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The Number of Ways to Write n as a Sum of ` Regular Figurate Numbers

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The Number of Ways to Write n as a Sum of ℓ Regular Figurate Numbers

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Honors Capstone Project in Mathematics

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Abstract

The elementary approaches to finding the number of integer solutions (x, y) to the equation $x^2 + y^2 = n$ with $n \in \mathbb{N}$ are well known. We first examine a generalization of this problem by finding the ways a natural number n can be written as a sum of two k-sided regular figurate numbers. The technique is then adapted to find the number of ways n can be written as a sum of ℓ regular figurate numbers.

Contents

1 Introduction

1.1 Figurate numbers

Regular figurate numbers have been an object of study since Fermat. Special cases of figurate numbers, triangular and square numbers were studied far before that [4].

Definition 1.1.1. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $k \geq 2$. Then the n^{th} k-sided regular figurate number $f_k(n)$ is defined recursively by $f_k(1) = 1$ and*

$$f_k(n) = f_k(n-1) + (k-2)n - (k-3).$$

Consider when $k = 3$. From the definition, $f_3(n) = f_3(n-1) + n + 0$. Since $f_3(1) = 1$ and $f_3(2) = 1 + 2$, $f_3(3) = 1 + 2 + 3$. Inductively, we see that

$$f_3(n) = n + \sum_{i=1}^{n-1} i = \sum_{i=1}^{n} i.$$

Since we can arrange $f_3(n)$ dots into a triangle with n dots on each side (see Figure 1), $f_3(n)$ are also known as the triangular numbers.

Similarly, $f_4(n)$ dots can be arranged into squares (see Figure 2) and $f_5(n)$ dots can be arranged into pentagons (see Figure 3). In fact, $f_k(n)$ dots will always give the k sided regular polygon with sides of length n. The formula gives $f_k(1) = 1$ and $f_k(2) = k$. Then, $f_k(n)$ is $f_k(n-1)$ dots with $k-2$ sides of length n added to it and $k-3$ subtracted from it for the overlap at corners. For a specific example, examine Figure 3. If we want to get $f_5(n)$, we start with with $f_5(n-1)$ and add 3 more sides of length n, and subtract 2 for the overlap at the bottom two corners.

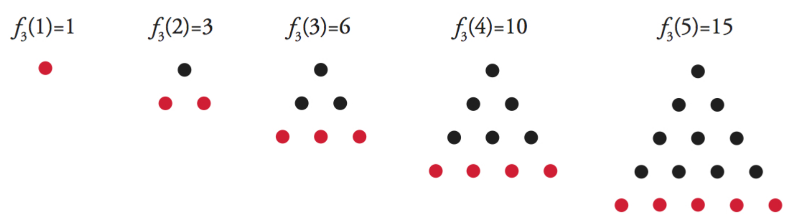

Figure 1: The first five $f_3(n)$, each arranged as a triangle. The numbers $f_3(n)$ are also known as triangular numbers. Note that $f_3(n) = f_3(n-1) + n$.

Theorem 1.1.2. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $k \geq 2$. Then*

$$f_k(n) = \frac{n\left[(k-2)n + (4-k)\right]}{2}.$$

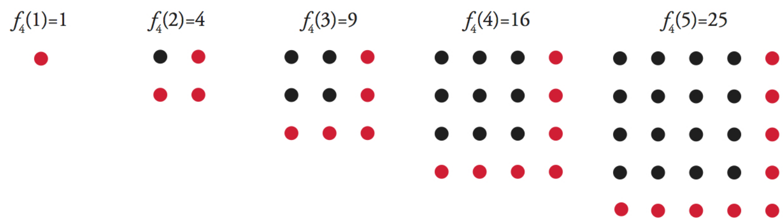

Figure 2: The first five $f_4(n)$, each as a square with sides of length n. By definition, $f_4(n) = f_4(n-1) + 2n - 1$.

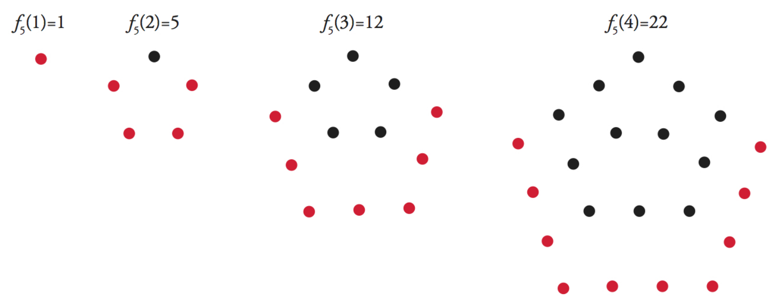

Figure 3: The first four "pentagonal" numbers. The nth pentagonal number has 5 sides of length n. Notice that $f_5(n) = f_5(n-1) + 3n - 2$.

Proof. Define

$$g_k(n) = \frac{n[(k-2)n + (4-k)]}{2}.$$

We show that $g_k(n) = f_k(n)$ by showing that they satisfy the same initial conditions and the same recurrence relation. First, notice that

$$g_k(1) = \frac{1 \cdot [(k-2) \cdot 1 + 4 - k]}{2} = \frac{2}{2} = 1 = f_k(1).$$

Now we need to show that $g_k(n) = g_k(n-1) + (k-2)n - (k-3)$. To check this, note that

$$\begin{aligned}
&g_k(n-1) + (k-2)n - (k-3) \\
&\qquad = \frac{(n-1)[(k-2)(n-1) + (4-k)]}{2} + (k-2)n - (k-3) \\
&\qquad = \frac{1}{2}(n^2(k-2) + n(2k + 4 - k - 2k)) \\
&\qquad = \frac{n[(k-2)n + (4-k)]}{2} \\
&\qquad = g_k(n).
\end{aligned}$$

Therefore $g_k(n) = f_k(n)$ and thus

$$f_k(n) = \frac{n\left[(k-2)n + (4-k)\right]}{2}.$$

$\square$

When $k = 4$, the formula gives

$$f_4(n) = \frac{n[(4-2)n + 4 - 4]}{2} = n^2;$$

thus the n^{th} "square" number is n squared. For $k = 3$ and $k = 5$ the formulas are

$$f_3(k) = \frac{n^2 + n}{2}$$

and

$$f_5(k) = \frac{3n^2 - n}{2}$$

respectively.

1.2 Big O notation

It turns out to be impossible to find a general explicit formula for certain functions in which we are interested, so we utilize *big O notation* to describe the growth of those functions in general.

Definition 1.2.1. *Let f be a function over the real numbers. We say that $f(x) = O(g(x))$ for some function on the real numbers g if there exist constants c and x_0 so that $|f(x)| \leq c|g(x)|$ for all $x \geq x_0$.*

For example, let x be a real number such that $x \geq 1$. Since $\sqrt{x} \geq 1$,

$$\begin{aligned}
\sqrt{2x} + \frac{1}{2} &\leq \sqrt{2}\sqrt{x} + \frac{1}{2}\sqrt{x} \\
&= \left(\sqrt{2} + \frac{1}{2}\right)\sqrt{x} \\
&= c\sqrt{x},
\end{aligned}$$

where $c = \sqrt{2} + \dfrac{1}{2}$. Therefore, $\sqrt{2x} + \frac{1}{2} = O(\sqrt{x})$. However, keep in mind that it is equally valid to say $\sqrt{2x} + \frac{1}{2} = O(x)$. We may do that because the big O is dependent on an inequality. In general, we seek the "smallest" possible $g(x)$ to give the best approximation.

Definition 1.2.2. *Let f, g, and h be functions on the real numbers. We say*

$$f(x) = h(x) + O(g(x))$$

if there exists c such that

$$|f(x) - h(x)| \leq c|g(x)|.$$

In this case, we think of $g(x)$ as an error term as long as

$$\lim_{x \to \infty} \frac{g(x)}{h(x)} = 0.$$

1.3 The Gauss Circle Problem

Definition 1.3.1. *Let $n \in \mathbb{N} \cup \{0\}$. Let $r_4(n)$ be the number of ways to write n as a sum of two squared nonnegative integers.*

Notice that if an integer n can be written as a sum of two squares, then the circle $x^2 + y^2 = n$ will pass through the point (h, k) where $h^2 + k^2 = n$. In this way, we gain a geometric representation of numbers that can be described as a sum of two squares (see Figure 4).

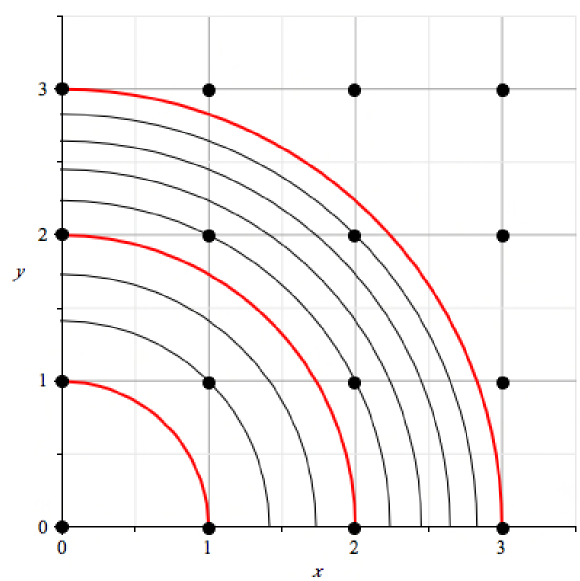

Figure 4: The circles $x^2 + y^2 = n$ with $n \leq 9$. Note that $r_4(n)$ is exactly the number of lattice points on a circle centered at the origin with radius $\sqrt{n}$.

Definition 1.3.2. *Let $n \in \mathbb{N} \cup \{0\}$. Then define $p_4(n)$ by $p_4(n) = \sum_{i=0}^{n} r_4(i)$.*

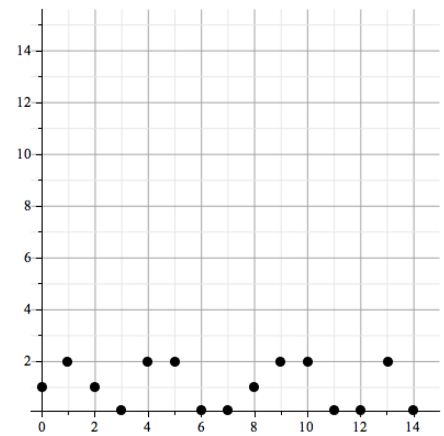

Figure 5: The first 14 values of $r_4(n)$. Notice that there is no discernible pattern.

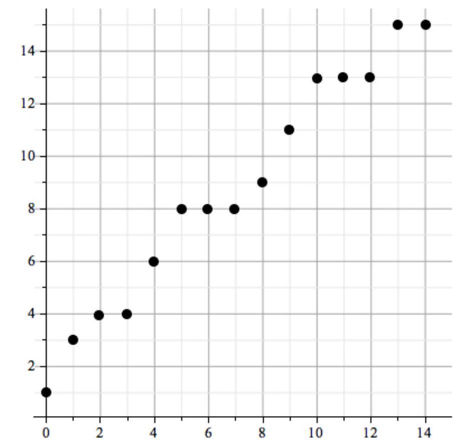

Figure 6: The first 14 values of $p_4(n)$. We create a patten out of the $r_4(n)$ by adding them together.

We study $p_4(n)$ because it is a more predictable function than $r_4(n)$. While $r_4(n)$ can vary wildly from one positive integer to the next, $p_4(n)$ has a noticeable increasing trend (see Figures 5 and 6).

If we set lattice points in $\mathbb{R}^2$ at every (h, k) where $h, k \in \mathbb{Z} \geq 0$, then $p_4(n)$ is the number of *first quadrant* lattice points inside a circle of radius $\sqrt{n}$.

We identify each lattice point with a square of area one, whose bottom left corner on a lattice point inside the circle. That is, the square for the lattice point (h, k) has its four corners at the points $(h + i, k + j)$ where $i, j \in \{0, 1\}$. From this point forward, we treat $p_4(n)$ as the area created by the lattice points of $p_4(n)$ by that method, (see Figure 7).

Lemma 1.3.3. *Let $n \geq 0$. Then* $\dfrac{\pi n}{4} \leq p_4(n) \leq \dfrac{\pi(\sqrt{n} + \sqrt{2})^2}{4}$.

Proof. The maximum linear distance between two points in a square with side 1 is $\sqrt{2}$. Since we are only working with the first quadrant, $p_4(n)$ is strictly greater than the area of the circle of radius $\sqrt{n}$ and strictly less than the area of the circle of radius $\sqrt{n} + \sqrt{2}$. $\square$

Theorem 1.3.4. *Let $n \geq 0$. Then* $p_4(n) = \dfrac{\pi n}{4} + O(\sqrt{n})$.

Proof. From Lemma 1.3.3, we have that $\dfrac{\pi n}{4} \leq p_4(n) \leq \dfrac{\pi(\sqrt{n} + \sqrt{2})^2}{4}$. In the

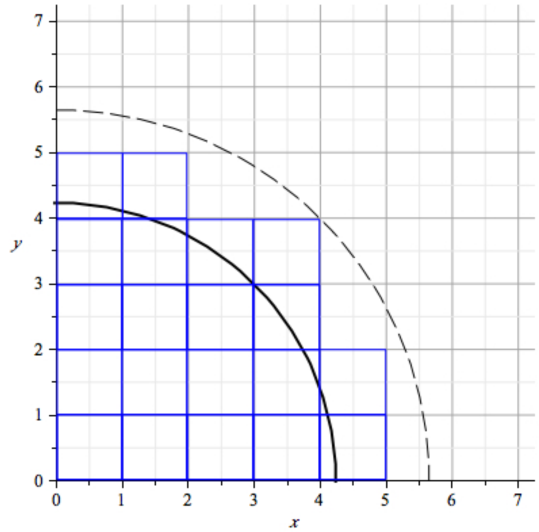

Figure 7: The solid black line gives a circle of radius $\sqrt{18}$ centered at (0,0), and the dotted black line gives a circle of radius $\sqrt{18}+\sqrt{2}$. Each blue square has is assigned to the lattice point at it's bottom left corner. The total area of the blue squares is $p_4(18) = 20$.

right hand inequality, we have that

$$\begin{aligned} p_4(n) &\leq \frac{\pi(\sqrt{n}+\sqrt{2})^2}{4} \\ &= \frac{\pi n + 2\pi\sqrt{2}\sqrt{n} + 2\pi}{4} \end{aligned}$$

which gives us

$$p_4(n) - \frac{\pi n}{4} \leq \frac{2\pi\sqrt{2}\sqrt{n} + 2\pi}{4}. \tag{1}$$

The left hand inequality gives

$$p_4(n) \geq \frac{\pi n}{4}$$

which implies that

$$\begin{aligned} p_4(n) - \frac{\pi n}{4} &\geq 0 \\ &\geq -\frac{2\pi\sqrt{2}\sqrt{n} + 2\pi}{4}. \end{aligned}$$

From this, and from statement (1) we know that

$$\left| p_4(n) - \frac{\pi n}{4} \right| \leq \frac{2\pi\sqrt{n} - 2\pi}{4},$$

which means that

$$p_4(n) = \frac{\pi n}{4} + O(\sqrt{n}).$$

$\square$

The Gauss Circle Problem is the quest for the smallest possible $g(x)$ so that

$$p_4(n) = \frac{\pi n}{4} + O(g(x)).$$

2 Sums of Two Figurate Numbers

The problem of determining $p_4(n)$ was first proposed by Gauss, who also found this relationship to the area of the circle. We generalize his solution from counting the sums of two squares, $f_4(m)$, to counting the sums of two regular figurate numbers, $f_k(m)$.

Definition 2.0.5. *Fix $n \geq 0$ and $k \geq 3$. Then $r_k(n)$ is the number of ways n can be written as a sum of two k-sided regular figurate numbers.*

In other words $r_k(n)$ counts nonnegative integer solutions (x, y) of

$$n = \frac{x[(k-2)x + 4 - k]}{2} + \frac{y[(k-2)y + 4 - k]}{2}. \tag{2}$$

Definition 2.0.6. *Fix $n \geq 0$ and $k \geq 3$. Then define $p_k(n)$ by*

$$p_k(n) = \sum_{i=0}^{n} r_k(i).$$

We again attach the bottom left corner of a square with area 1 to each lattice point in the first quadrant. We think of $p_k(n)$ as the sum of the areas of all of the squares attached to first quadrant lattice points inside the circle defined by equation (2).

2.1 Sums of two triangular numbers

We set $k = 3$ in equation (2) to get

$$\frac{x(x+1)}{2} + \frac{y(y+1)}{2} = n$$

and complete the square so that

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 2n + \frac{1}{2}.$$

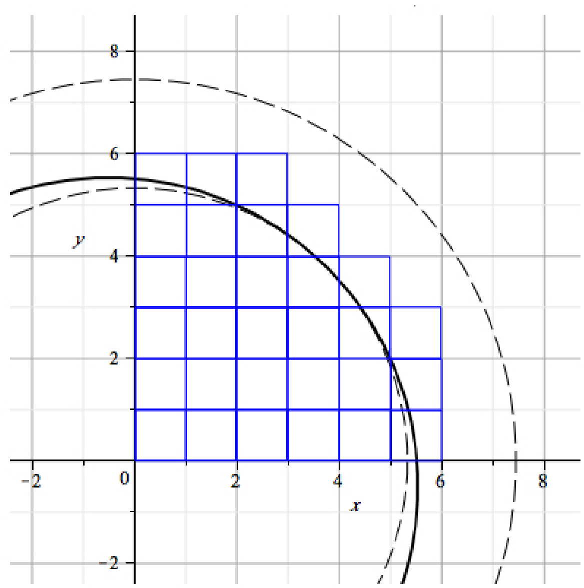

Figure 8: The solid black line gives a circle of radius $2 \cdot 18 + \frac{1}{2}$ centered at $(-\frac{1}{2}, -\frac{1}{2})$. Each blue square is assigned to a lattice point included in $p_3(18)$, so their sum is $p_3(18) = 30$. The dotted black line quarter circles enclose areas that are upper and lower bounds for $p_3(18)$.

Notice that this is a circle with radius $\sqrt{2n + \frac{1}{2}}$ and center $(-\frac{1}{2}, -\frac{1}{2})$. However, the radius is different from that of the Gauss Circle Problem, and the center of this circle is not at the origin.

Lemma 2.1.1. *Let $n \geq 2$ be given. Then*

$$\frac{\pi\left(\sqrt{2n+1/2} - \sqrt{1/2}\right)^2}{4} \leq p_3(n) \leq \frac{\pi(\sqrt{2n+1/2} + \sqrt{2})^2}{4}.$$

Proof. As in the proof of Lemma 1.3.3, this is geometrically clear. We bound the circle from $p_3(n)$ above and below by circles centered at $(0, 0)$. We compensate for the center being in the third quadrant by decreasing the radius of the lower bound by $\sqrt{\frac{1}{2}}$. We increase the radius of the upper bound by $\sqrt{2}$ for the same reason as in the proof of Lemma 1.3.3 (see Figure 8). $\square$

Theorem 2.1.2. *Let $n \geq 2$. Then $p_3(n) = \dfrac{\pi n}{2} + O(\sqrt{n})$.*

Proof. From Lemma 2.1.1, we have

$$\begin{aligned} p_3(n) &\leq \frac{\pi\left(\sqrt{2n+1/2}+\sqrt{2}\right)^2}{4} \\ &= \frac{\pi(2n+1/2)+2\pi\sqrt{2}\sqrt{2n+1/2}+2\pi}{4} \end{aligned}$$

so that

$$p_3(n) - \frac{\pi n}{2} \leq \frac{\pi(1/2)+2\pi\sqrt{2}\sqrt{2n-1/2}+2\pi}{4}. \tag{3}$$

Also from Lemma 2.1.1 we get

$$\begin{aligned} p_3(n) &\geq \frac{\pi\left(\sqrt{2n+1/2}-\sqrt{1/2}\right)^2}{4} \\ &= \frac{\pi(2n+1/2)-2\pi\sqrt{1/2}\sqrt{2n+1/2}+\pi/2}{4} \end{aligned} \tag{4}$$

so that

$$\begin{aligned} p_3(n) - \frac{\pi n}{2} &\geq \frac{\pi(1/2)-2\pi\sqrt{1/2}\sqrt{2n+1/2}+\pi/2}{4} \\ &\geq -\frac{\pi(1/2)+2\pi\sqrt{2}\sqrt{2n-1/2}+2\pi}{4}. \end{aligned}$$

From (3) and (4) we have that

$$\left|p_3(n) - \frac{\pi n}{2}\right| \leq \frac{\pi(1/2)+2\pi\sqrt{2}\sqrt{2n-1/2}+2\pi}{4}.$$

Thus

$$p_3(n) = \frac{\pi n}{2} + O\left(\sqrt{n}\right).$$

□

2.2 Sums of two pentagonal numbers

By Theorem 1.1.2, a pentagonal number is a regular figurate number of the form $\frac{x(3x-1)}{2}$ where $x \in \mathbb{N} \cup \{0\}$. We set $k = 5$ in equation (2) and complete the square to get

$$\left(x - \frac{1}{6}\right)^2 + \left(y - \frac{1}{6}\right)^2 = \frac{2n}{3} + \frac{1}{18}.$$

Lemma 2.2.1. *Let $n \geq 0$. Then*

$$\frac{\pi}{4}\left(\frac{2n}{3}+\frac{1}{18}\right) \leq p_5(n) \leq \frac{\pi}{4}\left(\sqrt{\left(\frac{2n}{3}+\frac{1}{18}\right)}+\sqrt{2}+\sqrt{1/2}\right)^2.$$

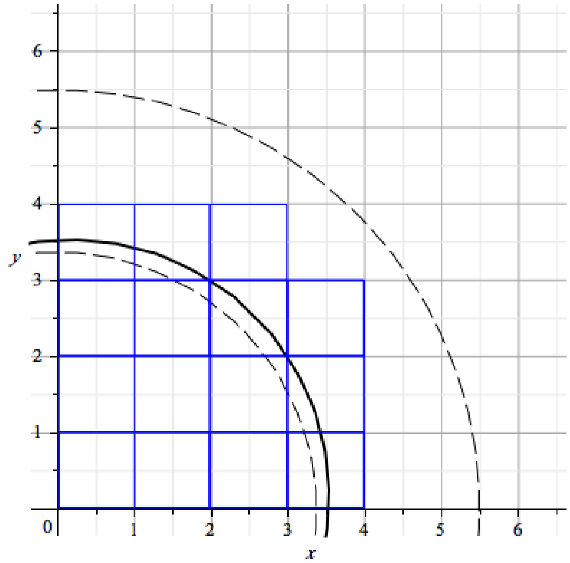

Figure 9: The solid black line gives a circle of radius $\sqrt{\frac{34}{3}+\frac{1}{18}}$ centered at $\left(\frac{1}{6},\frac{1}{6}\right)$. The total area of the blue squares is $p_5(17) = 15$. The dotted black line quarter circles enclose areas which are upper and lower bounds bounds for $p_5(17)$.

Proof. The bounds are geometrically clear. The center of this circle is at the point $\left(\frac{1}{6},\frac{1}{6}\right)$. As in the Gauss Circle Problem, we use the area of the quarter circle with this radius as a lower bound for $p_5(n)$. We can do this because the area of the quarter circle with this radius centered at the origin is less than the circle with the same radius centered at $\left(\frac{1}{6},\frac{1}{6}\right)$ (see Figure 9).

For the upper bound we increase the radius by $\sqrt{2}+\sqrt{\frac{1}{2}}$, which is $\frac{3\sqrt{2}}{2}$, where the $\sqrt{\frac{1}{2}}$ is an overcompensation for moving the center a distance of $\sqrt{\frac{1}{18}}$ into the first quadrant. □

Theorem 2.2.2. *Let $n \geq 1$. Then* $p_5(n) = \dfrac{\pi n}{6} + O(\sqrt{n})$.

Proof. We begin in the same way, with the right hand inequality in of Lemma 2.2.1;

$$\begin{aligned} p_5(n) &\leq \frac{\pi}{4}\left(\sqrt{\left(\frac{2n}{3}+\frac{1}{18}\right)}+\frac{3\sqrt{2}}{2}\right)^2 \\ &= \frac{\pi}{4}\left[\left(\frac{2n}{3}+\frac{1}{18}\right)+3\sqrt{2}\sqrt{\frac{2n}{3}+\frac{1}{18}}+\frac{9\cdot 2}{4}\right] \end{aligned}$$

which gives us

$$p_5(n) - \frac{\pi n}{6} \leq \frac{\pi}{4}\left(\frac{1}{18} + 3\sqrt{2}\sqrt{\frac{2n}{3} + \frac{1}{18}} + \frac{9}{2}\right). \tag{5}$$

We rearrange the left inequality from Lemma 2.2.1 to see that

$$p_5(n) \geq \frac{\pi}{4}\left(\frac{2n}{3} + \frac{1}{18}\right)$$

and subtract the leading term from both sides to get that

$$\begin{aligned} p_5(n) - \frac{\pi n}{6} &\geq \frac{\pi}{72} \\ &\geq -\frac{\pi}{4}\left(\frac{1}{18} + 3\sqrt{2}\sqrt{\frac{2n}{3} + \frac{1}{18}} + \frac{9}{2}\right). \end{aligned} \tag{6}$$

Finally, we combine (5) and (6) and see that

$$\left|p_5(n) - \frac{\pi}{6}\right| \leq \frac{\pi}{4}\left(\frac{1}{18} + 3\sqrt{2}\sqrt{\frac{2n}{3} + \frac{1}{18}} + \frac{9}{2}\right)$$

or

$$p_5(n) = \frac{\pi n}{6} + O\left(\sqrt{n}\right).$$

□

2.3 Sums of two regular figurate numbers

We use the same methodology as in the proof of Lemma 2.2.1 to find upper and lower bounds for $p_k(n)$.

Lemma 2.3.1. *Let $n \geq 1$ and $k \geq 4$. Then*

$$\frac{\pi}{4}\left[\frac{2n}{k-2} + 2\left(\frac{4-k}{2k-4}\right)^2\right] \leq p_k(n) \leq \frac{\pi}{4}\left(\sqrt{\frac{2n}{k-2} + 2\left(\frac{4-k}{2k-4}\right)^2} + \frac{3\sqrt{2}}{2}\right)^2.$$

Proof. We begin by completing the square of equation (2) to get

$$\left(x + \frac{4-k}{2k-4}\right)^2 + \left(y + \frac{4-k}{2k-4}\right)^2 = \frac{2n}{k-2} + 2\left(\frac{4-k}{2k-4}\right)^2.$$

Since $k \geq 4$ the circle is centered in the first quadrant and the lower bound can be the area of the quarter circle with the given radius.

To find an upper bound on the center we examine the behavior of the center $\left(-\frac{4-k}{2k-4}, -\frac{4-k}{2k-4}\right)$. First we see that the expression $-\frac{4-k}{2k-4}$ is strictly monotonically increasing in k since

$$-\frac{4-k}{2k-4} < -\frac{3-k}{2k-2} = -\frac{4-(k+1)}{2(k+1)-4}.$$

Then we take the limit as k goes to infinity to get

$$\lim_{k\to\infty} -\frac{4-k}{2k-4} = \frac{1}{2}.$$

Therefore, our upper bound must be adjusted not just by $\sqrt{2}$ as in the Gauss Circle Problem, but by $\sqrt{2}+\sqrt{\frac{1}{2}}$. $\square$

Theorem 2.3.2. *Let $n \geq 1$ and $k \geq 3$. Then $p_k(n) = \frac{\pi n}{2k-4} + O(\sqrt{n})$.*

Proof. Theorem 2.1.2 proves this when $k = 3$. If $k > 3$ we first manipulate the right inequality of Lemma 2.3.1 to get that

$$\begin{aligned}
p_k(n) &\leq \frac{\pi}{4}\left(\sqrt{\frac{2n}{k-2}+2\left(\frac{4-k}{2k-4}\right)^2}+\frac{3\sqrt{2}}{2}\right)^2 \\
&= \frac{\pi}{4}\left(\frac{2n}{k-2}+2\left(\frac{4-k}{2k-4}\right)^2+3\sqrt{2}\sqrt{\frac{2n}{k-2}+2\left(\frac{4-k}{2k-4}\right)^2}+\frac{9\cdot 2}{4}\right) \\
&= \frac{\pi n}{2k-4}+\frac{\pi}{2}\left(\frac{4-k}{2k-4}\right)^2+\frac{3\pi\sqrt{2}}{4}\sqrt{\frac{2n}{k-2}+2\left(\frac{4-k}{2k-4}\right)^2}+\frac{9\pi}{8}.
\end{aligned}$$

Thus

$$p_k(n) - \frac{\pi n}{2k-4} \leq \frac{\pi}{2}\left(\frac{4-k}{2k-4}\right)^2+\frac{\pi\sqrt{2}}{4}\sqrt{\frac{2n}{k-2}+2\left(\frac{4-k}{2k-4}\right)^2}+\frac{9\pi}{8}.$$

The left inequality of Lemma 2.3.1 gives

$$p_k(n) \geq \frac{\pi}{4}\left[\frac{2n}{k-2}+2\left(\frac{4-k}{2k-4}\right)^2\right],$$

so that

$$p_k(n) - \frac{\pi n}{2k-4} \geq \frac{\pi}{2}\left(\frac{4-k}{2k-4}\right)^2.$$

Note that $\frac{\pi}{2}\left(\frac{4-k}{2k-4}\right)^2$ is a positive number. Therefore

$$p_k(n) - \frac{\pi n}{2k-4} \geq -\frac{\pi}{2}\left(\frac{4-k}{2k-4}\right)^2-\frac{\pi\sqrt{2}}{4}\sqrt{\frac{2n}{k-2}+2\left(\frac{4-k}{2k-4}\right)^2}-\frac{9\pi}{8}.$$

Combining the results from above, we get

$$\left|p_k(n) - \frac{\pi n}{2k-4}\right| \leq \frac{\pi}{2}\left(\frac{4-k}{2k-4}\right)^2 + \frac{\pi\sqrt{2}}{4}\sqrt{\frac{2n}{k-2} + 2\left(\frac{4-k}{2k-4}\right)^2} + \frac{9\pi}{8}.$$

Thus,

$$p_k(n) = \frac{\pi n}{2k-4} + O\left(\sqrt{n}\right).$$

□

Corollary 2.3.3. *Let $k \geq 4$. Then $r_k(n) = 0$ for infinitely many n.*

Proof. First, notice that $r_k(n)$ is always a non-negative integer. Define the average value of $r_k(n)$ to be $\bar{r}_k(n)$ where

$$\bar{r}_k(n) = \frac{p_k(n)}{n}.$$

By Theorem 2.3.2,

$$\bar{r}_k(n) = \frac{\pi}{2k-4} + O\left(\frac{1}{\sqrt{n}}\right).$$

Notice that since $k \geq 4$, the leading term is less than 1. As n goes to infinity, the error goes to 0. Therefore,

$$\lim_{n\to\infty} \bar{r}_k(n) = \frac{\pi}{2k-4} < 1$$

therefore infinitely many $r_k(n)$ must be 0. □

3 Sums of Multiple Figurate Numbers

3.1 Sums of three regular figurate numbers

Definition 3.1.1. *Let $n \geq 0$ and $k \geq 3$ be given. Then $r_{k,3}(n)$ is the number of ways n can be written as a sum of three k-sided regular figurate numbers. In other words it counts positive integer solutions (x_1, x_2, x_3) of*

$$n = \frac{x_1[(k-2)x_1 + (4-k)]}{2} + \frac{x_2[(k-2)x_2 + (4-k)]}{2} + \frac{x_3[(k-2)x_3 + (4-k)]}{2}. \tag{7}$$

Definition 3.1.2. *Let $n \geq 0$ and $k > 2$. Define $p_{k,3}(n)$ by*

$$p_{k,3}(n) = \sum_{i=0}^{n} r_{k,3}(i).$$

For each lattice point (h_1, h_2, h_3) with $h_1, h_2, h_3 \in \mathbb{N} \cup \{0\}$ we attach a cube of volume 1 so that all vertices of the cube are of the form $(h_1 + j_1, h_2 + j_2, h_3 + j_3)$ where $j_i \in \{0, 1\}$. We think of $p_{k,3}(n)$ as the sum of the volume of each of the cubes attached to lattice points inside the sphere defined by equation (7).

Lemma 3.1.3. *Let $n \geq 0$ be given. Then*

$$\frac{\pi n^{\frac{3}{2}}}{6} \leq r_{4,3}(n) \leq \frac{\pi \left(\sqrt{n} + \sqrt{3}\right)^3}{6}.$$

Proof. The lower bound is the volume of the sphere in the first octant,

$$\frac{1}{8} \cdot \frac{4}{3} \pi \left(\sqrt{n}\right)^3.$$

The upper bound is adjusted by the maximum distance from one point of our lattice-counting-cubes to another, $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. $\square$

Theorem 3.1.4. *Let $n \geq 0$ be given. Then $p_{4,3}(n) = \frac{\pi}{6} n^{\frac{3}{2}} + O(n)$.*

Proof. We manipulate the right side of the Lemma 3.1.3 inequality to get

$$\begin{aligned} p_{4,3}(n) &\leq \frac{\pi \left(\sqrt{n} + \sqrt{3}\right)^3}{6} \\ &= \frac{\pi}{6} \left(n^{\frac{3}{2}} + 3n\sqrt{3} + 9\sqrt{n} + 3^{\frac{3}{2}}\right) \end{aligned}$$

so that

$$p_{4,3}(n) - \frac{\pi}{6} n^{\frac{3}{2}} \leq \frac{\pi}{6} \left(3n\sqrt{3} + 9\sqrt{n} + 3^{\frac{3}{2}}\right). \tag{8}$$

We rearrange the left inequality from Lemma 3.1.3 to get

$$\begin{aligned} p_{4,3}(n) - \frac{\pi}{6} n^{\frac{3}{2}} &\geq 0 \\ &\geq -\frac{\pi}{6} \left(3n\sqrt{3} + 9\sqrt{n} + 3^{\frac{3}{2}}\right). \end{aligned} \tag{9}$$

We put (8) and (9) together to get

$$\left| p_{4,3}(n) - \frac{\pi}{6} n^{\frac{3}{2}} \right| \leq \frac{\pi}{6} \left(3n\sqrt{3} + 9\sqrt{n} + 3^{\frac{3}{2}}\right).$$

Thus,

$$p_{4,3}(n) = \frac{\pi}{6} n^{\frac{3}{2}} + O(n).$$

$\square$

We use the same methodology as in Subsection 2.3 to find upper and lower bounds for $p_{k,3}(n)$.

Lemma 3.1.5. *Let $n \geq 1$ and $k \geq 4$. Then*

$$\frac{\pi}{6}\left[\frac{2n}{k-2}+3\left(\frac{4-k}{2k-4}\right)^2\right]^{\frac{3}{2}} \leq p_{k,3}(n) \leq \frac{\pi}{6}\left(\sqrt{\frac{2n}{k-2}+3\left(\frac{4-k}{2k-4}\right)^2}+\frac{3\sqrt{3}}{2}\right)^3.$$

Proof. The proof is analogous to that of Lemma 2.3.1. First we complete the square in equation (7) to get that

$$\left(x_1+\frac{4-k}{2k-4}\right)^2+\left(x_2+\frac{4-k}{2k-4}\right)^2+\left(x_3+\frac{4-k}{2k-4}\right)^2=\frac{2n}{k-2}+3\left(\frac{4-k}{2k-4}\right)^2.$$

The, the bounds follow geometrically as before. □

Theorem 3.1.6. *Let $n \geq 1$ and $k \geq 4$. Then*

$$p_{k,3}(n)=\frac{\pi}{6}\left(\frac{2n}{k-2}\right)^{\frac{3}{2}}+O(n).$$

We omit the proof in favor of the proof for Theorem 3.2.4, which is a generalization.

3.2 Sums of ℓ regular figurate numbers

Definition 3.2.1. *Let $n \geq 0$ and $k \geq 3$ be given. Then $r_{k,\ell}(n)$ is the number of ways n can be written as a sum of ℓ, k-sided regular figurate numbers, the positive integer solutions $(x_1,\dots,x_\ell)$ of*

$$n=\frac{x_1[(k-2)x_1+(4-k)]}{2}+\cdots+\frac{x_\ell[(k-2)x_\ell+(4-k)]}{2}. \tag{10}$$

Definition 3.2.2. *Let $n \geq 0$ and $k \geq 3$. Define $p_{k,\ell}(n)$ by*

$$p_{k,\ell}(n)=\sum_{i=0}^{n} r_{k,\ell}(i).$$

For each lattice point $(h_1,\dots,h_\ell)$ with $h_1,\dots,h_\ell \in \mathbb{N}\cup\{0\}$ we attach a hypercube of volume 1 so that all vertices of the cube are of the form $(h_1+j_1,\dots,h_\ell+j_\ell)$ where $j_i \in \{0,1\}$.

In the same way as before, we can think $p_{k,\ell}(n)$ as the sum of the volumes of each of the hypercubes attached to lattice points inside the hypersphere defined by equation (10). The maximum distance between two points in a unit hypercube is $\sqrt{\ell}$.

Lemma 3.2.3. *Let $n \geq 1$, $k \geq 4$ and $\ell \geq 2$ be given. If ℓ is even,*

$$\frac{\pi^{\frac{\ell}{2}}}{2^{\ell}\left(\frac{\ell}{2}\right)!}\left[\frac{2n}{k-2}+\ell\left(\frac{4-k}{2k-4}\right)^{2}\right]^{\frac{\ell}{2}} \leq p_{k,\ell}(n)$$

and

$$p_{k,\ell}(n) \leq \frac{\pi^{\frac{\ell}{2}}}{2^{\ell}\left(\frac{\ell}{2}\right)!}\left(\sqrt{\frac{2n}{k-2}+\ell\left(\frac{4-k}{2k-4}\right)^{2}}+\frac{3\sqrt{\ell}}{2}\right)^{\ell}.$$

If ℓ is odd,

$$\frac{\left(\frac{\ell-1}{2}\right)!\,\pi^{\frac{\ell-1}{2}}}{\ell!}\left[\frac{2n}{k-2}+\ell\left(\frac{4-k}{2k-4}\right)^{2}\right]^{\frac{\ell}{2}} \leq p_{k,\ell}(n)$$

and

$$p_{k,\ell}(n) \leq \frac{\left(\frac{\ell-1}{2}\right)!\,\pi^{\frac{\ell-1}{2}}}{\ell!}\left(\sqrt{\frac{2n}{k-2}+\ell\left(\frac{4-k}{2k-4}\right)^{2}}+\frac{3\sqrt{\ell}}{2}\right)^{\ell}.$$

Proof. We begin by completing the square in equation (10) to see that

$$\left(x_1+\frac{4-k}{2k-4}\right)^{2}+\cdots+\left(x_\ell+\frac{4-k}{2k-4}\right)^{2}=\frac{2n}{k-2}+\ell\left(\frac{4-k}{2k-4}\right)^{2}.$$

From [3], the volume of the ℓ-dimensional hypersphere with radius r is $\frac{\pi^{\frac{\ell}{2}}}{\left(\frac{\ell}{2}\right)!}r^{\ell}$, if ℓ is even, and $\frac{2^{\ell}\left(\frac{\ell-1}{2}\right)!\,\pi^{\frac{\ell-1}{2}}}{\ell!}r^{\ell}$, if ℓ is odd. Since we are restricted to positive solutions, we divide the volume by 2^{ℓ}.

We use the radius

$$r_{\text{lower}}=\sqrt{\frac{2n}{k-2}+\ell\left(\frac{4-k}{2k-4}\right)^{2}}$$

for the lower bound. For the upper bound, we use the radius

$$r_{\text{upper}}=\sqrt{\frac{2n}{k-2}+\ell\left(\frac{4-k}{2k-4}\right)^{2}}+\frac{\sqrt{\ell}}{2}+\sqrt{\ell}.$$

Indeed, as in the proof of Lemma 2.3.1 as k goes to infinity, the center is $(\frac{1}{2},\ldots,\frac{1}{2})$. We take into account this distance of $\frac{\sqrt{\ell}}{2}$ from the origin, and add an additional $\sqrt{\ell}$ for the maximum distance between two corners of one of volume-counting hypercube. □

Theorem 3.2.4. *Let $n \geq 1$, $k \geq 4$, and $\ell \geq 2$ be given. Then, if ℓ is even,*

$$p_{k,\ell}(n)=\frac{\pi^{\frac{\ell}{2}}}{2^{\ell}\left(\frac{\ell}{2}\right)!}\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}}+O\left(n^{\frac{\ell-1}{2}}\right)$$

and, if ℓ is odd,

$$p_{k,\ell}(n) = \frac{\left(\frac{\ell-1}{2}\right)!\pi^{\frac{\ell-1}{2}}}{2^\ell \ell!}\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}} + O\left(n^{\frac{\ell-1}{2}}\right).$$

Proof. We consider the even and odd cases separately for clarity. Let ℓ be even. Define

$$a = \frac{2n}{k-2} + \ell\left(\frac{4-k}{2k-4}\right)^2,$$

$$b = \frac{3\sqrt{\ell}}{2}$$

and

$$c = \frac{\pi^{\frac{\ell}{2}}}{2^\ell\left(\frac{\ell}{2}\right)!}.$$

We manipulate the right inequality from the even case of Lemma 3.2.3

$$\begin{aligned} p_{k,\ell}(n) &\leq c\left(\sqrt{\frac{2n}{k-2} + \ell\left(\frac{4-k}{2k-4}\right)^2} + \frac{3\sqrt{\ell}}{2}\right)^\ell \\ &= ca^{\frac{\ell}{2}} + c\sum_{i=1}^{\ell}\binom{\ell}{i}a^{\frac{\ell-i}{2}}b^i. \end{aligned}$$

We can expand $a^{\frac{\ell}{2}}$ with the binomial theorem

$$a^{\frac{\ell}{2}} = \left[\left(\frac{2n}{k-2}\right) + \sum_{i=1}^{\ell}\binom{\ell}{i}\left(\frac{2n}{k-2}\right)^{\ell-i}\ell^i\left(\frac{4-k}{2k-4}\right)^{2i}\right]^{\frac{1}{2}}$$

so that for n sufficiently large,

$$\begin{aligned} a^{\frac{\ell}{2}} &\leq \left[\left(\frac{2n}{k-2}\right)^\ell + \sum_{i=1}^{\ell}\binom{\ell}{i}\left(\frac{2n}{k-2}\right)^{\ell-1}\ell\left(\frac{4-k}{2k-4}\right)^2\right]^{\frac{1}{2}} \\ &= \left[\left(\frac{2n}{k-2}\right)^\ell + (2^\ell - 1)\left(\frac{2n}{k-2}\right)^{\ell-1}\ell\left(\frac{4-k}{2k-4}\right)^2\right]^{\frac{1}{2}}. \end{aligned}$$

From there, the triangle inequality gives

$$a^{\frac{\ell}{2}} \leq \left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}} + 2^{\frac{\ell}{2}}\ell^{\frac{1}{2}}\left(\frac{2n}{k-2}\right)^{\frac{\ell-1}{2}}\left(\frac{4-k}{2k-4}\right).$$

Therefore,

$$\begin{aligned} &p_{k,\ell}(n) - c\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}} \\ &\qquad \leq c\left[2^{\frac{\ell}{2}}\ell^{\frac{1}{2}}\left(\frac{2n}{k-2}\right)^{\frac{\ell-1}{2}}\left(\frac{4-k}{2k-4}\right) + \sum_{i=1}^{\ell}\binom{\ell}{i}a^{\frac{\ell-i}{2}}b^i\right]. \qquad (11) \end{aligned}$$

We rearrange the left inequality from the even case of Lemma 3.2.3 to get

$$\begin{aligned} p_{k,\ell}(n) &\geq ca^{\frac{\ell}{2}} \\ &\geq c\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}} \end{aligned}$$

so that

$$\begin{aligned} p_{k,\ell}(n) - c\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}} &\geq 0 \\ &\geq -c\left[2^{\frac{\ell}{2}}\ell^{\frac{1}{2}}\left(\frac{2n}{k-2}\right)^{\frac{\ell-1}{2}}\left(\frac{4-k}{2k-4}\right) + \sum_{i=1}^{\ell}\binom{\ell}{i}a^{\frac{\ell-i}{2}}b^i\right]. \quad (12) \end{aligned}$$

We put (11) and (12) together to get

$$\begin{aligned} &\left|p_{k,\ell}(n) - \frac{\pi^{\frac{\ell}{2}}}{2^{\ell}(\frac{\ell}{2})!}\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}}\right| \\ &\qquad\leq c\left[2^{\frac{\ell}{2}}\ell^{\frac{1}{2}}\left(\frac{2n}{k-2}\right)^{\frac{\ell-1}{2}}\left(\frac{4-k}{2k-4}\right) + \sum_{i=1}^{\ell}\binom{\ell}{i}a^{\frac{\ell-i}{2}}b^i\right]. \end{aligned}$$

So, when ℓ is even, we conclude that

$$p_{k,\ell}(n) = \frac{\pi^{\frac{\ell}{2}}}{2^{\ell}\left(\frac{\ell}{2}\right)!}\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}} + O\left(n^{\frac{\ell-1}{2}}\right).$$

Now let ℓ be odd. Set

$$a = \frac{2n}{k-2} + \ell\left(\frac{4-k}{2k-4}\right)^2,$$

$$b = \frac{3\sqrt{\ell}}{2}$$

as before, and set

$$c = \frac{(\frac{\ell-1}{2})!\pi^{\frac{\ell-1}{2}}}{2^{\ell}\ell!}.$$

We manipulate the right inequality from the odd case of Lemma 3.2.3

$$\begin{aligned} p_{k,\ell}(n) &\leq c\left(\sqrt{\frac{2n}{k-2} + \ell\left(\frac{4-k}{2k-4}\right)^2} + \frac{3\sqrt{\ell}}{2}\right)^{\ell} \\ &= ca^{\frac{\ell}{2}} + c\sum_{i=1}^{\ell}\binom{\ell}{i}a^{\frac{\ell-i}{2}}b^i \end{aligned}$$

$$p_{k,\ell}(n) - c\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}}$$
$$\leq c\left[2^{\frac{\ell}{2}}\ell^{\frac{1}{2}}\left(\frac{2n}{k-2}\right)^{\frac{\ell-1}{2}}\left(\frac{4-k}{2k-4}\right) + \sum_{i=1}^{\ell}\binom{\ell}{i}a^{\frac{\ell-i}{2}}b^i\right]. \tag{13}$$

We rearrange the left inequality from the odd case of Lemma 3.2.3 to get

$$\begin{aligned} p_{k,\ell}(n) &\geq ca^{\frac{\ell}{2}} \\ &\geq c\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}} \end{aligned}$$

so that

$$p_{k,\ell}(n) - c\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}} \geq 0$$
$$\geq -c\left[2^{\frac{\ell}{2}}\ell^{\frac{1}{2}}\left(\frac{2n}{k-2}\right)^{\frac{\ell-1}{2}}\left(\frac{4-k}{2k-4}\right) + \sum_{i=1}^{\ell}\binom{\ell}{i}a^{\frac{\ell-i}{2}}b^i\right]. \tag{14}$$

We put (13) and (14) together to get

$$\left|p_{k,\ell}(n) - \frac{(\frac{\ell-1}{2})!\pi^{\frac{\ell-1}{2}}}{2^\ell \ell!}\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}}\right|$$
$$\leq c\left[2^{\frac{\ell}{2}}\ell^{\frac{1}{2}}\left(\frac{2n}{k-2}\right)^{\frac{\ell-1}{2}}\left(\frac{4-k}{2k-4}\right) + \sum_{i=1}^{\ell}\binom{\ell}{i}a^{\frac{\ell-i}{2}}b^i\right].$$

So, when ℓ is odd, we conclude that

$$p_{k,\ell}(n) = \frac{(\frac{\ell-1}{2})!\pi^{\frac{\ell-1}{2}}}{2^\ell \ell!}\left(\frac{2n}{k-2}\right)^{\frac{\ell}{2}} + O\left(n^{\frac{\ell-1}{2}}\right).$$

□

References

[1] George E. Andrews, *Number Theory*. Philadelphia: Saunders, 1971.

[2] W. Duke, *Some old and new results about quadratic forms*, Notices. Amer. Math. Soc **44**, 1997, 190-196

[3] A.E. Lawrence, *The volume of an n-dimensional hypersphere.* Web. ⟨http://www-staff.lboro.ac.uk/ ∼ coael/hypersphere.pdf⟩.

[4] A. Weil, *Number Theory: An approach through history*, Birkhäuser, 1984.

4 Summary of Capstone Project

The problem of finding the number of integer solutions (x, y) to the equation $x^2 + y^2 = n$ with n as a positive integer is more than 200 years old [1]. This capstone examines some generalizations of this problem.

We examine elementary solutions to the sum of two figurate numbers, an expression motivated by a polygon with k sides of the same length. From there we adapt the technique to find the number of ways n can be written as a sum of ℓ regular figurate numbers, instead of only two.

Vocabulary

We define regular figurate numbers by a formula, and then give examples of how the formula is related to a polygon with k sides. The proof that they are the same is in the paper.

Definition. *The equation of the n^{th} figurate number with k sides is defined by*

$$f_k(n) = \frac{n\left[(k-2)n + (4-k)\right]}{2}.$$

As an example, consider when $k = 4$, the four sided figurate numbers which we call $f_4(n)$. We would like $f_4(n)$ to form a polygon with four sides of equal length. Computing gives

$$f_4(n) = \frac{n[(4-2)n + 4 - 4]}{2} = \frac{n(2n) - 0}{2} = n^2.$$

You might recognize $f_4(n)$ as n "squared" but it is important to see (check Figure 10) that these are literally squares. For each n, $f_4(n)$ gives a square with sides of length n.

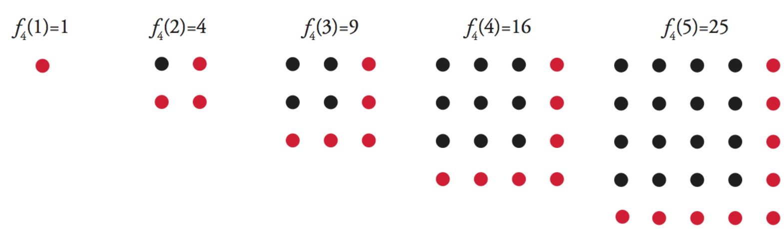

Figure 10: The first five four sided figurate numbers. Notice that $f_4(n)$ dots can be arranged in a square, with each of the four sides having a length of n. This is the rationale for calling $f_4(n)$ "square" numbers.

This geometric definition is actually the rationale behind the term regular figurate number. For every k, $f_k(n)$ dots forms a polygon with k sides each of length n. Setting $k = 3$ gives us the "triangular" numbers. When looking at the diagram, you should notice that each $f_k(n)$ is built on the previous figurate number, $f_k(n-1)$ in a specific way that is the same for all figurate numbers. For a more rigorous explanation consult Subsection 1.1 of the paper.

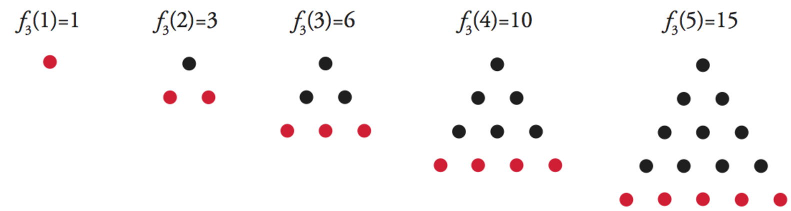

Figure 11: The first five $f_3(n)$ in dots, each arranged as a triangle. The n^{th} triangular number can be represented as a triangle with three sides of length n.

The next tool we will need to define is called "big O" notation. This is a way to talk about the relative size of a function as compared with an easier to understand function.

Definition. *Let f, g and h be functions on the real numbers. We say*

$$f(x) = h(x) + O(g(x))$$

if there exists c such that

$$|f(x) - h(x)| \leq c|g(x)|.$$

We read this "$f(x)$ is $h(x)$ plus big O of $g(x)$".

In other words, if the distance between two functions $f(x)$ and $h(x)$ is less than a constant times $g(x)$, we call $f(x)$ equal to $h(x) + O(g(x))$.

This notation is useful in cases where it is impossible to find an explicit formula for a function $f(x)$. The number of ways to write a positive integer n as a sum of some regular figurate numbers is one such function.

Definition. *Let n be greater than or equal to 0, and k greater than or equal to 3 be given. Then $r_{k,\ell}(n)$ is the number of ways n can be written as a sum of ℓ, k-sided regular figurate numbers. It counts the number of points $(x_1, \ldots, x_\ell)$ that satisfy*

$$n = f_k(x_1) + \cdots + f_k(x_\ell)$$

or

$$n = \frac{x_1[(k-2)x_1 + (4-k)]}{2} + \cdots + \frac{x_\ell[(k-2)x_\ell + (4-k)]}{2}.$$

The number of ways to write n as a sum of ℓ regular figurate numbers, $r_{k,\ell}$, is not easy predict. In fact, the function $r_{k,\ell}(n)$ cannot be written as a polynomial. As an example, consider when $k = 4$ and $\ell = 2$, the function $r_{4,2}(n)$. This is the number of ways a number n can be written as a sum of two squares. To find $r_{4,2}(n)$ we would calculate

$$
\begin{aligned}
&1 = 1^2 + 0^2 = 0^2 + 1^2; && 2 = 1^2 + 1^2; \\
&3 \neq a^2 + b^2 \text{ for all } a \text{ and } b; && 4 = 2^2 + 0^2 = 0^2 + 2^2; \\
&5 = 2^2 + 1^2 = 1^2 + 2^2; && 6 \neq a^2 + b^2 \text{ for all } a \text{ and } b
\end{aligned}
$$

so that

$$
\begin{aligned}
&r_{4,2}(1) = 2 && r_{4,2}(2) = 1 \\
&r_{4,2}(3) = 0 && r_{4,2}(2) = 2 \\
&r_{4,2}(5) = 1 && r_{4,2}(6) = 0.
\end{aligned}
$$

Since there isn't an explicit formula for $r_{k,\ell}(n)$, we add up all the $r_{k,\ell}(i)$ from $i = 0$ to n. We call this sum $p_{k,\ell}(n)$. As it is the primary object in this paper, we provide a formal definition here.

Definition. *Let $n \geq 0$ and $k \geq 3$. Define $p_{k,\ell}(n)$ by*

$$p_{k,\ell}(n) = \sum_{i=0}^{n} r_{k,\ell}(i).$$

It turns out that the function $p_{k,\ell}(n)$ is closely related to the volume of a hypersphere in ℓ dimensions. This capstone capitalizes on the inherent geometry of the problem to find a formula for $p_{k,\ell}(n)$ in big O notation. The geometric connection to $p_{k,\ell}(n)$ is fascinating, the interested reader should consult the figures in Section 1.3 for more information.

Results

As early as 1638, Fermat stated that every number n can be written as a sum of k, k-sided figurate numbers. That result was later proved by Cauchy [2]. The results from this paper point to that statement, but do not aid in proving it.

These are the two primary results from the paper.

Theorem 2.3.2 *Let $n \geq 1$ and $k \geq 3$. Then $p_{k,2}(n) = \dfrac{\pi n}{2k-4} + O(\sqrt{n})$.*

This is the result for sums of two figurate numbers, $p_{k,2}(n)$. It says that the error term is $O(\sqrt{n})$ regardless of which two regular figurate numbers are added together. However, the more sides each figurate number has, the smaller the leading term. So, as k increases, $p_{k,2}(n)$ decreases. The generalization of this theorem is the final result from the paper.

Theorem 3.2.4 *Let $n \geq 1$, $k \geq 4$, and $\ell \geq 2$ be given. Then, if ℓ is even,*

$$p_{k,\ell}(n) = \frac{\pi^{\frac{\ell}{2}}}{2^\ell (\frac{\ell}{2})!} \left(\frac{2n}{k-2} \right)^{\frac{\ell}{2}} + O\left(n^{\frac{\ell-1}{2}} \right)$$

and, if ℓ is odd,

$$p_{k,\ell}(n) = \frac{\left(\frac{\ell-1}{2}\right)! \pi^{\frac{\ell-1}{2}}}{2^\ell \ell!} \left(\frac{2n}{k-2} \right)^{\frac{\ell}{2}} + O\left(n^{\frac{\ell-1}{2}} \right).$$

In this theorem the same pattern holds. When ℓ is constant, an increase in k makes $p_{k,\ell}(n)$ smaller. This implies that a number n should be a sum of more k figurate numbers than $k-1$ figurate numbers. While it can not prove Fermat's statement, this theorem does give an overview of how arbitrary sums of figurate numbers behave.