2-1992

Fault-Detection in Networks

H. F. Mattson Jr

SU-CIS-92-04

Fault-Detection in Networks

H.F. Mattson, Jr.

February, 1992

School of Computer and Information Science
Syracuse University
Suite 4-116, Center for Science and Technology
Syracuse, NY 13244-4100

Fault-Detection in Networks[1]

by H. F. Mattson, Jr.

School of Computer and Information Science

Syracuse University

4-116 Center for Science & Technology

Syracuse, New York 13244-4100

email: jen@SUVM.acs.syr.edu/jen@SUVM.bitnet

Address for correspondence up to July 24, 1992:

On sabbatical, 1991–1992, at

Ecole National Supérieure des Télécommunications

46 rue Barrault, C-220

75634 Paris cédex 13, France

email: mattson@inf.enst.fr

June 2, 1992

Abstract. To find broken links in networks we use the cut-set space. Information on which nodes can talk, or not, to which other nodes allows reduction of the problem to that of decoding the cut-set code of a graph. Special classes of such codes are known to have polynomial-time decoding algorithms. We present a simple algorithm to achieve the reduction and apply it in two examples.

[1]To appear in *Discrete Mathematics.*

Suppose in a network used for communication it becomes impossible to send messages between certain pairs of nodes. How could one find the broken links?

The model considered here is a connected undirected graph in which the vertices stand for the nodes of the network and the edges for the communication links between nodes.

Notation. The graph $G = (V, E)$ has n edges, and $\mathbf{F} := GF(2)$. We are working in the space $\mathbf{F}^n$, in which the coordinate-places are identified with the edges (in other words, in $\mathbf{F}^E$.) Each subset of edges of G is the support of a unique element of $\mathbf{F}^n$. We will freely identify subsets of E with their corresponding elements of $\mathbf{F}^n$. For example, if $X,\ Y \subseteq E$, then $|X \cap Y|$ reduced mod 2 is the dot product of the elements of $\mathbf{F}^n$ corresponding to X and Y.

If M is a matrix over $\mathbf{F}$, $*M$ denotes the row-space of M and $M*$ the column-space.

By a **trail** in G we mean a walk with no repeated edges. A **circuit** is a trail in which the first and last vertices are the same. A trail starting at x and ending at y is called an x-y **trail**. The set of edges of a trail W is (identified with) a vector in $\mathbf{F}^n$, and we shall also denote it as W.

A simple coding-theoretic approach is useful in a special case of the problem mentioned at the outset. The circuit-space C of a graph, the vector space over $GF(2)$ spanned by the circuits considered as subsets of edges, was first viewed as a code by Kasami [6] in 1961. (See also [5] for other independent discoverers.) The cut-set space of the graph is the code $C^{\perp}$ orthogonal to C. $C^{\perp}$ consists of cut-sets and unions of mutually disjoint cut-sets [7].

Problem 1. In the first problem we consider, we assume that any failure of communication is caused by broken links. We suppose that for certain 2-subsets of nodes we know the "communication status" of the network; that is, between various nodes x, y we know that communication is possible, and for various others that it is not possible.

Suppose communication is not possible between nodes x and y. This means that the set B of all broken links includes some cut-set K of edges separating x and y.

We now make a simplifying assumption, that B includes only one cut-set. This assumption is reasonable if breaks in edges occur with a Poisson distribution and if we sample the communication status of pairs often enough to detect an x, y failure very soon after it first occurs.

Cut-sets have the following property.

Proposition 1 *Let K be a cut-set of G, and denote by G_1 and G_2 the components of $G-K$. Let x and y be vertices of G_1 and z a vertex of G_2. Then every x-y trail has dot-product 0 with K, and every x-z trail has dot-product 1 with K.*

Proof. If W denotes the x-y trail, then $|W \cap K|$ is even, because each edge of K in W causes a change to the other component. If W goes from x to z, then $|K \cap W|$ must be odd. □

This simple Proposition dictates our strategy. There is an unknown cut-set $K \subseteq B$, and we know the communication status of certain pairs of nodes. These pairs allow us to conclude that K is included in a proper subset of the cut-set code $C^\perp$, in fact, in a coset of a subcode of $C^\perp$. We may then decode according to an appropriate criterion dictated by the probability distribution of edge-breaks.

The incidence matrix M of G is a $0,1$ matrix with rows indexed by the vertices of G and columns by the edges. For each vertex v of G, $R(v) :=$ row v of M is the vector of $\mathbf{F}^n$ consisting of the edges of G with v as an endpoint. Sometimes we abuse the notation by writing v instead of $R(v)$. We know that $*M = C^\perp$. Since the sum of the rows of M is 0, $C^\perp$ is an $[n, k]$ code, where $k = |V| - 1$.

We shall use the following result.

Lemma 1 *Let x and y be any distinct vertices of G. Let W be any x-y trail. Then for all vertices $z \neq x, y$ $R(z) \cdot W = 0$; and $R(x) \cdot W = R(y) \cdot W = 1$.*

Proof. If $z \neq x, y$, then either W does not pass through z, or W uses two edges of $R(z)$ at each passage through z. □

We denote by T [J] the set of all pairs of nodes of G between which we know communication is possible [not possible] despite [because of] broken links. We shall write xTy for $(x, y) \in T$, and xJy for $(x, y) \in J$. Under our assumption, edges of $B - K$ have no effect on J. (See Example 2 below.)

A First Case

We begin with a simple case: $T := \{(x,y)\}$ and $J := \{(x,z)\}$. Our cut-set K is in $C^{\perp} = *M$, but for all x-y trails $W, K \cdot W = 0$ and for all x-z trails $W', K \cdot W' = 1$. So we consider only the subset S of $*M$ having these properties. (We do not use these trails for communication, so W' exists in the model G, not as an intact path in a network with broken edges.)

To find S, consider an x-y trail W. If v is any vertex other than x or y, then by the Lemma, $W \cdot R(v) = 0$. But $W \cdot R(x) = W \cdot R(y) = 1$. Therefore the set of all cut-sets having dot-product 0 with every x-y trail is a subset of $*M'$, where M' is M with $R(x)$ and $R(y)$ removed, but a new row, $R(x) + R(y)$, inserted. Thus $K \in *M'$.

Now consider an x-z trail W'. With all rows of M', other than $R(z)$ and $R(x) + R(y)$, W' has dot-product 0. We form a matrix M'' by removing these two rows and inserting their sum as a new row $R(x) + R(y) + R(z)$. The subspace of $C^{\perp}$ having dot-product 0 with all x-y trails and with all x-z trails is $*M''$. The subset of $C^{\perp}$ having dot-product 0 with all x-y trails and 1 with all x-z trails is a coset of the subcode $*M''$, namely,

$$R(z) + *M''.$$

The cut-set K that we seek is some element of this coset.

Comment. Notice that the passage from M to M' amounts to merging the two vertices x and y to make a new graph G' with incidence matrix M'. If there is an edge between x and y it disappears in G', and $*M'$ is in that case the subcode of $C^{\perp}$ of all vectors 0 on that edge. If we eliminate that column we have a shortened subcode of $C^{\perp}$.

M'' is the incidence matrix of the graph G'' obtained from the merger of the three vertices x, y, and z (again with the understanding that no loops are produced).

The General Case

We give here a running account of our algorithm. Our data consist of two relations T and J on V. We take J as symmetric.

Since communicability is reflexive, symmetric, and transitive, in Step 1 we use the polynomial-time union-find algorithm (UFA) [1, p. 110] to find the equivalence closure of T. The result is a partition P of V, among the cells of which appear as singletons the points of V not related by T to other points of V.

Step 1. Let $P = UFA(T, V)$.

Define $\pi : V \longrightarrow P$ by the rule

$$\forall x \in V, \quad x \in \pi(x).$$

Thus $\pi(x)$ is the cell of P containing x. We now adapt J to P.

Step 2. (i) While $\exists x, y, z, w \in V$ such that

xJy and wJz and $\pi(x) = \pi(w)$ and $\pi(y) \neq \pi(z)$

do $P := P - \{\pi(y), \pi(z)\}$,

$P := P \cup \{\pi(y) \cup \pi(z)\}$.

Note that π changes as P changes.

(ii) If $\exists x, y \in V$ such that

$\pi(x) = \pi(y)$ and xJy, then halt

with an error report: the unique cut-set

property does not hold.

The reason for Step 2(i) is that since xTw, both y and z are on the other side of the cut from x; hence we infer that y and z can talk to each other. Step 2 (ii) stops the procedure with a failure, which might occur if we do not read the data soon enough after a cut-set of broken edges exists.

Define $\overline{J}$ as the following relation on P:

$$\forall D, D' \in P, \ D\overline{J}D' \text{ iff } \exists x \in D, \ x' \in D' \text{ such that } xJx'.$$

Step 3. Define, using mod-2 summation, a new matrix M':

$\forall D \in P$, $R(D) := \sum_{x \in D} R(x)$ is a row of M'.

M' has no other rows.

M' is the incidence matrix of the graph G' we obtain from G when we merge all vertices that we know are able to talk to each other into a single vertex.

Because of Step 2 the relation $\overline{J}$ is now a *matching* on P, i.e., $\forall D, D', D'' \in P$ if $D\overline{J}D'$ and $D\overline{J}D''$, then $D' = D''$.

Step 4. Define the matrix M'' as follows:

(i) $M'' := M'$

(ii) $\forall D, D' \in P$, if $D\overline{J}D'$ then remove rows $R(D)$ and $R(D')$ from M'' and insert a new row $R(D) + R(D')$ into M''.

Step 5. Choose one of each pair $(D, D') \in \overline{J}$; i.e., choose a maximal set S of elements of P such that $\forall x,\ y \in S\ (x, y) \notin \overline{J}$ but x and y are the first coordinates of pairs in $\overline{J}$.

Step 6. Find K as an element of the coset Z, where

$$Z := \sum_{x \in S} R(x) + *M''.$$

Steps 4 and 5 take account of the information in $\overline{J}$. (To avoid redundancy, if Step 4 is done for $(D, D') \in \overline{J}$, then it should not be done for $(D', D) \in \overline{J}$.) The coset Z is a subset of the set Y of all elements of $C^{\perp}$ which have dot-product 0 with every x-y trail if $(x, y) \in T$, and dot-product 1 with every x-y trail if $(x, y) \in J$. Z is a proper subset of Y if we have used Step 2 (i) to modify P.

To find K in Step 6 requires a decoding procedure for the cut-set code $*M''$. This problem is NP-complete [2, II A]. If the graph G is planar, however, it has a dual graph D, the circuit code of which is the cut-set code of G. Since there is a polynomial-time algorithm for decoding the circuit code of a graph [8], [10], then there is one for cut-set codes of planar graphs. Recent papers [4, (6.6)], [9] have extended this result to other special classes of graphs. (To adapt [4, (6.6)] to our situation, set the "cost" function c to be 1 on the coordinate-positions of the received word u, and -1 on the other coordinate-positions. If v is a codeword, then

$$c(v) = wt(u) - d(u, v);$$

maximizing $c(v)$ over codewords v minimizes $d(u, v)$, producing a v closest to u.)

Problem 2. Here we do not know of any pairs in J, but we deliberately break edges.

If that produces pairs in J, we then proceed as in the first problem. Breaking an edge corresponds to puncturing the code $C^{\perp}$ at that coordinate, so we eliminate those columns from M. As shown in Example 2 below, it is sometimes necessary to choose more than one subset of edges to break in order to determine the unknown subset of broken edges.

Problem 3. Suppose nodes may also fail. Failure of node x could be viewed as a case of Problem 1 in which all edges incident to x had failed. But it would be simpler in this problem to check node x as soon as several pairs $(x, y_1), \ldots, (x, y_s)$ appeared in J.

Examples. Here are two examples at the level of puzzles. Both are taken from [3]. Because the graphs are small the only decoding procedure needed is exhaustion.

1. Consider the graph of Figure 1.

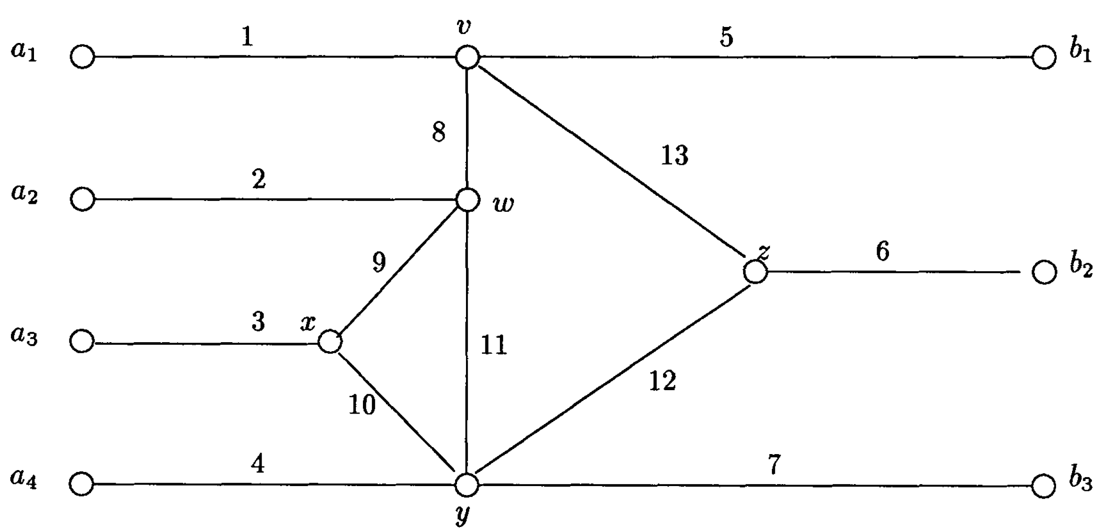

Figure 1

It has incidence matrix M_1 when the vertices are ordered as $a_1, \ldots, a_4, b_1, b_2, b_3, v, w, x, y, z$:

$$M_1 = \begin{array}{|c|c|} \hline I_7 & 0 \\ \hline \multicolumn{2}{|c|}{B} \\ \hline \end{array} ,$$

where B is the 5×13 matrix

$$B = \begin{array}{c|ccccccc|cccccc|}
\hline
v & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
w & & 1 & & & 0 & & & 1 & 1 & & 1 & & \\
x & & & 1 & & 0 & & & 0 & 1 & 1 & & & \\
y & & & & 1 & 0 & & 1 & 0 & & 1 & 1 & 1 & 0 \\
z & & & & & 0 & 1 & & & & & & 1 & 1 \\
\hline
\end{array}$$

We are given the data $a_1Tb_1, a_2Tb_1, a_3Tb_2, a_4Tb_3, a_1Jb_2$, and a_3Jb_1. The problem is to find the smallest set of broken edges that fit the data. This is a case of Problem 1.

Since the "a" and "b" vertices all have degree 1 and each can talk to another vertex, we infer that all edges to those vertices are unbroken. Hence we infer a_1Tv, a_2Tw, etc. The result of Step 1 is then

$$P = \underbrace{a_1, a_2, b_1, v, w}_{D_1} \Big| \underbrace{a_3, b_2, x, z}_{D_2} \Big| \underbrace{a_4, b_3, y}_{D_3} \; ;$$

and from the data we see that $D_1\bar{J}D_2$. The result of Step 3 is the 3×13 matrix $M_1' = 0B'$, where B' is the 3×5 matrix of columns $9, \ldots, 13$:

$$B' = \begin{array}{cccccc}
R(D_1) & 1 & 0 & 1 & 0 & 1 \\
R(D_2) & 1 & 1 & 0 & 1 & 1 \\
R(D_3) & 0 & 1 & 1 & 1 & 0
\end{array} .$$

Step 4 produces $R(D_1) + R(D_2) = R(D_3)$ as the first row, so the upshot is that

$$\begin{aligned}
Z_1 &= R(D_1) + *[R(D_3)] \\
&= \{R(D_1),\ R(D_1) + R(D_3)\} \\
&= \{\{9, 11, 13\},\ \{9, 10, 12, 13\}\}.
\end{aligned}$$

Thus the answer is that edges 9, 11, 13 are broken.

2. Now for an example of Problem 2. Consider the graph of Figure 2.

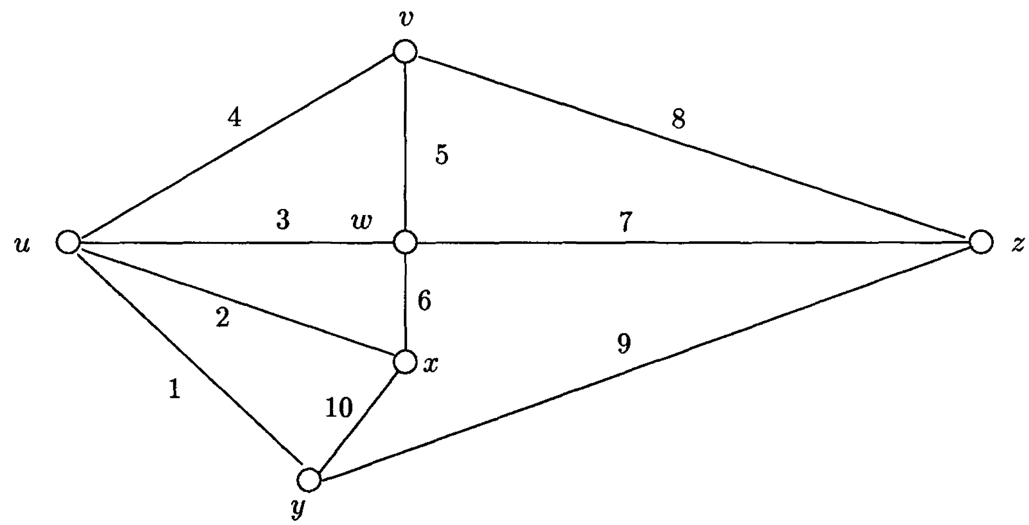

Figure 2

It has incidence matrix M_2:

$$M_2 = \begin{array}{c|cccccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline u & 1 & 1 & 1 & 1 & & & & & & \\ v & & & & 1 & 1 & & & 1 & & \\ w & & & 1 & & 1 & 1 & 1 & & & \\ x & & 1 & & & & 1 & & & & 1 \\ y & 1 & & & & & & & & 1 & 1 \\ z & & & & & & & 1 & 1 & 1 & \\ \hline \end{array}.$$

For data we have

(i) If we break edges 2, 3, 4, then uJz.

(ii) If we break edges 3, 4, then uTz.

(iii) If we break edges 3, 4, 8, 9, then uJz.

(iv) If we break edges 4, 8, 9, then uTz.

Problem: Find the smallest set of broken edges consistent with these data.

From (i) and (ii) we infer that edge 2 is unbroken, from (iv) and the graph that edge 7 is unbroken. Hence uTx and wTz. These inferences are the only conclusions we draw from (ii) and (iv).

In (i) and (iii) we have possibly different cut-sets, since the subsets of edges that we break are different. Therefore we treat them as different problems. But in both problems we know that wTz, so we have from Step 3

$$M_2' = \begin{array}{r|ccccccccc|} \hline u & 1 & 1 & 1 & 1 & & & & & \\ v & & & & 1 & 1 & & 1 & & \\ x & & 1 & & & & 1 & & & 1 \\ y & 1 & & & & & & & 1 & 1 \\ w+z & & & 1 & & 1 & 1 & 1 & 1 & \\ \hline \end{array} .$$

Subproblem (i): The datum is uJz, with incidence matrix M_2' punctured at columns 2, 3, 4. We do Step 4; the result is

$$M_i'' = \begin{array}{r|ccccccccc|} \hline v & & \square & \square & \square & 1 & & 1 & & \\ x & & \square & \square & \square & & 1 & & & 1 \\ y & 1 & \square & \square & \square & & & & 1 & 1 \\ u+w+z & 1 & \square & \square & \square & 1 & 1 & 1 & 1 & \\ \hline \end{array} .$$

$*M_i''$ is a $[6,3,2]$ code, since column 7 is 0. The coset Z_i is, with subscript $\square$ denoting puncturing on 2, 3, 4,

$$Z_i := R(u)_\square + *M_i'',$$

which has a unique leader of weight 1, namely, $R(u)_\square = 1\square\square\square00000$. It corresponds to edge 1.

Remembering that Z_i is the subset of $C_i^\perp$ of all elements separating u and z in the graph "of" M_i'', we see that edge 1 is at least a candidate as part of the solution to our original problem.

Subproblem (iii): The data are uTx and uJz with incidence-matrix M_2' punctured at columns 3, 4, 8, 9. We perform Step 3, adding rows u and x. The result is

$$M'_{iii} = \begin{array}{r|cccccccccc|} v & & & \square & \square & 1 & & & \square & \square & \\ u+x & 1 & & \square & \square & & 1 & & \square & \square & 1 \\ y & 1 & & \square & \square & & & & \square & \square & 1 \\ w+z & & & \square & \square & 1 & 1 & & \square & \square & \end{array} .$$

Since uJz, we add rows 2 and 4 of M'_{iii} to get from Step 4

$$M''_{iii} = \begin{array}{r|cccccccccc|} v & & & \square & \square & 1 & & & \square & \square & \\ y & 1 & & \square & \square & & & & \square & \square & 1 \\ u+x+w+z & 1 & & \square & \square & 1 & & & \square & \square & 1 \end{array} .$$

All the nonzero elements of $*M''_{iii}$ appear as rows of the matrix, which tells us that the coset $Z_{iii} := (R(w) + R(z))_{\square} + *M''_{iii}$ is

$$\{\{5,6\},\ \{6\},\ \{1,5,6,10\},\ \{1,6,10\}\}.$$

The leader is $\{6\}$, and it is a cut-set of the graph "of" M''_{iii} separating u and z. Thus $\{6\}$ is a candidate for part of our overall solution.

In fact, since $\{6\}$ solves (iii) and $\{1\}$ solves (i), their union $\{1,6\}$ must solve both problems. We then check that if edges 1 and 6 are broken, both (ii) and (iv) are satisfied. Therefore $\{1,6\}$ is a smallest solution. It is the only solution of size 2 because $\{5,6\}$ is not a solution to (i).

Remark. We invoke the algorithm as soon as we have enough elements of T and J, say at time t. Breaks occurring after time t will not affect the data (namely, T and J) on which the algorithm operates. Those breaks eventually contribute to data for the next running of the algorithm.

Future Work. We plan to consider questions of decoding and implementability in future work in this area.

Acknowledgments. It is a pleasure to acknowledge that it was a problem of Fujii [3] which led me to the ideas of this paper. My solution is Example 2. I am grateful to Lockwood Morris for help with my algorithm, and I particularly thank Gilles Zémor for valuable discussions on complexity and on the simplifying assumption used here. I thank ENST for its hospitality during the last part of my work on this paper.

References

[1] Aho, A. V., Hopcroft, J.E., and Ullman, J. D., *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading, 1974.

[2] Bruck, J., and Naor, M., "The hardness of decoding linear codes with preprocessing," *IEEE Trans. Inform. Theory*, vol. 36 (1990), pp. 381–385.

[3] Fujii, John, *Puzzles and Graphs*, National Council of Teachers of Mathematics, Washington, 1966.

[4] Grötschel, M. and Truemper, K., "Decomposition and optimization over cycles in binary matroids," *JCT* B, vol. 46, (1989), pp. 306–337.

[5] Hakimi, S. L., and Frank, H., Cut-set matrices and linear codes, *IEEE Trans. Inform. Theory* IT–11 (1965), pp. 457–458.

[6] Kasami, T., A topological approach to construction of group codes, *J. Inst. Elec. Communication Engrs.*, (Japan), vol 44, Sept. 1961, pp. 1316–1321.

[7] Liu, C. L., *Introduction to Combinatorial Mathematics*, McGraw-Hill, New York, 1968.

[8] Ntafos, S.C., and Hakimi, S.L., On the complexity of some coding problems, *IEEE Trans. Inform. Theory* IT-27 (1981), pp. 794–796.

[9] Solé, P., "Covering codes and combinatorial optimization, *LNCS* 539 (1991), pp. 426–434. *Proceedings Int'l Symp.* AAECC 9.

[10] Solé, P., and Zaslavsky, T., Covering radius and maximality of the cycle code of a graph, *Discrete Math.* (to appear).