Modification of the Magnetic Flux-Line Interaction at a Superconductor's Surface

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The pair interaction between magnetic flux lines in a semi-infinite slab of an anisotropic type-II superconductor in an external field is derived in the London limit. The case where the applied field is normal to the superconductor/vacuum interface is considered. The presence of stray fields near the surface leads to an additional contribution to the repulsive interaction between flux lines that vanishes exponentially with the distance from the interface. The pair interaction is used to obtain the continuum elastic energy of a distorted semi-infinite flux-line array. The presence of the superconductor/vacuum interface yields surface contributions to the compressional and tilt elastic constants.

7/92

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1. Introduction

The nature of the ordering of the magnetic flux array in the mixed state of high-temperature copper-oxide superconductors has received considerable experimental and theoretical attention in the last few years. It has been shown that fluctuation effects are important in these materials and can lead to a number of new phases or regimes, including entangled flux liquids, hexatic flux liquids and hexatic vortex glasses [1-4]. Most experiments probe the properties of the flux array indirectly by measuring bulk properties of the superconductors, such as transport, magnetization or mechanical dissipation. At present direct measurements of the microscopic order of the magnetic flux array are mainly limited to decoration experiments at low fields [5-7]. These experiments aim to extract information on the vortex line configurations in the bulk of the material by imaging the pattern of the magnetic flux lines as they emerge from the surface of the sample. It is clear that to interpret the experiments and assess whether one can indeed consider the surface patterns as representative of vortex line configurations in the bulk of the sample, one needs to quantitatively understand what are the relative effects of bulk versus surface interactions and disorder in determining the configuration of the vortex tips as they emerge at the surface. Almost thirty years ago Pearl showed that the interaction between flux-line tips at a superconductor-vacuum interface decays as $1/r_\perp$ at large distances, with $r_\perp$ the distance beween flux tips along the surface [8]. In contrast, the interaction between flux-line elements in bulk decays exponentially at large distances. For this reason Huse [9] recently questioned the assumption that is made implicitly in all experimental work that surface patterns are representative of flux lines configurations in the bulk. A proper interpretation of the flux decoration experiments clearly requires a detailed knowledge of the modification of the properties of flux-line arrays near the surface of the sample [10]. The presence of the surface modifies the pair interaction between flux lines and changes the long wavelength properties of flux-line liquids and lattices.

The motivation of this paper is the desire to provide a framework for a quantitative analysis of the decoration experiments. The central result of the paper is a coarse-grained hydrodynamic free energy that describes the long wavelength properties of flux-line liquids in a semi-infinite anisotropic superconductor occupying the half space $z < 0$. In a forthcoming publication [11] this free energy will be used as the starting point for evaluating the structural properties of the flux-line tips as they emerge at the surface of a superconducting slab.

Explicit expressions for the liquid elastic constants, the compressional modulus and the tilt modulus, are obtained in terms of the superconductor parameters. The wave vector-dependent elastic constants can be written as the sum of bulk and surface contributions. The bulk parts are the well-known nonlocal elastic constants of a bulk flux-line liquid. In addition, the modification of the flux-line interaction near the surface and the magnetic energy from the stray fields in the region above the superconductor/vacuum interface lead to a surface contribution to both the compressional and the tilt moduli. These surface contributions vanish exponentially with the distance from the interface. They do, however, affect the surface properties and structure of the flux array [11]. The compressional modulus of the flux-line liquid is enhanced in a surface layer near the interface by the stiffening of the repulsive interaction between the lines. At the interface the repulsive interaction between the flux tips decays as $1/r_\perp$ at large distances. This yields an additive surface contribution to the wave vector-dependent compressional modulus that diverges as $1/q_\perp$ at small wave vectors. In contrast, the surface contribution to the tilt modulus of the flux line liquid is always negative. This is because the magnetic field lines associated with each vortex fan out as the interface is approached. As a result, transverse magnetic field fluctuations associated with tilting the flux lines become less costly in energy. The presence of the superconductor/vacuum interface produces a practically incompressible, but very flexible surface layer of flux-line liquid.

Our first step to obtain the hydrodynamic free energy is the calculation of the magnetic energy of an assembly of flux lines in a semi-infinite superconductor sample in the London approximation. This calculation follows previous work by Brandt [12] on vortex interactions near the surface of isotropic superconductors. Here we have in mind applications to the CuO_2 superconductors and we consider the case where the interface is orthogonal to the c axis of the material. The magnetic field is also applied along the c axis, which is chosen as the z direction. The magnetic field energy is obtained in Section 2 by solving the London equation in the superconductor half space and Maxwell's equation in vacuum with the appropriate boundary conditions. The contribution to the magnetic energy from the superconductor half space is rewritten in a standard way as a pairwise additive interaction between flux lines. The contribution from the vacuum can be naturally recast in the form of a surface modification to the pair interaction between the vortices. Similar results for the flux-line interaction in a semi-infinite anisotropic superconductor were reported recently by Buisson et al. [13]. No details were given in that paper. Since we think the derivation itself of the interaction is instructive and illuminating, we briefly sketch the derivation here. In addition, the pair interaction is presented here in a form that, while equivalent to that of Ref. [13], is more transparent and more suited for approximations [11]. Using this flux-line energy as the starting point in Sections 3 and 4 we obtain the coarse grained hydrodynamic free energy that describes the long wavelength properties of a semi-infinite flux array. The modification of the compressional and tilt elastic constants due to surface effects are identified and discussed.

2. Magnetic energy of the flux-line array

We consider a semi-infinite CuO_2 superconductor sample that occupies the half space $z < 0$. The c axis is normal to the interface (i.e., along the z direction) and the sample is placed in a constant field H directed along the c axis, with $H_{c1} << H << H_{c2}$. The local magnetic induction $\mathbf{h}(\mathbf{r})$ can be found by solving London's equation in the superconductor ($z < 0$) and Maxwell's equations in vacuum ($z > 0$) with appropriate boundary conditions at the surface ($z = 0$). The equations to be solved for an anisotropic uniaxial superconductor are:

$$\begin{aligned} z<0: \quad & \mathbf{h} + \boldsymbol{\nabla} \times (\boldsymbol{\Lambda} \cdot \boldsymbol{\nabla} \times \mathbf{h}) = \phi_0 \sum_i \int_{-\infty}^{0} dz_1 \frac{d\mathbf{R}_i(z_1)}{dz_1} \delta^{(3)}(\mathbf{r} - \mathbf{R}_i(z_1)), \\ & \boldsymbol{\nabla} \cdot \mathbf{h} = 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} z>0: \quad & \boldsymbol{\nabla} \times \mathbf{h} = 0, \\ & \boldsymbol{\nabla} \cdot \mathbf{h} = 0, \end{aligned} \tag{2.2}$$

with the boundary condition,

$$\begin{aligned} z=0: \quad & \mathbf{h} \qquad \text{continous} \\ & [\hat{\mathbf{z}} \cdot (\boldsymbol{\nabla} \times \mathbf{h})]_{z=0^-} = 0. \end{aligned} \tag{2.3}$$

The flux line is parametrized in terms of the coordinate z and $\mathbf{R}_i(z) = \big(\mathbf{r}_i(z), z\big)$ denotes the position of a flux-line element at a heigth z below the planar superconductor/vacuum interface. The "line trajectories" $\mathbf{r}_i(z)$ are assumed to be single-valued functions of z since "backtracking" involves a large energy cost for $H > H_{c1}$. Below we will describe the tranverse fluctuations of the flux lines in terms of a three-dimensional line tangent vector $\mathbf{T}_i$, defined as,

$$\mathbf{T}_i(z) = \frac{d\mathbf{R}_i(z)}{dz} = \big(\mathbf{t}_i(z), z\big), \tag{2.4}$$

or in terms of its component $\mathbf{t}_i$ in the plane normal to the applied field, where,

$$\mathbf{t}_i(z) = \frac{d\mathbf{r}_i(z)}{dz}. \tag{2.5}$$

The elements of the symmetric tensor $\mathbf{\Lambda}$ are

$$\Lambda_{\alpha\beta} = \lambda_{ab}^2 \delta_{\alpha\beta} + (\lambda_c^2 - \lambda_{ab}^2)\hat{c}_\alpha \hat{c}_\beta, \tag{2.6}$$

where λ_{ab} and λ_c are the penetration lengths in the ab plane and along the c direction, respectively - $\hat{\mathbf{c}}$ is a unit vector along the c axis. In the Ginzburg-Landau theory the anisotropy is accounted for by introducing an effective-mass tensor for the superconducting electrons. For the high-T_c copper-oxides the effective-mass tensor is, to an excellent approximation, diagonal in the chosen coordinate system. Denoting by M_{ab} and M_c the effective masses describing the interaction in the ab plane and in the c direction, respectively, the anisotropy ratio is defined as $\gamma^2 = M_c/M_{ab}$, and $\lambda_c^2/\lambda_{ab}^2 = \gamma^2$. In the high-$T_c$ materials $\gamma^2 >> 1$.

The boundary conditions are simply the continuity of the field at the boundary and the condition that there is no current normal to the interface.

The solution of the equations can be obtained directly in terms of the partial Fourier tranform of the local induction,

$$\mathbf{h}(\mathbf{q}_\perp, z) = \int d\mathbf{r}_\perp e^{-i\mathbf{q}_\perp \cdot \mathbf{r}_\perp} \mathbf{h}(\mathbf{r}), \tag{2.7}$$

where $\mathbf{r} = (\mathbf{r}_\perp, z)$, with the result,

$$\mathbf{h}(\mathbf{q}_\perp, z) = \frac{\phi_0}{\lambda_{ab}^2} \sum_i \int_{-\infty}^0 dz_1 e^{-i\mathbf{q}_\perp \cdot \mathbf{r}_i(z_1)} \frac{\hat{\mathbf{z}} - i\hat{\mathbf{q}}_\perp}{q_\perp + \alpha} e^{z_1 \alpha} e^{-q_\perp z}, \tag{2.8}$$

for $z > 0$, i.e., in the vacuum half-space, and

$$\begin{aligned} \mathbf{h}(\mathbf{q}_\perp, z) = \frac{\phi_0}{2\lambda_{ab}^2} \sum_i \int_{-\infty}^0 dz_1 e^{-i\mathbf{q}_\perp \cdot \mathbf{r}_i(z_1)} \Big\{ & \Big[\hat{\mathbf{z}} + \hat{\mathbf{q}}_\perp \big(\hat{\mathbf{q}}_\perp \cdot \mathbf{t}_i(z_1)\big)\Big] \frac{e^{-\alpha|z-z_1|}}{\alpha} \\ & + \Big[\hat{\mathbf{z}} - \hat{\mathbf{q}}_\perp \big(\hat{\mathbf{q}}_\perp \cdot \mathbf{t}_i(z_1)\big)\Big] \frac{e^{\alpha(z+z_1)}}{\alpha} - [\hat{\mathbf{z}} q_\perp + i\hat{\mathbf{q}}_\perp \alpha] \, \frac{2e^{\alpha(z+z_1)}}{\alpha(q_\perp + \alpha)} \\ & + (\hat{\mathbf{z}} \times \hat{\mathbf{q}}_\perp) \, (\hat{\mathbf{z}} \times \hat{\mathbf{q}}_\perp) \cdot \mathbf{t}_i(z_1) \, \frac{e^{-\alpha_c|z-z_1|} - e^{\alpha_c(z+z_1)}}{\alpha_c} \Big\}, \end{aligned} \tag{2.9}$$

for $z < 0$, where

$$\alpha = \alpha(q_\perp) = \sqrt{q_\perp^2 + 1/\lambda_{ab}^2} \,, \tag{2.10}$$

and

$$\alpha_c = \alpha(\gamma q_\perp) = \sqrt{\gamma^2 q_\perp^2 + 1/\lambda_{ab}^2}\ . \tag{2.11}$$

Alternatively, one can find the local induction by using the method of images, as was done by Brandt for an isotropic superconductor [12]. If the line is straight, the field of a single flux line is identical to that evaluated many years ago by Pearl [8]. The field of many vortices is just the linear superposition of the field of a single vortex.

The total magnetic energy is given by,

$$U = U_v + U_s = \int_{z>0} d\mathbf{r}\ \frac{h^2}{8\pi} + \int_{z<0} d\mathbf{r}\frac{1}{8\pi}\ \big[h^2 + (\boldsymbol{\nabla}\times\mathbf{h})\cdot\boldsymbol{\Lambda}\cdot(\boldsymbol{\nabla}\times\mathbf{h})\big], \tag{2.12}$$

where the first term on the right hand side is the field energy from the vacuum half space and the second term is the energy from the fields and the supercurrents in the superconductor. The vacuum contribution U_v is immediately evaluated by substituting Eq. (2.8) in the first term of Eq. (2.12), with the result,

$$U_v = \frac{\phi_0^2}{8\pi\lambda_{ab}^2}\sum_{i,j}\int_{-\infty}^0 dz_1\int_{-\infty}^0 dz_2\int\frac{d\mathbf{q}_\perp}{(2\pi)^2}e^{-i\mathbf{q}_\perp\cdot\left[\mathbf{r}_i(z_1)-\mathbf{r}_j(z_2)\right]}\,\frac{e^{\alpha(z_1+z_2)}}{\lambda_{ab}^2 q_\perp(q_\perp+\alpha)^2}. \tag{2.13}$$

The contribution U_s from the half space occupied by the superconductor can be evaluated either by direct substitution of Eq. (2.9) or by first performing an integration by parts to obtain,

$$U_s = \frac{1}{8\pi}\int_{z<0} d\mathbf{r}\ \mathbf{h}\cdot\Big[\mathbf{h} + \boldsymbol{\nabla}\times(\boldsymbol{\Lambda}\cdot\boldsymbol{\nabla}\times\mathbf{h})\Big] + \frac{\lambda_{ab}^2}{8\pi}\int d\mathbf{r}_\perp\ \hat{\mathbf{z}}\cdot\big[\mathbf{h}\times(\boldsymbol{\nabla}\times\mathbf{h})\big]_{z=0^-}. \tag{2.14}$$

By substituting from the London equation (2.1) in the first term on the right hand side of Eq. (2.14), one then obtains,

$$U_s = \frac{\phi_0}{8\pi}\sum_i\int_{-\infty}^0 dz_1\ \mathbf{T}_i(z_1)\cdot\mathbf{h}\big(\mathbf{r}_i(z_1),z_1\big) + \frac{\lambda_{ab}^2}{8\pi}\int d\mathbf{r}_\perp\hat{\mathbf{z}}\cdot\big[\mathbf{h}\times(\boldsymbol{\nabla}\times\mathbf{h})\big]_{z=0^-}, \tag{2.15}$$

where $\mathbf{h}\big(\mathbf{r}_i(z_1),z_1\big)$ is the local field due to all the flux lines, as given in Eq. (2.9), evaluated at the location of the i-th line. After some manipulation of the surface contribution in Eq. (2.15), the total energy can be written as an integral over the superconductor half space of a pair interaction between flux lines,

$$U = \frac{\phi_0^2}{8\pi\lambda_{ab}^2}\sum_{i,j}\int_{-\infty}^0 dz_1\int_{-\infty}^0 dz_2\int\frac{d\mathbf{q}_\perp}{(2\pi)^2}e^{-i\mathbf{q}_\perp\cdot[\mathbf{r}_i(z_1)-\mathbf{r}_j(z_2)]}\Big\{\frac{e^{-\alpha|z_1-z_2|}}{2\alpha}(1+\mathbf{t}_i\cdot\mathbf{t}_j)$$
$$+\Big[\frac{e^{-\alpha_c|z_1-z_2|}}{2\alpha_c}-\frac{e^{-\alpha|z_1-z_2|}}{2\alpha}\Big](\hat{\mathbf{z}}\times\hat{\mathbf{q}}_\perp)\cdot\mathbf{t}_i\ (\hat{\mathbf{z}}\times\hat{\mathbf{q}}_\perp)\cdot\mathbf{t}_j$$
$$+\frac{e^{\alpha(z_1+z_2)}}{2\alpha}\Big[1-\mathbf{t}_i\cdot\mathbf{t}_j+\frac{2}{\lambda_{ab}^2 q_\perp(q_\perp+\alpha)}\Big]$$
$$-(\hat{\mathbf{z}}\times\hat{\mathbf{q}}_\perp)\cdot\mathbf{t}_i\ (\hat{\mathbf{z}}\times\hat{\mathbf{q}}_\perp)\cdot\mathbf{t}_j\Big[\frac{e^{-\alpha_c(z_1+z_2)}}{2\alpha_c}-\frac{e^{\alpha(z_1+z_2)}}{2\alpha}\Big]\Big\}, \tag{2.16}$$

where $\mathbf{t}_i = \mathbf{t}_i(z_1)$ and $\mathbf{t}_j = \mathbf{t}_j(z_2)$. The energy of Eq. (2.16) includes the self-energy of the vortices (terms with $i = j$ in the sum). The small distances divergence of the repulsive self-energy needs to be truncated by introducing a cutoff to account for the finite size of the vortex core. As discussed by Sudbo and Brandt [14], the proper cutoff is anisotropic. The terms in Eq. (2.16) that contain exponentials in $|z_1 - z_2|$ are identical to those obtained for the pair interaction between flux lines in bulk. In fact if one assumes that the superconductor extends over all space in the z direction, it is easily shown that the part of the interaction containing exponentials in $|z_1 - z_2|$ is simply identical to that obtained elsewhere for a bulk superconductor [15,16]. The terms containing exponentials in $(z_1 + z_2)$ arise from the fluctuations in the local induction due to the presence of the superconductor/vacuum interface. The corresponding contribution to the energy density is nonzero only within a layer of depth $\approx \lambda_{ab}$ near the surface. The total energy (2.16) of the flux array also includes the magnetic energy from the stray fields generated by the vortex ends outside the sample.

It is useful for the following to rewrite the interaction energy (2.16) in a form where the transverse (to $\hat{\mathbf{q}}_\perp$) components of the tangent vectors $\mathbf{t}_i$ are eliminated in terms of the magnitude of the vectors and their longitudinal components. This can be done by using $[(\hat{\mathbf{z}}\times\hat{\mathbf{q}}_\perp)\cdot\mathbf{t}_i]\ [(\hat{\mathbf{z}}\times\hat{\mathbf{q}}_\perp)\cdot\mathbf{t}_j] = \mathbf{t}_i\cdot\mathbf{t}_j - (\hat{\mathbf{q}}_\perp\cdot\mathbf{t}_i)\ (\hat{\mathbf{q}}_\perp\cdot\mathbf{t}_i)$. The terms containing only the longitudinal components of $\mathbf{t}_i$ are then integrated by parts, with the result,

$$U = \frac{\phi_0^2}{8\pi\lambda_{ab}^2}\sum_{i,j}\int_{-\infty}^0 dz_1\int_{-\infty}^0 dz_2\int\frac{d\mathbf{q}_\perp}{(2\pi)^2}e^{-i\mathbf{q}_\perp\cdot[\mathbf{r}_i(z_1)-\mathbf{r}_j(z_2)]}$$
$$\times\Big\{\frac{e^{-\alpha|z_1-z_2|}+e^{\alpha(z_1+z_2)}}{2\alpha}\Big(1-\frac{\alpha^2}{q_\perp^2}\Big)+\frac{e^{-\alpha_c|z_1-z_2|}+e^{\alpha_c(z_1+z_2)}}{2\alpha_c}\ \frac{\alpha_c^2}{q_\perp^2} \tag{2.17}$$
$$+\frac{e^{\alpha(z_1+z_2)}}{\lambda_{ab}^2 q_\perp\alpha(q_\perp+\alpha)}+\mathbf{t}_i\cdot\mathbf{t}_j\ \frac{e^{-\alpha_c|z_1-z_2|}-e^{\alpha_c(z_1+z_2)}}{2\alpha_c}\Big\}.$$

This form of the energy is equivalent to that given in Eq. (2.16) above and is a convenient starting point for the derivation of the elastic energy of a flux array described below.

To gain some physical insight on the effect that the presence of a superconductor/vacuum interface has on the flux-line interaction, it is useful to consider the interaction for the case of straight flux lines. In this case the two-dimensional position vectors $\mathbf{r}_i$ do not depend on z and the tangent vectors simply vanish. The integration over z_1 and z_2 can be carried out. The interaction energy of an array of rigid flux lines is then given by,

$$\begin{aligned} U &= \frac{1}{2}\sum_{i,j}\int \frac{d\mathbf{q}_\perp}{(2\pi)^2} e^{-i\mathbf{q}_\perp\cdot(\mathbf{r}_i-\mathbf{r}_j)} \int_{-\infty}^{0} dz \Big[V_B(q_\perp) + \frac{\phi_0^2}{4\pi\lambda_{ab}^2}\ \frac{e^{z\alpha}}{q_\perp \alpha(q_\perp+\alpha)}\Big] \\ &= \frac{1}{2}\sum_{i,j}\int \frac{d\mathbf{q}_\perp}{(2\pi)^2} e^{-i\mathbf{q}_\perp\cdot(\mathbf{r}_i-\mathbf{r}_j)} \Big[L\ V_B(q_\perp) + V_S(q_\perp)\Big], \end{aligned} \tag{2.18}$$

where L is the size of the superconductor sample in the z direction ($L >> \lambda_{ab}$)[17]. There are two contributions to the energy of Eq. (2.18). The first term is a bulk energy proportional to the size of the system in the z direction. The pair potential $V_B(q_\perp)$ is given by

$$V_B(q_\perp) = \frac{\phi_0^2}{4\pi}\frac{1}{1+q_\perp^2\lambda_{ab}^2}, \tag{2.19}$$

and it is the Fourier transform of the usual pair interaction per unit length between straight flux lines in bulk. The second term is a surface energy corresponding to the magnetic energy of the stray fields near the interface. It can be interpreted as a pair interaction between flux-line tips at the superconductor/vacuum interface, with

$$V_S(q_\perp) = \frac{\phi_0^2}{4\pi}\ \frac{1}{q_\perp(1+q_\perp^2\lambda_{ab}^2)^{3/2}[q_\perp\lambda_{ab} + \sqrt{1+q_\perp^2\lambda_{ab}^2}]}. \tag{2.20}$$

In the long wavelength limit the surface interaction becomes

$$V_S(q_\perp) \approx \frac{\phi_0^2}{4\pi}\frac{1}{q_\perp}. \tag{2.21}$$

Inverting the Fourier transform, one finds that the pair interaction between flux-line tips decays as $V_S(r_\perp) \approx \phi_0^2/4\pi^2 r_\perp$ at large distances ($r_\perp >> \lambda_{ab}$), as obtained many years ago by Pearl [8]. As pointed out by Huse [9], this is easily understood because each flux line spills a flux quantum ϕ_0 into the vacuum half-space when exiting the superconductor's surface. The interaction between flux tips is therefore the interaction between two magnetic monopoles of "charge" $\phi_0/2\pi$.

3. Elastic Energy

In this section we calculate the continuum elastic energy associated with long-wavelength deformations of the semi-infinite flux array. We confine ourselves to applied fields in the range $H_{c1} << H << H_{c2}$, corresponding to $\lambda_{ab} > d >> \xi_{ab}$. Here $d = \sqrt{n_0}$ is the average intervortex spacing in the xy plane, with $n_0 = \phi_0/B_0$ the equilibrium areal density of flux lines and B_0 the equilibrium induction in the superconductor. For $H >> H_{c1}$, $B_0 \sim H$. For the fields of interest the vortex cores do not overlap and the spatial variations in the order parameter outside the core can be neglected. The energy of an array of London vortices that was obtained in Section 2 is then the appropriate starting point for obtaining the continuum elastic energy of the flux array.

In order to calculate the energy associated with elastic distortions of the flux array, we follow Ref. [14] and write the two-dimensional position vector of the flux lines as,

$$\mathbf{r}_i(z) = \mathbf{r}_{i\mathrm{eq}} + \mathbf{u}_i(z), \tag{3.1}$$

where $\mathbf{r}_{i\mathrm{eq}}$ are the equilibrium positions and $\mathbf{u}_i(z)$ two-dimensional displacement vectors in the plane normal to the applied field. Strictly speaking the equilibrium positions in Eq. (3.1) should be the equilibrium positions of the flux lines in a semi-infinite superconductor in the presence of a surface. In general the equilibrium solution of the semi-infinite problem will differ in a surface layer from the usual Abrikosov solution of the bulk problem [12]. On the other hand, we are interested here in evaluating the long wavelength elastic energy in a regime where $d < \lambda_{ab}$ and the magnetic fields of the vortices overlap. We will then neglect below all corrections to the elastic energy due to the discreteness and the specific structure of the flux-line lattice and replace all lattice sums by integrals. It is then consistent to also neglect the deviations of the equilibrium flux-line positions near the surface from a regular Abrikosov lattice. As a result of this continuum approximation the elastic energy obtained below contains compressional and tilt elastic constants, but no shear modulus. To evaluate the shear energy one needs to carry out a more microscopic calculation that incorporates explicitly the discreteness of the flux lattice. Since our ultimate interest is in obtaining the hydrodynamic free energy of a semi-infinite flux-line liquid, the calculation of the shear modulus is beyond the scope of this paper.

We then assume that the equilibrium positions $\mathbf{r}_{i\mathrm{eq}}$ are everywhere those of a regular triangular Abrikosov lattice and expand the energy of Eq. (2.17) for small displacements from equilibrium, retaining terms up to quadratic in the displacements. The details are sketched in Appendix A. The terms linear in $\mathbf{u}_i$ vanish and one obtains,

$$U = U_{\mathrm{eq}} + \delta U, \tag{3.2}$$

with

$$\begin{aligned}\delta U = \frac{1}{2A}\sum_{\mathbf{q}_\perp}\int_{-\infty}^{0} dz_1 \int_{-\infty}^{0} dz_2 \frac{n_0}{A}\sum_{\mathbf{Q}}\Big\{\Big[&(\mathbf{q}_\perp+\mathbf{Q})_\alpha(\mathbf{q}_\perp+\mathbf{Q})_\beta G_1(|\mathbf{q}_\perp+\mathbf{Q}|;z_1,z_2)\\ &- Q_\alpha Q_\beta G_1(Q;z_1,z_2)\Big] u_\alpha(\mathbf{q}_\perp,z_1)u_\beta(-\mathbf{q}_\perp,z_2)\\ &+ G_2(\mathbf{q}_\perp;z_1,z_2)\partial_{z_1}\mathbf{u}(\mathbf{q}_\perp,z_1)\cdot\partial_{z_2}\mathbf{u}(-\mathbf{q}_\perp,z_2)\Big\},\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}G_1(q_\perp;z_1,z_2) =& \frac{\phi_0^2}{4\pi\lambda_{ab}^2}\Big\{\frac{e^{-\alpha|z_1-z_2|}+e^{\alpha(z_1+z_2)}}{2\alpha}\Big(1-\frac{\alpha^2}{q_\perp^2}\Big)+\frac{e^{-\alpha_c|z_1-z_2|}+e^{\alpha_c(z_1+z_2)}}{2\alpha_c}\frac{\alpha_c^2}{q_\perp^2}\\ &+\frac{e^{\alpha(z_1+z_2)}}{\lambda_{ab}^2 q_\perp\alpha(q_\perp+\alpha)}\Big\},\end{aligned} \tag{3.4}$$

and

$$G_2(q_\perp;z_1,z_2) = \frac{\phi_0^2}{4\pi\lambda_{ab}^2}\frac{e^{-\alpha_c|z_1-z_2|}-e^{\alpha_c(z_1+z_2)}}{2\alpha_c}. \tag{3.5}$$

Here $\mathbf{Q}$ are the reciprocal vectors of the triangular Abrikosov lattice and A is the area of the system in the plane normal to the applied field. Also, $\mathbf{u}(\mathbf{q}_\perp,z)$ is the lattice Fourier transform of the displacement, as defined in Appendix A.

Since we are only interested in the continuum limit here, we will not attempt to perform explicitly the sum over the reciprocal lattice vectors. To proceed, we separate out in Eq. (3.3) the $\mathbf{Q}=0$ term in the sum over $\mathbf{Q}$. This term gives the collective contribution to the energy that dominates for long-wavelength elastic deformations. In the remainder of the elastic energy, containing $\sum_{\mathbf{Q}\neq 0}$, we neglect $\mathbf{q}_\perp$ compared to $\mathbf{Q}$. This is because $|\mathbf{Q}|\geq k_{BZ}$, for $\mathbf{Q}\neq 0$, where $k_{BZ}=\sqrt{2\pi n_0}$ is the size of the first Brillouin zone, and we are interested in deformations of the flux array with $q_\perp << k_{BZ}$. The elastic energy can then be rewritten in a form that explicitly identifies a compressional and a tilt energy,

$$\begin{aligned}\delta U = \frac{1}{2A}\sum_{\mathbf{q}_\perp}\int_{-\infty}^{0} dz_1\int_{-\infty}^{0} dz_2\Big\{&\mathcal{B}(q_\perp;z_1,z_2)\ \mathbf{q}_\perp\cdot\mathbf{u}(\mathbf{q}_\perp,z_1)\ \mathbf{q}_\perp\cdot\mathbf{u}(-\mathbf{q}_\perp,z_2)\\ &\mathcal{K}(q_\perp;z_1,z_2)\partial_{z_1}\mathbf{u}(\mathbf{q}_\perp,z_1)\cdot\partial_{z_2}\mathbf{u}(-\mathbf{q}_\perp,z_2)\Big\},\end{aligned} \tag{3.6}$$

where $\mathcal{B}(q_\perp; z_1, z_2)$ and $\mathcal{K}(q_\perp; z_1, z_2)$ are the compressional and tilt elastic constants per unit length, given by,

$$\begin{aligned}\mathcal{B}(q_\perp; z_1, z_2) =& \frac{B_0^2}{4\pi\lambda_{ab}^2}\Big[\frac{e^{-\alpha|z_1-z_2|}+e^{\alpha(z_1+z_2)}}{2\alpha}\Big(1-\frac{\alpha^2}{q_\perp^2}\Big) \\ &+ \frac{e^{-\alpha_c|z_1-z_2|}+e^{\alpha_c(z_1+z_2)}}{2\alpha_c}\,\frac{\alpha_c^2}{q_\perp^2} + \frac{e^{\alpha(z_1+z_2)}}{q_\perp\lambda_{ab}^2\alpha(q_\perp+\alpha)}\Big],\end{aligned} \tag{3.7}$$

and

$$\mathcal{K}(q_\perp; z_1, z_2) = \mathcal{K}_0(z_1, z_2) + \frac{B_0^2}{4\pi\lambda_{ab}^2}\;\frac{e^{-\alpha_c|z_1-z_2|}-e^{\alpha_c(z_1+z_2)}}{2\alpha_c}. \tag{3.8}$$

In this continuum approximation the $\mathbf{Q}\neq 0$ part of the sum in Eq. (3.3) vanishes when the displacements do not depend on z, i.e., the lines are rigid. Therefore it does not contribute to the compressional modulus, but only to the tilt energy. The corresponding contribution is denoted by $\mathcal{K}_0(z_1, z_2)$ and is independent of $\mathbf{q}_\perp$ because in this term we have neglected $\mathbf{q}_\perp$ compared to $Q \geq k_{BZ}$. It is given by

$$\begin{aligned}\mathcal{K}_0(z_1, z_2) =& \frac{B_0\phi_0}{4\pi\lambda_{ab}^2}\frac{1}{2A}\sum_{\mathbf{Q}\neq 0}\Big\{\frac{e^{-\alpha(Q)|z_1-z_2|}-e^{\alpha(Q)(z_1+z_2)}}{2\lambda_{ab}^2\alpha^3(Q)} \\ &+ \frac{e^{-\alpha_c(Q)|z_1-z_2|}-e^{\alpha_c(Q)(z_1+z_2)}}{2\alpha_c(Q)} \\ &+ \frac{Q}{\lambda_{ab}^2\alpha^3(Q)[Q+\alpha(Q)]}\Big[e^{\alpha(Q)(z_1+z_2)} \\ &- \Theta(z_1-z_2)e^{z_2\alpha(Q)} - \Theta(z_2-z_1)e^{z_1\alpha(Q)}\Big]\Big\}.\end{aligned} \tag{3.9}$$

In the limit where the equilibrium areal density of vortices goes to zero, $\mathcal{K}_0(z_1, z_2)$ reduces to the tilt contant of a single flux line.

The elastic constants of a semi-infinite superconductor are, as usual, nonlocal in the xy plane and contain two types of nonlocal effects in the z direction. As in a bulk sample, the elastic constants depend on the vertical distance $|z_1 - z_2|$ between any two small volumes of the elastic flux array. This nonlocality reflects directly the range of the repulsive interaction and it occurs on the scale of the penetration lengths. In addition, the elastic constants of a semi-infinite superconductor depend on the distance of each deformed flux volume from the superconductor/vacuum surface. These surface effects yield the terms that depend exponentially on the distance of the deformed flux volume from the surface - the exponentials in (z_1+z_2) or z_1 and z_2 in Eqs. (3.7) - (3.9) . They are important within a surface layer of thickness determined by the penetration lengths. In Appendix B we display the expression for the elastic energy obtained by replacing the integrals over z_1 and z_2 by wavevector sums in the corresponding Fourier space. This expression is instructive because it contains wave vector-dependent elastic constants that naturally separate into the sum of the well-known wave vector-dependent elastic constants of a bulk flux-line array and surface contributions.

In order to gain some physical insight on the surface modification of the elastic constants of the flux array, it is useful to consider the elastic energy corresponding to two specific deformations: an isotropic compression and a uniform tilt of the flux array.

Isotropic Compression

Consider a pure isotropic compression of the flux array, where $\mathbf{q}_\perp \cdot \mathbf{u} \neq 0$, but $\mathbf{u}$ is independent of z. The corresponding elastic energy δU_{comp} is immediately obtained from (3.6) as,

$$\delta U_{comp} = \frac{1}{2A} \sum_{\mathbf{q}_\perp} \int_{-\infty}^{0} dz_1 \int_{-\infty}^{0} dz_2 \ \mathcal{B}(q_\perp; z_1, z_2) \ |\mathbf{q}_\perp \cdot \mathbf{u}(\mathbf{q}_\perp)|^2. \tag{3.10}$$

The z integrations can be carried out in Eq. (3.10) , with the result,

$$\delta U_{comp} = \frac{1}{2A} \sum_{\mathbf{q}_\perp} \left[L \ B_3(q_\perp) + B_2(q_\perp) \right] \ |\mathbf{q}_\perp \cdot \mathbf{u}(\mathbf{q}_\perp)|^2, \tag{3.11}$$

where

$$B_3(q_\perp) = c_L(q_\perp, q_z = 0) = \frac{B_0^2}{4\pi} \frac{1}{1 + q_\perp^2 \lambda_{ab}^2} \tag{3.12}$$

is the compressional modulus of a bulk flux-line array, as given in Eq. (B.5) and obtained before by other authours (see, for instance, Ref. [18,15]), evaluated at $q_z = 0$, and

$$B_2(q_\perp) = \frac{B_3(q_\perp)}{q_\perp \lambda_{ab}^2 \alpha (q_\perp + \alpha)} \tag{3.13}$$

is the compressional modulus of the two-dimensional array of flux-line tips at the superconductor's surface. In the long wavelength limit, i.e., for $q_\perp \lambda_{ab} << 1$, $B_2(q_\perp)$ reduces to the two-dimensional bulk modulus of an array of monopoles, interacting via a $1/r_\perp$ potential, $B_2(q_\perp) \simeq B_3(q_\perp)/q_\perp \simeq \phi_0^2/(4\pi^2 q_\perp)$ [9]. As discussed earlier the repulsive interaction between flux lines becomes stronger and long ranged near the superconductor/vacuum interface. This yields a corresponding increase of the compressional energy of the flux array.

Uniform Tilt

We now consider the energy corresponding to a uniform tilt of the flux array, that is a deformation with $(\hat{\mathbf{z}}\times\hat{\mathbf{q}}_\perp)\cdot\partial_z\mathbf{u}(\mathbf{q}_\perp,z)=\theta(\mathbf{q}_\perp)$, but $\hat{\mathbf{q}}_\perp\cdot\mathbf{u}=0$. The corresponding tilt energy is

$$\delta U_{tilt}=\frac{1}{2A}\sum_{\mathbf{q}_\perp}\int_{-\infty}^{0}dz_1\int_{-\infty}^{0}dz_2\ \mathcal{K}(q_\perp;z_1,z_2)|\theta(\mathbf{q}_\perp)|^2, \tag{3.14}$$

or, carrying out the integrations over the z variables,

$$\delta U_{tilt}=\frac{1}{2A}\sum_{\mathbf{q}_\perp}\left[L\ K_3(q_\perp)-K_2(q_\perp)\right]|\theta(\mathbf{q}_\perp)|^2, \tag{3.15}$$

where $K_3(q_\perp)=c_{44}(q_\perp,q_z=0)$ is the tilt coefficient of a bulk flux array, given in Eq. (B.6) and obtained before [14,15,18], evaluated at $q_z=0$,

$$K_3(q_\perp)=\frac{B_0^2}{4\pi}\frac{1}{1+q_\perp^2\lambda_{ab}^2\gamma^2}+\tilde{c}_{44}(q_z=0), \tag{3.16}$$

with

$$\tilde{c}_{44}(q_z=0)=n_0\Big(\frac{\phi_0}{4\pi\lambda_{ab}}\Big)^2\left[\frac{1}{\gamma^2}\ln(\gamma\kappa)-\frac{1}{2\gamma^2}\ln\left(1+\gamma^2\lambda_{ab}^2k_{BZ}^2\right)+\frac{1}{2(1+\lambda_{ab}^2k_{BZ}^2)}\right], \tag{3.17}$$

and $K_2(q_\perp)$ is the surface contribution to the tilt constant,

$$\begin{aligned}K_2(q_\perp)=&\frac{B_0^2\lambda_{ab}}{4\pi}\frac{1}{(1+q_\perp^2\lambda_{ab}^2\gamma^2)^{3/2}}\\&+n_0\lambda_{ab}\Big(\frac{\phi_0}{4\pi\lambda_{ab}}\Big)^2\Big[\frac{1}{\gamma^2\sqrt{1+\lambda_{ab}^2k_{BZ}^2}}+\frac{2}{3}\frac{1}{(1+\lambda_{ab}^2k_{BZ}^2)^{3/2}}\\&-\frac{1}{2(1+\lambda_{ab}^2k_{BZ}^2)}\ \frac{1}{\lambda_{ab}k_{BZ}+\sqrt{1+\lambda_{ab}^2k_{BZ}^2}}\\&+\frac{1}{2\sqrt{1+\lambda_{ab}^2k_{BZ}^2}}-\tan^{-1}\Big(\frac{1}{\lambda_{ab}k_{BZ}+\sqrt{1+\lambda_{ab}^2k_{BZ}^2}}\Big)\Big].\end{aligned} \tag{3.18}$$

The surface contribution to the tilt energy is *always negative.* This can be understood physically because the magnetic field lines associated with each vortex spread out as the interface is approached. The magnetic field fluctuations associated with each line are appreciable in an area that becomes very large near the interface. Consequently the energy cost for tilting the lines or inducing transverse line fluctuations decreases.

4. Flux Liquid Free Energy

In the range of applied fields of interest here, $H_{c1} << H << H_{c2}$, the properties of a flux-line liquid on length scales large compared to the intervortex spacing can be described in terms of two hydrodynamic fields, a microscopic areal density of vortices,

$$n_{\rm mic}(\mathbf{r}_\perp, z) = \sum_i \delta^{(2)}(\mathbf{r}_\perp - \mathbf{r}_i(z)), \tag{4.1}$$

and a microscopic "tangent" field in the plane perpendicular to the applied field,

$$\mathbf{t}_{\rm mic}(\mathbf{r}_\perp, z) = \sum_i \mathbf{t}_i(z)\ \delta^{(2)}(\mathbf{r}_\perp - \mathbf{r}_i(z)). \tag{4.2}$$

From these microscopic densities one constructs coarse-grained density fields $n(\mathbf{r}_\perp, z)$ and $\mathbf{t}(\mathbf{r}_\perp, z)$ by averaging (4.1) and (4.2) over a hydrodynamic volume centered at $\mathbf{r}_\perp$. As in the Landau's theory of phase transitions the long wavelength properties of the flux-line assembly can be described in terms of a coarse-grained free energy retaining only terms quadratic in the deviations $\delta n(\mathbf{r}_\perp, z) = n(\mathbf{r}_\perp, z) - n_0$ and $\mathbf{t}(\mathbf{r}_\perp, z)$ of the hydrodynamic fields from their equilibrium values. Because we are dealing with magnetic flux lines that cannot start or stop inside the medium, the density and tangent fields are not independent, but satisfy a "continuity" equation in the time-like variable z,

$$\partial_z n + \boldsymbol{\nabla}_\perp \cdot \mathbf{t} = 0. \tag{4.3}$$

This condition reflects the requirement of no magnetic monopoles. It can be implemented by introducing a vector potential or two-component "displacement field", $\mathbf{u}(\mathbf{r}_\perp, z)$, with,

$$\delta n = -n_0 \boldsymbol{\nabla}_\perp \cdot \mathbf{u}, \tag{4.4}$$

and

$$\mathbf{t} = n_0 \frac{\partial \mathbf{u}}{\partial z}. \tag{4.5}$$

It was shown earlier [19] that when the density and tangent fields are expressed in terms of the displacement vector $\mathbf{u}$, the hydrodynamic free energy of a flux-line liquid differs from the continuum elastic free energy of the Abrikosov flux-line lattice only for the absence in the former of a shear modulus. In other words it was found that in a bulk sample the hydrodynamic flux-line liquid free energy, when expressed in terms of the vector potential $\mathbf{u}$, is identical to the continuum elastic energy of a flux-line lattice, provided all effects due to the discreteness of the lattice - in particular the shear modulus - are neglected in the latter. Similarly, one can show that the compressional and tilt moduli of a semi-infinite flux-line liquid are identical to those obtained in Section 3. The hydrodynamic free energy of a flux-line liquid is then given by

$$F_L = \int \frac{d\mathbf{q}_\perp}{(2\pi)^2} \int_{-\infty}^{0} dz_1 \int_{-\infty}^{0} dz_2 \Big\{ \mathcal{B}(q_\perp; z_1, z_2)\ \delta n_v(\mathbf{q}_\perp, z_1)\ \delta n_v(-\mathbf{q}_\perp, z_2) \\ \mathcal{K}(q_\perp; z_1, z_2) \mathbf{t}(\mathbf{q}_\perp, z_1) \cdot \mathbf{t}(-\mathbf{q}_\perp, z_2) \Big\}, \tag{4.6}$$

where $\mathcal{B}(q_\perp; z_1, z_2)$ and $\mathcal{K}(q_\perp; z_1, z_2)$ are the compressional and tilt elastic constants per unit length, given in Eqs. (3.7) and (3.8), respectively. The statistical averages over (4.6) need to be evaluated with the constraint (4.3). If the constraint is incorporated in the free energy by expressing density and tangent fields in terms of the vector potential $\mathbf{u}$ according to Eqs. (4.4) and (4.5), the flux-liquid free energy and the elastic energy of the flux-line lattice only differ for the absence in the former of the shear modulus.

The flux-line liquid free energy given in Eq. (4.6) can also be obtained directly from the flux-line energy of Eq. (2.16) by the coarse-graining procedure described for instance in Ref. [15], without making any references to an equilibrium lattice of flux lines. Care has to be taken in dealing with the self-energy term, corresponding to the $i = j$ part of the pair interaction.

In a forthcoming paper we will propose an approximate form of the hydrodynamic free energy (4.6) obtained by assuming that the most importance source of spatial inhomogenieties in the z direction is the presence of the surface itself and by neglecting all other nonlocalities in the z direction. This approximate hydrodynamic free energy will be used there to analyze the interplay of bulk and surface forces in determining the structure of the flux-line tips as they emerge from the superconductor sample.

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Appendix A. Derivation of the Elastic Energy

When the energy given in Eq. (2.16) is expanded for small displacements of the flux lines from their equilibrium positions, the first nonvanishing correction to the equilibrium energy is quadratic in the displacements and it is given by,

$$\begin{aligned}\delta U = \frac{1}{2}\sum_{i,j}\int_{-\infty}^{0} dz_1 \int_{-\infty}^{0} dz_2 \frac{1}{A}\sum_{\mathbf{q}_\perp} e^{-i\mathbf{q}_\perp\cdot(\mathbf{r}_{i\mathrm{eq}}-\mathbf{r}_{j\mathrm{eq}})} \\ \times \Big\{ G_1(q_\perp; z_1, z_2) q_{\perp\alpha} q_{\perp\beta} \big[u_{i\alpha}(z_1)u_{j\beta}(z_2) - u_{i\alpha}(z_1)u_{i\beta}(z_1)\big] \\ + G_2(q_\perp; z_1, z_2)\partial_{z_1}\mathbf{u}_i(z_1)\cdot\partial_{z_2}\mathbf{u}_j(z_2)\Big\},\end{aligned} \tag{A.1}$$

where the elastic kernels $G_1(q_\perp; z_1, z_2)$ and $G_2(q_\perp; z_1, z_2)$ Hve been given in Eqs. (3.4) and (3.5) . It is convenient to expand the lattice displacements $\mathbf{u}_i$ in Fourier series according to,

$$\mathbf{u}_i(z) = \frac{1}{a_0}\sum_{\mathbf{k}\in BZ} \mathbf{u}(\mathbf{k}, z) e^{i\mathbf{k}\cdot\mathbf{r}_{i\mathrm{eq}}}, \tag{A.2}$$

where $a_0 = 1/n_0$ is the area of the primitive unit cell and the sum is restricted to the wavevectors of the first Brillouin zone, as indicated. Conversely, the Fourier coefficients $\mathbf{u}(\mathbf{k}, z)$ are given by

$$\mathbf{u}(\mathbf{k}, z) = \frac{1}{n_0}\sum_i \mathbf{u}_i(z) e^{-i\mathbf{k}\cdot\mathbf{r}_{i\mathrm{eq}}}. \tag{A.3}$$

The normalization of the Fourier expansion (A.2) has been chosen in such a way that the lattice Fourier amplitudes $\mathbf{u}(\mathbf{k}, z)$ as defined in Eq. (A.3) for $\mathbf{k}\in BZ$ are also the two-dimensional continuum Fourier transform of a coarse-grained density field,

$$\mathbf{u}(\mathbf{r}_\perp, z) = \frac{1}{n_0}\sum_i \mathbf{u}_i(z)\delta^{(2)}(\mathbf{r}_\perp - \mathbf{r}_{i\mathrm{eq}}), \tag{A.4}$$

according to,

$$\mathbf{u}(\mathbf{k}, z) = \int d\mathbf{r}_\perp e^{-i\mathbf{k}\cdot\mathbf{r}_\perp}\mathbf{u}(\mathbf{r}_\perp, z). \tag{A.5}$$

The δ-function in Eq. (A.4) is really a smeared-out two-dmensional δ-function with a finite spatial extent $\approx k_{BZ}$. After inserting the Fourier expansion (A.2) in Eq. (A.1) , the lattice sums can be carried out, using,

$$\sum_j e^{i\mathbf{q}_\perp\cdot\mathbf{r}_{i\mathrm{eq}}} = \frac{1}{N}\sum_{\mathbf{Q}} \delta_{\mathbf{q}_\perp,\mathbf{Q}}, \tag{A.6}$$

where $\mathbf{Q}$ are the reciprocal lattice vectors of the Abrikosov lattice, and

$$\sum_{\mathbf{k}\in BZ} e^{-i\mathbf{k}\cdot(\mathbf{r}_{i\mathrm{eq}}-\mathbf{r}_{j\mathrm{eq}})} = \delta_{ij}. \tag{A.7}$$

Finally, making use of the periodicity of the displacements in reciprocal space, $\mathbf{u}(\mathbf{k}+\mathbf{Q}) = \mathbf{u}(\mathbf{k})$, one obtains Eq. (3.3) .

Appendix B. Wavevector-Dependent Elastic Constants

It is useful to rewrite the elastic energy of Eq. (3.6) by taking the Fourier transform of the displacement vectors with respect to z,

$$\mathbf{u}(\mathbf{q}_\perp, q_z) = \int_{-\infty}^{0} dz\ e^{-iq_z z}\mathbf{u}(\mathbf{q}_\perp, z), \tag{B.1}$$

for $Im(q_z) > 0$. The elastic energy can then be written as

$$\begin{aligned}\delta U = \frac{1}{2A}\sum_{\mathbf{q}_\perp}\int_{-\infty}^{\infty}\frac{dq_z}{2\pi}\int_{-\infty}^{\infty}\frac{dq'_z}{2\pi}\Big\{&\mathcal{B}(q_\perp; q_z, q'_z)\ \mathbf{q}_\perp\cdot\mathbf{u}(\mathbf{q}_\perp, q_z)\ \mathbf{q}_\perp\cdot\mathbf{u}(-\mathbf{q}_\perp, q'_z)\\ &\mathcal{K}(q_\perp; q_z, q'_z)q_z q'_z\mathbf{u}(\mathbf{q}_\perp, q_z)\cdot\mathbf{u}(-\mathbf{q}_\perp, q'_z)\Big\}.\end{aligned} \tag{B.2}$$

The wave vector-dependent elastic constants are given by,

$$\begin{aligned}\mathcal{B}(q_\perp; q_z, q'_z) =& 2\pi\delta(q_z+q'_z)\ c_L(q_\perp, q_z) + \frac{B_0^2}{4\pi\lambda_{ab}^2}\Big\{\Big(1-\frac{\alpha^2}{q_\perp^2}\Big)\frac{1}{\alpha(\alpha-iq_z)}\frac{iq'_z}{\alpha^2+q_z'^2}\\ &+\frac{\alpha_c}{q_\perp^2(\alpha_c-iq_z)}\frac{iq'_z}{\alpha^2+q_z'^2}+\frac{1}{q_\perp\lambda_{ab}^2\alpha(\alpha+q_\perp)}\ \frac{1}{(\alpha-iq_z)(\alpha-iq'_z)}\Big\},\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}\mathcal{K}(q_\perp; q_z, q'_z) =& 2\pi\delta(q_z+q'_z)\ c_{44}(q_\perp, q_z) - \frac{B_0^2}{4\pi\lambda_{ab}^2}\frac{1}{\alpha_c-iq_z}\frac{1}{\alpha_c^2+q_z'^2}\\ &-\frac{B_0\phi_0}{4\pi\lambda_{ab}^2}\frac{1}{2A}\sum_{\mathbf{Q}\neq 0}\Big\{\frac{1}{\lambda_{ab}^2\alpha^2(Q)(\alpha(Q)-iq_z)}\frac{1}{\alpha^2(Q)+q_z'^2}\\ &+\frac{1}{\alpha_c(Q)_c-iq_z}\frac{1}{\alpha_c^2(Q)+q_z'^2}\\ &+\frac{Q}{\lambda_{ab}^2\alpha^3(Q)[Q+\alpha(Q)]}\Big[\frac{1}{\alpha(Q)-i(q_z+q'_z)}\Big(\frac{1}{\alpha(Q)-iq_z}\\ &+\frac{1}{\alpha(Q)-iq'_z}\Big)-\frac{1}{(\alpha(Q)-iq_z)(\alpha(Q)-iq'_z)}\Big]\Big\}.\end{aligned} \tag{B.4}$$

In this form the elastic constants are explicitly given by the sum of bulk and surface contributions. The bulk contributions are the usual ones, given by,

$$c_L(q_\perp, q_z) = \frac{B_0^2}{4\pi} \frac{1+\gamma^2\lambda_{ab}^2 q^2}{(1+\lambda_{ab}^2 q^2)(1+\lambda_{ab}^2 q_z^2+\gamma^2\lambda_{ab}^2 q_\perp^2)}, \tag{B.5}$$

with $q^2 = q_\perp^2 + q_z^2$, and

$$c_{44}(q_\perp, q_z) = \frac{B_0^2}{4\pi} \frac{1}{1+\lambda_{ab}^2 q_z^2+\gamma^2\lambda_{ab}^2 q_\perp^2} + \tilde{c}_{44}(q_z). \tag{B.6}$$

In Eq. (B.6), $\tilde{c}_{44}(q_z)$ is the contribution to the tilt coefficient arising from the large wavevector ($\mathbf{Q} \neq 0$) part of the lattice sum. The sum over the reciprocal lattice vectors is evaluated in the continuum limit by replacing it by an integral with appropriate cutoffs, according to,

$$\frac{1}{A}\sum_{\mathbf{Q}\neq 0} \to \int_{k_{BZ}\leq Q\leq 1/\xi_{ab}} \frac{d\mathbf{Q}}{(2\pi)^2}. \tag{B.7}$$

For $\kappa = \lambda_{ab}/\xi_{ab} >> 1$ one obtains,

$$\begin{aligned}\tilde{c}_{44}(q_z) =& n_0\Big(\frac{\phi_0}{4\pi\lambda_{ab}}\Big)^2\Big\{\frac{1}{\gamma^2}\ln(\gamma\kappa) - \frac{1}{2\gamma^2}\ln\big(1+\gamma^2\lambda_{ab}^2 k_{BZ}^2+\lambda_{ab}^2 q_z^2\big)\\ &+\frac{1}{2\lambda_{ab}^2 q_z^2}\Big[\ln\big(1+\lambda_{ab}^2 k_{BZ}^2+\lambda_{ab}^2 q_z^2\big) - \ln\big(1+\lambda_{ab}^2 k_{BZ}^2\big)\Big]\Big\}.\end{aligned} \tag{B.8}$$

Aside from numerical differences due to the details of the short-length scale cutoff, the expression for $\tilde{c}_{44}$ obtained here agrees with that given by D.S. Fisher [20] and, as discussed there, corrects an earlier result of Brandt and Sudbo [14]. Finally, in the limit where the density n_0 of flux lines vanishes - and therefore $k_{BZ} \to 0$ in Eq. (B.8) -, $\tilde{c}_{44}(q_z)/n_0$ reduces to the single-line tilt coefficient, $\tilde{\epsilon}_1(q_z)$, given by

$$\begin{aligned}\tilde{\epsilon}_1(q_z) &= \lim_{n_0\to 0}\tilde{c}_{44}(q_z)/n_0\\ &= \Big(\frac{\phi_0}{4\pi\lambda_{ab}}\Big)^2\Big[\frac{1}{\gamma^2}\ln(\gamma\kappa) - \frac{1}{2\gamma^2}\ln(1+\lambda_{ab}^2 q_z^2) + \frac{1}{2\lambda_{ab}^2 q_z^2}\ln(1+\lambda_{ab}^2 q_z^2)\Big].\end{aligned} \tag{B.9}$$

For $q_z = 0$, one obtains

$$\tilde{\epsilon}_1(0) = \Big(\frac{\phi_0}{4\pi\lambda_{ab}}\Big)^2\Big[\frac{1}{\gamma^2}\ln(\gamma\kappa) + \frac{1}{2}\Big]. \tag{B.10}$$

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