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Influence of Surface Tension on the Conical Miniscus of a Magnetic Fluid in the Field of a Current-Carrying Wire

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We study the influence of surface tension on the shape of the conical miniscus built up by a magnetic fluid surrounding a current-carrying wire. Minimization of the total energy of the system leads to a singular second order boundary value problem for the function $\zeta(r)$ describing the axially symmetric shape of the free surface. An appropriate transformation regularizes the problem and allows a straightforward numerical solution. We also study the effects a superimposed second liquid, a nonlinear magnetization law of the magnetic fluid, and the influence of the diameter of the wire on the free surface profile.

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I. INTRODUCTION

The shape of the free surface of a ferrofluid in a static magnetic field is one of the prominent examples for a non-trivial interplay between magnetic and hydrodynamic degrees of freedom in ferrohydrodynamics: On the one hand the magnetic stresses contribute to the force balance determining the surface profile whereas on the other hand the local magnetic field depends on this profile due to the magnetic boundary conditions at the surface of the magnetically permeable material [1]. A particularily popular setup is the conical meniscus of a ferrofluid surrounding a vertical current-conducting wire [1], see Fig. 1. Neglecting surface tension the conical shape can be determined analytically from the balance between gravitational and magnetic force [1].

In the present note we investigate theoretically the influence of surface tension on the shape of the conical meniscus. Although surface tension is known to modify free surface profiles in ferrohydrodynamics and its influence has been studied, e.g., for small gaps [2, 3] and capillaries [4, 5] to our knowledge no systematic investigation has been done so far for the conical meniscus problem. Clearly, surface tension is present in all experiments in ferrohydrodynamics and a full understanding of its impact on a basic experiment in the field is hence desirable.

As expected no analytical expression for the free surface profile can be derived when surface tension is included. The numerical determination can be reduced to a singular boundary value problem for an ordinary differential equation which is, however, not completely straightforward to solve. Still, using an appropriate transformation, accurate solutions can be obtained. The numerical effort is much smaller than in alternative procedures such as finite difference, Galerkin, or collocation methods which is for example used from commercial software like ANSYS or COMSOL Multiphysics [6, 7]

The paper is organized as follows. In section II we collect the basic equations describing the system. Section III contains the analysis of a somewhat idealized situation. In section IV we discuss the modification of the results if the idealizations of section III are removed. Finally section V provides some conclusions.

II. BASIC EQUATIONS

The setup to be investigated is sketched in Fig. 1. An infinitely long, straight wire of radius R oriented along the z-axis of a cylindrical coordinate system (r, ϕ, z) carries an electric current I. The wire is surrounded by two superimposed liquids. The lower one with density ϱ_1 is a ferrofluid with a given magnetization law $\vec{M} = \vec{M}(\vec{H})$. The upper one has density $\varrho_2 < \varrho_1$ and is a non-magnetic fluid. The interface between the two fluids is characterized by an interface tension $\sigma_{1,2}$. For $I = 0$ there is no magnetic field and the flat interface between the two fluids is taken as the $z = 0$–plane of the coordinate system. For $I \neq 0$ a magnetic field

$$\vec{H}(r) = \frac{I}{2\pi r}\, \vec{e_\phi} \tag{1}$$

builds up which induces a force density in the lower fluid. This gives rise to an axis-symmetric conical interface parametrized by a function $\zeta(r)$ the determination of which is the central aim. Note that the magnetic field (1) is everywhere tangential to the surface and is therefore independent of the detailed shape of the surface which makes the analysis of this case particularily transparent.

One way to determine $\zeta(r)$ is by minimizing the total energy E_{tot} of the system. It is convenient then to use the energy of the flat interface configuration as reference state and hence to minimize the energy difference between states with a non-trivial $\zeta(r)$ and $\zeta \equiv 0$. Equivalently one may start with the ferrohydrodynamic Bernoulli equation [1, 8].

The total energy difference E_{tot} is the sum of three parts, the gravitational, the surface, and the magnetic energy,

$$E_{\text{tot}} = E_{\text{g}} + E_{\text{s}} + E_{\text{m}}. \tag{2}$$

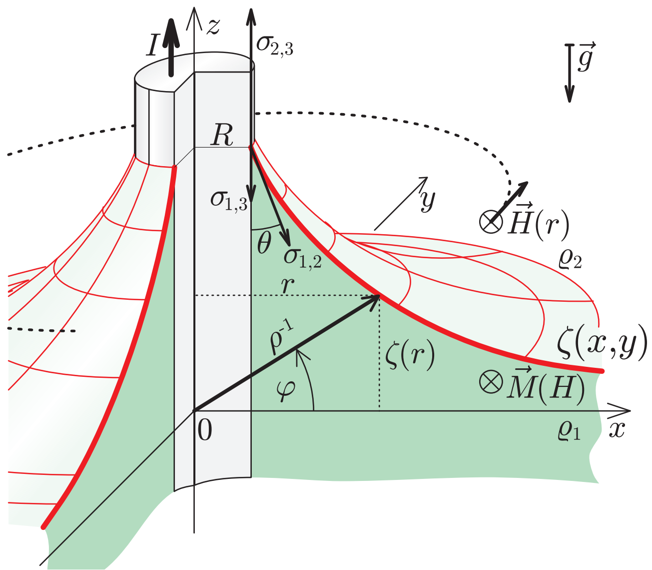

FIG. 1: Sketch of the setup and definition of the main variables. The free surface $\zeta(r)$ of the ferrofluid is axis-symmetric.

The different parts are given, respectively, by

$$E_{\mathrm{g}} = \int_V \mathrm{d}^3r\,(\varrho_1-\varrho_2)gz = \pi(\varrho_1-\varrho_2)g\int_R^\infty \mathrm{d}r\,r\,\zeta(r)^2 \quad (3)$$

$$E_{\mathrm{s}} = \int_{\partial V} \mathrm{d}^2r\,\sigma_{1,2} = 2\pi\sigma_{1,2}\int_R^\infty \mathrm{d}r\,r\left(\sqrt{1+\zeta'(r)^2}-1\right) \quad (4)$$

and [1, 9]

$$E_{\mathrm{m}} = -\int_V \mathrm{d}^3r\,\mu_0\int_0^{H(\vec{r})}\mathrm{d}\vec{H}\cdot\vec{M}(\vec{H})$$
$$= -2\pi\mu_0\int_R^\infty \mathrm{d}r\,r\,\zeta(r)\int_0^{H(r)}\mathrm{d}H\,M(H) \quad (5)$$

Here V denotes that part of the volume occupied by the ferrofluid for which $z>0$, ∂V is the area of the interface between the two fluids and μ_0 is the permeability of free space. Note that in the last equality of (5) we have assumed that the magnetization $\vec{M}$ and the magnetic field $\vec{H}$ are parallel as is the case in ferrofluids. Note also that $H(r)$ is given by (1).

The Euler-Lagrange equation

$$\frac{\delta E_{\mathrm{tot}}}{\delta\zeta(r)} = 0 \quad (6)$$

corresponding to the minimization of $E_{\mathrm{tot}}[\zeta(r)]$ is a nonlinear ordinary differential equation for the desired interface profile $\zeta(r)$. For a unique solution this equation has to be complemented by appropriate boundary conditions. These are

$$\lim_{r\to\infty}\zeta(r) = 0 \quad (7)$$

and

$$\zeta'(R) = \frac{\sigma_{1,3}-\sigma_{2,3}}{\sqrt{\sigma_{1,2}^2-(\sigma_{1,3}-\sigma_{2,3})^2}}, \quad (8)$$

where the prime denotes differentiation with respect to r. The second boundary condition results from the Young equation [10]

$$\sigma_{2,3} = \sigma_{1,3}+\sigma_{1,2}\cos\theta, \quad (9)$$

for the contact angle θ of the fluids at the wire and the fact that $\zeta'(R) = -\tan(\pi/2-\theta)$ (cf. Fig. 1).

III. SIMPLIFIED MODEL

We first analyse a somewhat simplified version of the boundary value problem derived in the last section. The simplifications are the following. We assume that the non-magnetic fluid is absent, $\varrho_2=0$, that the magnetization law of the ferrofluid is linear, $\vec{M}(H)=\chi\vec{H}$ where χ denotes the magnetic susceptibility, and that the diameter of the wire is negligible, $R=0$.

Since the magnetic field $H(r)$ diverges for $r\to 0$ so does the magnetic force density. In order to get a stable interface profile the magnetic force has to be counterbalanced by gravitation and surface tension which is possible only if

$$\lim_{r\to 0}\zeta(r) = \infty. \quad (10)$$

This equation replaces the boundary condition (8) in the present case. Consequently $\sigma_{1,3}$ and $\sigma_{2,3}$ are irrelevant and only $\sigma_{1,2}$ remains which will be denoted simply by σ in the present section.

Using the simplifying assumptions we find for the magnetic energy (5) the expression

$$E_{\mathrm{m}} = -\frac{\mu_0\,\chi\,I^2}{4\pi}\int_0^\infty \mathrm{d}r\,\frac{\zeta(r)}{r}, \quad (11)$$

which allows to explicitly perform the variation in eq.(6). As a result we get the following differential equation for $\zeta(r)$

$$g\varrho_1\zeta - \frac{\mu_0\chi I^2}{8\pi^2}\frac{1}{r^2} - \sigma\frac{\zeta'+\zeta'^3+r\zeta''}{(1+\zeta'^2)^{3/2}}\frac{1}{r} = 0. \quad (12)$$

It is convenient to introduce dimensionless quantities by measuring both r and ζ in units of

$$a = \sqrt[3]{\frac{\mu_0\chi I^2}{8\pi^2 g\varrho_1}} \quad (13)$$

and σ in units of $\varrho_1 g a^2$. Eq. (12) then acquires the form

$$\zeta - \frac{1}{r^2} - \sigma\frac{\zeta'+\zeta'^3+r\zeta''}{(1+\zeta'^2)^{3/2}}\frac{1}{r} = 0. \quad (14)$$

This equation is easily solved for $\sigma=0$ yielding the well-known hyperbola $\zeta(r)=1/r^2$ [1]. A perturbative solution for $\sigma\neq 0$ by expanding $\zeta(r)$ in powers of σ was found to yield satisfactory results only for values of σ

much smaller than those relevant in experiments. We therefore turned to a numerical solution.

Eq. (14) together with the boundary conditions (7) and (10) represents a singular boundary value problem of second kind (see, e.g., [11]). The main problem that makes a straightforward numerical solution inexpedient are the infinite intervals for r and ζ. We therefore transform both quantities according to (cf. also Fig. 1)

$$\zeta(\rho,\varphi)=\frac{\sin\varphi}{\rho}\quad,\quad r(\rho,\varphi)=\frac{\cos\varphi}{\rho} \tag{15}$$

and describe the interface profile by $\rho=\rho(\varphi)$. The boundary conditions (7) and (10) then transform into

$$\rho(\varphi=0)=0\quad\text{and}\quad\rho\left(\varphi=\frac{\pi}{2}\right)=0 \tag{16}$$

respectively which are much more convenient for the subsequent numerical solution. Substituting the derivatives in (14) according to

$$\zeta'=\frac{\mathrm{d}\zeta}{\mathrm{d}r}=\frac{\rho'\sin\varphi-\rho\cos\varphi}{\rho\sin\varphi+\rho'\cos\varphi}, \tag{17}$$

$$\zeta''=\frac{\mathrm{d}^2\zeta}{\mathrm{d}r^2}=-\frac{\rho^3(\rho+\rho'')}{(\rho\sin\varphi+\rho'\cos\varphi)^3}, \tag{18}$$

we find as differential equation for $\rho(\varphi)$

$$\left(\sin\varphi-\rho^3\sec^2\varphi\right)\left(\rho^2+\rho'^2\right)^{3/2}+\sigma\rho^2\left(\rho\rho'^2-\rho'^3\tan\varphi-\rho^2\rho'\tan\varphi+\rho^2\left(2\rho+\rho''\right)\right)=0. \tag{19}$$

Rewriting this equation in the form $\rho''(\varphi)=f(\varphi,\rho(\varphi),\rho'(\varphi))$ the boundary value problem (19), (16) can now easily be solved numerically using, e. g., the nonlinear finite-difference method described in [12]. As initial guess for the solution which is needed in this procedure the transformation of the analytical solution $\zeta_0(r)=1/r^2$ for $\sigma=0$ given by $\rho_0(\varphi)=\cos(\varphi)\sqrt[3]{\tan(\varphi)}$ may be used.

Fig.2 shows the resulting interface profile for parameter values corresponding to the ferrofluid EMG909 of Ferrotec [13]. For comparison the shape for $\sigma=0$ is also shown. It is clearly seen that the free surface profile is markedly modified by the influence of the surface tension. As expected the inclusion of surface tension makes the profile narrower since an additional force pointing radially inward builds up.

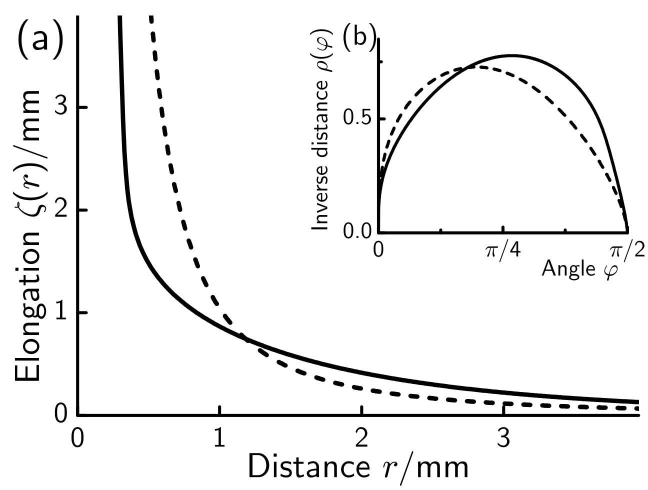

FIG. 2: (a) Free surface profile $\zeta(r)$ as determined numerically for $R=0$ and $I=30\,\mathrm{A}$ for the ferrofluid EMG909 with parameters $\varrho_1=1120\,\mathrm{kg\,m^{-3}}$, $\sigma=0.0259\,\mathrm{N\,m^{-1}}$, and $\chi=0.8$ (full line). Also shown is the result for $\sigma=0$ and otherwise identical parameters (dashed line). (b) Same profiles as in (a) in terms of the transformed function $\rho(\varphi)$.

IV. MORE REALISTIC SITUATIONS

The above solution of the idealized problem forms a convenient starting point for the discussion of the influence of those feature in the original setup that were neglected in the previous section.

A simple modification is the case in which the ferrofluid is superimposed by a non-magnetic liquid. Then the density ϱ_1 has to substituted by the density difference $\varrho_1-\varrho_2$ and the surface tension σ is to be replaced by the interface tension $\sigma_{1,2}$. The gravitational contribution to the total energy gets reduced and consequently larger displacements from the flat interface are to be expected. The situation can be analyzed quantitatively without further effort by mapping it to the case analyzed in the previous section. In fact the interface profile is again determined by eq. (14) with only the dimensionless units and the value of σ being modified. As an explicit example we show in Fig. 3(a) how the interface profile of Fig. 2 gets modified for $\varrho_2=10^3\,\mathrm{kg\,m^{-3}}$ and $\sigma_{1,2}=\sigma$.

Deviations form the linear magnetization law which become relevant in particular for small values of r where the magnetic field gets strong can also be dealt with. An improved approximation is provided by the Langevin relation [1],

$$M(H) = M_\mathrm{s}\left(\coth(\alpha) - \frac{1}{\alpha}\right) \quad , \quad \alpha = \frac{mH}{k_\mathrm{B}T}, \tag{20}$$

with the saturation magnetization of the fluid M_S, the magnetic moment m of a single ferromagnetic particle, the Boltzmann-constant k_B and the temperature T. The zero-field susceptibility is then given by $\chi = mM_\mathrm{S}/(3k_\mathrm{B}T)$. The density of the magnetic energy can again be determined analytically

$$\int_0^{H(\vec{r})} \mathrm{d}\vec{H} \cdot \vec{M}(\vec{H}) = \frac{M_\mathrm{S}k_\mathrm{B}T}{m} \ln \frac{\sinh\alpha}{\alpha}. \tag{21}$$

This expression gives rise to a somewhat more complicated equation for $\zeta(r)$, however, the overall procedure of section III remains valid. As shown in Fig. 3(a) the modification of the result for linear magnetization law is usually small in accordance with the fact that the fraction of the ferrofluid volume for which the field is large is rather small.

Finally, to elucidate the influence of a non-zero radius of the wire we consider the problem for $R > 0$ but with $\varrho_2 = 0$ and a linear magnetization law. The main qualitative difference is that instead of (10) we have to use (8) as boundary condition for small r. This implies that $\zeta(r)$ is not singular at the lower boundary. Hence already the simple transformation $r = 1/u$ is sufficient to get a numerically tractable boundary value problem for the function $\zeta(u)$. As initial guess a linear dependence $\zeta_0(u) = \lambda u$ may be used.

Note that λ fixes the value of $\zeta(r)$ at $r = R$ whereas the boundary condition (8) involves the derivative $\zeta'(r)$ at $r = R$. Therefore λ or equivalently $\zeta(R)$ has to be modified until the required value for $\zeta'(R)$ is obtained. This procedure leads to a monotonic relation between the prescribed contact angle θ and $\zeta(R)$, cf. Fig. 3(c). With the help this value for $\zeta(R)$ the complete surface profile $\zeta(r)$ can then be determined. Fig. 3(b) shows a collection of surface profiles with different $\zeta(R)$ and corresoponding contact angles θ. In the limiting case $R \to 0$, the surface profile is independent of the contact angle θ as demonstrated in Fig. 3(d) for a sequence of profiles with different values of R and a fixed contact angle $\theta = 45^\circ$.

V. CONCLUSIONS

In the present investigation we have determined the free surface profile of a magnetic fluid surrounding a vertical current carrying wire with special emphasis on the effects of interface and surface tension respectively. We have found that for experimentally relevant parameter values there is a strong influence of the surface tension giving rise to a more slender profile as compared to the well-studied case without surface tension [1]. A more general magnetization law including saturation as well as the prescription of the contact angle at the wire changes the profile only in the vicinity of the wire. A superimposed non-magnetic liquid changes the overall scale of the profile due to the reduction of the gravitational energy.

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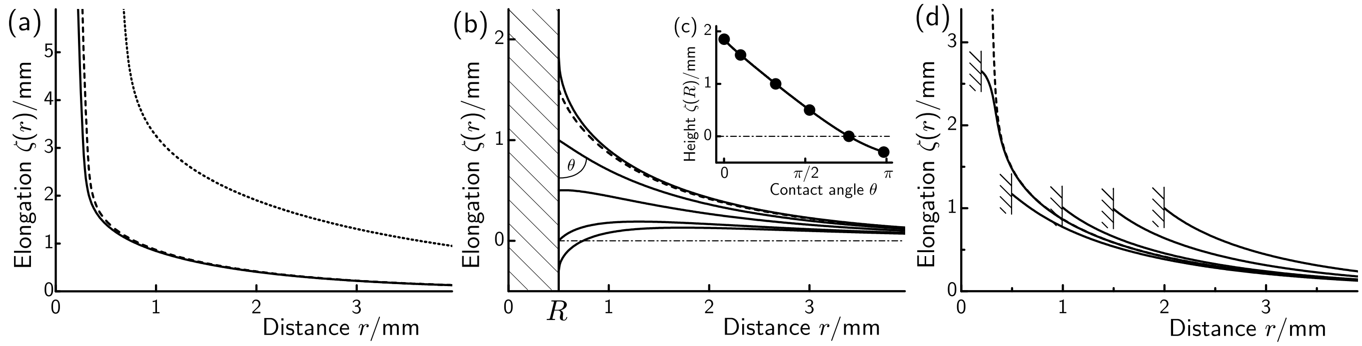

FIG. 3: Modifications of the idealized results of section 3. If not explicitly stated otherwise the parameters are the same as in Fig. 2. The dashed line shows always the same interface profile as the full line in Fig. 2. (a) The dotted line shows the interface profile if a non-magnetic fluid with density $\varrho_2 = 10^3\,\mathrm{kg\,m^{-3}}$ is superimposed to the ferrofluid. Also in (a), the solid line shows the interface profile for a ferrofluid with magnetization law of Langevin type with $M_\mathrm{s} = 16\,\mathrm{kA\,m^{-1}}$ and the same zero-field susceptibility χ as in Fig. 2. (b) Dependence of the interface profile on the contact angle θ for non-zero radius $R = 0.5\,\mathrm{mm}$. (c) Relation between the contact angle θ and the height of the interface at the wire $\zeta(R)$. Marked points correspond to the profiles shown in (b). (d) Sequence of surface profiles with fixed contact angle $\theta = 45°$ for decreasing diameter of the wire, $R = 2; 1.5; 1; 0.5; 0.2\,\mathrm{mm}$. For $R \to 0$ the profile of Fig. 2 is almost reproduced.