial := u5
end
```

Algorithm 1

3. DATA PARALLELISM

For simulation of a cellular automaton, the ideal *multicomputer architecture* is a square matrix of identical processor *nodes* (Fig. 5). Every node is connected to its nearest neighbors (if any) by four communication *channels.*

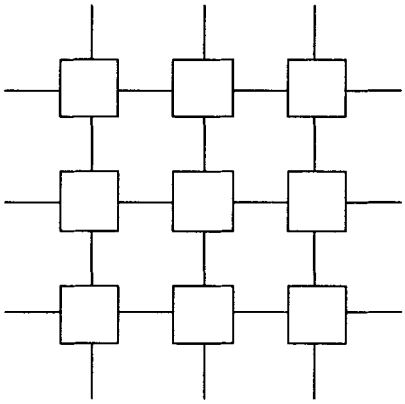

Fig. 5 Processor matrix

Figure 6 shows a grid with 36 interior elements divided into 9 subgrids. We now have a 3×3 matrix of nodes and a 3×3 matrix of subgrids. The two matrices define a one-to-one correspondence between subgrids and nodes. We will assign each subgrid to the corresponding node and let the nodes update the subgrids simultaneously. This form of distributed processing is called *data parallelism.*

–	1	0	1	0	1	0	–
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
–	0	1	0	1	0	1	–

Fig. 6 A subdivided grid

Every processor holds a 4×4 *subgrid* with 4 interior elements and 8 boundary elements (Fig. 7). Every boundary element holds either an interior element of a neighboring subgrid or a boundary element of the entire grid. (We will say more about this later.)

–	1	0	–
1	0	1	0
0	1	0	1
–	0	1	–

Fig. 7 A subgrid

4. PROCESSOR NODES

With this background, we are ready to program a cellular automaton that runs on a $q \times q$ *processor matrix.*

The *nodes* follow the same script (Algorithm 2).

```
procedure node(qi, qj: integer;
  n, s, e, w: channel);
var u: subgrid; k: integer;
begin
  newgrid(qi, qj, u);
  for k := 1 to steps do
    relax(qi, qj, n, s, e, w, u);
  output(qi, qj, e, w, u)
end
```

Algorithm 2

A node is identified by its row and column numbers (q_i, q_j) in the processor matrix, where

$$1 \leq q_i \leq q \quad \text{and} \quad 1 \leq q_j \leq q$$

Four communication channels labeled n, s, e, and w connect a node to its nearest neighbors (if any).

Every node holds a subgrid with $m \times m$ interior elements and $4m$ boundary elements (Fig. 7):

```
const m = ...;
type subrow = array [0..m+1] of state;
  subgrid = array [0..m+1] of subrow;
```

The grid dimension n is a multiple of the subgrid dimension m:

$$n = m * q$$

After initializing its subgrid, a node updates the subgrid a fixed number of times before outputting the final values. In numerical analysis, grid iteration is known as *relaxation.*

Node (q_i, q_j) holds the following subset of the complete grid $u[0..n+1, 0..n+1]$:

$$u[i_0..i_0 + m + 1, j_0..j_0 + m + 1]$$

where

$$i_0 = (q_i - 1)m \quad \text{and} \quad j_0 = (q_j - 1)m$$

The initialization of a subgrid is straightforward (Algorithm 3).

```
procedure newgrid(qi, qj:
  integer; var u: subgrid);
var i, i0, j, j0: integer;
begin
  i0 := (qi − 1)*m;
  j0 := (qj − 1)*m;
  for i := 0 to m + 1 do
    for j := 0 to m + 1 do
      u[i,j] := initial(i0+i, j0+j)
end
```

Algorithm 3

5. PARALLEL RELAXATION

In each time step, every node updates its own subgrid. The next value of an interior element is a function of its current value u_c, and the values u_n, u_s, u_e, and u_w of the four adjacent elements (Fig. 2). Every application of a cellular automaton requires a different *transition function* (Algorithm 4).

```
function next(uc, un, us,
  ue, uw: state): state;
begin next := ... end
```

Algorithm 4

Parallel relaxation is not quite as easy as it sounds. When a node updates row number 1 of its subgrid, it needs access to row number m of the subgrid of its northern neighbor (Fig. 6). To relax its subgrid, a node must share a single row or column with each of its four neighbors.

The solution to this problem is to let two neighboring grids *overlap* by one row or column vector. Before a node updates its interior elements, it exchanges a pair of vectors with each of the adjacent nodes. The overlapping vectors are kept in the boundary elements of the subgrids (Fig. 7). If a neighboring node does not exist, a boundary vector holds the corresponding boundary elements of the entire grid (Figs. 4 and 6).

The northern neighbor of a node outputs row number m to the node, which inputs it in row number 0 of its own subgrid (Fig. 7). In return, the node outputs row number 1 to its northern neighbor, which inputs it in row number $m + 1$ of its subgrid. Similarly, a node exchanges rows with its southern neighbor, and columns with its eastern and western neighbors (Fig. 5).

The *shared elements* raise the familiar concern about time-dependent errors in parallel programs. *Race conditions* are prevented by a rule of *mutual exclusion*: While

a node updates an element, another node cannot access the same element. This rule is enforced by an ingenious method [Barlow and Evans 1982].

Every grid element $u[i, j]$ is assigned a *parity*

$$(i + j) \textbf{ mod } 2$$

which is either *even* (0) or *odd* (1) as shown in Figs. 3 and 6. To eliminate tedious (and unnecessary) programming details, we assume that the subgrid dimension m is *even.* This guarantees that every subgrid has the same *parity ordering* of the elements (Figs. 6 and 7).

Parity ordering reveals a simple property of grids: The next values of the even interior elements depend only on the current values of the odd elements, and vice versa. This observation suggests a reliable method for parallel relaxation.

In each relaxation step, the nodes scan their grids twice:

First scan: The nodes exchange odd elements with their neighbors and update all even interior elements simultaneously.

Second scan: The nodes exchange even elements and update all odd interior elements simultaneously.

The key point is this: In each scan, the simultaneous updating of local elements depends only on shared elements with constant values! In the terminology of parallel programming, the nodes are *disjoint processes* during a scan.

The *relaxation* procedure uses a local variable to update elements with the same *parity* b after exchanging elements of the opposite parity $1 - b$ with its neighbors (Algorithm 5).

```
procedure relax(qi, qj: integer;
  n, s, e, w: channel; var u:
  subgrid);
var b, i, j, k, last: integer;
begin
  for b := 0 to 1 do
    begin
      exchange(qi, qj, 1 − b,
        n, s, e, w, u);
      for i := 1 to m do
        begin
          k := (i + b) mod 2;
          j := 2 − k;
          last := m − k
          while j <= last do
            begin
              u[i,j] := next(u[i,j],
                u[i−1,j], u[i+1,j],
                u[i,j+1], u[i,j−1]);
              j := j + 2
            end
        end
    end
end
```

Algorithm 5

6. LOCAL COMMUNICATION

The nodes communicate through *synchronous channels* with the following properties:

1. Every channel used connects exactly two nodes.

2. The communications on a channel take place one at a time.

3. A communication takes place when a node is ready to output a value through a channel and another node is ready to input the value through the same channel.

4. A channel can transmit a value in either direction between two nodes.

5. The four channels of a node can transmit values simultaneously.

These requirements are satisfied by *transputer* nodes programmed in *occam* [Cok 1991].

The identical behavior of the nodes poses a subtle problem. Suppose the nodes simultaneously attempt to input from their northern channels. In that case, the nodes will deadlock, since none of them are ready to output through these channels. There are several solutions to this problem. We use a method that works well for transputers.

Before the nodes scan elements of the same parity, they communicate with their neighbors in two phases (Fig. 8).

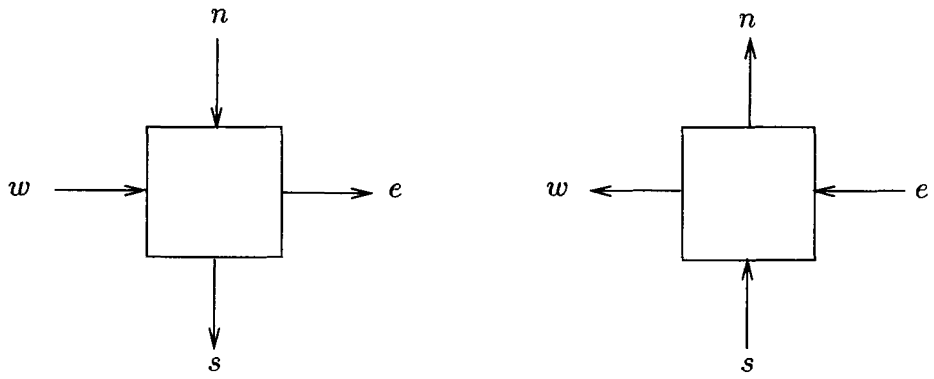

Fig. 8 Communication phases

In each phase, every node communicates simultaneously on its four channels as shown below. Phases 1 and 2 correspond to the left and right halves of Fig. 8.

Channel	Phase 1	Phase 2
n	input	output
s	output	input
e	output	input
w	input	output

Since every input operation on a channel is matched by a simultaneous output operation on the same channel, this *protocol* is *deadlock free.* It is also very *efficient*, since every node communicates simultaneously with its four neighbors.

Algorithm 6 defines the *exchange* of elements of parity b between a node and its four neighbors.

```
procedure exchange(qi, qj, b:
  integer; n, s, e, w: channel;
  var u: subgrid);
begin
  phase1(qi, qj, b, n, s, e, w, u);
  phase2(qi, qj, b, n, s, e, w, u)
end
```

Algorithm 6

Phase 1 is defined by Algorithm 7. The *if* statements prevent boundary nodes from communicating with nonexisting neighbors (Fig.5).

```
procedure phase1(qi, qj, b:
  integer; n, s, e, w: channel;
  var u: subgrid);
var k, last: integer;
begin
  k := 2 − b;
  last := m − b;
  while k <= last do
    begin
      parbegin
        if qi > 1 then n?u[0,k] |
        if qi < q then s!u[m,k] |
        if qj < q then e!u[k,m] |
        if qj > 1 then w?u[k,0]
      end;
      k := k + 2
    end
end
```

Algorithm 7

Phase 2 is similar (Algorithm 8).

```
procedure phase2(qi, qj, b:
  integer; n, s, e, w: channel;
  var u: subgrid);
var k, last: integer;
begin
  k := b + 1;
  last := m + b − 1;
  while k <= last do
    begin
      parbegin
        if qi > 1 then n!u[1,k] |
        if qi < q then s?u[m+1,k] |
        if qj < q then e?u[k,m+1] |
        if qj > 1 then w!u[k,1]
      end;
      k := k + 2
    end
end
```

Algorithm 8

We have used this protocol on a *Computing Surface* with transputer nodes. Since transputer links can communicate in both directions simultaneously, the two communication phases run in parallel. So every transputer inputs and outputs simultaneously through all four links!

If the available processors cannot communicate simultaneously with their neighbors, a sequential protocol must be used [Dijkstra 1982]. This is also true if the overhead of parallelism and communication is substantial. However, the replacement of one protocol by another should only change Algorithms 6–8 and leave the rest of the program unchanged.

7. GLOBAL OUTPUT

At the end of a simulation, the nodes output their final values to a *master* processor that assembles a complete grid. The boundary channels of the processor matrix are not used for grid relaxation (Fig. 5). We use the horizontal boundary channels to connect the nodes and the master into a *pipeline* for *global output* (Fig. 9).

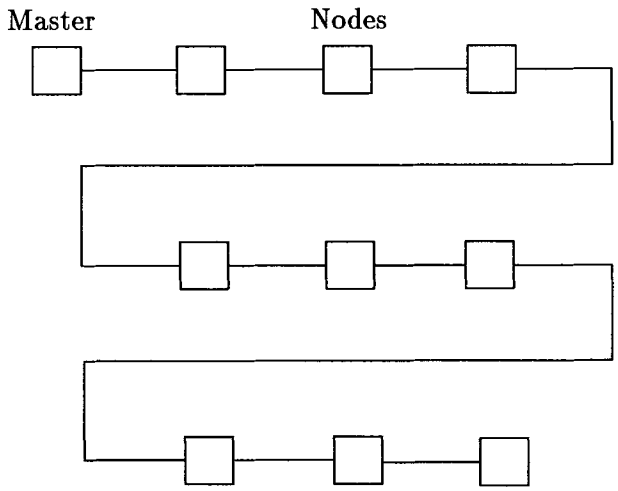

Fig. 9 Output pipeline

The boundary elements of the entire grid have known fixed values (Fig. 4). These elements are needed only during relaxation. The final output is an $n \times n$ matrix of interior elements only. Every element defines the final state of a single cell.

So we redefine the full grid, omitting the boundary elements:

```
type row = array [1..n] of state;
  grid = array [1..n] of row;
```

The *master* inputs the final grid row by row, one element at a time (Algorithm 9).

```
procedure master(inp: channel;
  var u: grid);
var i, j: integer;
begin
  for i := 1 to n do
    for j := 1 to n do
      inp?u[i,j]
end
```

Algorithm 9

The nodes use a common procedure to *output* interior elements in row order (Algorithm 10).

```
procedure output(qi, qj: integer;
  inp, out: channel; var u:
  subgrid);
var i, j: integer;
begin
  for i := 1 to m do
    begin
      for j := 1 to m do
        out!u[i,j];
      copy((q − qj)*m, inp, out)
    end;
  copy((q − qi)*m*n, inp, out)
end
```

Algorithm 10

Every row of elements is distributed through a row of nodes (Figs. 5 and 6). For each of its subrows, node (q_i, q_j) outputs the m interior elements and copies the remaining $(q - q_j)m$ elements of the same row from its eastern neighbor. This completes the output of the rows of elements, which are distributed through row q_i of the processor matrix. The node then copies the remaining $(q - q_i)m$ complete rows of n elements each.

A simple procedure is used to *copy* a fixed number of elements from one channel to another (Algorithm 11).

```
procedure copy(no: integer;
  inp, out: channel);
var k: integer; uk: state;
begin
  for k := 1 to no do
    begin
      inp?uk; out!uk
    end
end
```

Algorithm 11

In our program for the Computing Surface, we extended the copy procedure with parallel input/output. We also modified Algorithms 2 and 9 slightly to enable the program to output intermediate grids at fixed intervals.

8. PROCESSOR NETWORK

Figure 10 illustrates the *network* that ties the processors together. The network consists of a horizontal channel matrix h and a vertical channel matrix v.

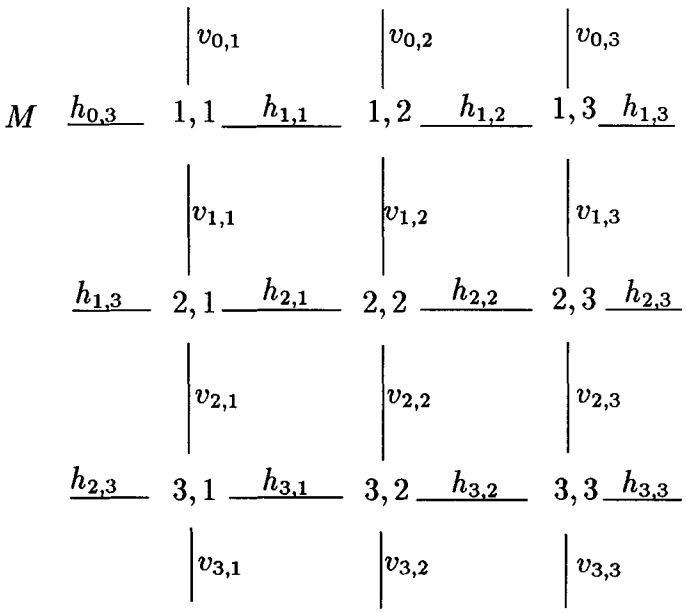

Fig. 10 Processor network

The following examples illustrate the abbreviations used:

M	master
3,2	node(3,2)
$v_{2,2}$	channel $v[2,2]$
$h_{3,1}$	channel $h[3,1]$

Algorithm 12 defines parallel *simulation* of a cellular automaton that computes a relaxed grid u. Execution of the parallel statements activates (1) the master, (2) the first column of nodes, and (3) the rest of the nodes.

```
procedure simulate(var u: grid);
type
  line = array [1..q] of channel;
  matrix = array [0..q] of line;
var h, v: matrix; i, j, k: integer;
begin
  parbegin
    master(h[0,q], u) |
    parfor k := 1 to q do
      node(k, 1, v[k−1,1], v[k,1],
        h[k,1], h[k−1,q]) |
    parfor i := 1 to q do
      parfor j := 2 to q do
        node(i, j, v[i−1,j], v[i,j],
          h[i,j], h[i,j−1])
  end
end
```

Algorithm 12

This completes the development of the model program. We will now demonstrate how easily the program can be adapted to different applications of cellular automata.

9. EXAMPLE: FOREST FIRE

A typical application of a cellular automaton is simulation of a *forest fire.* Every cell represents a *tree* that is either *alive, burning,* or *dead.* In each time step, the next state of every tree is defined by *probabilistic rules* similar to the ones proposed by Bak and Chen [1990]:

1. If a live tree is next to a burning tree, it burns; otherwise, it catches fire with probability p_1.

2. A burning tree dies.

3. A dead tree has probability p_2 of being replaced by a live tree.

Parallel simulation of a forest fire requires only minor changes of the model program:

1. The possible *states* are:

```
type state = (alive, burning, dead)
```

2. The *initial* states may, for example, be:

$$u_1 = u_2 = u_3 = u_4 = \text{dead} \quad u_5 = \text{alive}$$

3. Algorithm 4.1 defines state *transitions.*

```
function next(uc, un, us,
  ue, uw: state): state;
const p1 = 0.01; p2 = 0.3;
begin
  if uc = alive then
    if (un = burning) or
      (us = burning) or
      (ue = burning) or
      (uw = burning)
    then next := burning
    else if random <= p1
      then next := burning
      else next := alive
  else if uc = burning then
    next := dead
  else {uc = dead}
    if random <= p2
      then next := alive
      else next := dead
end
```

Algorithm 4.1

4. A *random number* generator is added.

10. EXAMPLE: LAPLACE'S EQUATION

A cellular automaton can also solve *Laplace's equation* for *equilibrium temperatures* in a square region with fixed temperatures at the boundaries. Every cell represents the temperature at a single point in the region. In each time step, the next temperature of every cell is defined by a simple *deterministic rule.*

Parallel simulation of heat flow requires the following changes of the model program:

1. The *states* are temperatures represented by reals.

2. A possible choice of *initial* temperatures is:

$$\begin{aligned} u_1 &= 0 \\ u_2 &= 100 \\ u_3 &= 100 \\ u_4 &= 0 \\ u_5 &= 50 \end{aligned}$$

3. Algorithm 4.2 defines the *next* temperature of an interior cell.

```
function next(uc, un, us,
  ue, uw: real): real;
var res: real;
begin
  res := (un + us + ue +
    uw)/4.0 − uc;
  next := uc + fopt*res
end
```

Algorithm 4.2

In steady state, the temperature of every interior cell is the average of the neighboring temperatures:

$$u_c = (u_n + u_s + u_e + u_w)/4.0$$

This is the discrete form of Laplace's equation. The *residual res* is a measure of how close the temperatures are to satisfying this equation. The correction of a temperature u_c is proportional to its residual.

4. A *relaxation factor* f_{opt} is added:

For a large square grid relaxed in parity order, the relaxation factor

$$f_{opt} = 2 - 2\pi/n$$

ensures the fastest possible convergence towards stationary temperatures. In numerical analysis, this method is called *successive overrelaxation* with *parity ordering.* The method requires n relaxation *steps* to achieve 3-figure accuracy of the final temperatures [Young 1954, Press et al. 1989].

The complete algorithm for parallel simulation of steady state heat flow is listed in the Appendix. The corresponding sequential program is explained in [Brinch Hansen 1992b]. Numerical solution of Laplace's equation on multicomputers is also discussed in [Barlow and Evans 1982, Evans 1984, Pritchard et al. 1987, Saltz et al. 1987, Fox et al. 1988].

11. COMPLEXITY

In each time step, every node exchanges overlapping elements with its neighbors in $O(m)$ time and updates its own subgrid in $O(m^2)$ time. The final output takes $O(n^2)$ time. The *parallel run time* required to relax an $n \times n$ grid n times on p processors is

$$T(n,p) = n(am^2 + O(m)) + O(n^2)$$

where a is a system-dependent constant of relaxation and

$$n = m\sqrt{p} \tag{1}$$

The complexity of parallel simulation can be rewritten as follows:

$$T(n,p) = n^2(an/p + O(1) + O(1/\sqrt{p}))$$

For $1 \leq p \ll n$, the communication times are insignificant compared to the relaxation time, and we have approximately

$$T(n,p) \approx an^3/p \quad \text{for } n \gg p \tag{2}$$

If the same simulation runs on a single processor, the *sequential run time* is obtained by substituting $p = 1$ in (2):

$$T(n,1) \approx an^3 \quad \text{for } n \gg 1 \tag{3}$$

The processor efficiency of the parallel program is

$$E(n,p) = \frac{T(n,1)}{p\,T(n,p)} \tag{4}$$

The nominator is proportional to the number of processor cycles used in a sequential simulation. The denominator is a measure of the total number of cycles used by p processors performing the same computation in parallel.

By (2), (3), and (4) we find that the parallel efficiency is close to one, when the problem size n is large compared to the machine size p:

$$E(n,p) \approx 1 \quad \text{for } n \gg p$$

Since this analysis ignores communication times, it cannot predict how close to one the efficiency is.

In theory, the efficiency can be computed from (4) by measuring the sequential and parallel run times for the same value of n. Unfortunately, this is not always feasible. When 36 nodes relax a 1500×1500 grid of 64-bit reals, every node holds a subgrid of $250 \times 250 \times 8 = 0.5$ Mbytes. However, on a single processor, the full grid occupies 18 Mbytes.

A more realistic approach is to make the $O(n^2)$ grid proportional to the machine size p. Then every node has an $O(m^2)$ subgrid of constant size independent of the number of nodes. And the nodes always perform the same amount of computation per time step.

When a *scaled simulation* runs on a single processor, the run time is approximately

$$T(m,1) \approx am^3 \quad \text{for } m \gg 1 \tag{5}$$

since $p = 1$ and $n = m$.

From (1), (3), and (5) we obtain

$$T(n,1) \approx p^{3/2}T(m,1) \quad \text{for } m \gg 1 \tag{6}$$

The computational rule we need follows from (4) and (6):

$$E(n,p) \approx \frac{\sqrt{p}\,T(m,1)}{T(n,p)} \quad \text{for } m \gg 1 \tag{7}$$

This formula enables us to compute the efficiency of a parallel simulation by running a scaled-down version of the simulation on a single node.

12. PERFORMANCE

We reprogrammed the model program in *occam 2* and ran it on a *Computing Surface* with T800 transputers configured as a square matrix with a master node [Inmos 1988, Meiko 1987, Trew and Wilson 1991]. The program was modified to solve *Laplace's equation* as explained in Section 10. The complete program is found in the Appendix.

Table I shows measured (and predicted) run times $T(n,p)$ in seconds for n relaxations of an $n \times n$ grid on p processors. In every run, the subgrid dimension $m = 250$.

Table I

p	n	$T(n,p)$	$E(n,p)$
1	250	278 (281)	1.00
4	500	574 (563)	0.97
9	750	863 (844)	0.97
16	1000	1157 (1125)	0.96
25	1250	1462 (1406)	0.95
36	1500	1750 (1688)	0.95

The predicted run times shown in parentheses are defined by (2) using

$$a = 18\ \mu s$$

The processor efficiency $E(n,p)$ was computed from (7) using the measured run times.

SUMMARY

We have developed a model program for parallel execution of cellular automata on a multicomputer with a square matrix of processor nodes. We have adapted the model program for simulation of forest fires and numerical solution of Laplace's equation for stationary heat flow. On a Computing Surface with 36 transputers the program performs 1500 relaxations of a 1500 × 1500 grid of 64-bit reals in 29 minutes with an efficiency of 0.95.

APPENDIX: COMPLETE ALGORITHM

The complete algorithm for parallel solution of *Laplace's equation* is composed of Algorithms 1–12.

```
const q = 6; m = 250 {even};
  n = 1500 {m*q};
type row = array [1..n] of real;
  grid = array [1..n] of row;

procedure laplace(var u: grid; u1, u2,
  u3, u4, u5: real; steps: integer);
const pi = 3.14159265358979;
type subrow = array [0..m+1] of real;
  subgrid = array [0..m+1] of subrow;
var fopt: real;

  procedure master(inp: channel;
    var u: grid);
  var i, j: integer;
  begin
    for i := 1 to n do
      for j := 1 to n do
        inp?u[i,j]
  end;

  procedure copy(no: integer;
    inp, out: channel);
  var k: integer; uk: real;
  begin
    for k := 1 to no do
      begin
        inp?uk; out!uk
      end
  end;
```

```
procedure output(qi, qj: integer;
  inp, out: channel; var u:
  subgrid);
var i, j: integer;
begin
  for i := 1 to m do
    begin
      for j := 1 to m do
        out!u[i,j];
      copy((q − qj)*m, inp, out)
    end;
  copy((q − qi)*m*n, inp, out)
end;

procedure phase1(qi, qj, b:
  integer; n, s, e, w: channel;
  var u: subgrid);
var k, last: integer;
begin
  k := 2 − b;
  last := m − b;
  while k <= last do
    begin
      parbegin
        if qi > 1 then n?u[0,k] |
        if qi < q then s!u[m,k] |
        if qj < q then e!u[k,m] |
        if qj > 1 then w?u[k,0]
      end;
      k := k + 2
    end
end;
```

```
procedure phase2(qi, qj, b:
  integer; n, s, e, w: channel;
  var u: subgrid);
var k, last: integer;
begin
  k := b + 1;
  last := m + b − 1;
  while k <= last do
    begin
      parbegin
        if qi > 1 then n!u[1,k] |
        if qi < q then s?u[m+1,k] |
        if qj < q then e?u[k,m+1] |
        if qj > 1 then w!u[k,1]
      end;
      k := k + 2
    end
end;

procedure exchange(qi, qj, b:
  integer; n, s, e, w: channel;
  var u: subgrid);
begin
  phase1(qi, qj, b, n, s, e, w, u);
  phase2(qi, qj, b, n, s, e, w, u)
end;

function initial(i, j: integer)
  : real;
begin
  if i = 0 then
    initial := u1
  else if i = n + 1 then
    initial := u2
  else if j = n + 1 then
    initial := u3
  else if j = 0 then
    initial := u4
  else
    initial := u5
end;
```

```
function next(uc, un, us,
  ue, uw: real): real;
var res: real;
begin
  res := (un + us + ue +
    uw)/4.0 − uc;
  next := uc + fopt*res
end;

procedure newgrid(qi, qj:
  integer; var u: subgrid);
var i, i0, j, j0: integer;
begin
  i0 := (qi − 1)*m;
  j0 := (qj − 1)*m;
  for i := 0 to m + 1 do
    for j := 0 to m + 1 do
      u[i,j] := initial(i0+i, j0+j)
end;

procedure relax(qi, qj: integer;
  n, s, e, w: channel; var u:
  subgrid);
var b, i, j, k, last: integer;
begin
  for b := 0 to 1 do
    begin
      exchange(qi, qj, 1 − b,
        n, s, e, w, u);
      for i := 1 to m do
        begin
          k := (i + b) mod 2;
          j := 2 − k;
          last := m − k
          while j <= last do
            begin
              u[i,j] := next(u[i,j],
                u[i−1,j], u[i+1,j],
                u[i,j+1], u[i,j−1]);
              j := j + 2
            end
        end
    end
end;
```

```
procedure node(qi, qj: integer;
  n, s, e, w: channel);
var u: subgrid; k: integer;
begin
  newgrid(qi, qj, u);
  for k := 1 to steps do
    relax(qi, qj, n, s, e, w, u);
  output(qi, qj, e, w, u)
end;

procedure simulate(var u: grid);
type
  line = array [1..q] of channel;
  matrix = array [0..q] of line;
var h, v: matrix; i, j, k: integer;
begin
  parbegin
    master(h[0,q], u) |
    parfor k := 1 to q do
      node(k, 1, v[k−1,1], v[k,1],
        h[k,1], h[k−1,q]) |
    parfor i := 1 to q do
      parfor j := 2 to q do
        node(i, j, v[i−1,j], v[i,j],
          h[i,j], h[i,j−1])
  end
end;

begin
  fopt := 2.0 − 2.0*pi/n;
  simulate(u)
end
```

ACKNOWLEDGEMENTS

It is a pleasure to acknowledge the constructive comments of Jonathan Greenfield and the flawless text editing of Elaine Weinman.

REFERENCES

BAK, P., and TANG, C. 1989. Earthquakes as a self-organized critical phenomenon. *Journal of Geophysical Research* **94**, B11, 15635-15637.

BAK, P., and CHEN, K. 1990. A forest-fire model and some thoughts on turbulence. *Physics Letters A* **147**, 5, 6, 297–299.

BARLOW, R. H., and EVANS, D. J. 1982. Parallel algorithms for the iterative solution to linear systems. *Computer Journal* **25**, 1, 56–60.

BERLEKAMP, E. R., CONWAY, J. H., and GUY, R. K. 1982. *Winning Ways for Your Mathematical Plays.* Vol. 2. Academic Press, New York, NY, 817–850.

BRINCH HANSEN, P. 1990. The all-pairs pipeline. School of Computer and Information Science, Syracuse University, Syracuse, NY.

BRINCH HANSEN, P. 1991a. A generic multiplication pipeline. School of Computer and Information Science, Syracuse University, Syracuse, NY.

BRINCH HANSEN, P. 1991b. Parallel divide and conquer. School of Computer and Information Science, Syracuse University, Syracuse, NY.

BRINCH HANSEN, P. 1992a. Parallel Monte Carlo trials. School of Computer and Information Science, Syracuse University, Syracuse, NY.

BRINCH HANSEN, P. 1992b. Numerical solution of Laplace's equation. School of Computer and Information Science, Syracuse University, Syracuse, NY.

COK, R. S. 1991. *Parallel Programs for the Transputer.* Prentice Hall, Englewood Cliffs, NJ.

DEWDNEY, A. K. 1984. Sharks and fish wage an ecological war on the toroidal planet Wa-Tor. *Scientific American* **251**, 6, 14–22.

DIJKSTRA, E. W. 1982. *Selected Writings on Computing: A Personal Perspective.* Springer-Verlag, New York, NY, 334–337.

EVANS, D. J. 1984. Parallel SOR iterative methods. *Parallel Computing* **1**, 3–18.

FOX, G. C., JOHNSON, M. A., LYZENGA, G. A., OTTO, S. W., SALMON, J. K., and WALKER, D. W. 1988. *Solving Problems on Concurrent Processors.* Vol. I. Prentice-Hall, Englewood Cliffs, NJ.

FRISCH, U., HASSLACHER, B., and POMEAU, Y. 1986. Lattice-gas automata for the Navier-Stokes equation. *Physical Review Letters* **56**, 14, 1505–1508.

GARDNER, M. 1970. The fantastic combinations of John Conway's new solitaire game "Life." *Scientific American* **223**, 10, 120–123.

GARDNER, M. 1971. On cellular automata, self-reproduction, the Garden of Eden and the game "Life." *Scientific American* **224**, 2, 112–117.

HOPPENSTEADT, F. C. 1978. Mathematical aspects of population biology. *Mathematics Today: Twelve Informal Essays*, L. A. Steen, ed., Springer-Verlag, New York, 297–320.

HWA, T., and KARDAR, M. 1989. Dissipative transport in open systems: an investigation of self-organized criticality. *Physical Review Letters* **62**, 16, 1813–1816.

INMOS, 1988. *Occam 2 Reference Manual.* Prentice-Hall, Englewood Cliffs, NJ.

MEIKO, 1987. *Computing Surface Technical Specifications.* Meiko Ltd, Bristol, England.

PRESS, W. H., FLANNERY, B. P., TEUKOLSKY, S. A., and VETTERLING, W. T. 1989. *Numerical Recipes in Pascal: The Art of Scientific Computing.* Cambridge University Press, Cambridge, MA.

PRITCHARD, D. J., ASKEW, C. R., CARPENTER, D. B., GLENDINNING, I., HEY, A. J. G., and NICOLE, D. A. 1987. Practical parallelism using transputer arrays. *Lecture Notes in Computer Science* **258**, 278–294.

SALTZ, J. H., NAIK, V. K., and NICOL, D. M. 1987. Reduction of the effects of the communication delays in scientific algorithms on message passing MIMD architectures. *SIAM Journal on Scientific and Statistical Computing* **8**, 1, s118–134.

TREW, A., and WILSON, G. eds. 1991. *Past, Present, Parallel—A Survey of Available Parallel Computer Systems.* Springer-Verlag, New York, NY.

ULAM, S. 1986. *Science, Computers, and People: From the Tree of Mathematics.* Birkhäuser, Boston, MA.

VON NEUMANN, J. 1966. *Theory of Self-Reproducing Automata.* Edited and completed by A. W. Burks, University of Illinois Press, Urbana, IL.

YOUNG, D. M. 1954. Iterative methods for solving partial difference equations of elliptic type. *Transactions of the American Mathematical Society* **76**, 92–111.