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Higher Derivative Gravitational Systems and Ghost Fields

Michele Fontanini
Syracuse University

Abstract

Effective Field Theory (EFT) is one of the most powerful theoretical tools in the hands of cosmologists, it allows them to come up with testable effective descriptions of the universe even when a fundamental theory is missing. EFT though, is not the only possible answer for pushing our knowledge beyond the limits of what has already been established. Applying EFT and other alternative methods has become an important part of a cosmologist's work, particularly in the last few years when a vast plethora of extensions of the *Standard Model of Cosmology* has been proposed and needs to be tested against experimental results. In this work we mainly investigate the limits of EFT in the context of cosmic acceleration, and the possibility of calculating corrections to the low energy standard cosmological results by re-interpreting the meaning of higher derivative terms in perturbation expansions.

Higher Derivative Gravitational Systems and Ghost Fields

by

Michele Fontanini

M.S. Università degli Studi di Trieste, 2005

B.S. Università degli Studi di Trieste, 2001

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Preface

Cosmology is currently undergoing an extremely exciting phase due to the increase in available data. In the words of J.A. Peacock [1] cosmology can be described as "... a subject that has the modest aim of understanding the entire universe and all its contents"; the questions that cosmology poses and tries to find an answer to are many and different in nature. It is thus natural that, having to explain the evolution of the cosmos or universe as a whole, cosmology has to overlap to many other branches of physics. The set of scales under examination in cosmology is so huge that connections to astrophysics, hydrodynamics, nuclear and particle physics are unavoidable and essential for a proper understanding of the subject. In this sense cosmology connects the largest and tiniest scales, with literally the whole universe in between.

Compared to roughly one hundred years ago, modern cosmology has two advantages. The first is that the amount of collected data has seen an unprecedented increase in the past years, reaching now a point (already familiar to other fields in physics such as particle physics) where modern advanced computers are essential to reduce the data and extract the physical information sought after. The second advantage comes from the huge body of theoretical work done in many branches of physics that now allows theorists to come up with sensible and quite elaborate cosmological models. In particular the progress made in field theory over the past half century is the main theoretical tool for modeling and understanding what we see in the sky.

The introduction of Einstein's theory of General Relativity almost a century ago, provided physicists with the first complete classical theoretical tool since Newton's works to describe the gravitational behavior of objects in the universe, and therefore to model the cosmos as a whole. Consequently, many possible cosmological models emerged, and evolved or disappeared once compared to increasingly precise experimental results. Eventually, the physics community synthesized what is now known as the ΛCDM model, the simplest model available today in general agreement with present data that attempts to explain the existence and structure of the Cosmic Microwave Background (CMB), the large scale structure of galaxy clusters, the distribution and abundance of light elements (hydrogen, helium, lithium and oxygen), and the currently ongoing accelerated expansion observed by studying the light from distant supernovae.

From an experimental point of view, cosmology poses incredibly hard problems to solve. In other branches of physics experiments can be prepared and performed in more or less controlled environments, in laboratories. In cosmology the only available "lab" is the observable universe – the only part of it we have access to –, and we have very little control (if any) on the phenomena occurring in it. In addition to all this, we experience the obvious difficulty of being stuck in one place and having direct access to a minute part of the system under examination. Nonetheless, as previously mentioned, observations made both from earth and from space have been able to unveil mysteries we could not have imagined just a century ago. Observations like the CMB – both its incredible uniformity at the 10^{-5} level, and the statistical properties of its tiny fluctuations – gave rise to a wide set of fundamental questions concerning the origin of the universe and the connection of particle physics in the very first instants of the life of the universe to what we see now after roughly 14 billion years.

Among the many interesting and intriguing questions that can be asked in the framework of cosmology, some used to be part of philosophical speculation rather than physical analysis, such as how the universe came to be, and what its ultimate fate is. Others are more closely related to the above mentioned observations collected over the years. We now know that most of the visible component of our universe is made of matter (as opposed to a matter-antimatter mixture), that this visible component makes up for just a fraction of the total amount of energy in the universe, while the main components (dubbed dark matter and dark energy) sum up to roughly 25% and 70% of the whole, respectively.

Even though we have only explored a tiny part of our galaxy, barely leaving our solar system with satellites, we can extract information about large scales in the universe by using accurate observations performed both from earth and from space. Our telescopes span almost all of the electromagnetic spectrum, from radio waves to hard gamma rays, and soon gravitational waves as well. The data we have collected so far suggests that we live in a highly spatially flat homogeneous and isotropic universe (at large scales).

By looking at distant objects, at the behavior of large scale structure such as galaxy clusters and the CMB, we can infer that the evolution of the universe has not always been the same. We can infer that a period of exponential expansion occurred somehow close to the "origin of time", what we call the Hot Big Bang or initial singularity. The latter is a state in which our semiclassical models cannot apply owing to the extremely high energy densities, thus requiring a full quantum gravitational theory for a proper description of the physics involved. After the Big Bang an inflationary accelerated era smoothed out the universe while seeding the origin of structure by stretching quantum fluctuations in the gravitational field to cosmological scales. Later radiation, the leading source of energy after inflation, dominated the behavior of the expansion until it diluted away leaving the universe evolving to a matter dominated epoch with its characteristic expansion rate. Eventually the universe entered the ongoing vacuum energy dominated era, a new phase of accelerated expansion.

Although huge progress have been made in providing an answer to many cosmological questions, many issues are still unsolved. For instance the origin of the current accelerated expansion, the puzzling value of the vacuum energy that seems to have a gravitational strength 120 orders of magnitude smaller than expected, the details connected to the inflationary phase (in particular its end and the subsequent transition to a radiation epoch which is crucial in determining how the structure we see today formed afterwards), and many others.

As one can imagine, going into the details of any of the problems we have mentioned would require a substantial amount of work and time. This work will therefore focus on a very restricted volume in the space of open questions we have seen above (and all those we have omitted), as a contribution to the collective advance in the field of cosmology. In particular we have focused on techniques and methods that can be applied to general problems like the validity of our descriptions of the very early universe during inflation, as well as late time cosmology and the plausible necessity of infrared modification of GR. Along with these technical aspects, we will also present some of the physical consequences of their application to real models both from a theoretical and possibly from an experimental perspective.

In the next chapters we will deal with the validity of perturbation theory, pushing the UV boundaries to find how far we can trust our predictions, and how some of the widely accepted assumptions – such as the choice of an adiabatic vacuum for perturbations – can be partially justified and made more robust. This part is based on published work in collaboration with Cristian Armendariz-Picon, Riccardo Penco, and Mark Trodden [2], where each author equally contributed to the study. We will then approach the fundamental problem of higher derivative descriptions of physical systems that is brought forth by the interest in searching for a quantizable extension of GR. We studied in some detail this issue suggesting the idea and making a major contribution in developing it in collaboration with Mark Trodden. The results we obtained have been published in [3]. We will also investigate the possible observational signature of a completely alternative approach to the problem of late time acceleration. We will in fact present some of the properties of models that abandon the solid grounds given by the (well motivated) assumptions of homogeneity and isotropy. This last chapter is related to a publication proposed and developed in cooperation with Eric J. West and Mark Trodden [4] where, together with the second author, we performed the bulk of the calculations and coding.

Chapter 1

Introduction

Gravity is described by the classical theory of General Relativity (GR) introduced by Einstein almost a century ago [5]. Even though GR is a classical theory and its quantum completion has yet to be found, it provides a powerful tool to study a variety of physical phenomena ranging from black holes to stars, galaxies, clusters of galaxies, up to the acceleration of the universe. In this introduction we will briefly summarize some of the features of GR from a field theory point of view, and we will present some of the possible extensions focusing on the effective field theory approach. we will conclude this chapter with a review of the canonical treatment for higher derivatives systems which will be necessary to go deeper in the discussion in chapter 3.

1.1 A Very Brief Review of General Relativity

The theory of General Relativity is a classical theory introduced to reconcile the results of special relativity with Newtonian gravity starting from the simple and powerful concepts of general coordinate invariance and the equivalence principle. Probably the biggest leap of insight that Einstein took in proposing his GR theory was the idea of interpreting the gravitational force that attracts any two bodies in the universe as a geometrical property of spacetime, connecting then the dynamics of gravitating bodies to the properties of a curved manifold.

In GR the physical field that describes the gravitational interactions is the metric $g_{\mu\nu}$, a rank two symmetric tensor that satisfies Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{1.1}$$

where G is Newton's constant, Λ the cosmological constant, $T_{\mu\nu}$ the energy-momentum tensor, and $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ the Einstein tensor. The curvature tensors encode the geometric properties of the spacetime described by $g_{\mu\nu}$, and are defined starting from the Riemann tensor via contractions with the metric

$$R^{\alpha}{}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\nu\mu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\beta\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta} \ , \tag{1.2}$$

where repeated indices are summed over. Consequently, the Ricci tensor and scalar are respectively given by

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \ , \tag{1.3}$$

and

$$R = g^{\mu\nu} R_{\mu\nu} \ . \tag{1.4}$$

Everywhere here and in the rest of this work we will use Greek letters to indicate spacetime indices that run from 0 to 3, while we will use Latin indices running over 1, 2, 3 for spatial coordinates.

The equations presented above can be considered the starting point for GR, but here it is more convenient for us to derive them from an action principle by requiring that the action for a gravitational system is extremal on the physical configuration $\bar{g}_{\mu\nu}$

$$\delta S|_{\bar{g}_{\mu\nu}} = 0 \ . \tag{1.5}$$

The general action S is composed explicitly by

$$\begin{aligned} S &= S_{GR} + S_{GHY} + S_m \\ &= \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - 2\Lambda) + \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K + \int_{\mathcal{M}} d^4x\sqrt{-g}\mathcal{L}_{matter} \ , \end{aligned} \tag{1.6}$$

where $\kappa = 8\pi G$, $\mathcal{M}$ is the manifold on which the metric $g_{\mu\nu}$ is defined, $g = \det(g_{\mu\nu})$ the determinant of the metric, Λ a (possibly zero) cosmological constant, $\partial\mathcal{M}$ the boundary of the manifold, h the determinant of the induced metric on the boundary, and K its extrinsic curvature. The last integral containing $\mathcal{L}_{matter}$ includes all the matter fields which are assumed to be minimally coupled to the metric. The energy-momentum tensor is then defined as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{g}\mathcal{L}_{matter})}{\delta g^{\mu\nu}} \ . \tag{1.7}$$

The presence of the Gibbons-Hawking-York term [6, 7], the second integral in (1.6), is necessary to have a well defined variational principle when the manifold considered is not closed. In fact, variating the above action (1.6) second derivatives acting on $\delta g^{\mu\nu}$ appear, and after integration by parts they lead to boundary terms that, unless a new variational principle is defined, do not vanish. To obtain Einstein equations (1.1), is then necessary to add the boundary "counterterm" S_{GHY}. In what follows we will often neglect this surface term assuming that the relevant fields and their derivatives vanish on the boundary (when it is appropriate), or simply implicitly assuming that the correct boundary term has to be added to the action to obtain a well defined variational principle.

To try connecting to both the language and the contents that we will present later in this work, it is interesting to consider GR from a field theoretical point of view. In fact, thanks to the developments of field theory during the 1940's and 50's, GR has been shown to be the only non-trivially interacting massless helicity 2 theory. Following the logic of [8, 9], we recall that from the field theory point of view degrees of freedom are carried by fields, and the excitations of such fields are particles that in flat four dimensional spacetime are classified by their spin. In particular, since fermions cannot build up classical coherent states, long distance interactions have to be described by bosonic degrees of freedom, and therefore are described by fields of integer spin. Since a bosonic field φ satisfies the Klein-Gordon equation $(\Box - m^2)\varphi = 0$ with solutions that decay with the distance r from a localized source as $\sim \frac{1}{r}e^{-mr}$, the role of mediator of long range forces has to be played by massless fields to avoid the exponential suppression.

Massless particles, which do not have a proper "rest frame", are characterized by their helicity h (the projection of spin along the direction of motion, which is the relevant Casimir invariant) rather than spin. In four dimensions there are four possible cases, labeled by $h = 0, 1, 2$ and $h \geq 3$. Helicity 0 is described by a scalar field, and many possible interactions that preserve Lorentz symmetry can be written, leading to a plethora of possible non-trivial interacting theories. Helicity 1 leads to Maxwell's action for a vector field, and requiring consistent self interaction for such a vector leads to the non-abelian gauge theories, two of which describe the strong and weak interactions. Finally, helicity 2 implies essentially GR when consistent self-interactions are required [8, 10–16]. For higher helicities there is no self interaction that can be written [17], and so the story ends.

The connection to a spin-2 field is more evident once the action for gravity is expanded in terms of perturbations around a background solution (usually taken to be the flat background, Minkowski space). Using standard notation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \tag{1.8}$$

the quadratic action $S^{(2)}$ for the perturbations $h_{\mu\nu}$ in vacuum can then be found by expanding in a series in perturbations

$$\begin{aligned} S_{GR} &= S^{(0)} + S^{(1)} + S^{(2)} + \ldots \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-\tilde{g}}(\tilde{R} - 2\tilde{\Lambda}) + \frac{1}{4\kappa}\int \Big(d^4x - \frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \partial_\mu h_{\nu\alpha}\partial^\nu h^{\mu\alpha} \\ &\quad -\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\alpha h \partial^\alpha h \Big) + \ldots \ , \end{aligned} \tag{1.9}$$

where, in the first integral, tildes remind us that quantities depend only on the background metric ($\eta_{\mu\nu}$ in this case), $S^{(1)} = 0$ being proportional to the field equations for the background, and indices in the last integral are raised and lowered using the background metric.

The field theory description of gravity is particularly useful in modern cosmology, and is of course an essential step toward a quantum version of it. Very often a semi-classical point of view is taken, and quantum fields are studied in a classical GR background (either flat or curved). This is the approach taken in the next chapters where we consider small corrections or perturbations propagating on a classical background.

1.1.1 Cosmological Solutions

Modern Cosmology is built on the idea that the universe is spatially isotropic and homogeneous at large scales. From a theoretical standpoint this corresponds to the Copernican principle that states that the universe is pretty much the same everywhere and there is no special point. Observationally, physicists in the last few decades have accumulated an enormous amount of evidence that points to a high degree of isotropy at large scales, from the incredibly smoothness of the CMB [18], to the uniform distribution of galaxies observed in galaxy surveys such as [19]. Therefore, unless one is ready to believe that our position is special in the universe, isotropy with respect to us can be extended to isotropy with respect to any point, implying homogeneity. It has to be said that there are alternative cosmological models [20–22] that relax this last assumption, and allow for inhomogeneous universes such as the Lemaître-Tolman-Bondi model. We will say more about this in chapter 4, as well as relaxing to some grade the isotropy assumption.

Of course isotropy and homogeneity are not realized at all scales, one has just to think about the obvious example of a star and the almost empty space around it. On the other hand, when the global evolution and global characteristics of the universe (cosmology) are under examination, the local details do not play an important role. There have been works in which the possibility that for example the late time acceleration be a product of inhomogeneities was studied [23–29]. Even though a final agreement has not been reached, we believe that evidence points toward the validity of the assumptions that small scales inhomogeneities cannot drive the evolution at large scales [23, 24].

Spatial homogeneity and isotropy imply that the universe can be foliated into spacelike slices [30], such that each three-dimensional slice is maximally symmetric. In turn, this allows us to write the metric as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + a^2(t) \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) , \end{aligned} \tag{1.10}$$

where $a(t)$ is the scale factor, t the cosmological or physical time, r and Ω the radial and angular coordinates respectively, and where k is a negative, zero or positive constant that selects the curvature of the spatial slices (respectively often called open, flat and closed cases). The form of the metric in (1.10) is often referred to as Friedmann-Lemaître-Robertson-Walker (FLRW).

Once the above ansatz is made, Einstein equations and their trace reduce to the Friedmann equations. Assuming that the matter content can be modeled by a perfect fluid, in comoving coordinates the energy-momentum tensor can then be written as

$$T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p) , \tag{1.11}$$

where ρ is the energy density and p the pressure of the fluid. At this point, it is always possible to define a quantity

$$w = \frac{p}{\rho} , \tag{1.12}$$

that, when constant, defines the equation of state for the perfect fluid. Most of cosmologically interesting cases fall into this category, in fact a fluid with $w = 0$ is a pressureless fluid, also called dust, and a very good approximation of the low density non-relativistic matter in the universe. The case $w = \frac{1}{3}$ describes elecromagnetic radiation and relativistic particles, while $w = -1$ corresponds to vacuum energy or, in other words, a contribution like Λ in (1.1). In fact, moving it to the right hand side of the equation, and therefore considering it part of $T_{\mu\nu}$, Λ behaves as a negative pressure perfect fluid.

Using the FLRW metric (1.10) and the above assumption for the energy-momentum tensor, the $\mu\nu = 00$ component of Einstein equations becomes

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) \ , \tag{1.13}$$

where an overhead dot represents time derivative. Similarly the $\mu\nu = ij$ component reads

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) \ . \tag{1.14}$$

Eliminating the second time derivative from the ij equation using the 00 one, it is possible to rewrite the two Friedmann equations in the canonical form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \ , \tag{1.15}$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}(\rho + 3p) \ . \tag{1.16}$$

Solutions to the above give the scale factor $a(t)$ as a function of time, and fully specify the evolution of the universe. It is interesting to note that different kind of fluids will evolve differently. In particular it is possible to use the equation for the conservation of energy to solve for the energy density of different fluids as a function of the scale factor. Conservation of energy can be written as

$$\nabla_\mu T^\mu{}_\nu = 0 \ , \tag{1.17}$$

which implies

$$\frac{\dot{\rho}}{\rho} = -3(1 + w)\frac{\dot{a}}{a} \ . \tag{1.18}$$

With w constant, the latter can be integrated to get

$$\rho \propto a^{-3(1+w)} \ , \tag{1.19}$$

from which it is easy to see that in an expanding universe ($\dot{a} > 0$) the energy density for the different kind of fluids evolve as

$$\rho_{matter} \propto a^{-3} , \tag{1.20}$$

$$\rho_{radiation} \propto a^{-4} , \tag{1.21}$$

$$\rho_{\Lambda} \propto a^{0} . \tag{1.22}$$

This implies that in an expanding universe even a tiny amount of vacuum energy will eventually dominate since it is the only one that is not diluted by the expansion.

Observations [31] suggest we live in a spatially flat universe, which means that in the equations seen above such as the Friedmann equation (1.15) k should be set to zero. One can still keep k as a free parameter when comparing with experimental results, and interpret it for example as a fictitious fluid with energy density $\rho_{curvature} = -\frac{3k}{8\pi G a^2}$ (compare to (1.15)) which again dilutes with the expansion of the universe.

1.1.2 Acceleration and Dark Components

Our understanding of the data collected in the last few decades in modern Cosmology tells us that what we see in the sky is only a minute part of the matter and energy budget of our universe; this is in fact mostly made by Dark Energy (DE) and Dark matter (DM) in rough proportions of 71% and 25% of the total respectively [18]. As the names suggest we know very little about these dark components. Together with the mechanisms to reproduce the observed striking uniformity of the Cosmic Microwave Background such as inflation, DE and DM pose the biggest challenges for cosmologists.

Dark Energy is the elusive energy component responsible for the accelerating expansion of the universe at late times, an important observational discovery found studying Type IA Supernovae [32–34]. DE was in fact introduced relatively recently to explain the observed Hubble diagram for these exploding stars that allow for measurements at high redshifts ($z \lesssim 2$). Aside from its effect of accelerating the expansion of the universe, very little is known about DE; it is most commonly modeled by adding the constant Λ to the Einstein-Hilbert action (1.6) describing gravity [35, 36]. From the gravitational point of view the presence of the cosmological constant poses no problem, but as soon as one tries to connect it to a theory of fields its extremely small value becomes what has been called “the worst fine tuning problem in physics”. From a low energy field theory point of view in fact, the cosmological constant should receive contributions from the zero energies of all fields, and should then be quadratically divergent rather than almost zero, leaving physicists with 120 orders of magnitudes to fine tune away. In this theoretical framework then, the nature of the cosmological constant is somehow disturbing, thus leading to the many attempts made in the last decade to explain the observed amount of cosmic acceleration with other mechanisms such as phantom fields [37, 38], modifications of gravity with extra fields [39–42], extra dimensional theories [43–45] and others.

Dark Matter was introduced much earlier [46] to explain the rotational curves of spiral galaxies, and the mass to light ratio in galaxy clusters. Assuming it really is matter that only reacts weakly, mostly through gravitational interactions, we can say that DM most likely is an open door toward one of the many possible extensions of the Standard Model of particle physics. Dark Matter properties are studied using both astrophysical and particle physics observations, making use of experiments that probe a very wide set of scales, again from particle physics scales to galaxy clusters.

Models of modified gravity have been proposed to explain the above mentioned observations [47] in order to avoid the need of introducing new unknown particles, but even though the debate is still open, evidence is favoring the particle solution to the problem [48]. Despite the efforts though, DM is still elusive and its fundamental nature unknown.

Another important paradigm of modern cosmology is the idea that an inflationary epoch happened in the early universe stretching the physics of the tiniest scales to cosmological distances (see [49] for a review), making the sky we observe a projection of the microscopic physics at early times. As a consequence, the quantum properties of the inflaton, the field responsible for the exponential expansion (taking the simplest single field scenario as an example), play a role in determining the primordial seeds for large scales structure. Single field inflation is the simplest proposed scenario but not the only viable one, many models have been built to reproduce a scale invariant spectrum, and in general to match the observational constraints posed by the now very accurate measurements of temperature fluctuations in the Cosmic Microwave Background [50, 51].

1.2 GR Extensions in the Effective Field Theory Approach

In the previous section we have quickly reviewed the basis of General Relativity and its application to cosmology. While it is true that GR is widely considered to be the low energy correct description of gravitational interactions, it is also true that it cannot be considered to be the end of the story. When quantum mechanics enters into play, one immediately realizes that GR carries some problem, namely it is not a renormalizable theory. Moreover, while it is true that GR has been tested on a wide range of scales, from sub-millimeter scales in laboratories to cluster of galaxies scales via weak lensing measurements, there is no reason to believe it must be the correct theory at all scales. In other words, in the ultraviolet (short scales, high energies) we know thanks to quantum mechanics that GR needs some form of completion, and it could also be that in the infrared (extremely large scales and low energies) GR may not be the correct full description for gravitation.

We have also seen that GR is from the field theory point of view the theory that describes the propagation of a massless spin-2 particle, the graviton. We can then use the techniques accumulated in more than half a century to deal with the problems seen above. Namely we can use the effective field theory approach [52] to parametrize our ignorance about the fundamental theory that we know has to reduce to GR at the energies and scales at which GR has been tested. Theories such as String Theory or Loop Quantum Gravity are supposed to be the high energy complete versions of a more fundamental description of nature. This said, we are still far from being able to integrate these theories down to observable energy scales and connect them to classical results from GR or field theory (and semiclassical field theory). EFT is a powerful way to surpass the limits of the classical and semiclassical theories we have to describe observations, allowing us to parametrize the effect of wide classes of unknown more fundamental laws of nature and to compare them with observational data.

In a nutshell, the effective field theory approach consists in finding corrections to the lower states of a model by adding a series of all the possible operators compatible with the symmetries of the low energy theory. On dimensional grounds, the resulting corrections to physical observables coming from higher dimensional operators must be proportional to the ratio of the external momenta or energies that characterize the process to the energy cutoff, scale at which these contributions become important and the series approximation breaks down. In other words one assumes that the fundamental unknown theory can be described by an action which can be expanded in series of some relevant small parameter. The series is then cut at some energy scale, the cutoff scale, with the lowest order operators appearing in the expansion describing the theory one wants to extend, and all the other operators considered as corrections to the "ground state" theory. From a practical point of view, one does not know the fundamental theory, and has therefore to guess which operators could appear in the expansion. The rule is simple, since EFT is about finding corrections to some low energy model, the operators appearing in the expansion cannot introduce any substantial change, like changing the symmetries or introducing new degrees of freedom, unless this happens outside of the validity of the expansion, namely above the cutoff. Thus all the operators compatible with the low energy symmetries must be considered.

A typical example of where EFT is used is in the context of late time acceleration. The present accelerated expansion of the universe, mainly accepted to be the effect of the presence of a nonzero cosmological constant, that in turn corresponds to a negative pressure fluid, could very well be due to the effect of some other physical process. A modification to the propagator of the graviton at large scales could in fact simulate this effect. For example, the DGP model [53] described by the action

$$S_{DGP} = M^3 \int d^5X \sqrt{G} R_{(5)} + M_p^2 \int_{brane} d^4x \sqrt{|g|} R, \tag{1.23}$$

produces the usual four dimensional gravity as the result of the effective description of a five dimensional theory. In the five dimensional language a particular value for the curvature is assigned on the four dimensional boundary that corresponds to the four dimensional brane containing the standard model fields. In order to compare to experiments, the five dimensional theory is intrgrated down obtaining an effective four dimensional theory.

As already mentioned, the EFT description plays an important role in going beyond the background classical solutions for gravitational systems (and fields in general), especially given the fact that we are still lacking a good proposal for a quantum theory of gravity. More in general, higher dimensional operators, possibly containing more than two time derivatives acting on a field, may be used in the action describing a physical system like gravity. In the context of EFT this poses no additional problem, since the higher order operators are required to add only small corrections to the initial "ground state" theory. It has to be noted that in general the presence of more than two time derivatives acting on a field (possibly after integration by parts) in a Lagrangian formulation of a theory leads to a set of problems that can be traced back to 1850 with the famous theorem by Ostrogradski [54] which will be discussed later in section 1.3.1. If the operators containing higher derivatives are taken as part of the system and not as an artifact of the series expansion as in EFT, then most systems containing higher derivatives lead to equations of motion with unstable solutions for the fields they try to describe. This is already true for classical systems, and it gets even more problematic when one tries to move to quantization, leading to a loss of unitariety, negative norm states, and in general infinities that cannot be cured. Another way to consider the problem [55, 56] is to map a higher derivative theory into one that contains more fields but that is described by operators with up to second time derivatives (possibly after integration by parts in the action). We can show this with an example. Taking the action

$$S = \int d^4x - \frac{1}{2}\phi(\Box - m_1^2)(\Box - m_2^2)\phi \ , \tag{1.24}$$

by applying nonlinear transformations, that usually involve second or higher derivatives of the original fields in the definition of the new ones,

$$\psi_1 = \frac{(\Box - m_2^2)\phi}{\sqrt{m_2^2 - m_1^2}} \ , \quad \psi_2 = \frac{(\Box - m_1^2)\phi}{\sqrt{m_2^2 - m_1^2}} \ , \tag{1.25}$$

the order of the terms appearing in the action is reduced to the canonical value but one or more of the newly defined fields appear with the wrong sign (opposite with respect to the other fields) for the kinetic part. This fields are dubbed "ghosts".

$$S = \int d^4x \left(-\frac{1}{2}\psi_2(\Box - m_2^2)\psi_2 + \frac{1}{2}\psi_1(\Box - m_1^2)\psi_1 \right) \ . \tag{1.26}$$

We will say more on the subject in the next section 1.3.2, and later on in chapter 3, when we consider corrections coming from sixth order operators in the Euclidean path integral formulation.

For now it suffices to point out an interesting fact, which is that not all systems with higher derivatives in the action contain ghosts, and that higher derivatives operators appear not only in series expansions as in EFT, but also quite naturally in braneworld models. Recently a whole class of such systems has been studied, they contain scalar fields described by an action with higher derivatives that nevertheless does not lead to the existence of extra ghostly degrees of freedom. In four dimensions the Lagrangians for the extra π scalars are built via the nonzero Lovelock invariants [57][1]

$$
\begin{aligned}
\mathcal{L}_1 &= \pi \, , && (1.27)\\
\mathcal{L}_2 &= [\pi^2] \, , && (1.28)\\
\mathcal{L}_3 &= [\pi^2][\Pi] - [\pi^3] \, , && (1.29)\\
\mathcal{L}_4 &= \frac{1}{2}[\pi^2][\Pi]^2 - [\pi^3][\Pi] + [\pi^4] - \frac{1}{2}[\pi^2][\Pi^2] \, , && (1.30)\\
\mathcal{L}_5 &= \frac{1}{6}[\pi^2][\Pi]^3 - \frac{1}{2}[\pi^3][\Pi]^2 + [\pi^4][\Pi] - [\pi^5] + \frac{1}{3}[\pi^2][\Pi^3] - \frac{1}{2}[\pi^2][\Pi][\Pi^2] \\
&\quad + \frac{1}{2}[\pi^3][\Pi^2] \, . && (1.31)
\end{aligned}
$$

Many characteristics of these fields are linked to a symmetry, the Galilean symmetry in field space, hence the name for these scalar fields "Galileons" [58]. Among other nice properties, these fields often carry information about the space-time symmetries of the bulk space [59] when descending from a higher dimensional theory. Moreover, thanks to the presence of derivatives, these models are suitable to encode screening such as the Vainshtein mechanism [60] that allows a scalar field to mediate an extra force among particles without ruining local test of gravity.

We have thus seen that it is quite natural to look for extensions of the theory of General Relativity. Of course, from a practical point of view it may be sufficient to just look for corrections at the scales accessible by experiments, but it is clear that there can be rich physics involved in possible extensions to GR, or even completely different theories that reduce to GR at the right energies.

1.3 Higher Derivative Systems: Canonical Treatment

In the previous section we have mainly focused on the idea that extensions to a theory like GR can often be parametrized with a series expansion. The terms in the

[1] With the standard notation $\Pi_{\mu\nu} \equiv \partial_\mu \partial_\nu \pi$, $[\Pi^n] \equiv Tr\,(\Pi^n)$ and $[\pi^n] \equiv \partial\pi \cdot \Pi^{n-2} \cdot \partial\pi$.

expansion describe the effective low energy behavior, and corrections of models the form of which may or may not be known a priori. Once higher derivative terms are considered though, and as we have seen before this happens quite generally, one can also take a more radical point of view and try to give physical meaning to them. This introduces new degrees of freedom and requires considering runaway solutions that may be of physical interest in cosmology where the evolution of the universe breaks time symmetry [61]. In this section we will review some of the basis for considering higher derivatives, showing what the main problems related to them are, and how such problems arise even in extremely simple models. Later on, in chapter 3, we will discuss what one gains by considering higher derivatives in a non-EFT approach, and how to overcome, when possible, the difficulties here presented.

1.3.1 Ostrogradski's Theorem

When dealing with higher derivative systems, a fundamental result on the subject must be taken in consideration. This is a theorem in classical mechanics by Ostrogradski [54] that shows the presence of an instability in the Hamiltonian function associated to a system described by a Lagrangian containing more than one time derivative acting on the fields. Ostrogradski's result is quite general and has been studied and reviewed by many authors [62–67], but to keep things simple we will present it here for a one dimensional point particle.

We can start reminding ourselves of the usual case of a one dimensional system with no explicit dependence on time. This is described by a Lagrangian that can be written as

$$L = L(q, \dot{q}) \ , \tag{1.32}$$

where q is the position and overhead dot represents a time derivative. The equations describing the evolution of the system are then given by the Euler-Lagrange equations, namely

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \ . \tag{1.33}$$

When the system is non-degenerate, which is $\frac{\partial L}{\partial \dot{q}}$ depends on $\dot{q}$ [2], then the above equations take the form

$$\ddot{q} = F(q, \dot{q}) \ , \tag{1.34}$$

this implies that a solution will require two pieces of information, two initial conditions such as the value of the coordinate and its first derivative at some initial time. Counting the number of initial conditions needed to have a well defined Cauchy problem also gives the dimensionality of the phase space. The evolution of the system is then described by a trajectory in a two dimensional phase space in the two canonical coordinates Q and P defined via

$$Q \equiv q \ , \quad P \equiv \frac{\partial L}{\partial \dot{q}} \ , \tag{1.35}$$

relations that can be reversed thanks to the assumption of non-degeneracy. The Hamiltonian is then obtained by Legendre transforming with respect to $\dot{q}$ and reads

$$H(Q, P) \equiv P\dot{q}(Q, P) - L(q(Q, P), \dot{q}(Q, P)) \ . \tag{1.36}$$

We can now take a look at the simplest generalization of the above canonical example; a system described by a non-degenerate Lagrangian

$$L(q, \dot{q}, \ddot{q}) \ . \tag{1.37}$$

Being non-degenerate now means that $\frac{\partial L}{\partial \ddot{q}}$ depends on $\ddot{q}$ and therefore the Euler-Lagrange equations

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0 \ , \tag{1.38}$$

give

$$q^{(4)} = F(q, \dot{q}, \ddot{q}, q^{(3)}) \ . \tag{1.39}$$

[2] The non degeneracy condition is not necessary for applying the theorem, it is assumed here for the sake of simplicity. A general treatment can be found in the references given before.

The number of initial conditions needed has now doubled, and so has the dimensionality of the phase space. The choice for the canonical variables suggested by Ostrogradski is

$$Q_1 \equiv q \ , \qquad Q_2 \equiv \dot{q} \tag{1.40}$$

$$P_1 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \ , \qquad P_2 \equiv \frac{\partial L}{\partial \ddot{q}} \tag{1.41}$$

It is important to note at this point that even though the phase space is four dimensional, the Lagrangian only depends on three independent variables, q, $\dot{q}$, and $\ddot{q}$. This fact combined with non degeneracy allows us to find the inverse relations that give the configuration space variables in terms of three of the phase space ones and use them to write an Hamiltonian for the system as

$$\begin{aligned} H(Q_1, Q_2, P_1, P_2) &\equiv \sum_{i=1}^{2} P_i q^{(i)} - L \ , \\ &= P_1 Q_2 + P_2 \ddot{q}(Q_1, Q_2, P_2) \\ &\quad - L(q(Q_1, Q_2, P_2), \dot{q}(Q_1, Q_2, P_2), \ddot{q}(Q_1, Q_2, P_2)) \ . \end{aligned} \tag{1.42}$$

It is immediately obvious in the form above that the Hamiltonian we have found carries some complications. In fact, it is linear in the momentum P_1, and therefore unbounded from below. Considering that the original Lagrangian does not contain any explicit dependence on time, the Hamiltonian (1.42) is a conserved Noether current and represents the energy of the system.

Moving to the more general case of an arbitrary number of derivatives, we can consider a Lagrangian $L(q, \dot{q}, \ldots, q^{(N)})$. The Euler-Lagrange equations for such system will be

$$\sum_{i=0}^{N} \left(-\frac{d}{dt}\right)^{i} \frac{\partial L}{\partial q^{(i)}} = 0 \ , \tag{1.43}$$

and in the case of non-degenerate systems they will take the form

$$q^{(2N)} = F\left(q, \dot{q}, \ldots, q^{(2N-1)}\right) \ . \tag{1.44}$$

Consequently a solution will require $2N$ initial conditions and the phase space will be $2N$ dimensional. Repeating the procedure above, and using Ostrogradski's canonical variables

$$Q_i \equiv q^{(i-1)} \ , \quad P_i \equiv \sum_{j=i}^{N} \left(-\frac{d}{dt}\right)^{j-1} \frac{\partial L}{\partial q^{(j)}} \ , \quad i = 1, \dots N \ , \tag{1.45}$$

it is possible to solve for $q^{(N)}$ in terms of P_N and the Q_i, obtaining an Hamiltonian

$$\begin{aligned} H(Q_i, P_i) &\equiv \sum_{j=1}^{N} P_i q^{(i)} - L \ , \\ &= P_1 Q_2 + P_2 Q_3 + \cdots + P_{N-1} Q_N + P_N q^{(N)}(Q_i, P_N) - L(Q_i, P_N) \end{aligned} \tag{1.46}$$

As before, the Hamiltonian is the energy of the system and being linear in $P_1, \dots P_{N-1}$ it is unbounded from below. In particular, the instability related to the lack of a lower bound is present over almost half of the phase space.

1.3.2 Applications

More in general, and keeping in mind the kind of problem that we will face later on, the procedure outlined by the simple version of Ostrogradski's theorem presented in the previous section can be used to transform actions containing higher derivatives into canonical second order actions containing more fields. Following the works of [56, 68, 69], we can start from an action

$$S = \int d^4x \sqrt{-g} f(\phi, \nabla^2\phi, \dots, \nabla^{2k}\phi) \ , \tag{1.47}$$

where the scalar ϕ could represent a collection of fields ϕ_i. We go on assuming that f cannot be further reduced via integration by parts, so that the highest number of derivatives $2k$ cannot be integrated into a surface contribution. Moreover, for simplicity, we also assume non degeneracy, which in this case means that (writing $f = f(\chi_0, \dots, \chi_k)$, with $\chi_i = \nabla^{2i}\phi$) $\frac{\partial f}{\partial \chi_k}$ is a function of χ_k. It follows then that the equations of motion will be of $(4k)$th order.

The next step is to introduce auxiliary fields through Lagrange multipliers to eliminate all the higher derivative terms. The action can then be rewritten as

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\,\big(f(\chi_0,\ldots,\chi_k)+\lambda_0(\phi-\chi_0)+\lambda_1(\nabla^2\chi_0-\chi_1)+\ldots \\ &\quad +\lambda_k(\nabla^2\chi_{k-1}-\chi_k)\big)\ . \end{aligned} \tag{1.48}$$

Note that at this point the λ_0 term appears just to enforce the substitution $\phi \to \chi_0$, and we leave it there just for convenience of comparison with other results in literature.

The equation for χ_k reads

$$\lambda_k = \frac{\partial f}{\partial \chi_k}(\chi_0,\ldots,\chi_k)\ , \tag{1.49}$$

it is algebraic, and eliminating $\chi_k(\chi_0,\ldots,\chi_{k-1},\lambda_k)$ from the action using equation (1.49) will not introduce higher derivatives on the other fields χ_i [3]. The action then becomes

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\,\big(\bar f(\chi_0,\ldots,\chi_{k-1},\lambda_k)+\lambda_0(\phi-\chi_0)+\lambda_1(\nabla^2\chi_0-\chi_1)+\ldots \\ &\quad +\lambda_k(\nabla^2\chi_{k-1}-\chi_k(\chi_0,\ldots,\chi_{k-1},\lambda_k))\big)\ , \end{aligned} \tag{1.50}$$

which is already a second order action for the χ_i and can be put in canonical form via the substitution

$$\chi_{i-1}=\varphi_i+\psi_i \quad \lambda_i=\varphi_i-\psi_i\ , \quad i=1,\ldots,k\ , \tag{1.51}$$

in fact the terms containing derivatives will become, after integration by parts,

$$\lambda_i\nabla^2\chi_{i-1}\to -\left(\nabla\varphi_i\right)^2+\left(\nabla\psi_i\right)^2\ . \tag{1.52}$$

Notice the opposite sign for the two fields which shows the ghost instability that in the Hamiltonian formalism of the original Ostrogradski theorem appeared as lack of lower bound for the energy. The full action then reads

$$S=\int d^4x\sqrt{-g}\left(\sum_{i=1}^{k}\left[-\left(\nabla\varphi_i\right)^2+\left(\nabla\psi_i\right)^2\right]-V(\varphi_i,\psi_i,\chi_0)\right)\ , \tag{1.53}$$

[3] As is discussed in [55], the solution for χ_k is in general non-unique, and different branches will require separated analysis.

where the potential V reabsorbed the remaining terms not containing derivatives, and the constraint in λ_0 has been used to eliminate ϕ.

It is interesting to note that the same argument goes through even when the scalar ϕ is not a fundamental field. For instance one could substitute in the action (1.47) $\phi \to R$; the procedure would change only in few details to take in consideration that R already contains two derivatives acting on the fundamental field, the metric. Similarly for models where other scalar contractions of the curvature appear, such as $R_{\mu\nu}R^{\mu\nu}$ and so on [55, 56].

Chapter 2

Higher Order Corrections in EFT During Inflation

2.1 Introduction

We have seen in the previous introductory chapter what the effective field theory approach consists of. We can now move to its application in the context of inflation, with the idea of pushing the limits of its validity to find out the boundaries beyond which we cannot trust perturbation theory. What we are interested in is to probe the early time and short wavelength regime to find an upper limit for the cutoff of the chosen theory, namely perturbations of GR coupled to an inflaton. As required by EFT, we are going to consider all possible corrections from a general series of operators compatible with the symmetries of the low energy theory. The corresponding series of corrections is then built order by order with the highest contribution from each term in the series. We can then examine the series to find at which energy it stops converging and so extrapolate a value for the maximum cutoff of the theory.

Before diving into the calculation described above we can contextualize this work by saying something about inflation and evolution of perturbations during inflation. One of the main successes of inflation [70–73] is the explanation of the origin of structure [74–78]. During slow-roll (when the inflaton "slowly rolls" down an almost flat region of its potential), the Hubble radius remains nearly constant, while cosmological modes are constantly pushed out of the horizon. Thus, local processes determine the amplitude and properties of perturbations at sub-horizon scales, which are transfered to cosmologically large distances by the accelerated expansion. In that sense, the sky is the screen upon which inflation has projected the physics of the microscopic universe.

In the standard single field inflationary scenario, the primordial perturbations seeded during inflation arise from quantum-mechanical fluctuations of the inflaton around its homogeneous value. Hence, their properties directly depend on the quantum state of the inflaton perturbations. Conventionally, this is taken to be a state devoid of quanta in the asymptotic past, raising the crucial question of whether we can trust cosmological perturbation theory – and its quantum nature – at such early times [79].

As we pointed out in the introduction, according to our present understanding, quantum field theories and general relativity are merely low energy descriptions of a more fundamental theory of quantum gravity. In the case of inflation, the leading terms in the corresponding effective Lagrangian, what we called the "ground state" of the theory, are the Einstein-Hilbert term plus the inflaton kinetic term and potential, which we will describe more in detail below. In the EFT treatment, these terms are accompanied by all other possible operators compatible with the symmetries of the theory, namely, general covariance and any other symmetry of the inflaton sector. Higher dimensional operators are suppressed by powers of an energy scale M, which we will assume to be of the order of the reduced Planck mass, $M = M_{\rm p} \equiv (8\pi G)^{-1/2}$, and they are therefore expected to be negligible at sufficiently small momenta, or sufficiently long wavelengths. Note however that this does not imply that we can simply discard high-momentum modes from the low-energy theory. In a gauge theory in flat space for instance, a momentum cutoff breaks gauge invariance and is thus incompatible with the symmetries of the theory. Similarly, in a curved spacetime, the definition of properly renormalized generally covariant field operators requires subtractions that involve all the momentum modes of the fields [80]. The effective theory is a useful low-energy approximation simply because, on dimensional grounds, the corrections to any observable introduced by the higher-dimensional operators must be proportional to ratios of the external momenta or energies that characterize the process to the energy scale M.

Our goal in this chapter is to show how to determine the three-momentum scale Λ at which higher-dimensional operators from the EFT expansion significantly modify the dispersion relation of cosmological modes. This because beyond that scale we cannot trust the free sector of the theory, and cosmological perturbation theory breaks down. Since the dispersion relation of a mode is what sets its mean square amplitude, we identify such a breakdown with the point at which the corrections to the power spectrum caused by higher-dimensional operators become dominant.

In Minkowski spacetime, the scale at which effective corrections to observable quantities become important roughly coincides with the scale that suppresses the non-renormalizable operators in the effective action. For instance, in the presence of such terms, the propagator of a massless particle with (off-shell) momentum k^μ can be cast as an expansion of the form [8]

$$\Delta(k_\mu, k'_\mu) = \frac{1}{k_\mu k^\mu}\left(c_0 + c_2\frac{k_\mu k^\mu}{M^2} + c_4\frac{(k_\mu k^\mu)^2}{M^4} + \cdots\right)\delta(k_\mu - k'_\mu), \tag{2.1}$$

where the c_n are coefficients of order one that typically depend on logarithms of $k_\mu k^\mu$. Lorentz-invariance implies that the corrections must be a function of the scalar $k_\mu k^\mu$, while Poincare symmetry implies that they must conserve four-momentum. From the structure of the corrections, it is clear that the expansion breaks down around $k_\mu k^\mu = M^2$.

On the other hand, it is crucial to realize that the three-momentum scale Λ at which corrections to the power spectrum become dominant does not need to equal the fundamental scale M. On short time-scales and distances, an inflating spacetime can be regarded as flat. Hence, our previous result in Minkowski space suggests that cosmological perturbation theory is valid as long as $k_\mu k^\mu \ll M^2 = M_{\mathrm{p}}^2$. On shell, the four-momenta of cosmological perturbations are light-like, $k_\mu k^\mu \equiv -k_0^2 + \mathbf{k}\cdot\mathbf{k} = 0$. Thus, naïvely substituting in equation (2.1) we would find that corrections are not only independent of the three-momentum $\mathbf{k}$, but also that they are actually zero. As we shall see though, the evolution of the inflaton leads to small but finite violations of the Lorentz symmetry even in the short-wavelength limit, which are imprinted on the power spectrum as $\mathbf{k}$-dependent corrections.

The phenomenological imprints of trans-Planckian physics on the primordial spectrum of perturbations, and the implications of a finite cutoff Λ on the spatial momentum of cosmological modes have been extensively studied [81–107]. These articles mostly study corrections to the power spectrum in the long-wavelength limit $|\mathbf{k}/a| \equiv |\mathbf{k}_{ph}| \ll H$, at late times, which is the regime directly accessible by experimental probes. In this article we focus instead on the short-wavelength regime $|\mathbf{k}_{ph}| \gg H$, at early times, since we are interested in determining how far into the ultraviolet cosmological linear perturbation theory applies. At short wavelengths, the power spectrum can be cast again as a derivative expansion of the form

$$\langle \delta\varphi^*(\mathbf{k})\delta\varphi(\mathbf{k})\rangle = \frac{1}{2|\mathbf{k}|}\left(\alpha_0 + \alpha_2\frac{\mathbf{k}_{ph}\cdot\mathbf{k}_{ph}}{M_{\mathrm{p}}^2} + \alpha_4\frac{(\mathbf{k}_{ph}\cdot\mathbf{k}_{ph})^2}{M_{\mathrm{p}}^4} + \cdots\right), \tag{2.2}$$

with coefficients α_i that depend on slow-roll parameters and the dimensionless ratio H/M_{p}. The analytic corrections to the leading result $1/2|\mathbf{k}|$ arise from tree-level diagrams with vertices from higher-dimensional operators. We only consider tree-level diagrams here, since we expect loop diagrams to simply introduce a logarithmic dependence of the dimensionless coefficients α_i on scale, though we have not verified this explicitly. Cosmological perturbation theory fails (in our restricted sense) when the expansion in powers of $|\mathbf{k}|$ breaks down, namely, when all the terms become of the same order,

$$|\mathbf{k}_{ph}| \approx M_{\mathrm{p}}\sqrt{\frac{\alpha_{2n}}{\alpha_{2n+2}}} \equiv \Lambda\ . \tag{2.3}$$

As we shall show, the ratios $\alpha_{2n}/\alpha_{2n+2}$ are all quite large and of the same order, so the effective cutoff Λ significantly differs from $M_{\rm p}$. In a slightly different context, a similar analysis has been applied to the bispectrum in [108]. The terms that yield the leading (momentum-independent) corrections to the primordial spectrum have been discussed in [109]. Note by the way that there are many different ways in which perturbation could break down. The authors of [110] argue for instance that in a nearly de Sitter universe certain second order perturbations may be as important as linear ones, which also implies a failure of linear perturbation theory.

Before we show the detailed calculations, we will review in the next section the basics of tensor and scalar fluctuations, describe the relevant background to our problem, and set up a description of perturbation theory in cosmology. Then in section 2.3 and 2.4 we will compute the squared amplitude of tensor and then scalar perturbations deriving the results mentioned above. The scalar analysis will retrace the calculations and results of the tensorial counterpart, with few extra complications due to the mixing of the scalar excitations of the metric and the inflaton ones. To close the present chapter we will discuss possible implications of our results in section 2.5.

2.2 Setting the Scene – Cosmological Perturbation Theory

2.2.1 The Inflating Background

Our starting point is a standard single-field inflation model. At sufficiently late times, the inflaton and gravity must be described by a low-energy effective action, whose leading terms are dictated by general covariance and the field content,

$$S_0 = \int d^4x\sqrt{-g}\left[\frac{M_{\rm p}^2}{2}R - \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)\right]. \tag{2.4}$$

In an effective field theory context, the action should also contain additional terms suppressed by powers of a dimensionful scale, which we assume here to be of the order of the reduced Planck mass M_{p}. Our goal is to determine the point beyond which such higher-dimensional operators produce corrections to the two-point function of cosmological perturbations that cannot be neglected. Our considerations can be readily generalized to cases in which the suppression scale of the higher-dimensional operators is not the Planck mass, but any other scale.

If the potential $V(\varphi)$ is sufficiently flat, at least in a certain region in field space, there exist inflationary solutions, along which a homogeneous scalar field $\varphi(\eta, \mathbf{x}) = \varphi_0(\eta)$ slowly rolls down the potential and spacetime is spatially homogeneous, isotropic and flat[1] ,

$$g^{(0)}_{\mu\nu} \equiv a^2(\eta)\eta_{\mu\nu}, \tag{2.5}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and η denotes conformal time. This metric is related to the one we have seen before when in the introduction we talked about homogeneity and isotropy. In fact, it can be obtained by taking equation (1.10) in the case $k = 0$ and performing a conformal transformation

$$dt \to a(\eta)d\eta \ . \tag{2.6}$$

A model-independent measure of the slowness of the inflation is given by the slow-roll parameter

$$\epsilon \equiv -\frac{H'}{aH^2}, \tag{2.7}$$

where $H \equiv a'/a^2$ is the Hubble parameter and a prime denotes a derivative with respect to conformal time. During slow-roll, ϵ is nearly constant, and to lowest order in slow-roll parameters its time derivative can be neglected. From here on we will work keeping only the leading non-vanishing order in the slow-roll expansion.

[1]Strictly speaking, inflation generates an *almost* perfectly flat spacetime. However, tiny departures from perfect flatness will not play any role in what follows, since we will be interested in the small-scale regime at which even a spatially curved spacetime looks flat.

2.2.2 Cosmological Perturbations

Let us now consider cosmological perturbations around the homogeneous and isotropic background described above. Writing $\varphi = \varphi_0 + \delta\varphi$ and $g_{\mu\nu} = g^{(0)}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, \mathbf{x})$, and substituting into equation (2.4), we can expand the action S_0 up to the desired order in the fluctuations $\delta\varphi$ and $\delta g_{\mu\nu}$,

$$S_0[\varphi, g_{\mu\nu}] = \delta_0 S_0 + \delta_1 S_0 + \delta_2 S_0 + \cdots . \tag{2.8}$$

The lowest order term $\delta_0 S_0$ does not contain any fluctuations and describes the inflating background; the linear term $\delta_1 S_0$ vanishes because it corresponds to the first variation of the action along the background solution, and the quadratic part of the action $\delta_2 S_0$ describes the free dynamics of the perturbations. The latter is what we need in order to calculate the primordial spectrum of fluctuations. To quadratic order, tensor and scalar perturbations are decoupled, so we may study them separately.

Tensor Perturbations

Tensor perturbations are described by a transverse and traceless tensor

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right]. \tag{2.9}$$

The tensor h_{ij} itself can be decomposed in plane waves of two different polarizations, which again decouple at quadratic order. We shall hence focus on just one of them,

$$h_{ij}(\eta, \mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e_{ij}(\mathbf{k})\, h_k(\eta)\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.10}$$

where the $h_k(\eta)$ are the corresponding mode functions, and $e_{ij}(\mathbf{k})$ denotes the normalized graviton polarization tensor, $e^i{}_j e^j{}_i = 1$ (we raise and lower spatial indices with the Kronecker delta). Note that for later convenience we work in a toroidal universe of volume $V = L^3$; hence, the spatial wave numbers have components $k_i = n_i \cdot (2\pi/L)$, where the n_i are arbitrary integers.

Substituting the expansion (2.10) into the action (2.4), and using the background equations of motion, we may then express the free action $\delta_2 S_0$ as

$$\delta_2 S_0 = \frac{1}{2}\int d\eta \sum_{\mathbf{k}} \left[v_k' v_{-k}' - \left(k^2 - \frac{a''}{a}\right) v_k v_{-k} \right] , \tag{2.11}$$

where the scalar variable v_k is defined as

$$v_k = a\, M_{\rm p}\, h_k \ . \tag{2.12}$$

Thus, in terms of v_k the action for tensor perturbations takes the form of an harmonic oscillator with time-dependent frequency.

Scalar Perturbations

In spatially flat gauge, the perturbed metric reads

$$ds^2 = a^2(\eta)\left[-(1+2\phi)d\eta^2 + 2\partial_i B dx^i d\eta + \delta_{ij} dx^i dx^j\right] \ . \tag{2.13}$$

On first inspection there appear to be three independent scalar variables: ϕ, B and the inflaton perturbation $\delta\varphi$. However, Einstein equations impose constraints on both ϕ and B. Solving the corresponding Fourier transformed equations to leading order in the slow-roll expansion, one finds (see e.g. [111])

$$\begin{aligned} \phi_k &= \sqrt{\frac{\epsilon}{2}}\frac{\delta\varphi_k}{M_{\rm p}} \ , \\ B_k &= \sqrt{\frac{\epsilon}{2}}\frac{\delta\varphi_k'}{M_{\rm p} k^2} \ . \end{aligned} \tag{2.14}$$

Consequently, there is only one physical scalar degree of freedom, and scalar perturbations can be described by just one variable. A convenient choice that is particularly useful for quantizing scalar perturbations is the Mukhanov variable [112], which in spatially flat gauge takes the simple form

$$v_k = a\, \delta\varphi_k \ . \tag{2.15}$$

Using relations (2.14) and (2.15), we may express $\delta_2 S_0$ in terms of v_k only. For constant ϵ, that is, to leading order in the slow-roll expansion, the resulting action is also given by equation (2.11). This agreement greatly simplifies the analysis, because it allows us to use the same set of propagators to describe both scalar and tensor fluctuations.

To leading order in the slow-roll expansion, the mode functions of both scalar and tensor perturbations hence satisfy the same equation of motion during inflation. Varying the action (2.11) with respect to v_{-k} we obtain

$$v_k'' + \left[k^2 - \frac{a''}{a}\right] v_k = 0 \ , \tag{2.16}$$

which has a unique solution for appropriate initial conditions. The conventional choice is the Bunch-Davies or adiabatic vacuum, whose mode functions obey

$$v_k(\eta) \quad \xrightarrow{|k\eta|\gg 1} \quad \frac{e^{-ik\eta}}{\sqrt{2k}} \left[1 + \mathcal{O}\left(\frac{1}{k\eta}\right)\right] \ . \tag{2.17}$$

Because we are only interested in the sub-horizon limit, this is all we need to know about the mode functions. In particular, because the behavior of the mode functions in the short-wavelength limit does not depend on the details of inflation, our results are also insensitive to the particular form of the inflaton potential.

2.2.3 Quantum Fluctuations and the *in-in* Formalism

In order to study the properties of cosmological modes in the short-wavelength regime, we concentrate on the two-point function of the field v,

$$\langle v^*(\eta, \mathbf{k}) v(\eta, \mathbf{k})\rangle \equiv \langle 0, in| v^*(\eta, \mathbf{k}) v(\eta, \mathbf{k}) |0, in\rangle \ , \tag{2.18}$$

where $|0, in\rangle$ is the quantum state of the perturbations, which we assume to be the Bunch-Davies vacuum. The two-point function characterizes the mean square amplitude of cosmological perturbation modes, and differs from the power spectrum just by a normalization factor. Note that in an infinite universe, the two-point function is proportional to a momentum-conserving delta function, which in a spatially compact universe is replaced by a Kronecker delta.

In the *in-in* formalism (see [113] for a clear and detailed exposition) the two-point function can be expressed as a path integral,

$$\langle v^*(\eta,\mathbf{k})v(\eta,\mathbf{k})\rangle = \int \mathcal{D}v_+\mathcal{D}v_-\, v_+^*(\eta,\mathbf{k})v_-(\eta,\mathbf{k})\exp\left(iS_{\rm free}[v_+,v_-]\right)\exp\left(iS_{\rm int}[v_+]\right)\exp\left(-iS_{\rm int}[v_-]\right) \quad (2.19)$$

where $S_{\rm free}$ is quadratic in the fields, and $S_{\rm int}$ contains not just the remaining cubic and higher order terms in the action, but also any other quadratic terms we may decide to regard as perturbations. Note that there are two copies of the integration fields v_- and v_+, because we are calculating expectation values, rather than *in-out* matrix elements. This path integral expression is very useful to perturbatively expand the expectation value in powers of any interaction. In particular, each contribution can be represented by a Feynman diagram, with vertices drawn from the terms in $S_{\rm int}$ and propagators determined by the free action $S_{\rm free}$. In our case, the latter are given by

$$= \int \mathcal{D}v_+\mathcal{D}v_-\, v_+^*(\eta,\mathbf{k})v_+(\eta',\mathbf{k})\exp(iS_{\rm free}) \approx \frac{e^{-ik|\eta-\eta'|}}{2k}, \quad (2.20)$$

$$= \int \mathcal{D}v_+\mathcal{D}v_-\, v_-^*(\eta,\mathbf{k})v_-(\eta',\mathbf{k})\exp(iS_{\rm free}) \approx \frac{e^{ik|\eta-\eta'|}}{2k}, \quad (2.21)$$

$$= \int \mathcal{D}v_+\mathcal{D}v_-\, v_+^*(\eta,\mathbf{k})v_-(\eta',\mathbf{k})\exp(iS_{\rm free}) \approx \frac{e^{ik(\eta-\eta')}}{2k}, \quad (2.22)$$

which we quote here just in the sub-horizon limit. Note that to first order in $S_{\rm int}$ there are two vertices, one that contains powers of v_+ and one that contains powers of v_-; the associated coefficients just differ by an overall sign[2].

As a simple example, let us calculate the value of the two-point function in the short-wavelength limit to zeroth order in the interactions. Using the definition (2.19)

[2] The quadratic action $S_{\rm free}$ enforces $v_+(\vec{k}) = v_-(\vec{k})$ at time η. Hence, we could replace $v_+^*(\eta,\mathbf{k})v_-(\eta,\mathbf{k})$ by $v_+^*(\eta,\mathbf{k})v_+(\eta,\mathbf{k})$ or $v_-^*(\eta,\mathbf{k})v_-(\eta,\mathbf{k})$ inside the path integral (2.19). Our choice removes the apparently ill-defined corrections we otherwise obtain when higher-order time derivatives act on the time-ordered products in equations (2.20) and (2.21). These ill-defined corrections can also be eliminated by field-redefinitions, a procedure that leads to the same corrections we find using our choice of field insertions.

and equation (2.22), we find

$$\langle v^*(\eta, \mathbf{k}) v(\eta, \mathbf{k})\rangle = \bullet\!\!\rule[0.5ex]{3em}{0.4pt}\;{-}\,{-}\,{-}\,{-}\!\bullet \approx \frac{1}{2k} \qquad (|k\eta| \gg 1), \tag{2.23}$$

which is the well-known and standard short-wavelength limit result. In this regime, the two-point function is hence the inverse of the dispersion relation, since the latter determines the appropriate boundary conditions for the mode functions, as in equation (2.17).

In the next two sections we use the path integral (2.19) to calculate the corrections to the two-point function coming from higher-order operators in the action. These can be interpreted as corrections to the dispersion relation, even though in the presence of such terms the mode equations generally contain higher order time derivatives. In any case, a significant disagreement between the calculated two-point function and the lowest order result (2.23) points to the lack of self-consistency of our quantization procedure, and signals the breakdown of cosmological perturbation theory.

2.3 The Limits of Perturbation Theory: Tensors

The lowest order action (2.4) contains the leading terms that describe the dynamics of the inflaton and its perturbations. However, as we have noted, in an EFT approach the action generically contains all possible terms compatible with general covariance and any other symmetry of the theory. Here, for simplicity, we assume invariance under parity, an approximate shift symmetry of the inflaton, and a discrete $\mathbb{Z}_2$ symmetry $\varphi \to -\varphi$. Thus, all possible effective corrections to the action (2.4) can be built from the metric $g_{\mu\nu}$, the Riemann tensor $R_{\mu\nu\lambda\rho}$, the covariant derivative ∇_μ and an even number of scalar fields φ. In what follows, we consider these additional terms and compute the corrections they induce on the two-point function of tensor perturbations in the short-wavelength limit. This allows us to determine the regime in which additional terms in the action cannot be neglected, and hence, the range over which cosmological perturbation theory is applicable. The reader not interested in technical details may skip directly to section 2.3.3, where we collect and summarize our results.

2.3.1 Dimension Four Operators

Before tackling the general problem, we begin our analysis by considering the simplest corrections, namely all dimension four operators, which will appear in the action multiplied by dimensionless coefficients. On dimensional grounds, we expect these to yield corrections to the two-point function that are suppressed by only two powers[3] of M_{p}. These operators will also help us to illustrate our formalism and discuss some of the important issues related to our calculation.

Any generally covariant dimension four effective correction must be of the form

$$S_1 \equiv S_\alpha + S_\beta = \int \sqrt{-g} \left(\alpha R^2 + \beta C^2\right), \tag{2.24}$$

where C^2 is the square of the Weyl tensor,

$$C^2 = R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2, \tag{2.25}$$

and the dimensionless couplings α and β are assumed to be of order one. Note that we have ignored total derivatives like the Gauss-Bonnet term, since they do not lead to any corrections in perturbation theory. The Levi-Civita tensor cannot appear in the action because we assume invariance under parity.

We start by substituting the perturbed metric (2.9) into equation (2.24) and expanding up to second order in h_{ij}. Using the modified background equations and equation (2.12) to express the tensor perturbations in terms of the variable v, we

[3] Dimension six operators quadratic in φ also contribute at this order; we consider them later.

obtain in the sub-horizon limit

$$\delta_2 S_\alpha = \frac{\alpha}{2M_{\rm p}^2}\sum_{\mathbf{k}}\int d\eta'\left\{-\frac{6a''}{a^3}v_k\left[v''_{-k}+k^2 v_{-k}\right]-\frac{6a''}{a^3}\left[v''_k+k^2v_k\right]v_{-k}\right\}, \tag{2.26}$$

$$\delta_2 S_\beta = \frac{\beta}{M_{\rm p}^2}\sum_{\mathbf{k}}\int d\eta'\left\{\frac{1}{a}\left[v''_k+k^2v_k\right]-2\,aH\left(\frac{v_k}{a}\right)'\right\}\times \left\{\frac{1}{a}\left[v''_{-k}+k^2v_{-k}\right]-2\,aH\left(\frac{v_{-k}}{a}\right)'\right\}. \tag{2.27}$$

From these expressions, it is easy to derive the rules for the vertices

$$\text{———}\!\times\!\text{———} \approx \frac{i\alpha}{M_{\rm p}^2}\int_{-\infty}^{\eta}d\eta'\left\{-\frac{6a''}{a^3}\left(\overrightarrow{\partial}^2_{\eta'}+k^2\right)-\left(\overleftarrow{\partial}^2_{\eta'}+k^2\right)\frac{6a''}{a^3}\right\}$$

$$\text{- - -}\!\times\!\text{- - -} = -\;\text{———}\!\times\!\text{———} \tag{2.28}$$

$$\text{———}\!\times\!\text{———} \approx \frac{2i\beta}{M_{\rm p}^2}\int_{-\infty}^{\eta}d\eta'\left\{\left(\overleftarrow{\partial}^2_{\eta'}+k^2\right)-\overleftarrow{\partial}_{\eta'}2\,aH\right\}\frac{1}{a^2}\left\{\left(\overrightarrow{\partial}^2_{\eta'}+k^2\right)-2\,aH\overrightarrow{\partial}_{\eta'}\right\}$$

$$\text{- - -}\!\times\!\text{- - -} = -\;\text{———}\!\times\!\text{———}\quad, \tag{2.29}$$

where the arrows indicate the propagator on which the derivatives act (because the vertex is quadratic, two propagators meet at the vertex.)

We are now ready to consider the correction due to the square of the Ricci scalar. The first order correction to the two-point function is given by the sum of the following two graphs,

$$\bullet\text{———}\!\times\!\text{——- - -}\bullet \approx \frac{i\alpha}{M_{\rm p}^2}\int_{-\infty}^{\eta}d\eta'\left\{\frac{i\delta(\eta-\eta')}{2k}\frac{6a''}{a^3}\right\}\approx -\frac{\alpha}{2k}\frac{12H^2}{M_{\rm p}^2},$$

$$\bullet\text{——- -}\!\times\!\text{- - - - -}\bullet = \left(\bullet\text{———}\!\times\!\text{——- - -}\bullet\right)^*, \tag{2.30}$$

where we have used the fact that $a''/a^3 \approx 2H^2$ to lowest order in slow-roll. Notice that the operator $\left(\overrightarrow{\partial}^2_{\eta'}+k^2\right)$ acting on the time-ordered propagators (2.20) or (2.21) produces a delta function, since both are Green's functions. On the other hand, when the same operator acts on the propagator (2.22) we get zero, because the latter is a regular solution of the free equation of motion (2.16) in the sub-horizon limit. This remark will turn out to be very useful when studying higher dimension operators.

We can now consider the correction due to the square of the Weyl tensor. In this case, the first order contribution is given by the sum of the following two graphs

$$= -\frac{2i\beta}{M_{\mathrm{p}}^2}\int_{-\infty}^{\eta} d\eta' \left\{\delta(\eta-\eta')aH + H^2 e^{2ik(\eta'-\eta)}\right\}$$

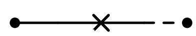
$$\approx -\frac{\beta}{2k}\left\{\frac{4iHk_{ph}}{M_{\mathrm{p}}^2} + \frac{2H^2}{M_{\mathrm{p}}^2}\right\}$$

$$= \left(\quad \right)^* . \qquad (2.31)$$

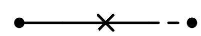
Notice that the imaginary parts cancel once we sum the two graphs. This result is quite general and ensures that only corrections with even powers of k_{ph} appear.

In conclusion, we have found that the leading corrections due to dimension four operators result in a two-point function which in the short-wavelength limit has the form

$+ \Big($



$+$



$$+ \text{c.c.}\Big) \approx \frac{1}{2k}\left[1 - (24\alpha + 4\beta)\frac{H^2}{M_{\mathrm{p}}^2}\right]. \qquad (2.32)$$

Thus, when H becomes of order M_{p}, these corrections become as important as the leading result, and standard cosmological perturbation theory ceases to be applicable, as was expected.

2.3.2 Higher Dimension Operators

We would now like to consider a generic operator of dimension $2d+4$, suppressed by a factor of order $1/M_{\mathrm{p}}^{2d}$. However, it turns out that considering directly corrections to the action (2.11) for the perturbations is a much more efficient approach than starting from generally covariant effective terms added to the Lagrangian (2.4), particularly if we are interested in identifying the dominant corrections in the sub-horizon limit. Hence, we shall focus directly on modifications to the action for the perturbations. A related approach has been described in [52].

Dimensional analysis implies that any operator of dimension $2d+4$, quadratic in the dimensionless tensor perturbations h_{ij} and proportional to $2f$ powers of the inflaton field φ must contain $2d-2f+4$ derivatives ∂_μ acting on h_{ij}, $\varphi_0(\eta)$ and $a(\eta)$. The derivatives can be distributed and contracted using the Minkowski metric in many different ways[4], but each of these terms can be schematically represented as

$$M_{\mathrm{p}}^{-2d-2}\left(\partial^{2n+m+p}\,[a,\varphi_0]\right)\,\left(\partial^{2q+m+r}\,v\right)\,\left(\partial^{2s+p+r}\,v\,\right), \tag{2.33}$$

where we have used equation (2.12), and $\partial^i[a,\varphi_0]$ is just a symbol that represents any combination of i derivatives acting on a's and φ_0's. One such term would have $2n+m+p$ derivatives acting on one or more factors of a or φ_0, $2q+m+r$ derivatives acting on one field v and $2s+p+r$ acting on the other v. In particular, $2n$ of the derivatives acting on the scale factor or the background field are contracted among themselves while m and p of them are contracted with derivatives acting on, respectively, the first and second field v. The derivatives acting on the fields v are organized in a similar way.

Let us illustrate this notation by considering a term with $p=s=f=0$, $m=n=q=1$ and $r=2$. Dimensional analysis implies that $d=3$, and thus the explicit form of such a term would be

$$M_{\mathrm{p}}^{-8}\partial_{2+1+0}\,[a]\,\partial_{2+1+2}\,v\,\partial_{0+0+2}\,v=M_{\mathrm{p}}^{-8}\partial_\mu\partial^\mu\,\partial_\nu[a]\,\partial_\lambda\partial^\lambda\,\partial^\nu\,\partial_\alpha\partial_\beta\,v\,\partial^\alpha\partial^\beta\,v\,, \tag{2.34}$$

where $\partial_\mu\partial^\mu\partial_\nu[a]$ denotes all possible ways to construct a term with three derivatives of the scale factor, with the given tensor structure.

[4]The reader may think that derivatives could be contracted not only among each other with the Minkowski metric, but also by using the additional tensor structure provided by the metric perturbations $\delta g_{\mu\nu}/a^2=h_{ij}\eta_{\mu i}\eta_{\nu j}$. However, it turns out that $\left(\delta g_{\mu\nu}/a^2\right)\partial^\nu a=h_{ij}\eta_{\mu i}\partial^j a=0$ and, since h_{ij} is transverse,

$$\partial^\mu\left(\delta g_{\mu\nu}/a^2\right)=\eta_{\nu j}\partial^i h_{ij}=0\,.$$

Thus, we get a non-vanishing contribution only when we contract derivatives with the Minkowski metric while the factors $\eta_{\mu i}\eta_{\nu j}$ are contracted among each other yielding an irrelevant overall factor.

The first step to estimate the leading correction due to a term of the form (2.33) is realizing that this can always be expressed as a linear combination of terms of the form

$$M_{\mathrm{p}}^{-2d-2}\partial^{2j+l}\,[a,\varphi_0]\,\partial^{2m+l}v\,\partial_\mu\partial^\mu v, \tag{2.35}$$

plus, possibly, a term with no derivatives acting on v, which in any case gives a contribution that is always subdominant in the sub-horizon limit. A proof of this goes as follows. We want to integrate by parts every term of the form

$$\partial^{\,2n+m+p}\,[a,\varphi_0]\,\partial^{\,2q+m+r}\,v\,\partial^{\,2s+p+r}\,v \tag{2.36}$$

in order to express it as a linear combination of terms like

$$\partial^{2j+l}\,[a,\varphi_0]\,\partial^{2m+l}v\,\Box\,v \tag{2.37}$$

plus, possibly, a term with no derivatives acting on v. Notice that, for notational convenience, we have defined $\Box\equiv\partial_\mu\partial^\mu$. Of course, if the index q (or s) in equation (2.36) is not zero, we can easily integrate by parts $2q+m+r-2$ $(2s+p+r-2)$ times to get only terms of the form of that in equation (2.37). Therefore, in what follows we only consider terms with $q=s=0$. When this is the case, we can always integrate by parts an appropriate number of times to get only terms for which $m=p$. Thus, without loss of generality, we can restrict ourselves to considering terms of the form

$$\partial^{\,2n+m+p}\,[a,\varphi_0]\,\partial^{\,m+r}\,v\,\partial^{\,p+r}\,v, \qquad\qquad (m=p)\ . \tag{2.38}$$

The derivatives acting on v that are contracted with derivatives acting on a or φ_0 can be systematically eliminated by repeated integrations by parts:

$$\begin{aligned}\partial^{\,2n+m+p}\,[a,\varphi_0]\,\partial^{\,m+r}\,v\,\partial^{\,p+r}\,v &\sim -\partial^{\,2n+m+(p-1)}\,[a,\varphi_0]\,\partial^{\,m+r}\,v\,\partial^{\,(p-1)+r}\,\Box\,v\\ &+\frac{1}{2}\partial^{\,2(n+1)+(m-1)+(p-1)}\,[a,\varphi_0]\,\partial^{\,(m-1)+(r+1)}\,v\,\partial^{\,(p-1)+(r+1)}\,v\ ,\end{aligned} \tag{2.39}$$

where we have denoted equivalence up to integration by parts with $\sim$. The first term on the right hand side can be cast in the form (2.37) by integrating by parts $(p-1)+r$

times, while the second one is of the form (2.38) with n and r (p and m) increased (decreased) by one. By iterating this procedure, we eventually obtain terms of the form

$$\partial^{2n}\,[a,\varphi_0]\,\partial^r\,v\,\partial^r\,v\ , \tag{2.40}$$

where now n and r have changed. Again, we can integrate by parts and obtain

$$\partial^{2n}\,[a,\varphi_0]\,\partial^r\,v\,\partial^r\,v \sim -\partial^{2n}\,[a,\varphi_0]\,\partial^{r-1}\,v\,\partial^{r-1}\,\Box\,v + \frac{1}{2}\partial^{2(n+1)}\,[a,\varphi_0]\,\partial^{(r-1)}\,v\,\partial^{(r-1)}\,v\ .$$

The first term on the right hand side can be re-written as (2.37) after $r-1$ integrations by parts, while the second term has the form (2.40) with n (r) increased (decreased) by one. Thus, by repeating this procedure we obtain many terms of the form (2.37) and we are eventually left with a term without derivatives acting on v. This completes our proof. In what follows, we therefore restrict ourselves to terms of the form (2.35).

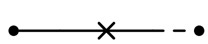
Dimensional analysis requires that the indexes j, l and m in equation (2.35) obey

$$j + l + m = d - f + 1\ . \tag{2.41}$$

Furthermore, since $d^n\varphi_0/d\eta^n \propto \sqrt{2\epsilon}\,M_{\rm p}\,a^n H^n$ and $d^n a/d\eta^n \propto a^{n+1}H^n$, each field φ_0 yields a factor of $M_{\rm p}$, while each derivative acting on it or on the scale factor results in a factor of H to leading order in slow-roll. Finally, the l partial derivatives ∂_μ acting on v that are contracted with derivatives acting on a or φ_0 can be turned into derivatives with respect to η only. Thus, (2.35) can be re-written as

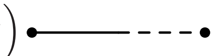
$$M_{\rm p}^{-2d+2f-2} f(a) H^{2j+l}\,\Box^m \partial_\eta^l v\,\Box\,v\ , \tag{2.42}$$

where we have defined $\Box \equiv \partial_\mu\partial^\mu$; the corresponding correction to the two-point function is schematically given by



$$= \frac{i}{M_{\rm p}^{2d-2f+2}}\int_{-\infty}^{\eta} d\eta' f(a) H^{2j+l} \times \tag{2.43}$$

$$\left(\overleftarrow{\Box}^m\,\overleftarrow{\partial}^{\,l}_{\eta'}\,\overrightarrow{\Box} + \overleftarrow{\Box}\,\overrightarrow{\partial}^{\,l}_{\eta'}\,\overrightarrow{\Box}^m\right)$$



plus the complex conjugate of this graph. Because (2.22) satisfies the free equation of motion, this correction is non-vanishing only when the index m is equal to zero, and in this case we obtain

$$
\begin{aligned}
&= \frac{1}{M_{\mathrm{p}}^{2d-2f+2}} \int_{-\infty}^{\eta} d\eta' f(a) H^{2j+l} \delta(\eta-\eta') \frac{(ik)^l}{2k} \\
&= \frac{f(a)}{2k} \frac{H^{2j+l}(ik)^l}{M_{\mathrm{p}}^{2d-2f+2}}.
\end{aligned}
\tag{2.44}
$$

The leading correction in the short-wavelength limit is the one with the maximum number of powers of k. According to equation (2.41), this maximum number simply equals $d - f + 1 \equiv l_{max}$, and it corresponds to the case in which $j = m = 0$. Thus, if $d-f$ is odd, l_{max} is even and the leading correction is simply given by

$$
\delta\langle v^*(\mathbf{k})v(\mathbf{k})\rangle \propto \frac{1}{2k}\left(\frac{H}{M_{\mathrm{p}}}\right)^{d-f+1}\left(\frac{k_{ph}}{M_{\mathrm{p}}}\right)^{d-f+1} \qquad (d-f \text{ odd}) , \tag{2.45}
$$

since each factor of k/M_{p} must be accompanied by a factor of a to render the spatial momentum physical. On the other hand, if $d-f$ is even, l_{max} as defined above is odd and the term with the highest number of powers of k is purely imaginary. As we have seen in the previous section, such a term disappears when we add the contribution from the complex conjugate graph. Therefore, the leading correction corresponds to the largest *even* value of l, which turns out to be $l_{max} = d - f$, and is therefore given by

$$
\delta\langle v^*(\mathbf{k})v(\mathbf{k})\rangle \propto \frac{1}{2k}\left(\frac{H}{M_{\mathrm{p}}}\right)^{d-f+2}\left(\frac{k_{ph}}{M_{\mathrm{p}}}\right)^{d-f} \qquad (d-f \text{ even}) . \tag{2.46}
$$

Equations (2.45) and (2.46) represent the main results of this section: they express the leading corrections to the two-point function (in the sub-horizon limit) associated with a generic operator of dimension $2d+4$ containing $2f$ powers of the inflaton field. Since we have assumed an approximate shift invariance of the inflaton, the total number of derivatives, $2d-2f+2$, must be greater or equal than the number of fields $2f$, which in turn implies that $d \geq f$. Thus, we can label all the possible corrections

Table 2.1: Leading corrections to the gravitational wave two-point functions in the short-wavelength limit.

$d-f$	Leading correction	$d-f$	Leading correction
0	H^2/M_p^2	4	$H^6k_{ph}^4/M_\mathrm{p}^{10}$
1	$H^2k_{ph}^2/M_\mathrm{p}^4$	5	$H^6k_{ph}^6/M_\mathrm{p}^{12}$
2	$H^4k_{ph}^2/M_\mathrm{p}^6$	6	$H^8k_{ph}^6/M_\mathrm{p}^{14}$
3	$H^4k_{ph}^4/M_\mathrm{p}^8$	7	$H^8k_{ph}^8/M_\mathrm{p}^{16}$

with the non-negative index $d-f$. Their magnitude is given in Table 2.1 for the first eight values of $d-f$. Note that corrections with $d-f=0$ arise from the operators identified by Weinberg in [109]. The leading momentum-dependent corrections are given by operators with $d-f=1$.

So far, we have calculated the largest possible corrections to the two-point function in the sub-horizon limit given a certain value of $d-f$. However, the reader might still wonder whether such terms can be actually obtained from a covariant action. Employing the same technique we used to study the impact of the lowest order terms, it is indeed possible to show – after some rather lengthy calculations – that the following family of covariant terms generates this kind of contributions,

$$\begin{aligned} d-f=0: &\qquad R^{\mu\nu}\,R_{\mu\nu} \\ d-f=1: &\qquad (\nabla^\alpha R^{\mu\nu})\,\nabla_\alpha R_{\mu\nu} \\ d-f=2: &\qquad (\nabla^\alpha\nabla^\beta R^{\mu\nu})\,\nabla_\alpha\nabla_\beta R_{\mu\nu} \\ \vdots\quad &\qquad\qquad \vdots \qquad\qquad . \end{aligned} \tag{2.47}$$

It can be also verified that the $d-f=1$ term yields a correction to the two-point function proportional to the slow-roll parameter ϵ, and given the structure of this family of operators, we anticipate the remaining terms to share the same slow-roll suppression.

2.3.3 The Three-Momentum Scale Λ

The corrections to the two-point function are functions of two dimensionful parameters, H and k_{ph}. For our purposes, it is convenient to organize these corrections in powers of k_{ph}. Thus, following Table 2.1, and reintroducing the subleading terms that we previously neglected, we find that the two-point function is

$$\langle v^*(\mathbf{k})v(\mathbf{k})\rangle \approx \frac{1}{2k}\left[\left(1+\alpha_{20}\frac{H^2}{M_\mathrm{p}^2}+\cdots\right)+\left(\alpha_{22}\frac{H^2}{M_\mathrm{p}^2}+\alpha_{42}\frac{H^4}{M_\mathrm{p}^4}+\cdots\right)\frac{k_{ph}^2}{M_\mathrm{p}^2}+\right.$$
$$\left.+\left(\alpha_{44}\frac{H^4}{M_\mathrm{p}^4}+\alpha_{64}\frac{H^6}{M_\mathrm{p}^6}+\cdots\right)\frac{k_{ph}^4}{M_\mathrm{p}^4}+\cdots\right]. \quad (2.48)$$

The coefficient α_{20} is of order one, while all the α_{nn} with $n \geq 2$ are of order ϵ, as the family of covariant terms (2.48) suggests. At the end of Section 2.4 we provide further evidence supporting this claim.

In order for Equation (2.48) to be a valid perturbative expansion, every correction term must be much smaller than one. Because α_{20} is of order one, this implies the condition

$$\frac{H}{M_\mathrm{p}} \ll 1, \quad (2.49)$$

which must hold for all values of k_{ph}. Equation (2.48) then shows that if condition (2.49) is satisfied, the corrections to the two-point function remain small even for $k_{ph} \approx M_\mathrm{p}$. In fact, to leading order in H/M_p we can rewrite equation (2.48) as

$$\langle v^*(\mathbf{k})v(\mathbf{k})\rangle \approx \frac{1}{2k}\left[1+\alpha_{22}\frac{k_{ph}^2}{\Lambda^2}+\alpha_{44}\frac{k_{ph}^4}{\Lambda^4}+\cdots\right], \quad (2.50)$$

where we have introduced the effective cutoff

$$\Lambda \approx \frac{M_\mathrm{p}^2}{H}. \quad (2.51)$$

Equations (2.50) and (2.51) are the main result we were aiming at, given that they show the three-momentum scale at which the perturbative expansion fails. For $k_{ph} \ll \Lambda$, all the corrections are strongly suppressed and can thus be neglected. However, at $k_{ph} \approx \Lambda$, all the corrections become of the same order ϵ, the asymptotic series breaks down, and the effective theory ceases to be valid. As we discuss below, this value of Λ should be understood as an upper limit on the validity of cosmological perturbation theory.

To conclude this section, let us briefly comment on the effects of terms that break the shift symmetry. Because the only difference is that these terms contain undifferentiated scalars, any such correction can be cast as a generally covariant term that respects the symmetry, multiplied by a power of the dimensionless ratio φ/M_p. Hence, these terms introduce corrections to the two-point function of the form we have already discussed, but with coefficients α_{ij} that can now depend on arbitrary powers of the background field φ_0,

$$\alpha_{ij} = \alpha_{ij}^{(0)} + \alpha_{ij}^{(1)} \frac{\varphi_0}{M_\mathrm{p}} + \alpha_{ij}^{(2)} \left(\frac{\varphi_0}{M_\mathrm{p}} \right)^2 + \cdots . \tag{2.52}$$

Therefore, in the absence of any mechanism or symmetry that keeps the coefficients $\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \ldots$ small (e.g. an approximate shift symmetry), such an expansion looses its validity for $\varphi_0 > M_\mathrm{p}$, regardless of the value of k_{ph}. If, on the other hand, equation (2.52) is a sensible expansion, and $\alpha_{ij}^{(0)}$ is much greater than $\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, ...$, then we can effectively assume that the shift-symmetry is exact, and perturbation theory breaks down again at $k_{ph} \approx \Lambda$.

2.3.4 Loop Diagrams and Interactions

Our analysis so far has concentrated only on tree-level corrections to the two-point function, which arise just from the quadratic terms in the action. Cubic and higher order interactions also contribute to the two-point function, but their contribution is obscured by the appearance of divergent momentum integrals in loops. Even in a non-renormalizable theory like general relativity, at any order in the derivative expansion it is still possible to cancel these divergences by renormalizing a finite number of parameters, provided that all terms consistent with the symmetries of the theory are included in the Lagrangian [8]. In practice, this cancellation is due to the presence of appropriate counter-terms in the Lagrangian. For this reason, in the present context, divergent integrals in loop diagrams are rather harmless. They yield corrections of the same structure as tree-level diagrams, modulo a (mild) logarithmic running of their values with scale [114]. Hence, we do not expect this type of contributions to drastically change our conclusions, though we should emphasize that this is just an expectation. Thus, strictly speaking, equation (2.51) is just an upper limit for the scale beyond which the corrections to the two-point function remain small and one can trust cosmological perturbation theory.

2.4 The Limits of Perturbation Theory: Scalars

We now turn our attention to corrections to the two-point function of scalar perturbations. Despite some complications that are particular to the this sector, the method developed in the previous section can be easily extended to scalars.

To this end, let us consider the action $S = S_0 + \lambda S_1$, where S_0 is the action (2.4) describing a scalar field minimally coupled to Einstein gravity, while S_1 is a generic generally covariant correction suppressed by a coupling $\lambda \sim 1/M_{\rm p}^{2d}$. As we pointed out in the previous section, S_1 generically involves contractions of the Riemann tensor $R_{\mu\nu\lambda\rho}$ and the covariant derivative ∇_μ as well as the scalar field φ. In order to compute the resulting first order contribution to the two-point function for $v = a\,\delta\varphi$, we insert the perturbed metric (2.13) and the perturbed inflaton field into the action S and expand up to second order in $\delta\varphi, \phi$ and B. We then express the metric perturbations in terms of $\delta\varphi$ using equations (2.14) and, finally, in terms of v using the definition (2.15). Note that, even though the relations (2.14) were derived by solving the constraints associated with the unperturbed action for the perturbations $\delta_2 S_0$, corrections to these relations due to $\delta_2 S_1$ do not contribute at first order in λ. To show this, let us expand ϕ and B in powers of the coupling

$$\phi = \phi_0 + \lambda\phi_1 + \mathcal{O}(\lambda^2), \qquad B = B_0 + \lambda B_1 + \mathcal{O}(\lambda^2), \tag{2.53}$$

where ϕ_0 and B_0 satisfy the unperturbed relations (2.14) and (2.14) respectively. By expanding the full quadratic action for the perturbations $\delta_2 S\,[\delta\varphi,\phi,B]$ to first order in λ we obtain

$$\begin{aligned}\delta_2 S\,[\delta\varphi,\phi,B] \;\approx\;& \delta_2 S_0\,[\delta\varphi,\phi_0,B_0]+\lambda\int\left[\frac{\delta(\delta_2 S_0)}{\delta\phi}\right]_{\lambda=0}\phi_1+\lambda\int\left[\frac{\delta(\delta_2 S_0)}{\delta B}\right]_{\lambda=0}B_1\\ &+\lambda\,\delta_2 S_1[\delta\varphi,\phi_0,B_0]+\mathcal{O}(\lambda^2).\end{aligned} \tag{2.54}$$

However, the second and the third term vanish because they contain the unperturbed constraint equations evaluated at ϕ_0 and B_0, which by assumption are solutions to the constraints. Therefore, as long as we are interested in first order results, we can safely use the unperturbed solutions ϕ_0 and B_0 given by equations (2.14).

Expanding the leading action S_0 to quadratic order in the perturbations, we obtain the free action (2.11). The additional quadratic terms stemming from S_1 must be appropriately contracted expressions containing partial derivatives of the perturbation variable v, the scale factor a and the background field φ_0. However, in the case of scalar perturbations, the field v can arise from fluctuations of the scalar field, $\delta\varphi=v/a$, or from fluctuations of the metric,

$$\delta g^{\mu\nu}=-\frac{a\sqrt{2\epsilon}}{M_{\rm p}}\left[2v_k\delta^\mu{}_0\delta^\nu{}_0+\frac{ik_j}{k^2}\left(v_k'-aHv_k\right)\left(\delta^\mu{}_j\delta^\nu{}_0+\delta^\mu{}_0\delta^\nu{}_j\right)\right]\equiv\frac{a\sqrt{2\epsilon}\,\mathcal{V}^{\mu\nu}}{M_{\rm p}}\,. \tag{2.55}$$

Thus, unlike the case of tensor perturbations, $\delta g^{\mu\nu}$ provides an additional tensor structure that can be used to contract derivatives. We now show that such contractions yield terms where the derivatives acting on v or a are contracted with $\eta^{\mu\nu}$. This means that the argument in the previous section can be applied to scalar perturbations as well, yielding essentially the same results. We note that terms which contain only fluctuations coming directly from the scalar field do not present this problem, and can be easily written as in equation (2.33).

Let us first consider terms with only one factor of $\mathcal{V}^{\mu\nu}$. In this case, $\mathcal{V}^{\mu\nu}$ can be contracted either with $\eta_{\mu\nu}$, leading to $\mathcal{V}^{\mu\nu}\eta_{\mu\nu}=2\,v$, or with two derivatives $\partial_\mu\partial_\nu$, resulting in

$$\partial_\mu \partial_\nu \mathcal{V}^{\mu\nu} = 2\left(\frac{\partial_\mu \partial^\mu a}{a} v - \frac{\partial_\mu a\, \partial^\mu a}{a^2} v + \frac{\partial_\mu a\, \partial^\mu v}{a}\right), \tag{2.56}$$

$$\partial_\mu a\, \partial_\nu \mathcal{V}^{\mu\nu} = \partial_\mu a\, \partial^\mu v + \frac{\partial_\mu a\, \partial^\mu a}{a} v, \tag{2.57}$$

$$\partial_\mu \partial_\nu [a, \varphi_0]\, \mathcal{V}^{\mu\nu} = 2\left(\partial_\mu \partial^\mu [a, \varphi_0]\right) v\ , \tag{2.58}$$

where, again, the square brackets in the last line mean that the derivatives can act on *one or more* factors of a or φ_0. Thus, terms with only one factor of $\mathcal{V}_{\mu\nu}$ do not present any problem since, as anticipated, all the derivatives are contracted with the inverse of the Minkowski metric.

Corrections which contain two factors of $\mathcal{V}^{\mu\nu}$, and are not products of terms in (2.56 – 2.58), can always be recast as

$$\mathcal{V}_{\mu\nu}\mathcal{V}^{\mu\nu} = 2v^2 + \frac{2}{k^2}\left[\partial_\mu v\, \partial^\mu v - \frac{\partial_\mu a}{a}\, \partial^\mu (v^2) + \frac{\partial_\mu a\, \partial^\mu a}{a^2} v^2\right], \tag{2.59}$$

$$\begin{aligned}\partial^\mu \mathcal{V}_{\mu\nu} \partial_\lambda \mathcal{V}^{\lambda\nu} = -\frac{1}{k^2}\partial_\mu v' \partial^\mu v' + \frac{1}{k^2}\Bigg[&2\frac{\partial_\mu a\, \partial^\mu v}{a}\partial_\nu \partial^\nu v - 2\left(\frac{\partial_\mu \partial_\nu a}{a} - \frac{\partial_\mu a\, \partial_\nu a}{a^2}\right) v\, \partial^\mu \partial^\nu v \\ &- \frac{\partial_\mu a\, \partial^\mu a}{a^2}\partial_\nu v\, \partial^\nu v + \left(\frac{\partial_\mu \partial^\mu a}{a} - \frac{\partial_\mu a\, \partial^\mu a}{a^2}\right)\frac{\partial_\nu a}{a}\, \partial^\nu (v^2)\Bigg].\end{aligned} \tag{2.60}$$

All the terms inside the square brackets become negligible in the sub-horizon limit, since their contribution is suppressed by an extra factor of $1/k^2$. The only term in which some derivatives are not contracted with the Minkowski inverse metric is the first one in equation (2.60). However, the two derivatives with respect to conformal time result in a factor of k^2 which is precisely canceled by the extra factor $1/k^2$, and for all practical purposes such a term is equivalent to $\partial_\mu v\, \partial^\mu v$.

Therefore, we have demonstrated that terms quadratic in the scalar fluctuations can be schematically written as in equation (2.33). The remainder of the analysis then proceeds as for tensor perturbations, and effective corrections to scalar perturbations are thus also subdominant in the regime

$$H \ll M_\mathrm{p} \quad \text{and} \quad k_{ph} \ll \Lambda \sim \frac{M_\mathrm{p}^2}{H}\ . \tag{2.61}$$

Before concluding, we would like to address again whether the operators that we have considered can be actually obtained from generally covariant terms. In the case of scalar perturbations, it is indeed possible to show – after further rather lengthy calculations – that the following family of covariant terms generates the kind of corrections shown in Table 2.1,

$$\begin{aligned}
d-f=0: &\qquad R^{\mu\nu}\,(\nabla_\mu\,\varphi)\,\nabla_\nu\,\varphi \\
d-f=1: &\qquad R^{\mu\nu}\,(\nabla_\mu\,\varphi)\,\nabla_\nu\,\nabla_\gamma\,\nabla^\gamma\,\varphi \\
d-f=2: &\qquad (\nabla^\alpha\nabla^\beta R^{\mu\nu})\,(\nabla_\alpha\nabla_\mu\,\varphi)\,\nabla_\beta\nabla_\nu\,\varphi \\
d-f=3: &\qquad (\nabla^\alpha\nabla^\beta R^{\mu\nu})\,(\nabla_\alpha\nabla_\mu\,\varphi)\,\nabla_\beta\nabla_\nu\,\nabla_\gamma\,\nabla^\gamma\,\varphi \\
\vdots\qquad &\qquad\qquad \vdots \qquad\qquad .
\end{aligned} \tag{2.62}$$

In order to illustrate how this happens, let us consider for example the $d-f=1$ term. It contains, among many other terms a factor

$$a^2\,R^{\mu\nu}\,\partial_\mu\,\delta\varphi\,\partial_\nu\,\partial_\gamma\,\partial^\gamma\,\delta\varphi \supset \frac{2\epsilon}{a^6}\,\partial^\mu a\,\partial^\nu a\,\partial_\mu\,v\,\partial_\nu\,\partial_\gamma\,\partial^\gamma\,v \sim -\frac{2\epsilon}{a^6}\,\partial^\mu a\,\partial^\nu a\,\partial_\mu\,\partial_\nu\,v\,\Box\,v + ... \tag{2.63}$$

where, in the last step, we have neglected a subdominant contribution in the short-wavelength limit. The last term in (2.63) indeed generates a correction proportional to $H^2k_{ph}^2/M_\mathrm{p}^4$ and it is suppressed by one factor of the slow-roll parameter. It is relatively easy to verify that the corrections generated by the other members of the family (2.62) have the same slow-roll suppression, which strongly supports the assumption we made in the context of tensor perturbations.

2.5 Discussion

The connection, through cosmological inflation, between physics on the smallest scales, described by quantum field theory, and that on the largest scales in the universe is one of the most profound aspects of modern cosmology. However, since inflation takes place at such early epochs, and magnifies fluctuations of such small wavelengths, it is important to establish the regime of validity of the usual formalism – that of semiclassical gravity, with quantum field theory assumed valid, and coupled to the minimal Einstein-Hilbert action – at those scales.

On general grounds, we expect the canonical approach to break down at ultra-short distances, where the operators that arise in an EFT treatment of the coupled metric-inflaton system become relevant. In this chapter we have calculated the impact of these higher-dimensional operators on the power spectrum at short wavelengths. In this way, we have been able to probe the regime in which the properties of the perturbations deviate from what is conventionally assumed. From a purely theoretical standpoint, these considerations are important if we are to understand the limits of applicability of cosmological perturbation theory. From an observational standpoint, cosmic microwave background measurements are becoming so precise that we may hope to use them to identify the signatures of new gravitational or field theoretic physics.

Our analysis has focused on tree-level corrections to the spectrum. Because we have essentially considered all possible generally covariant terms in the effective action, we expect to have unveiled the form of all possible corrections that are compatible with the underlying symmetries of the theory. It is however possible that loop diagrams yield additional corrections that we have not considered. In any case, our results indicate that cosmological perturbation theory does not apply all the way to infinitesimally small distances, $k_{ph} \to \infty$, and that, indeed, there is a physical spatial momentum Λ (or a physical length $1/\Lambda$) beyond which cosmological perturbation ceases to be valid. The scale at which perturbation theory breaks down has to be lower than

$$\Lambda \sim \frac{M_{\mathrm{p}}^2}{H}, \tag{2.64}$$

which, because of existing limits on the scalar to tensor power spectrum ratio [115], is at least 10^4 times the Planck scale.

These results have significant implications for the impact of trans-Planckian physics on the primordial spectrum of primordial perturbations, which typically is at most of order H/Λ [95]. Substituting the upper limit of Λ we have found, we obtain corrections of the order of H^2/M_{p}^2, which are likely to remain unobservable [99]. This value of Λ also solves a problem that was noticed in [116], namely, that in the presence of a Planckian cutoff, cosmological perturbations do not tend to decay into the Bunch-Davis vacuum (or similar states). In particular, to lowest order in perturbation theory, the transition probability from an excited state into the Bunch-Davis vacuum is significantly less than one for $\Lambda = M_{\mathrm{p}}$, but proportionally larger if Λ is given by (2.64). Ultimately, a large decay probability is what justifies the choice of the Bunch-Davies vacuum as the preferred initial state for the perturbations at scales below the cutoff, since, as we have found, our theories certainly lose their validity at momentum scales above the spatial momentum Λ.

Chapter 3

Eucidean Path Integral Formalism and 6^{th} Order Corrections

3.1 Introduction

We have seen in the previous chapter how the effective field theory approach can be applied to problems such as determining the limits of validity of perturbation theory in the early universe. We can now take a more radical point of view, still looking for small corrections of the underlying cosmological system (here the description of late time acceleration), but now giving physical meaning to some higher derivative terms via the Euclidean path integral formulation.

One of the main motivation for considering higher order systems comes from fairly recent developments in modeling late time acceleration. The suggestion that the accelerated expansion of the universe may be explained by an infrared modification of gravity has fueled renewed interest in higher derivative theories and their associated pathologies [117–124]. Corrections to the Einstein-Hilbert action, when coming from pure gravity, must be constructed with covariant contractions of powers of the Riemann tensor. When this happens though, the corrections may contain four or more time derivatives acting on the physical field, the metric. We have seen in the introduction in section 1.3.1 that, a few special cases aside, systems with more than two time derivatives can be described via a classically equivalent Lagrangian that quite generically contains ghosts – degrees of freedom with the wrong sign for the kinetic term. These ghosts lead to catastrophic instabilities if they appear in the perturbative spectrum and hence have to be removed or "cured". It follows then that unless a scheme to deal with the ghosts is chosen, the models containing them become unsuitable to describe physical phenomena.

The best known and standard way to make sense of theories with higher derivatives is through the effective field theory approach described and utilized in the previous chapters. We recall that, from the EFT point of view, terms that contain higher derivatives appear from an expansion of an unknown UV-complete theory. This expansion, by definition, is supposed to provide an accurate description of the full theory only at low energies, and the physical degrees of freedom are assumed to be only those that appear in the ground state of the theory [52]. Classically, the presence of extra solutions to the field equations that correspond to the existence of ghosts is then considered as an artifact of the effective theory, due to the truncation of an infinite series. If then, for instance, it is possible to push the masses of these degrees of freedom beyond the cutoff – the energy scale below which the effective field theory is trusted – the ghosts can be ignored.

Effective field theory then provides a safe way to deal with higher derivatives. They arise naturally and are considered an artifact, so can be ignored unless they introduce unwanted corrections below the cutoff. This is the point of view taken in chapter 2, where back-substitution has been enforced. In other words, in the previous chapter the propagators were defined by the quadratic free action, and so only described the propagation of the degrees of freedom of the low energy second order theory. At the same time, higher time derivatives have been treated as interactions – vertices – correcting the propagation of the physical degrees of freedom.

The story seems to be complete, and there seems to be very little need to go beyond these ideas, especially since difficulties like ghost fields lurk around the corner. This is of course, unless one thinks that higher derivative terms in the action may have a physical meaning. We cite here a couple of these alternative procedures that have been proposed to deal with higher derivative terms [125, 126]. In these works the authors apply new procedures to usual systems finding results that differ from the canonical EFT treatment, and that have quite deep philosophical consequences. In the first case for instance, it is shown that quantum systems described by a higher derivative action may not have a classical smooth limit (in their description), while retaining the ability of matching observational constraints. In the second case instead, the main purpose is to try to get closer to a quantizable and renormalizable theory of gravity.

Quite in general, taking the bottom-up approach from GR to a more fundamental theory of quantum gravity, higher derivatives appear and going beyond the parametrization of our ignorance of such a theory requires making physical sense of these objects.

We can then see that there are very good motivations to think of higher order terms, and also to look for something else than EFT. In fact it has to be said that EFT, relying on a series expansion also lacks the ability of describing runaway solutions which in general are troublesome, but that may make sense in a cosmological framework [61].

In this chapter we will focus on the prescription introduced by Hawking and Hertog [126], who demonstrated that the Euclidean path integral formulation of the quantum theory allows one to define a probability distribution for a scalar field that appears in the Lagrangian with four time derivatives. In particular their work tried to make sense of the ghosts without incorporating them into the full physical theory, but by assuming that they correspond to some infinities that – like in many other theories – need to be integrated away in some smart way to avoid losing the information they hide. Adopting this point of view, the Euclidean path integral approach could be seen as a substitute of EFT and nothing more, since its main purpose is still to find corrections to the lower order theory. Either way an interesting result arises; in fact the two methods do not always agree allowing for a possibility of testing their efficacy and validity with experiments, and therefore moving from speculation to physics.

It will be more clear later, when the details of the calculations are shown, that if the higher order terms are considered as corrections to the second order action for the field, the results calculated in the Euclidean path integral approach could lead, in principle, to a different physical result. This happens because corrections to the probability for the fields may display a different dependence on the "coupling constant" than the equivalent corrections calculated via EFT. Here by "coupling constant" we mean the parameter that controls the strength of the higher order term in the action. For example, in higher order theories of gravity here considered, the "coupling constant" contains appropriate inverse powers of the cutoff scale. A different dependence on the behavior of such corrections may shift the energy at which they become important. However, at fourth order, the analysis in [127] of a term proportional to the Weyl tensor squared demonstrated that in a de Sitter background there is no discrepancy between the Euclidean path integral procedure and the effective field theory one.

In this chapter, we explore further whether the Euclidean path integral approach can be extended to apply generally, to higher derivative orders, and study whether observational differences from the effective field theory approach can be realized in practice. Since the study of a general higher derivative correction to the Einstein-Hilbert action is prohibitively complicated, we therefore focus here on a nontrivial correction beyond 4th order, namely the 6th order term $\nabla_\alpha R_{\mu\nu}\nabla^\alpha R^{\mu\nu}$. As we will see, this is sufficient to draw interesting conclusions. We demonstrate how to apply, at least in principle, the Euclidean path integral prescription to sixth order terms. However, the question of whether it can be applied to a specific system such as General Relativity plus fourth and sixth order corrections is highly dependent on the choice of background. In particular, a Minkowski background always admits choices for the "coupling constants" that yield a well defined Euclidean theory. On the other hand a de Sitter background, due to its explicit time dependence, introduces some complications, since the simple requirements to apply the whole prescription are not met. Nevertheless, as happened in the fourth order case, we shall see that this does not preclude the possibility of finding a viable result.

The chapter is organized as follows. Section 3.2 is devoted to a brief review of the Euclidean path integral approach, and a discussion of its generalization to sixth order for a certain class of quadratic Lagrangians. In section 3.3 we derive the perturbed action for tensorial modes coming from a sixth order action about two backgrounds, Minkowski and de Sitter. We then solve for the classical solutions and perform the canonical procedure to build the path integral in the Lagrangian formulation. Finally, in section 3.4 we comment on the results and present our conclusions. As done before, we use t and η to denote cosmological and conformal times respectively, and denote the time derivative with respect to them with an overdot $d/dt \equiv \dot{(\ )}$, with the difference between t and η being clear from the context. After a Wick rotation the time coordinate is described by a real parameter that for both cosmic and conformal times we call τ, and the derivative with respect to it is represented by a prime sign $d/d\tau \equiv (\ )'$. Conformal time is only used when the de Sitter background is taken into consideration.

3.2 Review of the Hawking-Hertog Formalism

We can start by reviewing the idea behind the Euclidean path integral procedure and discussing the ways in which the fourth order case differs from the usual second order treatment.

Following the lines of what we have seen in the introduction in sections 1.2, 1.3.1, and 1.3.2, in a second order theory the propagator for a field ϕ defined by a Lagrangian

L_ϕ can be found computing a path integral between the initial and final configurations

$$\langle(\phi_f;t_f)|(\phi_i;t_i)\rangle = \int_{\phi_i}^{\phi_f} d[\phi(t)] \exp\left[iS[\phi]\right] \ , \tag{3.1}$$

where the action for the field is given by

$$S[\phi] = \int_{t_i}^{t_f} dt' L(\dot{\phi}, \phi, t') \ . \tag{3.2}$$

Here $\phi_\times$ represents the state of the field at time $t_\times$. A system described by a quadratic Lagrangian with a higher number of time derivatives can be transformed into a second order system via nonlinear transformations[1]; for instance a fourth order system with Lagrangian

$$L = -\frac{1}{2}\phi\left(\frac{d^2}{dt^2} - m_1^2\right)\left(\frac{d^2}{dt^2} - m_2^2\right)\phi, \tag{3.3}$$

can be recast as

$$L = \frac{1}{2}\psi_1\left(\frac{d^2}{dt^2} - m_1^2\right)\psi_1 - \frac{1}{2}\psi_2\left(\frac{d^2}{dt^2} - m_2^2\right)\psi_2 \ , \tag{3.4}$$

where ψ_1 and ψ_2 are defined via

$$\psi_1 = \frac{(\frac{d^2}{dt^2} - m_2^2)\phi}{\sqrt{m_2^2 - m_1^2}} \ , \quad \psi_2 = \frac{(\frac{d^2}{dt^2} - m_1^2)\phi}{\sqrt{m_2^2 - m_1^2}} \ . \tag{3.5}$$

In the canonical treatment of higher order systems, this transformed Lagrangian (3.4) is the starting point and the system is viewed as a multi-field one, where at least one of the newly defined second order fields is a ghost. In the case at hand it is easy to note that ψ_1 has the wrong sign for the kinetic term, playing the role of the ghost field.

The propagator is given by a path integral over both fields

$$\langle(\psi_{2f},\psi_{1f};t_f)|(\psi_{2i}\psi_{1i};t_i)\rangle = \int_{(\psi_2,\psi_1)_i}^{(\psi_2,\psi_1)_f} d[\psi_2(t)]d[\psi_1(t)] \exp\left[iS[\psi_2,\psi_1]\right] \ . \tag{3.6}$$

Note that via the definitions of ψ_1 and ψ_2, this functional integration can be interpreted as integrating over the original field ϕ and its second time derivative. However,

[1] We can ignore spatial dependence for the moment or, equivalently, we think of the field ϕ as a particular Fourier mode.

as pointed out in [126], this choice presents a problem. For second order systems the propagator obeys the composition law

$$G(\phi_3, \phi_1) = \int d[\phi_2] G(\phi_3, \phi_2) G(\phi_2, \phi_1) \ , \tag{3.7}$$

where $G(\phi_3, \phi_1)$ is the propagator between the two states "1" and "3", and "2" represents an intermediate state. When one joins the fields above and below the intermediate time t_2, the value of the field $\phi(t_2)$ is fixed, but its time derivative is not, resulting in a jump in $\dot{\phi}(t_2)$, which in turn corresponds to a delta function in the value of $\ddot{\phi}(t_2)$. Unfortunately, in the original fourth order action (3.3) the second time derivative appears quadratically, and hence the original composition law of the path integral is lost and infinities arise. This argument applies quite generally to higher derivative systems as we have seen in section 1.3.1.

In contrast to using the two new fields that are combinations of the original field and its second derivative, the alternative procedure proposed in [126] to deal with fourth order systems requires the fundamental variables to be the field and its first time derivative. This choice is motivated by the need to retain the continuity properties of the path integral formulation, as described above[2]. However, this point of view introduces a different problem; initial and final states are then described in terms of ϕ and $\dot{\phi}$, which behave much like position and momentum for a particle in quantum mechanics. The proposed procedure consists into rotating the system to Euclidean time, and then integrating out the $\dot{\phi}$ dependence in the definition of probabilities, thus obtaining well defined quantum mechanical observables at the price of a loss of unitarity. This procedure has always been possible in the special cases studied in the literature so far.

We can give a summary of the practical procedure that schematically reads:

1. From the fourth order action S perform a Wick rotation to obtain the Euclidean action S^E,

[2]We refer to the original article [126] for a longer discussion that would distract us here from the main point.

2. Derive the Euclidean equations of motion and corresponding solutions,

3. Use the Euclidean version of the path integral to find the propagator for ϕ with boundary conditions on ϕ and ϕ',

4. Define a "wavefunctional" as the propagator from a vacuum state at minus infinity in Euclidean time,

5. Find the modulus squared of the wavefunctional, or probability amplitude, which gives the probability that a quantum fluctuation leads to a state with specified ϕ and ϕ',

6. Finally, and crucially, trace over ϕ' before returning to real time. Note that if one were to rotate back to Lorentzian time before integrating, the probability would be ill defined, reflecting the existence of the ghost degree of freedom.

There is no magic trick behind all this, since taming the ghost by integrating over the infinities that it introduces happens at the price of a violation of unitarity[3]. The Euclidean formulation of the path integral together with the requirement that the fields die off at Euclidean infinity ensures that the fields remain bounded in real time. This is similar to using a final boundary condition to remove runaway solutions from systems that would otherwise contain them.

Let us examine this procedure in the specific case of the 4th order system discussed earlier. Using t for Lorentzian and τ for Euclidean time, rescaling the field ϕ, the action is

$$S = \int dt \left(\frac{\alpha^2}{2}\ddot{\phi}^2 - \frac{1}{2}\dot{\phi}^2 + \frac{m^2}{2}\phi^2 \right) , \tag{3.8}$$

where $\alpha^2/2$ is an arbitrary small parameter, the "coupling constant" mentioned earlier. After a Wick rotation $t \to i\tau$ the action becomes

$$S^E \equiv \int d\tau \left(\frac{\alpha^2}{2}\phi''^2 + \frac{1}{2}\phi'^2 + \frac{m^2}{2}\phi^2 \right) , \tag{3.9}$$

[3]The interested reader can refer to the original paper [126] for a detailed discussion of this issue.

so that $iS \to -S^E$ in the path integral. When S^E is positive definite the path integral converges giving a well defined Euclidean quantum theory. The resulting equations of motion take the form $D_4\phi = 0$, where

$$D_4 = \frac{1}{2}\left(\alpha^2\frac{\mathrm{d}^4}{\mathrm{d}\tau^4} - \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + m^2\right) \ , \tag{3.10}$$

and admit solutions

$$\phi(\tau) = A_1\sinh(\lambda_1\tau) + A_2\cosh(\lambda_1\tau) + A_3\sinh(\lambda_2\tau) + A_4\cosh(\lambda_2\tau) \ , \tag{3.11}$$

where λ_1 and λ_2 are given by

$$\lambda_1 = \sqrt{\frac{1}{2\alpha^2}(1-\sqrt{1-4m^2\alpha^2})} \ , \qquad \lambda_2 = \sqrt{\frac{1}{2\alpha^2}(1+\sqrt{1-4m^2\alpha^2})} \ . \tag{3.12}$$

The path integral for the propagator from state (ϕ_1, ϕ_1') at Euclidean time $-T$ to the state (ϕ_2, ϕ_2') at Euclidean time 0 is then

$$\begin{aligned}\langle(\phi_0,\phi_0';0)|(\phi_T,\phi_T';-T)\rangle &= \int_{(\phi_T,\phi_T')}^{(\phi_0,\phi_0')} d[\phi(\tau)]\exp\left[-S^E[\phi]\right] \\ &= e^{-S^E[\phi_{\mathrm{cl}}]}\int_{(0,0)}^{(0,0)} d[\varphi(\tau)]\exp\left[-S^E[\varphi]\right] \ ,\end{aligned} \tag{3.13}$$

where we have used the decomposition $\phi = \phi_{\mathrm{cl}} + \varphi$, with ϕ_{cl} the classical solution of the Euclidean equations of motion for the appropriate boundary conditions.

The wavefunctional for a state described by (ϕ_0, ϕ_0') at time $\tau = 0$ is then defined via (3.13) as

$$\Psi_0[\phi_0,\phi_0'] \equiv \lim_{T\to\infty}\langle(\phi_0,\phi_0';0)|(\phi_T,\phi_T';-T)\rangle \ , \tag{3.14}$$

which yields

$$\Psi_0[\phi_0,\phi_0'] = N\exp\left[-A\phi_0'^2 + B\phi_0\phi_0' - C\phi_0^2\right] \ . \tag{3.15}$$

The values of the coefficients A, B, and C, can be found by imposing the vanishing of the field and its first derivative at infinity as boundary conditions

$$(\phi_T,\phi_T') \to (0,0), \quad -T \to -\infty \ , \tag{3.16}$$

and are given by

$$
\begin{aligned}
A &= \frac{\sqrt{1-4m^2\alpha^2}}{2(\lambda_2-\lambda_1)} \\
B &= \frac{2m^2\alpha - m}{\alpha(\lambda_2-\lambda_1)^2} \\
C &= \frac{m\sqrt{1-4m^2\alpha^2}}{2\alpha(\lambda_2-\lambda_1)} \ ,
\end{aligned}
\tag{3.17}
$$

with the values for λ_i found above. We will discuss the normalization N below in the next section.

The next step is to define a probability

$$
\bar{P}[\phi_0, \phi_0'] \equiv \Psi_0 \Psi_0^* = N^2 \exp\left[-2A\left(\phi_0'^2 + \frac{m}{\alpha}\phi_0^2\right)\right] \ . \tag{3.18}
$$

As we have already mentioned, this would not provide a well defined probability if rotated back to Lorentzian time, since $A > 0$ and the rotation would introduce $(-i)^2 = -1$ in front of the ϕ_0' term. Therefore one rotates back to real time only after integrating over ϕ', to yield

$$
P[\phi_0] = \sqrt{\frac{m}{\pi\alpha}\frac{\sqrt{1-4m^2\alpha^2}}{(\lambda_2-\lambda_1)}} \exp\left[-\frac{m}{\alpha}\frac{\sqrt{1-4m^2\alpha^2}}{(\lambda_2-\lambda_1)}\phi_0^2\right] \ , \tag{3.19}
$$

which, as $\alpha \to 0$ simplifies to

$$
P[\phi_0] = \sqrt{\frac{m}{\pi}\left(1 + m\alpha + \ldots\right)} \exp\left[-m\left(1 + m\alpha + \ldots\right)\phi_0^2\right] \ . \tag{3.20}
$$

How might this procedure be extended to an arbitrary higher order system with a quadratic Lagrangian? Since ultimately we wish to consider higher order terms as corrections to the propagation of the degrees of freedom of a second order Lagrangian, we seek a way to generalize this procedure so that an integration over all the extra degrees of freedom is performed in order to obtain the final results. Although some of the original motivations presented in [126] for taking fourth order terms seriously are lost in this approach, this point of view is nonetheless consistent with the proposed procedure since it corresponds to tracing over the unobserved degrees of freedom.

Guided by the need for a composition law for the path integral, we are led to consider the metric perturbation γ_{ij} and its derivatives γ'_{ij}, and γ''_{ij} (rather than γ, γ'' and γ^{VI}) as the dynamical degrees of freedom in a sixth order Lagrangian for Gravity. The rest of the procedure developed in [126] is then unmodified, and in principle the only difficulties that appear are those associated with the explicit calculation of the normalization function for the wavefunctional.

3.2.1 Normalization of the Path Integral

We show here how normalize the path integral in the example above. The generalization to the sixth order system that will be presented later is straightforward.

In equation (3.15), N is a normalization factor found by calculating the path integral over the field φ. There has been some debate in the literature about how to actually calculate this normalization function. We will follow the approach of Zerbini and Di Criscienzo, in particular the work [128], where a powerful theorem by Forman [129] is used to calculate the regularized path integral. We sketch here the main points of the procedure. The normalization constant N can be formally calculated from equation (3.13) as

$$N = \sqrt{\frac{2\pi}{\mathrm{Det} D_4}}\ , \tag{3.21}$$

where D_4 is the operator that defines the equations of motion as in equation (3.10), with the boundary conditions from the path integral in (3.13). Forman's theorem, a generalization of Gel'fand-Yaglom and Levit-Smilansky theorems [130, 131], connects the value of the determinant of such an elliptic operator to the determinant of another operator $\bar{D}$ that satisfies in general

$$\begin{aligned} D_4 &= P_0(\tau)\frac{d^n}{d\tau^n} + \mathcal{O}\left(\frac{d^{n-2}}{d\tau^{n-2}}\right) \\ \bar{D} &= P_0(\tau)\frac{d^n}{d\tau^n} + \mathcal{O}\left(\frac{d^{n-2}}{d\tau^{n-2}}\right)\ , \end{aligned} \tag{3.22}$$

where in our case $n = 4$, and the polynomial $P_0(\tau)$ is just a constant. In a nutshell, the theorem connects the determinant of two different operators defined on the same interval $\tau \in [-T, 0]$, with the same "highest order behavior", and possibly different boundary conditions. The main result of the theorem then reads

$$\frac{\mathrm{Det}D_4}{\mathrm{Det}\bar{D}} = \frac{\det(M + NY_{D_4}(0))}{\det(\bar{M} + \bar{N}Y_{\bar{D}}(0))} \ , \tag{3.23}$$

where M, N and their barred counterparts are $(n-1)\times(n-1)$ matrices that encode the boundary conditions, and $Y_{D_4}(\tau)$ and $Y_{\bar{D}}(\tau)$ propagate the kernel $h(\tau)$ (defined by $D_4 h = 0$) and its first $(n-1)$ time derivatives from initial time at $-T$ to τ

$$\begin{pmatrix} h(\tau) \\ \dots \\ h^{(n-1)}(\tau) \end{pmatrix} = Y_{D_4}(\tau) \begin{pmatrix} h(-T) \\ \dots \\ h^{(n-1)}(-T) \end{pmatrix} \tag{3.24}$$

Thus to find the determinant of D_4 it is sufficient to calculate the determinant of another operator, in particular one for which the spectrum is known and the determinant can be calculated explicitly. The easiest choice is then to take $\bar{D} = D_4$ and simply select a different set of boundary conditions that give a much simpler problem to solve. In particular one can choose what corresponds to the "usual" choice for the path integral integration, namely

$$\begin{aligned} \phi(0) = 0, \qquad & \phi''(0) = 0 \ , \\ \phi(-T) = 0, \qquad & \phi''(-T) = 0 \ , \end{aligned} \tag{3.25}$$

as opposed to the "physical" choice that corresponds to the actual boundary conditions for our system

$$\begin{aligned} \phi(0) = 0, \qquad & \phi'(0) = 0 \ , \\ \phi(-T) = 0, \qquad & \phi'(-T) = 0 \ . \end{aligned} \tag{3.26}$$

The spectrum for D_4 (or equivalently $\bar{D}$) with the boundary conditions (3.25) is known, and reads

$$\mathrm{spec}(\bar{D}) = \left\{ \lambda_n = \left(\frac{\pi n}{T}\right)^2 + m^2 + \alpha^2 \left(\frac{\pi n}{T}\right)^4 \ , n \in \mathbf{N} \right\} \tag{3.27}$$

By writing the boundary conditions as

$$M \begin{pmatrix} h(0) \\ \dots \\ h^{(n-1)}(0) \end{pmatrix} + N \begin{pmatrix} h(-T) \\ \dots \\ h^{(n-1)}(-T) \end{pmatrix} = 0 \ , \tag{3.28}$$

the matrices M, N, and the barred ones can be taken to be

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ,$$

$$\bar{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad \bar{N} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \tag{3.29}$$

and Y_{D_4} is given by (solving equation (3.24))

$$Y_{D_4}(\tau) = \begin{pmatrix} u_1(\tau) & u_2(\tau) & u_3(\tau) & u_4(\tau) \\ u_1'(\tau) & u_2'(\tau) & u_3'(\tau) & u_4'(\tau) \\ u_1''(\tau) & u_2''(\tau) & u_3''(\tau) & u_4''(\tau) \\ u_1'''(\tau) & u_2'''(\tau) & u_3'''(\tau) & u_4'''(\tau) \end{pmatrix} , \tag{3.30}$$

where $D_4 u_j = 0$ for $j = 0, \dots 4$, satisfying the boundary conditions

$$\begin{aligned} u_1(-T) = 1 \ , \qquad & u_j(-T) = 0 \ , \qquad j \neq 1 \ , \\ u_2'(-T) = 1 \ , \qquad & u_j'(-T) = 0 \ , \qquad j \neq 2 \ , \\ u_3''(-T) = 1 \ , \qquad & u_j''(-T) = 0 \ , \qquad j \neq 3 \ , \\ u_4'''(-T) = 1 \ , \qquad & u_j'''(-T) = 0 \ , \qquad j \neq 4 \ . \end{aligned} \tag{3.31}$$

By putting everything together we finally have the determinant needed for the normalization

$$\begin{aligned}\mathrm{Det}D_4 \quad &= \quad \frac{\alpha^3}{mT^2}\frac{1}{1-2\alpha^2m^2}\left[(1-2\alpha m)\sinh^2\left(\frac{T}{2\alpha}\sqrt{1+2\alpha m}\right)\right.\\ &\left.-(1+2\alpha m)\sinh^2\left(\frac{T}{2\alpha}\sqrt{1-2\alpha m}\right)\right] . \end{aligned} \tag{3.32}$$

3.3 Sixth Order Corrections

Since fourth order corrections have already been analyzed in [127], we focus here on calculating the corrections to the tensor part of the two point function coming from a sixth order term.

3.3.1 Expanding the Action

Our goal is to take a convenient contraction of Riemann tensors and their derivatives, and to expand it to quadratic order in perturbations about a conformally flat background. We will then study the action for the perturbations around two important backgrounds – Minkowski space and de Sitter space.

We focus on one of the simplest covariant terms that contains six time derivatives and is quadratic in metric perturbations,

$$\nabla_\alpha R_{\mu\nu}\nabla^\alpha R^{\mu\nu} \ . \tag{3.33}$$

The total action we start from therefore consists of the Einstein-Hilbert term, a cosmological constant, two distinct fourth order contributions and the term above

$$S=\int d^4x\sqrt{-g}\left[M_{\rm p}^2\left(\frac{R}{2}+\Lambda\right)+\lambda^2R^2-\alpha^2C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}-\frac{\beta^2}{M_{\rm p}^2}\nabla_\alpha R_{\mu\nu}\nabla^\alpha R^{\mu\nu}\right] , \tag{3.34}$$

where $\Lambda=0$ for a Minkowski background, and is nonzero for a de Sitter one. While this action is quite general, we shall henceforth ignore the R^2 term; its presence does not affect the result as we have explicitly checked, and as one would expect since it merely corresponds to an additional massive scalar degree of freedom. This can be seen by changing frame via a conformal transformation of the metric.

Writing the flat FLRW metric in terms of conformal time η, the perturbed metric is then

$$\begin{aligned} g_{\mu\nu} &= g^0_{\mu\nu} + h_{\mu\nu} \\ &= e^{2\rho}\left(\eta_{\mu\nu} + \delta^i_\mu \delta^j_\nu \gamma_{ij}\right) \ , \end{aligned} \tag{3.35}$$

where the scale factor $e^{\rho(\eta)}$ is equal to one for Minkowski space and equal to $(-H\eta)^{-1}$ in de Sitter space. The next step is to insert the perturbed metric in the action (3.34) and keep up to quadratic terms. We can save some time by recalling that the perturbation γ_{ij} is traceless and divergenceless $\gamma_{ii} = \partial_i \gamma_{ij} = 0$, and so it follows that the first variation of the volume element, being proportional to the trace of γ, vanishes leaving only

$$\delta^2 S = \int d^4x \left[\delta^2\sqrt{-g}\mathcal{L} + \sqrt{-g}\left(\frac{M_{\rm p}^2}{2}\delta^2 R + \lambda^2\delta^2 R^2 - \alpha^2\delta^2 C^2 - \frac{\beta^2}{M_{\rm p}^2}\delta^2\left(\nabla_\alpha R_{\mu\nu}\right)^2\right)\right] \ , \tag{3.36}$$

where $\mathcal{L}$ in the first term only contains background quantities.

The first few terms involving the Ricci Scalar are standard, and the Weyl squared term was already calculated in [127], and can be written as

$$\frac{1}{2}\delta^2 C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = \frac{1}{2}e^{-4\rho}\left(\ddot{\gamma}_{ij}\ddot{\gamma}_{ij} + 2\ddot{\gamma}_{ij}\gamma_{ij,nn} + 4\dot{\gamma}_{ij}\dot{\gamma}_{ij,nn} + \gamma_{ij,nn}\gamma_{ij,mm}\right) \ . \tag{3.37}$$

The only missing term is then the sixth order one. We can start with the expansion for the Christoffels

$$\begin{aligned} \Gamma^\alpha{}_{\mu\nu} &= (\Gamma^0)^\alpha{}_{\mu\nu} + \frac{1}{2}(g^0)^{\alpha\lambda}\left(\nabla_\mu h_{\nu\lambda} + \nabla_\nu h_{\mu\lambda} - \nabla_\lambda h_{\mu\nu}\right) \\ &\quad -\frac{1}{2}h^{\alpha\lambda}\left(\nabla_\mu h_{\nu\lambda} + \nabla_\nu h_{\mu\lambda} - \nabla_\lambda h_{\mu\nu}\right) + \dots \ , \end{aligned} \tag{3.38}$$

where the background metric $g^0_{\mu\nu}$ is used to rise or lower indices and to define the covariant derivatives. Second we look at the Riemann tensor that gives

$$\delta R^\alpha{}_{\beta\mu\nu} = \nabla_\mu \delta\Gamma^\alpha{}_{\nu\beta} - \nabla_\nu \delta\Gamma^\alpha{}_{\mu\beta} \ , \tag{3.39}$$

$$\delta^2 R^{\alpha}{}_{\beta\mu\nu} = \nabla_\mu \delta^2 \Gamma^{\alpha}{}_{\nu\beta} - \nabla_\nu \delta^2 \Gamma^{\alpha}{}_{\mu\beta} + 2\delta\Gamma^{\alpha}{}_{\mu\lambda}\delta\Gamma^{\lambda}{}_{\nu\beta} - 2\delta\Gamma^{\alpha}{}_{\nu\lambda}\delta\Gamma^{\lambda}{}_{\mu\beta} \; , \tag{3.40}$$

for its first and second variation respectively. The above can be contracted to find the corresponding variations for the Ricci tensor

$$\delta R_{\mu\nu} = -\frac{1}{2}\nabla_\mu \nabla_\nu h - \frac{1}{2}\nabla^2 h_{\mu\nu} + \nabla^\alpha \nabla_{(\mu} h_{\nu)\alpha} \; . \tag{3.41}$$

Finally, in terms of γ and using the traceless and divergenceless conditions we can write

$$\begin{aligned}
\delta\left(\nabla_\alpha R_{\mu\nu}\right) = {} & \frac{1}{2}\delta^0_\alpha \delta^i_\mu \delta^j_\nu \left[\gamma^{III}_{ij} + 2\dot\gamma_{ij}\left(\ddot\rho - 2\dot\rho^2\right) - \dot\gamma_{ij,kk} + 2\gamma_{ij}\left(\rho^{III} + 2\dot\rho\ddot\rho - 4\dot\rho^3\right)\right. \\
& \left. + 2\dot\rho\gamma_{ij,kk} \;\right] \\
& + \frac{1}{2}\delta^k_\alpha \left[\left(\delta^0_\mu \delta^i_\nu + \delta^i_\mu \delta^0_\nu\right)\left(-\dot\rho\ddot\gamma_{ik} + 2\dot\gamma_{ik}\left(\ddot\rho - 2\dot\rho^2\right) + \dot\rho\gamma_{ik,jj} - 4\gamma_{ik}\left(\dot\rho^3 - \dot\rho\ddot\rho\right)\right)\right. \\
& \left. + \delta^i_\mu \delta^j_\nu \left(\ddot\gamma_{ij,k} + 2\dot\rho\dot\gamma_{ij,k} - \gamma_{ij,kll}\right)\right] \; ,
\end{aligned} \tag{3.42}$$

and the second variation

$$\begin{aligned}
\delta^2(\nabla_\alpha R_{\mu\nu}\nabla^\alpha R^{\mu\nu}) = {} & \frac{1}{2}e^{-6\rho}\Big[-(\gamma^{III})^2 + \ddot\gamma^2(-10\dot\rho^2 + 4\ddot\rho) + 3\ddot\gamma^2_{,i} - 3\dot\gamma^2_{,ij} + 2\dot\gamma^2_{,i}(\ddot\rho + 2\dot\rho^2) \\
& + \dot\gamma^2(4\rho^{IV} - 4\dot\rho\rho^{III} - 20\ddot\rho^2 + 44\dot\rho^2\ddot\rho - 48\dot\rho^4) + 6\gamma^2_{,ij}(\dot\rho^2 - \ddot\rho) \\
& + \gamma^2_{,ijl} + \gamma^2_{,i}(-4\rho^{IV} - 24\dot\rho^4 - 8\ddot\rho^2 + 72\dot\rho^2\ddot\rho - 4\dot\rho\rho^{III}) \\
& + \gamma^2(-8\rho^{VI} + 336\dot\rho^4\ddot\rho + 64\rho^{IV}\ddot\rho + 56\ddot\rho^3 - 48\dot\rho^2\rho^{IV} - 688\dot\rho^2\ddot\rho^2 \\
& - 304\dot\rho^3\rho^{III} + 36(\rho^{III})^2 + 56\dot\rho\rho^V + 104\dot\rho\ddot\rho\rho^{III} + 72\dot\rho^6)\Big] \; .
\end{aligned} \tag{3.43}$$

We have now all the ingredients to obtain the full expansion for the action that will be the starting point for the calculations in the next sections,

$$
\begin{aligned}
S_\gamma = \int d\eta d^3x \Bigg[& -\frac{\beta^2}{2M_{\mathrm{p}}^2} e^{-2\rho} \Big[-(\gamma^{III})^2 + \ddot{\gamma}^2(-10\dot{\rho}^2 + 4\ddot{\rho}) + 3\ddot{\gamma}_{,i}^2 + 2\dot{\gamma}_{,i}^2(\ddot{\rho} + 2\dot{\rho}^2) \\
& -3\dot{\gamma}_{,ij}^2 + \dot{\gamma}^2(4\rho^{IV} - 4\dot{\rho}\rho^{III} - 20\ddot{\rho}^2 + 44\dot{\rho}^2\ddot{\rho} - 48\dot{\rho}^4) + \gamma_{,ijl}^2 + 6\gamma_{,ij}^2(\dot{\rho}^2 - \ddot{\rho}) \\
& +\gamma_{,i}^2(-4\rho^{IV} - 24\dot{\rho}^4 - 8\ddot{\rho}^2 + 72\dot{\rho}^2\ddot{\rho} - 4\dot{\rho}\rho^{III}) \\
& +\gamma^2\big[-8\rho^{VI} + 336\dot{\rho}^4\ddot{\rho} + 64\rho^{IV}\ddot{\rho} + 56\ddot{\rho}^3 - 48\dot{\rho}^2\rho^{IV} - 688\dot{\rho}^2\ddot{\rho}^2 - 304\dot{\rho}^3\rho^{III} \\
& +36(\rho^{III})^2 + 56\dot{\rho}\rho^V + 104\dot{\rho}\ddot{\rho}\rho^{III} + 72\dot{\rho}^6\big]\Big] - \alpha^2\left(\ddot{\gamma}^2 - 2\dot{\gamma}_{,i}^2 + \gamma_{,ij}^2\right) \\
& +\lambda^2\big[6\dot{\gamma}^2(\dot{\rho}^2 + \ddot{\rho}) - 6\gamma_{,i}^2(\dot{\rho}^2 + \ddot{\rho}) + \gamma^2(-12\rho^{IV} + 72\dot{\rho}^2\ddot{\rho} + 12\dot{\rho}\rho^{III} - 6\ddot{\rho}^2 - 18\dot{\rho}^4)\big] \\
& +\frac{M_{\mathrm{p}}^2}{2} e^{2\rho}\big[\dot{\gamma}^2 - \gamma_{,i}^2 - \gamma^2(\dot{\rho}^2 + 2\ddot{\rho})\big]\Bigg] .
\end{aligned}
\tag{3.44}
$$

Note that the background equations have not been used in this derivation.

Before we specialize to the two backgrounds of interest, we comment on boundary terms. The action (3.34) is classically equivalent to a whole class of actions that differ from it only by boundary terms. Since we are interested in field configurations localized in space we can drop terms on the spatial boundaries, taken to be at infinity. However temporal boundary terms cannot be neglected. In what follows we investigate the behavior of fluctuations described by the quadratic action (3.44) which is obtained from any classically equivalent covariant action by the addition of appropriate boundary terms. We invoke the requirement of a positive definite Euclidean action (at least in a Minkowski background, as will be discussed later) as a guideline to finding a well defined starting point. In fact, once the quadratic action for the perturbations is found, the procedure followed in this chapter is unique. Boundary terms that play a similar role to the York Gibbons Hawking term in General Relativity are then added to the covariant action (3.34) above, or any classical equivalent, whenever necessary to find the corresponding quadratic action for the perturbations and their first three derivatives.

3.3.2 Minkowski Background

Performing a Wick rotation to imaginary time, and focusing on a Minkowski background, for which $e^{\rho(\eta)} = 1$, the full sixth order action (3.44) reduces to

$$\begin{aligned} S_M^E &= -\int d\tau d^3x \left[-\frac{\beta^2}{2M_\mathrm{p}^2}\left(\gamma'''^2 + 3\gamma''^2_{,i} + 3\gamma'^2_{,ij} + 6\gamma^2_{,ijl}\right) - \alpha^2\left(\gamma''^2 + 2\gamma'^2_{,i} + \gamma^2_{,ij}\right) \right. \\ & \left. - \frac{M_\mathrm{p}^2}{2}\left(\gamma'^2 + \gamma^2_{,i}\right)\right] , \end{aligned} \tag{3.45}$$

where, for simplicity, we have omitted the indices on, and the argument of the perturbation $\gamma_{ij}(\eta)$. It is convenient to treat the problem in momentum-space by performing a Fourier transform on γ

$$\gamma_{ij}(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \epsilon^s_{ij}(\vec{k})\gamma^s_{\vec{k}}(\eta) e^{i\vec{k}\cdot\vec{x}} , \tag{3.46}$$

where the polarization tensor satisfies

$$\begin{aligned} \epsilon_{ii} = k^i\epsilon_{ij} &= 0 , \\ \epsilon^*_{ij}(\vec{k}) &= \epsilon_{ij}(-\vec{k}) , \\ \epsilon^s_{ij}(\vec{k})\epsilon^{r*}_{ij}(\vec{k}) &= 2\delta^{sr} . \end{aligned} \tag{3.47}$$

In order to avoid confusion through notation, we will drop all the unnecessary indices. The action for the k-mode then becomes

$$\begin{aligned} S^E_{M\,k} &= \int d\tau \left[-\frac{\beta^2}{M_\mathrm{p}^2}\left(-|\gamma'''|^2 + 3k^2|\gamma''|^2 - 3k^4|\gamma'|^2 + 6k^6|\gamma|^2\right) \right. \\ & \left. -2\alpha^2\left(|\gamma''|^2 - 2k^2|\gamma'|^2 + |\gamma|^2\right) + M_\mathrm{p}^2\left(|\gamma'|^2 - k^2|\gamma|^2\right)\right] , \end{aligned} \tag{3.48}$$

where we have used the notation $|\gamma^{(n)}|^2 \equiv \frac{d^n}{d\tau^n}\gamma\frac{d^n}{d\tau^n}\gamma^*$. Varying this action with respect to γ^* yields the Euclidean equations of motion

$$D_6^M \gamma(\eta) = 0 , \tag{3.49}$$

with

$$D_6^M \equiv \frac{d^6}{d\tau^6} - \left(3k^2 + \frac{2\alpha^2 M_\mathrm{p}^2}{\beta^2}\right)\frac{d^4}{d\tau^4} + \left(3k^4 + 4k^2\frac{\alpha^2 M_\mathrm{p}^2}{\beta^2} + \frac{M_\mathrm{p}^4}{\beta^2}\right)\frac{d^2}{d\tau^2} - \left(k^6 + 2k^4\frac{\alpha^2 M_\mathrm{p}^2}{\beta^2} + k^2\frac{M_\mathrm{p}^4}{\beta^2}\right) \tag{3.50}$$

Solutions to these equations can easily be written in terms of exponentials as

$$\gamma_{cl}^M(\tau) = c_{11}e^{\lambda_1\tau} + c_{12}e^{-\lambda_1\tau} + c_{21}e^{\lambda_2\tau} + c_{22}e^{-\lambda_2\tau} + c_{31}e^{\lambda_3\tau} + c_{32}e^{-\lambda_3\tau}\ , \tag{3.51}$$

with λ_1, λ_2, and λ_3 given by

$$\begin{aligned}\lambda_1 &= k\ ,\\ \lambda_{2,3} &= \sqrt{k^2 + \frac{M_\mathrm{p}^2\alpha^2}{\beta^2} \pm \sqrt{\frac{M_\mathrm{p}^4\left(\alpha^4-\beta^2\right)}{\beta^4}}}\ .\end{aligned} \tag{3.52}$$

Following the procedure highlighted in the previous section we now define a wave-functional that describes the probability amplitude of being in a state characterized by γ_0, γ_0' and γ_0''

$$\begin{aligned}\Psi_{0M}^E[\gamma_0,\gamma_0',\gamma_0''] &= Ne^{-S_M^E[\gamma_{cl}]}\\ &= N\exp\Bigg[-\frac{1}{2M_\mathrm{p}^2}\left(A_{00}\gamma_0^*\gamma_0 + A_{01}\gamma_0^*\gamma_0' + A_{02}\gamma_0^*\gamma_0'' + A_{10}\gamma_0'^*\gamma_0\right.\\ &\left.+A_{11}\gamma_0'^*\gamma_0' + A_{12}\gamma_0'^*\gamma_0'' + A_{20}\gamma_0''^*\gamma_0 + A_{21}\gamma_0''^*\gamma_0' + A_{22}\gamma_0''^*\gamma_0''\right)\Bigg] .\end{aligned} \tag{3.53}$$

The coefficients A_{jl} are functions of the three λ_i, and we present their explicit forms in the appendix A.1. It is, in fact, possible to calculate the normalization factor N as seen in section 3.2.1 however, since this does not change our result, for simplicity we shall ignore the contributions coming from N in what follows, until a normalization for the probability is needed.

A probability distribution for γ_0 can then be defined integrating over γ_0'' and γ_0' and by rotating back to Lorentzian time

$$P^E[\gamma_0] \equiv \int d[\gamma_0']\int d[\gamma_0'']\Psi_{0M}^E\Psi_{0M}^{E*} \to P[\gamma_0]\ , \tag{3.54}$$

where the arrow implies rotating clockwise in the complex plane to Lorentzian time. The normalized probability expanded for $k \ll M_{\rm p}$ then gives

$$\begin{aligned}P[\gamma_0] &= \left(M_{\rm p}\sqrt{\frac{k}{\pi}}+\dots\right)\exp\left[-kM_{\rm p}^2\left(1+\frac{k\left(2\alpha^2+\beta\right)}{M_{\rm p}}\times\right.\right.\\ &\left.\left.\frac{1}{\left(\sqrt{\alpha^2-\sqrt{\alpha^4-\beta^2}}+\sqrt{\alpha^2+\sqrt{\alpha^4-\beta^2}}\right)}+\dots\right)|\gamma_0|^2\right]. \qquad (3.55)\end{aligned}$$

Interestingly, we have encountered no difficulties in extending the Euclidean path integral prescription to our sixth order term in a Minkowski background. This straightforward extension suggests that it may be possible to extend the procedure to any system with $2n$ derivatives.

3.3.3 The de Sitter Background

We now repeat the above calculation in a de Sitter background. As we shall see, the explicit time-dependence of the background introduces crucial differences in this case. Setting $\Lambda > 0$ and the scale factor to be $e^\rho = (-H\eta)^{-1}$, the action in Euclidean time and Fourier space becomes

$$\begin{aligned}S^E_{dS\ k} = \int d\tau\Bigg[&\beta^2\frac{H^2}{2M_{\rm p}^2}\tau^2\left[\gamma'''^2+\gamma''^2\left(3k^4+6\frac{k^2}{\tau^2}+\frac{8}{\tau^4}\right)+\gamma^2\left(k^6+8\frac{k^2}{\tau^4}\right)\right]\\ &-\alpha^2\left(\gamma''^2+2k^2\gamma'^2+k^4\gamma^2\right)+\frac{M_{\rm p}^2}{4H^2\tau^2}\left(\gamma'^2+k^2\gamma^2\right)\Bigg]. \qquad (3.56)\end{aligned}$$

Note that if we started without the sixth order term (i.e. set $\beta = 0$) we would have the action presented in [127], which is not positive definite. Nevertheless, the authors of [127] showed that this does not prevent one from following the Euclidean path integral procedure and obtaining a well defined result. We will therefore adopt the same point of view here and, although we realize that we are dealing with a non positive definite Euclidean action, proceed as planned to see if a meaningful result can be obtained.

It can also be noted that in principle we could obtain a positive definite action if we started from a different form for equation (3.34). There, in fact, the signs of α^2 and β^2 have been chosen arbitrarily. If we were to change the signs though, the results presented in section 3.3.2 would not stand. We choose to keep the sign conventions so that the validity of the method is preserved in a Minkowski background.

Defining, for simplicity, $z = -k\tau$, the Euclidean equations of motion become

$$D_6^{dS}\gamma(z) = 0 \ , \tag{3.57}$$

with

$$\begin{aligned} D_6^{dS} \equiv \frac{d^6}{dz^6} &+ \frac{6}{z}\frac{d^5}{dz^5} + \left(-3 + \frac{C_1}{z^2}\right)\frac{d^4}{dz^4} - \frac{12}{z}\frac{d^3}{dz^3} + \left(3 + \frac{(4-2C_1)}{z^2} + \frac{C_2}{z^4}\right)\frac{d^2}{dz^2} \\ &+ \left(\frac{6}{z} - 2\frac{C_2}{z^5}\right)\frac{d}{dz} - \left(1 - \frac{C_1}{z^2} + \frac{C_2}{z^4}\right) \ , \end{aligned} \tag{3.58}$$

where

$$C_1 = 2\left(\frac{\alpha M_{\rm p}}{\beta H}\right)^2 \tag{3.59}$$

$$C_2 = 8 + 24\left(\frac{\lambda M_{\rm p}}{\beta H}\right)^2 + \frac{1}{2}\left(\frac{M_{\rm p}^2}{\beta H^2}\right)^2 \ . \tag{3.60}$$

Solutions to these equations can be found by factorizing the sixth order differential operator[4] D_6^{dS}, and can be written in terms of exponentials and Bessel functions as

$$\begin{aligned} \gamma_{cl}(z) = {} & A_1\left[\sinh(z) - z\cosh(z)\right] + A_2\left[z\sinh(z) - \cosh(z)\right] + A_3 z^{\frac{3}{2}}\mathrm{J}_{\lambda_1}(-iz) \\ & + A_4 z^{\frac{3}{2}}\mathrm{Y}_{\lambda_1}(-iz) + A_5 z^{\frac{3}{2}}\mathrm{J}_{\lambda_2}(-iz) + A_6 z^{\frac{3}{2}}\mathrm{Y}_{\lambda_2}(-iz) \ , \end{aligned} \tag{3.61}$$

where J and Y are respectively Bessel functions of first and second kind. Recalling that z takes values in $(0, +\infty)$ with $+\infty$ being the past infinity boundary, in order to find the wavefunctional we need to apply a set of boundary conditions analogous to the one described earlier, namely

$$\left.\begin{aligned} \gamma(z) &\to 0 \\ \gamma'(z) &\to 0 \\ \gamma''(z) &\to 0 \end{aligned}\right\} \quad z \to +\infty \quad \text{and} \quad \left.\begin{aligned} \gamma(z) &\to \gamma_0 \\ \gamma'(z) &\to \gamma_0' \\ \gamma''(z) &\to \gamma_0'' \end{aligned}\right\} \quad z \to +z_0 \ . \tag{3.62}$$

[4]For details see the appendix A.2.

The relevant classical solution of the equations of motion is therefore

$$\gamma_{cl}(z) = B_1\,(1+z)\,e^{-z} + B_2 z^{\frac{3}{2}}\mathrm{H}^{(2)}_{\lambda_1}\,(-iz) + B_3 z^{\frac{3}{2}}\mathrm{H}^{(2)}_{\lambda_2}\,(-iz)\ , \tag{3.63}$$

where $\mathrm{H}^{(2)}$ represents the Hankel function of the second kind, and the coefficients B_i contain the dependence on z_0 and on the boundary conditions γ_0, γ_0', and γ_0''.

To calculate the wavefunction it is sufficient to rewrite the action as

$$S^E_{dSk} = [\text{surface terms}] + \int_{-\infty}^{\tau_0} d\tau\gamma D_6^{dS}\gamma\ , \tag{3.64}$$

so that on the classical path only the first set of terms survives, with the contribution from the integral term being zero. Since we are ultimately interested in integrating over γ_0'' and γ_0' it is convenient to collect terms and write the wavefunctional schematically as

$$\begin{aligned}\Psi_0^{dS} \quad = \quad & N\exp\Bigg[-i\frac{k^3}{D}\left(A_{00}\gamma_0^*\gamma_0 + A_{01}\gamma_0^*\gamma_0' + A_{02}\gamma_0^*\gamma_0'' + A_{10}\gamma_0'^*\gamma_0 + A_{11}\gamma_0'^*\gamma_0'\right.\\ & \left. + A_{12}\gamma_0'^*\gamma_0'' + A_{20}\gamma_0''^*\gamma_0 + A_{21}\gamma_0''^*\gamma_0' + A_{22}\gamma_0''^*\gamma_0''\right)\Bigg]\ .\end{aligned} \tag{3.65}$$

The analytic dependence of the coefficients A_{ij} and D on the parameters α, β, and H/M_{p} appearing in the action is somewhat complicated and not very instructive, and so we do not display this here.

To make progress analytically we now introduce an approximation scheme, taking α, λ (if the R^2 term is considered) and β to be of order unity, with $H/M_{\mathrm{p}} \ll 1$ playing the role of the small parameter in a series expansion. Beside the reasonable choices for the parameters in the action, an extra assumption is needed to simplify the calculation. We assume that $\beta^2 < 2\alpha^4$, allowing us to approximate the frequencies λ_1 and λ_2 and the Hankel functions. With these approximations the associated probability takes a form similar to that of equation (3.65), with the same kinds of terms and different coefficients. In particular, focusing on the coefficient of $\gamma_0''^*\gamma_0''$, which we require to have a negative real part in order to proceed with the integration, we find

$$\bar{\bar{P}}[\gamma_0, \gamma_0', \gamma_0''] \equiv N N^* \exp\left[-\frac{\alpha^2 k^3 \tau^4 \left(1+\sqrt{1-\frac{\beta^2}{2\alpha^4}}\right)}{k^2\tau^2 - 1} \gamma_0''^* \gamma_0'' + \dots \right] . \tag{3.66}$$

$\bar{\bar{P}}$ is not yet the probability we are looking for, since integration over γ_0'' and γ_0' is still needed. The bars are a reminder of this fact, counting the maximum number of derivatives acting on γ_0. From equation (3.66) we note that gaussian integration over the real and imaginary parts of γ_0'' can be performed only if $(k\tau)^2 > 1$. Recalling that $k^2\eta^2 = k^2/(aH)^2$, with a being the scale factor, considering $k^2\tau^2 > 1$ means that the treatment can be considered valid for subhorizon modes.

With the above assumptions both the integrations over γ_0'' and γ_0' can be performed, and after rotating back to Lorentzian time the full final result is reported in the appendix A.2. Before we can say we have found a probability for γ_0, one last check is necessary: the coefficient of $|\gamma_0|^2$, in Lorentzian time, has to be negative in order to have a well defined (normalizable) probability. We check this by expanding the argument of the exponential as a series in H/M_p, keeping only the leading contribution

$$P_L[\gamma_0] = \tilde{N}\tilde{N}^* \exp\left[\frac{M_\mathrm{p}^2}{H^2} \left(-\frac{k^3\left(1+2\sqrt{\frac{2\alpha^4}{\beta^2}}\right)}{2\left(1+k^2\eta^2\right)} + O\left(\frac{H}{M_\mathrm{p}}\right)\right) \gamma_0^* \gamma_0 \right] , \tag{3.67}$$

where the symbol L is a reminder that we have rotated back to Lorentzian time. We can see that the probability can be integrated over all values of $|\gamma_0|$ giving a sensible extension of the method in [126] to the sixth order case. This may be compared with the equivalent form for the probability in GR,

$$P_{GR}[\gamma_0] = |\hat{N}|^2 \exp\left[-\frac{k^3 M_\mathrm{p}^2}{2H^2\left(1+k^2\eta^2\right)} |\gamma_0|^2 \right] \tag{3.68}$$

Finally, from the probability distribution we obtain the two point function for the tensorial perturbations γ_0 in the sixth order case

$$\langle |\gamma_0|^2 \rangle \simeq \left(\frac{H}{M_\mathrm{p}}\right)^2 \frac{1+k^2\eta^2}{k^3\left(1+2\sqrt{\frac{2\alpha^4}{\beta^2}}\right)} . \tag{3.69}$$

3.4 Discussion

The Euclidean path integral prescription is a method to integrate out the infinities appearing in higher derivative theories with ghosts and extract meaningful probability distributions for the non-ghost degrees of freedom. In this chapter we have reviewed the original fourth order version of the method and have shown how to extend this to a sixth order system in a Minkowski background and in a time dependent one – de Sitter. The two cases are treated separately since we have shown that a time dependent background, even if highly symmetric, introduces some difficulties. The Euclidean action is in fact not positive definite, raising doubts about the validity of the underlying quantum theory. Fortunately, as in the fourth order case, this does not prevent us from finding a sensible result.

With higher order gravity in mind, in this chapter we have examined an action containing GR, a sixth order term and two fourth order ones, with relative strengths set by the Planck mass and their relative mass dimension. We have found that the Euclidean path integral prescription can be applied to find corrections to the probability distribution of the tensorial perturbations about both Minkowski and de Sitter backgrounds. The corrections we have found are at least of order one in the de Sitter case, depending on the values of the parameters appearing in the action. Therefore the results pose stringent constraints on either the validity of the approach, or the presence of the covariant sixth order term considered.

It is important to be clear about the assumptions made throughout this chapter. The first one has already been mentioned, and concerns the validity of the quantum theory when the Euclidean action is not positive definite. However, note that we could have performed the whole calculation in Lorentzian signature, and the present procedure is merely an ad hoc prescription for rotating to Euclidean signature only when needed to integrate over ghosts. A second problem arises due to the fact that we have chosen $\ddot{\gamma}$ as one of our dynamical variables. This is somewhat in contrast with the original idea of preserving the continuity properties of the path integral. We leave to future studies the analysis of the effect of this particular choice of dynamical variables. Third, we have considered the simplest possible scheme for taking the limit in which the higher order terms become less important in the action; with this choice the behaviors of the fourth and sixth order terms are locked together. A general approximation scheme in which the two terms may go to zero independently and introduce different corrections requires further study. Finally, note that we have only considered one specific sixth order term in the covariant action for gravity. Although a full calculation is needed, we do not expect the other sixth order terms to conspire and drastically change the results found here.

Chapter 4

Anisotropic Cosmologies: Possible Signatures in Parallax Experiments

4.1 Introduction

We have seen in the previous chapters how one may tackle some of the problems related to cosmic acceleration during inflation and at late times, and the methods that can be applied to slightly modify the models that describe our understanding of the cosmological evolution. We want here to consider some completely alternative approach to the problem of late time acceleration, and the related signatures in future experiments. The possibly more radical main idea in this chapter is to keep GR as is, and to relax one of the fundamental assumptions we talked about in chapter 1, namely isotropy.

In our opinion, a particularly interesting aspect of this analysis is that the measurement we discuss here belongs to a new and revolutionary class of experiments, which could be called *real time cosmology*. We are in fact talking about repeated measurements of the relative position of distant objects – quasars – with respect to each other (cosmic "parallax"), with time baselines of the order of a decade or less. An astonishing idea that can only exist thanks to the latest progress in experimental techniques. Before we get too excited about this possibility we have to warn the reader that our analysis suggests that upcoming experiments are not quite at the level required to perform such measurements. We feel though that the required precision will definitely be available in the short future, and that such "real time" experiments will play an important role in constraining alternative proposals to the standard model of cosmology.

As already pointed out, homogeneity and isotropy are the cornerstones of the standard cosmological model, providing not only a tremendous simplification of General Relativity, but remarkable agreement with all observations. Homogeneity is supported by the observed galaxy distribution from large-scale structure surveys, while isotropy is supported by, in particular, the deep spatial uniformity of the temperature of the CMB. Nevertheless, the paradigm-changing observation of cosmic acceleration, now more than a decade old, has forced cosmologists to re-examine even their most cherished assumptions, including the correctness of GR as we have discussed in some detail in previous chapters, a vanishing cosmological constant, and, more recently, the fundamental ideas of spatial homogeneity and isotropy.

The primary evidence for the accelerating universe comes from the unexpected dimming of type Ia supernovae [32, 132–138], as measured through their light curves. The connection to cosmic acceleration though, requires the assumptions of homogeneity and isotropy, and thus this raises the possibility that abandoning one of these principles may allow for the appearance of accelerated expansion without actual acceleration itself.

Of course, the usual cosmological FLRW metric is so simple by virtue of its underlying symmetries. Abandoning these leads to a correspondingly more complicated form for the metric. It is convenient therefore to begin by studying toy models. One class of these that has shown some promise in this direction are the *Lemaître-Tolman-Bondi* (LTB) metrics, in which we are assumed to live inside a spherically symmetric underdense region of spacetime (or "void") embedded in an otherwise spatially flat and homogeneous Einstein-de Sitter universe [139–145]. Such a spacetime is manifestly inhomogeneous, due to the void, and on its own violates any strong version of the Copernican principle, since we must live inside this void in order to account for the observed supernovae dimming. Nevertheless, it has been shown that these models can provide a satisfactory fit to the luminosity distance-redshift relation of type Ia supernovae and the position of the first peak in the CMB [143, 146]. Thus, LTB models have been suggested as a possible solution to the problem of cosmic acceleration, obviating the need for quintessence fields, modifications of gravity, or a cosmological constant, and considerable effort has been devoted to constraining them [147–151].

Beyond the usual cosmological tests of homogeneity and isotropy, it has recently been suggested that these models could be further constrained by precision measurements of the evolution of the angular positions of distant sources. Following the authors of [152], we refer to this effect as cosmic parallax. The expansion of an FLRW universe is isotropic for all observers, and so cosmic parallax would not be observed. Of course although our universe is very close to an FLRW universe, it is not exactly so. For instance, on large scales bound structures may acquire small peculiar velocities, giving rise to a slight deviation from observed isotropic expansion. However, to any observer living off-center inside the void of an LTB universe, cosmic evolution itself is anisotropic and is an additional source of cosmic parallax. For sufficiently off-center observers the cosmic parallax due to anisotropic expansion would dominate over the contribution from peculiar velocities. Cosmic parallax could therefore provide an interesting test of void models. Upcoming sky surveys such as GAIA [153] may be able to initiate a measurement of this effect, requiring only that a similar survey be completed 10 years later in order to complete the measurement. The absence of cosmic parallax beyond what is expected from peculiar velocities would put an upper bound on how far our galaxy could be from the center of the void in otherwise allowed LTB models, where isotropy is restored. For example, the authors of [152] argue that GAIA may map sufficiently many quasars, with enough accuracy, so that two such surveys spaced 10 years apart could detect the additional LTB contribution to cosmic parallax if the Milky Way is more than 10 megaparsecs from the center of a 1 Gpc void. If after a decade no additional contribution were found, that would constrain the Milky Way to lie unnaturally close to the center of such a void.

On the other hand, detection of a contribution to the cosmic parallax not arising from peculiar velocities would indicate that the expansion of the universe is anisotropic from our vantage point. In LTB models this would be due to living away from the center of the void, and the strength of the additional contribution would be related to this distance. But cosmic parallax not attributable to peculiar velocities is a generic feature of any cosmological model with anisotropic expansion – an observation also made in [152]. For example, spacetimes of the Bianchi type would exhibit an additional contribution to cosmic parallax around every point. These homogeneous and anisotropic spacetimes have recently been invoked during inflation to explain anomalies in the CMB [154–161][1], and during late-time cosmology to describe the expansion driven by anisotropic dark energy [163, 164]. Nevertheless, the fact that motivations exist for studying both LTB and Bianchi spacetimes, and that both may exhibit contributions to cosmic parallax that are not attributable to peculiar velocities, raises the question of how we might interpret any additional cosmic parallax signal. Although an observed signal would provide evidence for deviations from an FLRW universe, such deviations could be due to deviations from spatial homogeneity (as in LTB models) or deviations from isotropy (as in Bianchi models). It is therefore of interest to consider how cosmic parallax in Bianchi models differs from LTB models. This is what we will try to do in this chapter in the context of parallax observations.

The structure of the chapter is as follows. In section 4.2 we briefly review the results of [152]. In section 4.3 we describe the kinds of anisotropic models that we will focus on; namely a subclass of Bianchi Type I models. Since we know from

[1]Such theories may not, however, be without problems [162].

observations of the CMB that the universe is very nearly isotropic at the time of last scattering, we describe how we restrict ourselves to Bianchi Type I metrics which pass existing cosmological tests. In section 4.4 we derive the geodesic equations for these spacetimes and discuss our numerical techniques for integrating them. We then present the cosmic parallax signal we find for these models and discuss how it compares to the cosmic parallax in void models.

4.2 Cosmic Parallax in an LTB Void

We begin by briefly reviewing how cosmic parallax manifests itself in LTB models, as discussed in [152]. Here and throughout the rest of the chapter, unless otherwise indicated, by cosmic parallax we mean cosmic parallax due to anisotropic expansion about an observer. The LTB metric is given by [20–22]

$$ds^2 = -dt^2 + \frac{|R'(t,r)|^2}{1+\beta(r)}dr^2 + R^2(t,r)d\Omega^2 \ , \tag{4.1}$$

where $R(t,r)$ is a position-dependent scale factor, $\beta(r)$ is related to the curvature of the spatial slices, and $()' \equiv \partial/\partial r$. The Einstein equations relate $R(t,r)$ to $\beta(r)$ and an additional arbitrary function of integration $\alpha(r)$. Specifying $\alpha(r)$, $\beta(r)$, and an initial condition for $R(t,r)$ completely determines the spacetime. In models with an underdense region, or "void", surrounded by an overdense region, $\alpha(r)$ and $\beta(r)$ roughly correspond to the width of the void and the gradient of the boundary between the inner and outer regions, respectively, and will be specified below.

The LTB metric describes a region of spacetime that is isotropic about the origin but inhomogeneous with respect to the radial direction. Therefore distant galaxies appear to be receding at the same rate in all directions for observers located at the origin. On the other hand, observers located away from the center of the void could in principle observe anisotropic recession. One way to observe this effect [152] is to measure how the angle between the positions of two distant sources evolves over time. This difference is referred to as the cosmic parallax. To study this, one considers null geodesics in an LTB universe, obeying

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 \ . \tag{4.2}$$

Here $\Gamma^\mu{}_{\rho\sigma}$ are the Christoffel symbols, λ is an affine parameter along null geodesics, and the four-velocities $u^\mu \equiv \frac{dx^\mu}{d\lambda}$ satisfy $u^\mu u_\mu = 0$. The goal will be to solve these equations to determine the null geodesics along which light travels from various sources to an observer, and to do this for two different observation times.

Following [152], we work in spherical-polar coordinates (t, r, θ, ϕ) for which the origin coincides with the center of the void (labeled "O" in figure 4.1). Without

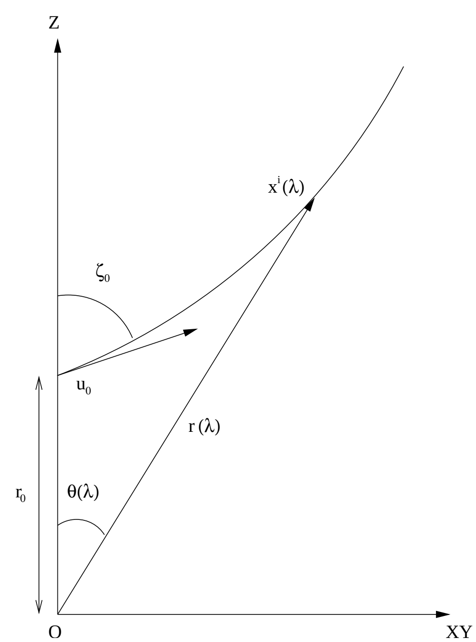


Figure 4.1: In the LTB model the observer is located at a distance r_0 from the center of the void along the axis of symmetry. Each point along the geodesic is described by the spatial coordinates $(\theta(\lambda), r(\lambda))$ or equivalently by $x^i(\lambda)$

loss of generality we choose the observer to lie on the polar axis at a coordinate distance r_0 along it. Spherical symmetry about the observer is now broken but the remaining cylindrical symmetry applied to (4.2) allows us to neglect the polar angle ϕ dependence. The system then reduces to three second-order geodesic equations, or equivalently six first-order equations. Applying the null geodesic condition further reduces the system to five independent first-order equations for t, r, θ, $p \equiv dr/d\lambda$ and the redshift z as [147, 152]

$$\begin{aligned}
\frac{dt}{d\lambda} &= -\sqrt{\frac{(R')^2}{1+\beta}p^2 + \frac{J^2}{R^2}} && (4.3)\\
\frac{dr}{d\lambda} &= p && (4.4)\\
\frac{d\theta}{d\lambda} &= \frac{J}{R^2} && (4.5)\\
\frac{dz}{d\lambda} &= \frac{(1+z)}{\sqrt{\frac{(R')^2}{1+\beta}p^2 + \frac{J^2}{R^2}}}\left[\frac{R'\dot{R}'}{1+\beta}p^2 + \frac{\dot{R}}{R^3}J^2\right] && (4.6)\\
\frac{dp}{d\lambda} &= 2\dot{R}'p\sqrt{\frac{p^2}{1+\beta} + \left(\frac{J}{RR'}\right)^2} + \frac{1+\beta}{R^3R'}J^2 + \left(\frac{\beta'}{2+2\beta} - \frac{R''}{R'}\right)p^2 \ , && (4.7)
\end{aligned}$$

where $J \equiv R^2 d\theta/d\lambda = J_0$, is constant along the geodesic.

To completely specify the system we require five initial conditions, which we provide for convenience at the initial observation event, and denote with a subscript “0”. Since one would like to specify initial conditions in terms of physically measurable quantities, we consider the angle ξ_0 between the polar axis and the line of sight along an incoming photon trajectory arriving at the observer (see figure 4.1). This angle ξ_0 coincides with the coordinate angle θ when $r_0 = 0$, but in general it is given by [147]

$$\cos\xi_0 = -\frac{R'(t,r)p}{\frac{dt}{d\lambda}\sqrt{1+\beta(r)}} \ . \tag{4.8}$$

This expression can be used to express J_0 and p_0 in terms of t_0, r_0, and ξ_0, via

$$\begin{aligned}
J_0 &= R(t_0,r_0)\sin\xi_0\\
p_0 &= \frac{\cos\xi_0}{R'(t_0,r_0)}\sqrt{1+\beta(r_0)} \ . && (4.9)
\end{aligned}$$

Therefore, the system is completely determined by specifying t_0, r_0, θ_0, z_0, and ξ_0.

Clearly, our coordinate choice means that $\theta_0 = 0$, and our conventional definition of redshift yields $z_0 = 0$. Following [152] we choose $r_0 = 15\text{Mpc}$, which is the largest value consistent with the CMB dipole [147]. What remains is to specify a direction on the sky and the time of observation to uniquely determine a geodesic that terminates at the spacetime event of observation. This picks out an initial geodesic along which light from a distant source travels to reach the observer.

Given the redshift and line of sight angle of a source at an initial time, we can determine the trajectory of light from that source at a later time in the following way. As just mentioned, the line of sight angle ξ_0 picks out an initial geodesic, terminating at the initial observation event. The observed redshift determines how far backwards in time along this initial geodesic the source lies. We extract the comoving coordinates of the source by numerically integrating backwards along this initial geodesic. Once the comoving coordinates of the observer and source are determined, and the interval of time between observations is specified, we solve a boundary-value problem to determine the final geodesic. We then extract the line of sight angle of this final geodesic ξ_f.

After repeating this procedure for two sources we have four angles: the input angles ξ_{a0} and ξ_{b0} and the output angles ξ_{af} and ξ_{bf}. We may then compute the difference

$$\Delta\gamma \equiv \gamma_f - \gamma_0 = (\xi_{af} - \xi_{bf}) - (\xi_{a0} - \xi_{b0}) \ , \tag{4.10}$$

which is the main quantity of interest, hereafter referred to as the cosmic parallax.

In [152] this quantity is calculated for LTB models characterized by the functions

$$\begin{aligned} \beta(r) &= \left(H_0^{\mathrm{OUT}}\right)^2 r^2 \frac{\Delta\alpha}{2}\left(1 - \tanh\frac{r - r_{\mathrm{vo}}}{2\Delta r}\right) & (4.11)\\ \alpha(r) &= \left(H_0^{\mathrm{OUT}}\right)^2 r^3 - r\beta(r) \ , & (4.12)\end{aligned}$$

which characterize the smooth transition from the inner underdense region to the outer, higher density region. The quantities $\Delta\alpha$, r_{vo}, Δr, and H_0^{OUT} are free parameters of the model that can be tuned to fit CMB and supernovae measurements. Following [152] and [143], we choose a model which is in good agreement with SNIa observations and the location of the first peak of the CMB, namely $\Delta\alpha = 0.9$,

$r_{\rm vo} = 1.3\,{\rm Gpc}$, $\Delta r = 0.4\,r_{\rm vo}$, and $H_0^{\rm OUT} = 51\,{\rm km\,s^{-1}\,Mpc^{-1}}$. Although this Hubble parameter outside the void seems to be in conflict with the measured Hubble Key project value of $72 \pm 8\,{\rm km\,s^{-1}\,Mpc^{-1}}$ [165], one should keep in mind that the measurements made to determine this value are all made at less than a few hundred megaparsecs ($z \approx 0.1$), whereas the size of the void in this model is about 1 Gpc ($z \approx 0.2$). The Hubble parameter inside the void in this model turns out to be $65\,{\rm km\,s^{-1}\,Mpc^{-1}}$, which is consistent with the Hubble Key project value. Recently the SHOES Team [166] was able to reduce the uncertainty on the value of the Hubble parameter to $H_0 = 74.2 \pm 3.6\,{\rm km\,s^{-1}\,Mpc^{-1}}$, raising more doubts about the validity of this model. Although LTB models have partly motivated this study, our purpose is not to discuss their validity. At worst, they will serve as useful toy models to study the characteristics and magnitude of the parallax signal.

The authors of [152] numerically compute the cosmic parallax for two different models. They find an angle-dependent signal with maximal value roughly $0.18\,\mu$as for sources at redshift of 1. It is claimed that this signal is within reach of the forthcoming Gaia mission, provided Gaia produces a sky map of the expected number of quasars (roughly $500,000$) with final positional error less than $30\,\mu$as, and that such measurements are repeated after 10 years.

4.3 Bianchi Type I Models

As we have discussed, anisotropic expansion of the universe around a given observer contributes to cosmic parallax. In the case of LTB models, this may allow us to constrain the distance of our galaxy from the center of the void. The further we are from the center of the void, the more anisotropic the universe would look to us. Of course, LTB spacetimes are not the only ones that can give rise to anisotropic expansion around a point. This raises the question of how we might interpret any observation of an anomalously large cosmic parallax, since such an observation would not itself be evidence that we live in an LTB universe. To understand this therefore, we study cosmic parallax in alternative anisotropic cosmologies and compare our results with those obtained in [152].

For simplicity we consider anisotropic spacetimes that are spatially homogeneous [167, 168]. In particular, we focus on a subclass of Bianchi Type I spacetimes, the metric for which may be written as

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2 \ . \tag{4.13}$$

In general, $a_1(t)$, $a_2(t)$, and $a_3(t)$ are independent scale factors, describing how the three spatial directions scale with time, which reduces to the standard FLRW case when $a_1(t) = a_2(t) = a_3(t)$. We specialize to the case when only one of the scale factors differs from the others, say $a_1(t) = a_2(t) \neq a_3(t)$, in which case the expansion is axisymmetric. We do so because we want to compare our results to observations made by an off-center observer in an LTB void. Such an observer will experience axisymmetric cosmic expansion, and so the most direct comparison will be to an axisymmetric Bianchi-I universe. There is, however, an important difference in the symmetries around observers in these two spacetimes. In the axisymmetric Bianchi-I universe there is an additional plane of symmetry normal to the axis of symmetry. The same is not true for an off-center observer in an LTB universe. To see this, it is sufficient to consider the extremal case of an observer outside the void, who can obviously distinguish the two directions along the polar axis: toward the void and away from it. This suggests that cosmic parallax in these two types of anisotropic models will differ at least qualitatively, if not in magnitude.

Setting $a_1 = a_2 = a(t)$ and $a_3 = b(t)$ in (4.13), the Einstein equations become

$$H_a^2 + 2H_aH_b = 8\pi G\rho \tag{4.14}$$

$$2\dot{H}_a + 3H_a^2 = -8\pi GP_z \tag{4.15}$$

$$\dot{H}_a + H_a^2 + \dot{H}_b + H_b^2 + H_aH_b = -8\pi GP_x \ , \tag{4.16}$$

where an overdot denotes a derivative with respect to t, $P_x = P_y$ and P_z are anisotropic pressures in the different directions, and we have defined the Hubble parameters $H_a \equiv \dot{a}/a$ and $H_b \equiv \dot{b}/b$. The conservation of energy equation in this case is

$$\dot{\rho} = -2H_a(\rho + P_x) - H_b(\rho + P_z)\ . \tag{4.17}$$

The observational success of FLRW cosmology places tight constraints on how anisotropic the universe can be. In order to restrict ourselves to solutions that remain close to an FLRW cosmology, we split each of these exact equations into an FLRW part which evolves according the FLRW equations of motion, and a non-FLRW part which we require to remain small, in a sense that we will now make clear. A similar approach was used in [169]. We define

$$H_a(t) = \bar{H}(t) + \epsilon f(t) \tag{4.18}$$

$$H_b(t) = \bar{H}(t) + \epsilon g(t) \tag{4.19}$$

$$\rho(t) = \bar{\rho}(t) + \epsilon r(t) \tag{4.20}$$

$$P_x(t) = P_y(t) = \bar{P}(t) \tag{4.21}$$

$$P_z(t) = \bar{P}(t) + \epsilon s(t) \tag{4.22}$$

where overbars denote the FLRW quantities and ϵ is a small perturbative parameter for which we will determine an upper bound later. Substituting these definitions into equations (4.14)-(4.17) and collecting powers of ϵ gives the zeroth-order (or background) equations, which are just the usual ones of the FLRW metric, and the first-order equations

$$2\bar{H}(f + 2g) = 8\pi G r \tag{4.23}$$

$$6\bar{H}f + 2\dot{f} = -8\pi G s \tag{4.24}$$

$$3\bar{H}(f + g) + \dot{f} + \dot{g} = 0 \tag{4.25}$$

$$\dot{r} = -3\bar{H}r - \bar{\rho}(2f + g) - \bar{H}s \tag{4.26}$$

These constitute four equations in four variables, but only three of these equations are independent. To close the system we need additional information, which we obtain by assuming an equation of state of the form

$$s(t) = \sigma r(t) \tag{4.27}$$

where the parameter σ is taken to be constant. Note that this is analogous to $\bar{P} = w\bar{\rho}$, except that it relates the anisotropic component of the pressure to the non-FLRW correction of the energy density. The value of σ will be important in determining whether the non-FLRW parts of equations (4.23)-(4.26) grow or decay in time.

Realistic models will be those for which the amount of anisotropy is sufficiently small in the past and present. Assuming the anisotropy is set (for example by inflation) to be sufficiently small at some early epoch, the question then is whether the anisotropy grows or not. In our set-up this corresponds to asking whether the non-FLRW parts of equations (4.23)-(4.26) grow or not, and if so, how quickly. We will see that σ governs the general behavior of the non-FLRW quantities, but for a given σ, the details will depend on the background (FLRW) solution. Since we are integrating from the present up to redshifts of order 1, our background is well described by the ΛCDM model. Using this background we can analytically find the asymptotic behavior of the non-FLRW quantities for different values of the equation state parameter σ.

The background energy density and pressure for ΛCDM are

$$\begin{aligned} \bar{\rho} &= \bar{\rho}_m + \rho_\Lambda \\ \bar{P} &= -\rho_\Lambda \ , \end{aligned}$$

where $\bar{\rho}_m$ is the background energy density of matter and ρ_Λ is the effective energy density of the cosmological constant. The background equations become

$$3\bar{H}^2 = 8\pi G(\rho_\Lambda + \bar{\rho}_m) \tag{4.28}$$

$$2\dot{\bar{H}} + 3\bar{H}^2 = 8\pi G\rho_\Lambda \tag{4.29}$$

$$\dot{\bar{\rho}}_m = -3\bar{H}\bar{\rho}_m \ . \tag{4.30}$$

The solutions can be written simply

$$\bar{H}(t) = A\coth\left(\frac{3}{2}At\right) \tag{4.31}$$

$$\bar{\rho}_m = \rho_\Lambda\left[\sinh\left(\frac{3}{2}At\right)\right]^{-2} \tag{4.32}$$

$$A = \sqrt{\frac{8\pi G\rho_\Lambda}{3}} . \tag{4.33}$$

Using (4.27) and (4.23) in (4.26) then gives

$$\dot{r} = -\left[(3+\sigma)A\coth\left(\frac{3}{2}At\right) + \frac{12A\pi G}{\sinh\left(\frac{3}{2}At\right)}\right] r(t) , \tag{4.34}$$

which can be integrated to find

$$r(t) = c_1\cosh\left(\frac{3}{2}At\right)\left[\sinh\left(\frac{3}{2}At\right)\right]^{-(3+\frac{2}{3}\sigma)} . \tag{4.35}$$

Using the equation of state (4.27) immediately gives $s(t)$. Equations (4.24) and (4.25) can then be integrated to find the remaining solutions (for $\sigma \neq -9/2$)

$$f(t) = \frac{A}{\left[\sinh\left(\frac{3}{2}At\right)\right]^2}\left(c_2 + \frac{3c_1}{2\left[\sinh\left(\frac{3}{2}At\right)\right]^{\frac{2}{3}\sigma}}\right) \tag{4.36}$$

$$g(t) = -f(t) - \frac{Ac_2}{\left[\sinh\left(\frac{3}{2}At\right)\right]^2} . \tag{4.37}$$

Now we consider the asymptotic behavior of these solutions for $t \to \infty$ and $t \to 0$ in turn.

The behavior as $t \to \infty$ is

$$r(t) \sim s(t) \sim c_1\left[\sinh\left(\frac{3}{2}At\right)\right]^{-2(1+\frac{1}{3}\sigma)} . \tag{4.38}$$

$$f(t) \sim -g(t) \sim c_1\left[\sinh\left(\frac{3}{2}At\right)\right]^{\mathrm{Max}\left[-2,-2(1+\frac{1}{3}\sigma)\right]} . \tag{4.39}$$

Here we see that $\sigma = -3$ is the boundary between growing and decaying solutions. If we require all of the non-FLRW quantities to decay as $t \to \infty$, then we must restrict ourselves to $\sigma > -3$.

As $t \to 0$, the situation is slightly more complicated. The behavior of $f(t)$ and $g(t)$ have a part that depends on the value of σ and a part that does not. The part that does not behaves as

$$c_2 \frac{A}{\left[\sinh\left(\frac{3}{2}At\right)\right]^2} \sim c_2 t^{-2} \;, \quad t \to 0 \;,$$

whereas the part that depends on σ behaves as

$$c_1 \frac{3A}{2\left[\sinh\left(\frac{3}{2}At\right)\right]^{2(1+\frac{1}{3}\sigma)}} \sim c_1 t^{-2(1+\frac{1}{3}\sigma)} \;, \quad t \to 0 \;.$$

We require that the expansion history is close to FLRW in the far past, which amounts to demanding that $|f/\bar{H}|$, $|g/\bar{H}|$, etc remain $\lesssim \mathcal{O}(1)$ as $t \to 0$. From the first term, this requires that $c_2 = 0$, while the second term requires that $\sigma < -3/2$. Therefore solutions with decaying anisotropy have an equation of state parameter lying in the range $-3 < \sigma < -3/2$. Turning to $r(t)$ and $s(t)$, as $t \to 0$ we have

$$r(t) \sim s(t) \sim c_1 \left[\sinh\left(\frac{3}{2}At\right)\right]^{-(3+\frac{2}{3}\sigma)} \;. \tag{4.40}$$

When $\sigma > -3$ these solutions diverge as $t \to 0$, but if $\sigma < -3/2$ they diverge slower than $\bar{\rho}$ and $\bar{P}$, respectively. So the condition

$$-3 < \sigma < -\frac{3}{2} \tag{4.41}$$

will ensure that all non-FLRW quantities remain small in the far past, as required. We will restrict ourselves to these models in the rest of this chapter.

What remains is to fix c_1 and the constant A in the solutions (4.35)-(4.37). We do this by imposing observational constraints on the models of interest. First we impose a condition at the surface of last scattering. For an anisotropically expanding universe to be viable it must at the very least predict angular variations in the temperature of the CMB no bigger than 10^{-5}. We can estimate the maximum temperature difference at the time of last scattering by

$$\begin{aligned} \Delta T_0 &= |T_0^{xy} - T_0^z| \\ &= \left| T_{\rm lss}\left(\frac{a_{\rm lss}}{a_0}\right) - T_{\rm lss}\left(\frac{b_{\rm lss}}{b_0}\right) \right| \;, \end{aligned} \tag{4.42}$$

where T^z is the temperature along the axis of symmetry and T^{xy} is the temperature in the transverse plane; subscripts "0" and "lss" refer to quantities today and at the last scattering surface, respectively. Recall that $b(t)$ is the scale factor along the axis of symmetry and $a(t)$ is the scale factor in the transverse plane. From the definitions of H_a and H_b, we have

$$a(t) = a_0 \exp \int_{t_0}^{t} (\bar{H}(t') + \epsilon f(t'))dt' \tag{4.43}$$

$$b(t) = b_0 \exp \int_{t_0}^{t} (\bar{H}(t') + \epsilon g(t'))dt' \ . \tag{4.44}$$

Using these, and choosing $a_0 = b_0 = 1$, we can re-write the expression for ΔT_0 to first order in ϵ as

$$\Delta T_0 = T_{\rm lss} \left(\frac{b_{\rm lss}}{b_0}\right) \left|1 - \left(\frac{a_{\rm lss}}{b_{\rm lss}} \frac{b_0}{a_0}\right)\right| \simeq 2\epsilon T_0^z \left|\int_{t_0}^{t_{\rm lss}} f(t)dt\right| \ , \tag{4.45}$$

where we have used the fact that $g(t) = -f(t)$, which is a consequence of requiring $c_2 = 0$. Inserting the solution for $f(t)$ gives

$$\frac{\Delta T_0}{T_0^z} \simeq 3A\epsilon c_1 \int_{t_0}^{t_{\rm lss}} \left[\sinh\left(\frac{3}{2}At\right)\right]^{-2(1+\frac{\sigma}{3})} dt \ . \tag{4.46}$$

Demanding the left-hand side to be at most 1.3×10^{-6} [170] yields one condition on the product ϵc_1 and A for a given equation of state parameter σ.

In order to break the degeneracy between ϵc_1 and A, we need one further condition. Equation (4.46) already requires the difference between H_a and H_b to be small – well within the accepted uncertainty in the measured value of the Hubble parameter [166]. We find it convenient to choose to set the arithmetic average of the Hubble parameters in the three directions equal to the observed value.

$$\begin{aligned} H_{\rm obs} &= \frac{2H_a + H_b}{3} \\ &= \bar{H} + \frac{\epsilon}{3} f(t) \ , \end{aligned} \tag{4.47}$$

where again we have used the fact that when $c_2 = 0$, $g(t) = -f(t)$. Alternative choices, such as setting H_a or H_b equal to the measured value of the Hubble parameter, would not change the order of magnitude of ϵc_1 and A and would leave the final result essentially unaltered. Inserting the solution for $f(t)$ into equation (4.47) gives a second condition on ϵc_1 and A

$$\epsilon c_1 = \frac{2}{A}(H_{\text{obs}} - \bar{H}) \left[\sinh\left(\frac{3}{2}At\right)\right]^{2(1+\frac{\sigma}{3})} \quad . \tag{4.48}$$

We can (numerically) solve this equation for A in terms of ϵc_1 and then substitute it back into the first condition (4.46) to obtain an upper bound on ϵc_1. By taking the maximal allowed value for ϵc_1, we can then find A using (4.48). For $\sigma = -2$ we find $A \simeq 62$(km/s)/Mpc and $\epsilon c_1 \simeq 1.3 \times 10^{-6}$. We always take c_1 to be $\mathcal{O}(1)$, and so in this case we choose $c_1 = 1.3$ and $\epsilon = 10^{-6}$.

In this way we can fully determine solutions to the non-FLRW quantities for a given equation of state parameter σ. By restricting ourselves to $-3 < \sigma < -3/2$ we have chosen to focus on models for which these solutions decay in the distant future and which diverge slower than the respective background quantities in the far past as one approaches the initial singularity. These models seem to be the most conservative realizations of a Bianchi-I cosmology, in the sense that they are the easiest to make consistent with observations, or alternatively, the most difficult to rule out. One might try to push the boundaries slightly, for example by considering models with anisotropies that approach a constant in the distant future rather than vanishing. It may be that such models can be carefully tuned to match observations. We do not consider these more general models here, since our main interest is not model-building, but rather to explore a general effect (cosmic parallax) arising in a Bianchi-I universe.

4.4 Geodesics and Parallax in Axisymmetric Bianchi-I Models

As in section 4.2, in order to analyze cosmic parallax we need to find null geodesics in the spacetime that connect an observer and various sources at two different times. As before, letting latin indices run over $1, 2, 3$ and $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, the non-zero Christoffel symbol components for the Bianchi-I metric in (4.13) are

$$\Gamma^0{}_{ij} = a_i^2 H_i \delta_{ij} \ , \quad \Gamma^i{}_{0j} = H_i \delta^i{}_j \ ,$$

where no sum on the index i is implied and H_i is defined as above. The four geodesic equations are then

$$\frac{d^2 t}{d\lambda^2} = -\sum_i H_i \left(a_i \frac{dx^i}{d\lambda} \right)^2 \tag{4.49}$$

$$\frac{d^2 x^i}{d\lambda^2} = -2H_i \frac{dt}{d\lambda} \frac{dx^i}{d\lambda} \ , \tag{4.50}$$

with the null geodesic condition, $u^\mu u_\mu = 0$, becoming

$$\left(\frac{dt}{d\lambda} \right)^2 = \sum_i \left(a_i \frac{dx^i}{d\lambda} \right)^2 \ . \tag{4.51}$$

As before, $u^\mu \equiv dx^\mu / d\lambda$. We specialize to the axisymmetric case by setting $a_1 = a_2 = a(t)$ and $a_3 = b(t)$, and also $H_1 = H_2 = H_a$ and $H_3 = H_b$. The scale factors, $a(t)$ and $b(t)$, and Hubble parameters, $H_a(t)$ and $H_b(t)$, are fixed after choosing σ and solving the full set of equations as in the previous section.

After fixing the background Bianchi-I spacetime, equations (4.49) and (4.50) yield four second-order differential equations and one constraint equation in four dependent variables. To solve this system we must in principle specify initial conditions for the four dependent variables as well as initial velocities (derivatives with respect to λ) giving a total of eight conditions. However, using the constraint equation the system can be reduced to seven independent first-order equations.

Considering the $u^\mu(t) \equiv dx^\mu(t)/d\lambda$ as functions of time along the geodesic, equations (4.50) can be integrated immediately to give

$$
\begin{aligned}
u^i(t) &= u_0^i \, e^{-2\int_{t_0}^{t} H_i(t')dt'} \\
&= u_0^i \, a_i^{-2}(t) \ ,
\end{aligned}
\tag{4.52}
$$

which can then be used in equation (4.51) to give

$$
\left(\frac{dt}{d\lambda}\right)^2 = \sum_i \left(\frac{u_0^i}{a_i(t(\lambda))}\right)^2 \ . \tag{4.53}
$$

As in the LTB case it is again useful to find an expression for the redshift as it will be one of our observational inputs. To find this expression, as usual, we consider two photons emitted from a source at times t and $t+\tau$, respectively, where τ is taken to be infinitesimally small. The trajectory of the first photon is described by equation (4.53), while to first order in τ, the trajectory of the second photon is described by the geodesic equation

$$
\left(\frac{dt}{d\lambda}\right)^2 + 2\frac{dt}{d\lambda}\frac{d\tau}{d\lambda} = \sum_i \left(u^i a_i\right)^2 (1 + 2\tau H_i) \ . \tag{4.54}
$$

Here the variation in time corresponds to a time-delay, and so to a change in geodesic, and not to a change of the time coordinate along a fixed geodesic. Since the $u^i(\lambda)$ are directional derivatives along a given geodesic, they remain unaffected by time variations (derivatives) in obtainig this equation. Using the standard definition of redshift, $1 + z(\lambda_{\rm em}) \equiv \tau(\lambda_{\rm ob})/\tau(\lambda_{\rm em})$, we find the relation

$$
\frac{d\log(1+z(\lambda_{\rm em}))}{d\lambda_{\rm em}} = -\frac{1}{\tau(\lambda_{\rm em})}\frac{d\tau(\lambda_{\rm em})}{d\lambda_{\rm em}} \ , \tag{4.55}
$$

where $\lambda_{\rm em}$ and $\lambda_{\rm ob}$ are the values of the affine parameter at the emission and observation event, respectively. Using this in (4.54) gives

$$
\begin{aligned}
\frac{d\log(1+z)}{d\lambda} &= -\frac{\sum_i \left(u^i a_i\right)^2 H_i}{\sqrt{\sum_i \left(u^i a_i\right)^2}} \\
&= \frac{d\log\sqrt{\sum_i \left(u^i a_i\right)^2}}{d\lambda} \ .
\end{aligned}
\tag{4.56}
$$

In the last step we have also made use of (4.52) and the chain rule

$$\frac{d}{d\lambda}(u^i(\lambda)a_i(\lambda)) = -\frac{dt}{d\lambda}(u^i(t)a_i(t)H_i(t)) .$$

With the initial condition $z(\lambda_0) = 0$, integrating (4.56) then gives

$$(1+z) = \frac{\sqrt{\sum_i (u^i a_i)^2}}{\sqrt{\sum_i (u_0^i)^2}} . \tag{4.57}$$

As in section 4.2, we would like to express our results not only in terms of redshift but also in terms of angles, since these are the actual observables. In general there are four pieces of data for each object in the sky, namely the time of observation, two angles with respect to an arbitrary coordinate system, and the observed redshift of the source. In the case of cylindrical symmetry one angle is enough. In contrast with LTB spacetimes, not much is gained in this case by rewriting the geodesic equations in terms of angles and redshift. Instead, we numerically integrate the equations in the above coordinates and then express the results in terms of angles and redshift.

Our procedure for computing the cosmic parallax for these models is analogous to that in section 4.2. We work in local Cartesian coordinates (t, x_1, x_2, x_3) in which the observer is located at the origin. As seen above, the system of four second-order differential equations plus a constraint reduces to seven independent first-order equations, some of which can be integrated immediately by hand. We are then left with the problem of fixing initial conditions for our complete set of equations

$$\frac{dt}{d\lambda} = -\sqrt{\sum_i (u^i a^i)^2} \tag{4.58}$$

$$u^i(\lambda) = \frac{u_0^i}{a_i^2} \tag{4.59}$$

$$\frac{dx^i}{d\lambda} = u^i , \tag{4.60}$$

where the subscripts "$_0$" refer to quantities at the initial observation event, corresponding to $\lambda = 0$. To close the system requires seven initial conditions that unfortunately cannot be all specified at the observation event. By construction we have

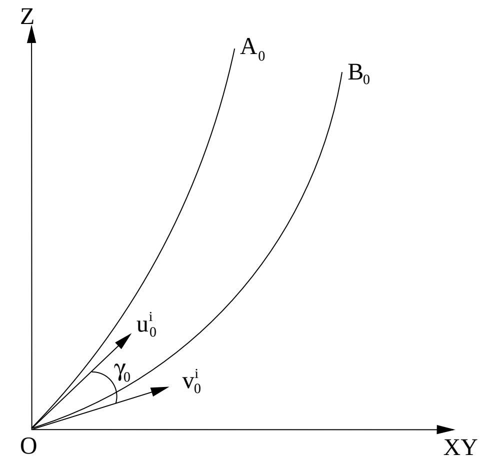


Figure 4.2: Initial angle γ_0 defined by the velocity vectors at observing time for two sources

${x^i}_0 = 0$, leaving four remaining conditions, three of which are obtained by specifying t_0 and two initial spatial velocities in terms of measurable quantities (angles), while the last one is given by the observed redshift. We integrate backwards along the initial geodesic until reaching the desired redshift and then find the comoving coordinates of the source. To find the final geodesic that connects the same source with the observer at a later time $t_0 + \Delta t$, we solve the corresponding boundary-value problem (namely, to find solutions of the null geodesic equations with two fixed endpoints). We then find the velocities along the final geodesics at the time of observation.

This procedure is repeated for two sources, yielding four sets of velocities: one set for each initial geodesic and one set for each final geodesic. The velocities u^i and v^i along two geodesics at the same observing time (see figure 4.2) are related to the angle γ between them by

$$\cos(\gamma) = \frac{\sum_i a_i^2 u^i v^i}{\sqrt{\left(\sum_j (a_j u^j)^2\right)\left(\sum_k (a_k v^k)^2\right)}} \,. \tag{4.61}$$

We then calculate the cosmic parallax $\Delta\gamma$, as defined in (4.10), by taking the difference of the angle between the two sources at the two different times. Finally, we plot the cosmic parallax as a function of the polar coordinate θ for one of the sources. By

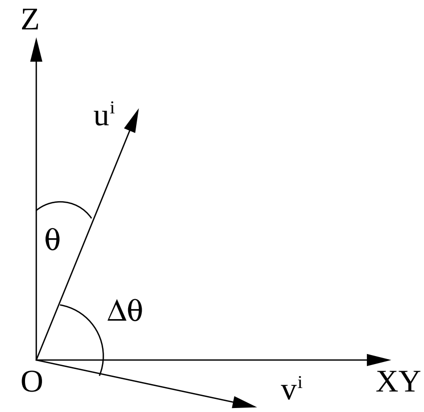


Figure 4.3: Location of sources as seen by an observer at the measuring event. The vertical axis points along the axis of symmetry. We are considering sources at equal redshifts of $z = 1$ and in a plane defined by a fixed value of the polar angle ϕ.

convention we choose the second, or trailing, source as our reference, as shown in figure 4.3. We find the parallax for two hypothetical sources with the same redshift, initially separated by a given angle on the sky, and for a given Δt and ϵ. In figure 4.4 we plot the parallax for two hypothetical sources at $z = 1$, with an initial separation of 90 degrees on the sky, for $\epsilon = 10^{-6}$ and various values of Δt. For $\Delta t = 500$ years and $\epsilon = 10^{-6}$ we find this maximal value to be of the order of 6×10^{-14} radians. Of course, our true goal is to find the maximal signal for reasonable time scales, say $\Delta t \sim 10$ years, but unfortunately numerical noise dominates over the signal for Δt of that magnitude. One can see from the first quadrant of figure 4.4 that by $\Delta t = 20$ years the signal-to-noise ratio becomes very poor, making it difficult to extract trustworthy predictions. Therefore, although we can directly compute the signal for this model at $\Delta t = 10$ years, we prefer to calculate the cosmic parallax for several values of Δt between 5 and 500 years and interpolate the value at 10 years. Since our primary goal is to find an order of magnitude estimate for the effect, this approach should be acceptable. Figure 4.5 shows the values of the cosmic parallax between two sources for decreasing values of Δt for a specific direction θ in the sky (keeping all other values fixed). For each value of θ, we use a linear fit passing through the origin (because

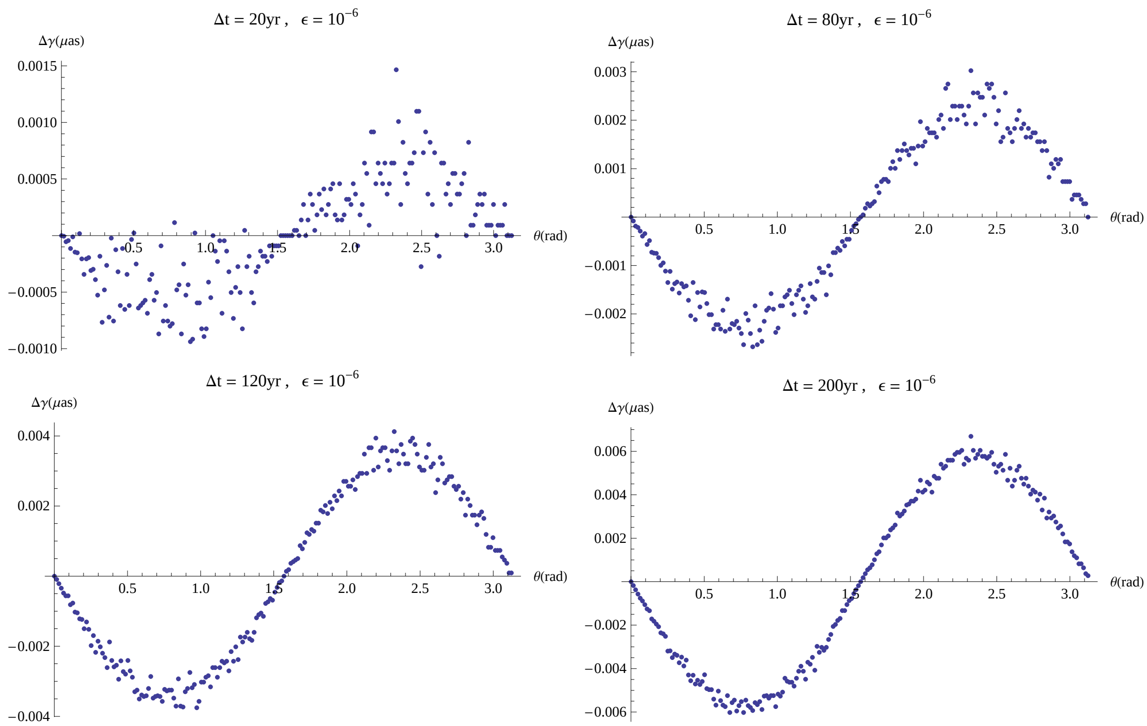


Figure 4.4: A sequence of cosmic parallax signals for different values of Δt and fixed $\epsilon = 10^{-6}$. Top row, left to right: Δt = 20yrs, Δt = 80yrs. Bottom row, left to right: Δt = 120yrs, Δt = 200yrs. The signal-to-noise ratio becomes smaller as Δt becomes smaller.

cosmic parallax must vanish for $\Delta t = 0$) to find the interpolated value at 10 years. We could just as well perform an extrapolation by omitting data below some cutoff, say $\Delta t = 50$ years. This does not have any noticeable effects on our results.

To check the consistency of this procedure we repeat the same analysis, except now we keep the time interval fixed, namely $\Delta t = 10$ years, and vary the amount of anisotropy by considering values of ϵ between 10^{-6} and 10^{-4}. Also in this case we find a linear dependence of the parallax angle on the varied parameter. In figure 4.6 we plot the angular dependence of the parallax for different values of anisotropy, and in figure 4.7 we plot the parallax as a function of anisotropy for a fixed direction.

After repeating the interpolation for all our data points for both the time and the anisotropy dependence we show our main result in figure 4.8. The two methods

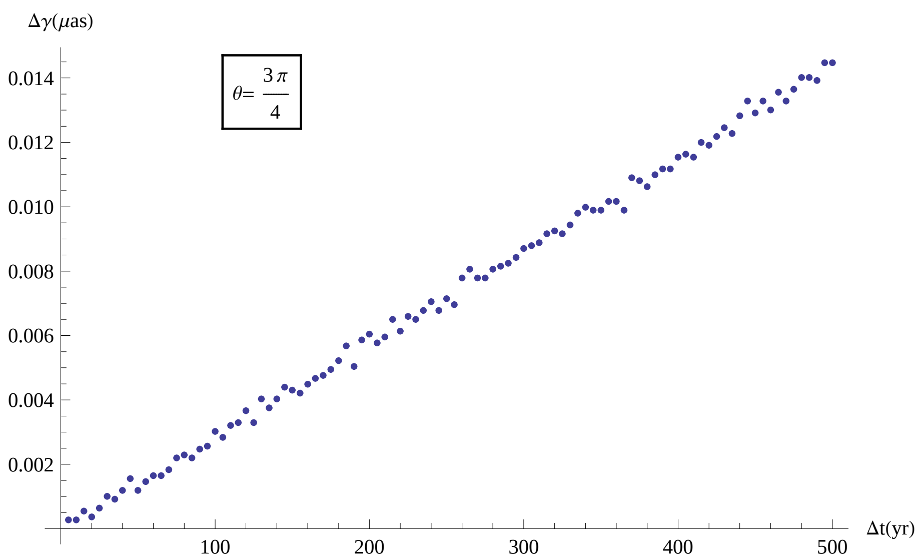


Figure 4.5: Cosmic parallax as a function of Δt. Here $\epsilon = 10^{-6}$ and sources are at $z = 1$, separation angle $(\Delta\theta)$ of 90 degrees.

give consistent results, strengthening our confidence in the correctness of the linear interpolation procedure.

For the largest allowed values of ϵ consistent with the observed anisotropy of the CMB (i.e., $\epsilon \sim 10^{-6}$) and for time delays on the order of 10 years, we find that the maximal cosmic parallax is on the order of 10^{-15} radians, or equivalently 10^{-4} microarcseconds. This is three orders of magnitude smaller than the maximal cosmic parallax seen by [152] for LTB models. It is also three orders of magnitude smaller than the expected level of cosmic parallax from peculiar velocities alone in a ΛCDM universe [171]. In other words, the contribution to cosmic parallax in this model due to anisotropic expansion is sub-dominant to the contribution from peculiar velocities[2].

The qualitative behavior of the cosmic parallax in this model is also quite different from that of LTB models as put forth in [152]. Both LTB and Bianchi models show

[2]Here we assume that the Bianchi-I models we consider have roughly the same peculiar velocity-redshift relation as an FLRW universe.

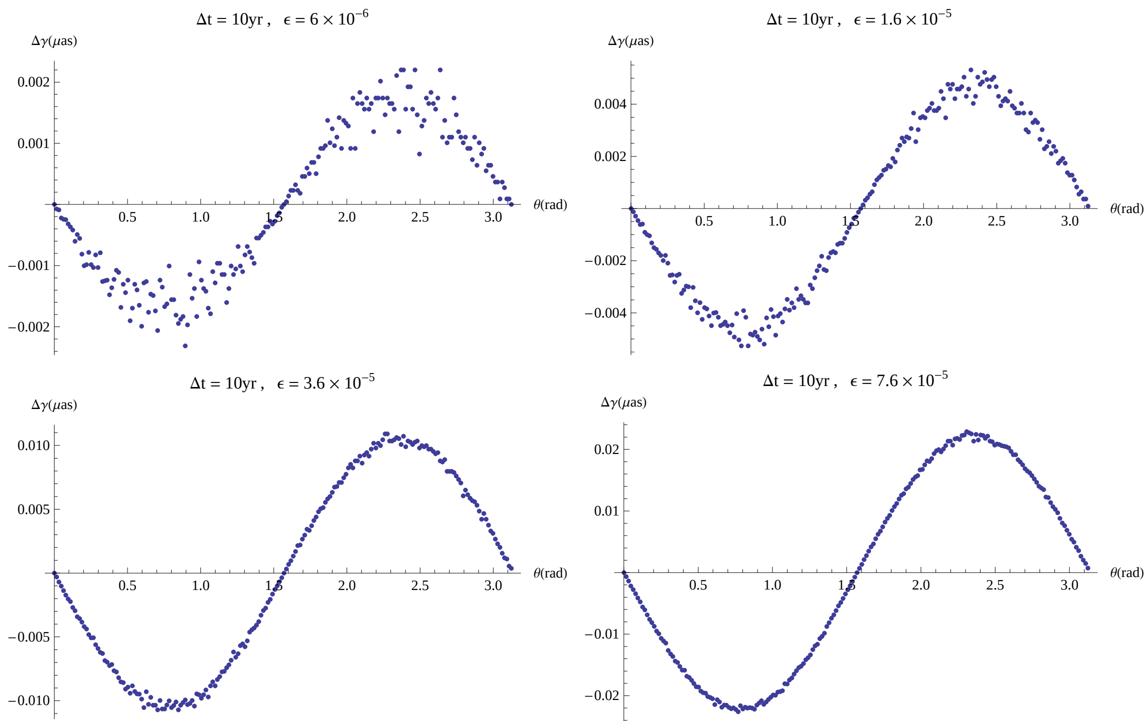


Figure 4.6: A sequence of cosmic parallax signals for different values of ϵ at $\Delta t = 10$yrs. Top row, left to right: $\epsilon = 6\,10^{-6}$, $\epsilon = 1.6\,10^{-5}$. Bottom row, left to right: $\epsilon = 3.6\,10^{-5}$, $\epsilon = 7.6\,10^{-5}$.

a sinusoidal (or at least quasi-sinusoidal) cosmic parallax. However, whereas LTB models exhibit a 2π-periodic behavior, here we see that Bianchi models exhibit a π-periodic behavior. This is to be expected due to the symmetries of the two types of spacetimes, as alluded to earlier in the chapter. The LTB spacetime is axisymmetric about an off-center observer but is not plane-symmetric about the plane normal to the symmetry axis. If we align the z-axis along the symmetry axis, then in spherical-polar coordinates this amounts to saying that the spacetime is invariant under changes in ϕ (the azimuthal angle) but has no symmetries under (nontrivial) changes in θ (the polar angle). In other words, one would expect cosmic expansion to be 2π-periodic in θ, which is just what was seen for the cosmic parallax in these models in [152]. The Bianchi spacetimes, on the other hand, are both axisymmetric and plane-symmetric about the plane normal to the axis of symmetry. So one would expect

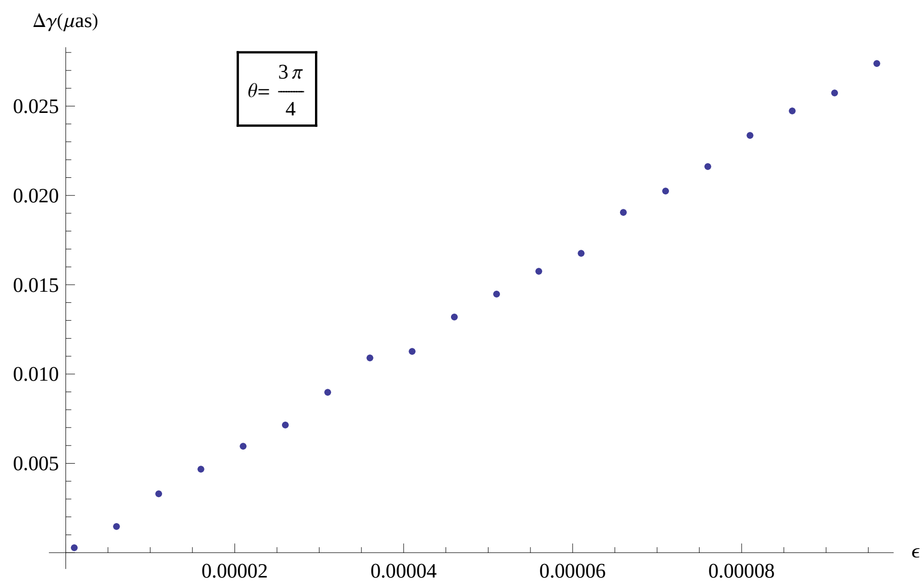


Figure 4.7: Cosmic parallax as a function of ϵ for a fixed value of $\Delta t = 10$yrs.

cosmic expansion to be π-periodic in θ, which is what we see for the cosmic parallax in these models.

Although here we have only considered a particular model, preliminary investigations into other models suggest that these results are robust. For example, one might consider models with other constant values for the equation of state parameter σ, a time-varying rather than constant σ, or an equation of state that takes a different form than equation (4.27). None of these modifications seem to affect the order of magnitude of the cosmic parallax (which, due to the symmetry of the metric, is the only free parameter). What this suggests is that the contribution to the cosmic parallax from viable Bianchi-I models is much smaller than the contribution from viable LTB-void models. If the observed cosmic parallax deviates from what is expected in an FLRW universe, it is unlikely that this is due to our living in a Bianchi-I spacetime.

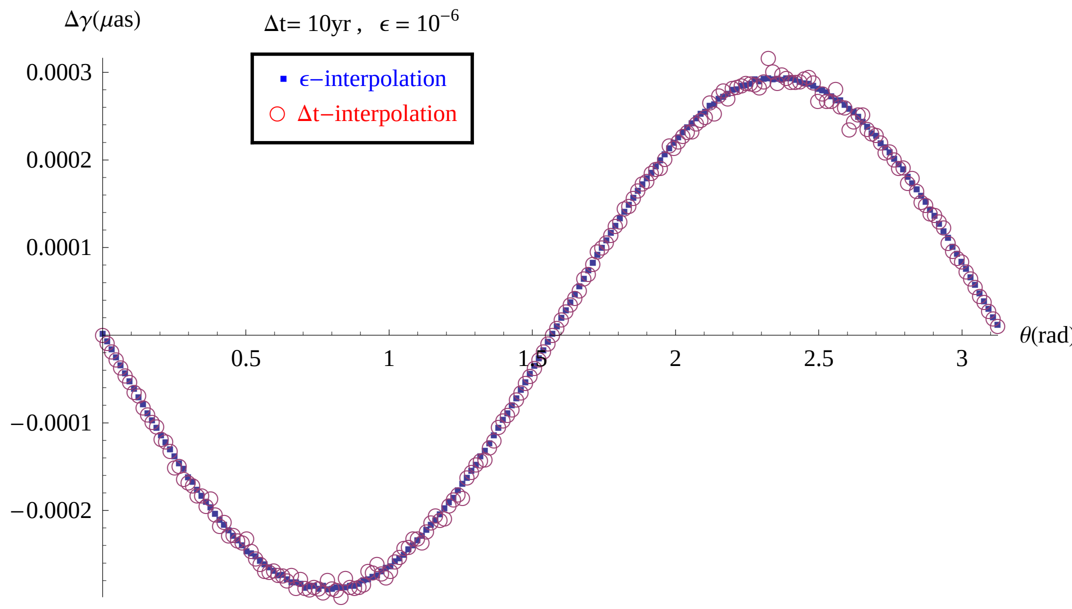


Figure 4.8: Estimate of the cosmic parallax for $\Delta t = 10$ years, $\epsilon = 10^{-6}$, and for sources at $z = 1$ with a separation angle of 90 degrees, plotted as a function of the angle between the trailing source and the symmetry axis (i.e., the angle between the leading source and the symmetry axis is advanced by 90 degrees). The two sets of plotted data correspond to interpolated values found using 100 trials for Δt spread from $5 - 500$ years, and interpolated values found using 20 trials in ϵ in the $10^{-6} - 10^{-4}$ range, respectively.

4.5 Discussion

In order for the standard FLRW cosmology to agree with observations, the expansion of the universe must be accelerating. So far, all suggested mechanisms to drive such acceleration involve new physics; either the existence of exotic new components of the cosmic energy budget, modifications to Einstein's theory of gravity, or a cosmological constant. Alternatively, it has been suggested that the phenomenon of cosmic acceleration is due to interpreting results in the FLRW model, when in fact the correct underlying cosmic geometry could be that of a void model, such as that idealized by an LTB spacetime. These models are relatively simple and make a host of predictions that can be used to either test or constrain them. One prediction that may soon be testable is cosmic parallax. For observers located 10 Mpc from the center of a 1 Gpc LTB void, this effect would have a magnitude of the order of $10^{-1}\mu$as per decade for sources at redshift 1. Since the effect in an FLRW universe (due to peculiar velocities) is expected to be roughly the same order of magnitude, one might hope to subtract the signal due to peculiar velocities from the signal due to anisotropic expansion about a point. Measuring this additional contribution to the cosmic parallax would clearly indicate a departure from FLRW.

However, what is less clear is how we might interpret detection of an additional contribution to cosmic parallax. Here we have examined an axisymmetric Bianchi-I universe as an alternative explanation for any cosmic parallax component that is not attributable to peculiar velocities. We find that for a class of models whose anisotropy is consistent with the observed temperature anisotropies of the CMB and Hubble expansion today, the maximum amount of cosmic parallax is three orders of magnitude smaller than the maximal signal in LTB models. Perhaps more importantly, the maximum effect is also three orders of magnitude smaller than the expected level of cosmic parallax from peculiar velocities in an FLRW universe. Although we have focused our discussion in this chapter on a particular model, we have found these results to be fairly model-independent. Thus it seems unlikely that measurements of cosmic parallax can constrain Bianchi-I models that are not already ruled out by the CMB or other cosmological observations. Therefore, while cosmic parallax will be nonzero for any anisotropic expansion, the magnitude of the effect suggested in [152] appears to be significantly larger, and qualitatively different than in the class of models considered here.

Conclusions

General Relativity and Field Theory are the basics tools cosmologists can use to describe the universe. In this framework, fields describing matter as well as corrections to the background geometry are treated as quantum objects living in an otherwise unperturbed classical background fully described by some classical GR solution. Luckily enough, this treatment seems to be sufficient to take into account the main features of cosmological systems, as encoded in the standard model of cosmology.

During most of the evolution of the universe, GR does an incredibly good job in providing the right background for all the physical phenomena that cosmologists and astro-physicists are interested in. It only fails where by definition it is expected to fail, namely when a full quantum theory of gravity is required as the times under exam are too close to the initial singularity. The need for a more fundamental theory though, is not the only reason why physicists venture into the research of alternative descriptions and models. The more data, the more special features of our universe emerge, and suggest that the simplest description may not be suitable to provide an explanation of what we see in the sky.

In order to go beyond the simplest description, there are only few possible steps that can be taken. One way to proceed is to consider modifications of the theory of gravity, and propose alternative descriptions such as scalar-tensor theories, bouncing cosmologies, or even inhomogeneous models. Each of these ideas has been proposed to address some specific problem or feature such as late time acceleration, inflation with a proper end and reheating of the universe, and so on. Another way to proceed which we have discussed here is via the parametrization of the unknown by using the effective field theory approach. The great advantage of this bottom-up point of view is the wide scope of application, as it can be used for calculating predictions for all of the above mentioned problems, as well as any other physical features for which nonlinearities are not strong and a series expansion makes sense – a necessary assumption to obtain sensible results for corrections of the "zeroth-order theory".

Chapter 2 of this work describes the application of the effective field theory approach to the study of the validity of perturbation theory during the inflationary early stages of the universe. The aim was finding when perturbations cease to be such and the formalism breaks down. In other words we have looked for an estimate of the upper momentum cutoff for the inflating theory when the background considered is not fully Lorentz-invariant. We have found that the perturbative expansion holds to higher physical momenta than naïvely expected.

In chapter 3, we adopted a more radical point of view in modifying gravity, and allowed for the existence of extra degrees of freedom. Via an intricate regularization scheme we have calculated the corrections to the GR description of late time acceleration due to the extra degrees of freedom. In particular we have considered the effect of a sixth order term to check the consistency of the Euclidean path integral method at orders higher than four. We have found that in an accelerated background ambiguities emerge and that, if this method is applied to all orders, perturbations grow at higher orders leading to a divergent result.

Finally in chapter 4, we considered an alternative approach to the interpretation of supernovae dimming. We have studied an anisotropic universe described by a Bianchi-I model and its signature on a parallax proposed experiment. We have then compared the results of our calculations and simulations with similar studies for LTB models that share some similarities with the one considered, and showed that the signatures of these models are in principle different in shape. We have also discussed the magnitude of cosmic parallax for viable anisotropic models, and showed it to be orders of magnitude smaller than in the case of LTB.

While in the work we have restricted our attention to particular problems and areas of cosmology, such as early and late time acceleration, we point out that the analyzed methods apply in general to almost any topic in cosmology. As we tried to stress in chapter 1 introduction, the EFT approach can be applied to any system either to integrate out some degrees of freedom that do not participate in the process under consideration (see for example heavy fields that at low energies effectively work as possibly classical spectators for the physics of other lighter ones), or to parametrize what is assumed to exist in the spectrum of a more fundamental theory at higher energies. Higher dimensional theories are a typical example. In these models extra dimensions beside the four that we experience in everyday life have some effect on the physics that we observe. In order to calculate such effects one has to integrate out the extra dimensional components, at least at the energies at which lab experiments do not give sensible deviations from the usual four dimensional physics. For instance any braneworld theory makes use of EFT calculations, and so do bouncing cosmologies such as Ekpyrosis. The same applies to models with extra scalar fields in the gravitational sector of the theory such as DGP, Galileons and many others.

When gravity is involved, higher derivatives often appear and and requires to decide whether or not to consider as physical the degrees of freedom that they may introduce. It follows that methods like the one analyzed in chapter 3 are of general interest when alternatives to EFT are taken into consideration. It is then important to investigate the robustness and the limits of applicability of such methods as difficulties may emerge once they are applied to general systems, as shown in chapter 3 for the case of the Euclidean path integral.

Finally, it can be noted that even cosmic parallax, and the analysis presented in chapter 4 are quite general and may be applied to any model that predicts anisotropy. It only requires minor code tuning for the simulation to take into account the particular equations for the geodesics of the chosen cosmology.

As a way of conclusion, this work has been devoted to studying some of the most commonly used mathematical tools in cosmology along with their application to open problems such as cosmic acceleration, and the investigation of their limits of applicability. The two-folded objective was trying to go beyond the effective field theory on the one side and studying the regularization of infinities from higher derivative ghosts on the other. Apart from the results we have found and discussed, this work is meant to serve as a basis for future research and as a deeper analysis of fundamental cosmological concepts.

Appendix A

Wavefunctionals and Probabilities

A.1 Minkowski Background: Wavefunctional and Probability

The explicit form for the coefficients appearing in the definition of the wavefunctional, equation (3.53), can be cast in terms of λ_i (3.52) as follows

$$\begin{aligned}
A_{00} &= \lambda_1\lambda_2\lambda_3\left(-2M_{\mathrm{p}}^2\alpha^2+\beta^2\left(-3k^2+\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_2\lambda_3+\lambda_1\lambda_2+\lambda_1\lambda_3\right)\right)\\
A_{01} &= 2\alpha^2M_{\mathrm{p}}^2\left(2k^2+\lambda_2\lambda_3+\lambda_1\lambda_2+\lambda_1\lambda_3\right)+\beta^2\left(3k^4-\lambda_2^3\lambda_3-\lambda_2^2\lambda_3^2-\lambda_1^3\left(\lambda_2+\lambda_3\right)\right.\\
&\quad +M_{\mathrm{p}}^4-\lambda_1\left(\lambda_2+\lambda_3\right)\left(-3k^2+\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)+\lambda_2\left(3k^2\lambda_3-\lambda_3^3\right)\Big)\\
&\quad -\lambda_1^2\left(\lambda_2+\lambda_3\right)^2\\
A_{02} &= -2\alpha^2M_{\mathrm{p}}^2\left(\lambda_1+\lambda_2+\lambda_3\right)+\beta^2\left(\lambda_1^3+\lambda_2^3+\lambda_2^2\lambda_3+\lambda_2\lambda_3^2+\lambda_3^3+\lambda_1^2\left(\lambda_2+\lambda_3\right)\right.\\
&\quad \left.-3k^2\left(\lambda_1+\lambda_2+\lambda_3\right)+\lambda_1\left(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2\right)\right)
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
A_{10} &= -\beta^2\lambda_1\lambda_2\lambda_3\left(\lambda_1+\lambda_2+\lambda_3\right)\\
A_{11} &= \beta^2\left(\lambda_1+\lambda_2\right)\left(\lambda_1+\lambda_3\right)\left(\lambda_2+\lambda_3\right)\\
A_{12} &= 2M_{\mathrm{p}}^2\alpha^2-\beta^2\left(-3k^2+\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3\right)
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
A_{20} &= \beta^2 \lambda_1 \lambda_2 \lambda_3 \\
A_{21} &= -\beta^2 \left(\lambda_2 \lambda_3 + \lambda_1 \lambda_2 + \lambda_1 \lambda_3\right) \\
A_{22} &= \beta^2 \left(\lambda_1 + \lambda_2 + \lambda_3\right) \ .
\end{aligned} \tag{A.3}$$

The traced probability in Lorentzian time then reads

$$\begin{aligned}
P[\gamma_0] = N N^* \exp \Bigg[& \frac{1}{4M_{\rm p}^2 \beta^2 (\lambda_1 + \lambda_2 + \lambda_3)} \Bigg[-4\beta^2 \lambda_1 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 + \lambda_3) \left[-2M_{\rm p}^2 \alpha^2 \right. \\
& \left. + \beta^2 \left(-3k^2 + \lambda_1^2 + \lambda_2^2 + \lambda_2 \lambda_3 + \lambda_3^2 + \lambda_1 (\lambda_2 + \lambda_3) \right) \right] \\
& + \left[-2M_{\rm p}^2 \alpha^2 (\lambda_1 + \lambda_2 + \lambda_3) + \beta^2 \left(\lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_2^2 \lambda_3 + \lambda_2 \lambda_3^2 + \lambda_1^2 \lambda_2 + \lambda_1^2 \lambda_3 \right. \right. \\
& \left. \left. + \lambda_1 (\lambda_2 + \lambda_3)^2 - 3k^2 (\lambda_1 + \lambda_2 + \lambda_3) \right) \right]^2 \Bigg] |\gamma_0|^2 \Bigg]
\end{aligned} \tag{A.4}$$

A.2 de Sitter Background: Classical Solutions and Probability

The equations of motion in a de Sitter background, equation (3.57), admit solutions in terms of Bessel functions. To find the general solution shown in the text, equation (3.61), it is convenient to search for a factorization of the full sixth order differential operator D_6 defined in equation (3.58). D_6 can be split into a fourth order operator acting on a second order operator via

$$D_6[z]\gamma_{cl}(z) = D_4[z] D_2[z] \gamma_{cl}(z) \ , \tag{A.5}$$

where

$$\begin{aligned}
D_4[z] = & \frac{1}{z^2}\frac{d^4}{dz^4} - \left(\frac{2}{z^2} + \frac{1}{4z^4} \left(25 - 4\lambda_i^2 - 8 \frac{M_{\rm p}^2}{H^2} \frac{\alpha^2}{\beta^2} \right) \right) \frac{d^2}{dz^2} \\
& + \frac{1}{2z^5} \left(25 - 4\lambda_i^2 - 8 \frac{M_{\rm p}^2}{H^2} \frac{\alpha^2}{\beta^2} \right) \frac{d}{dz} + \frac{1}{z^2} - 2 \frac{M_{\rm p}^2}{H^2} \frac{\alpha^2}{\beta^2 z^4} + \frac{25 - 4\lambda_i^2}{4z^4} \\
& + \frac{1}{16 z^6 \beta^2 H^4} \left[8M_{\rm p}^4 - 8H^2 M_{\rm p}^2 \left(\alpha^2 - 48\Lambda^2 - 4\alpha^2 \lambda_i^2 \right) \right. \\
& \left. + \beta^2 H^4 \left(153 - 104\lambda_i^2 + 16\lambda_i^4 \right) \right] \ ,
\end{aligned} \tag{A.6}$$

$$D_2[z] = z^2 \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \left(z^2 + \lambda_i^2 - \frac{9}{4} \right) \ . \tag{A.7}$$

Here, to avoid confusion, we have replaced the coefficient λ^2 of the R^2 term in the action with Λ^2, while λ_i is a parameter in the decomposition. There are then three independent choices of the parameter λ_i, namely

$$\lambda_1 = \frac{3}{2}\,, \tag{A.8}$$

$$\lambda_2 = \frac{1}{2}\sqrt{13 + \frac{\left(-4\alpha^2 M_{\rm p}^2 - 2\sqrt{4H^4\beta^4 + M_{\rm p}^4\left(4\alpha^4 - 2\beta^2\right) - 24H^2M_{\rm p}^2\beta^2\left(\alpha^2 + 4\lambda^2\right)}\right)}{H^2\beta^2}}\,, \tag{A.9}$$

$$\lambda_3 = \frac{1}{2}\sqrt{13 + \frac{\left(-4\alpha^2 M_{\rm p}^2 + 2\sqrt{4H^4\beta^4 + M_{\rm p}^4\left(4\alpha^4 - 2\beta^2\right) - 24H^2M_{\rm p}^2\beta^2\left(\alpha^2 + 4\lambda^2\right)}\right)}{H^2\beta^2}}\,, \tag{A.10}$$

with these choices we obtain the six solutions of (3.61).

Once the classical solution is given, it is possible to calculate the associated value of the Euclidean action, find a wavefunctional as discussed in the text, and after tracing over the unobserved γ_0'' and γ_0', and rotating back to Lorentzian time, eventually find a probability for γ_0.

The full form of the non-normalized probability is then

$$\begin{aligned}
P[\gamma_0] = N\exp\Bigg[\frac{1}{D}\Bigg(&k^2\alpha^3 M_{\rm p}\Bigg(3H^4k\beta^4\eta r_2^5\left(Hk^3\beta\eta^3 + \alpha\left(1+k^2\eta^2\right)M_{\rm p}r_2\right)\\
&+H^3\beta^3 r_1 r_2^4\left(H^2\beta^2\left(18+9k^2\eta^2+4k^4\eta^4\right) + Hk\alpha\beta\eta\left(6+7k^2\eta^2+k^6\eta^6\right)M_{\rm p}r_2\right)\\
&+H\beta r_1^3 r_2^2\Big(H^4\beta^4\left(18+45k^2\eta^2+19k^4\eta^4\right) + H^2k^4\alpha^2\beta^2\eta^4M_{\rm p}^2r_2^2\left(21+9k^2\eta^2+k^4\eta^4\right)\\
&+2Hk\alpha^3\beta\eta\left(12+17k^2\eta^2+5k^4\eta^4+k^6\eta^6\right)M_{\rm p}^3r_2^3 + \alpha^4\left(1+k^2\eta^2+k^4\eta^4+k^6\eta^6\right)M_{\rm p}^4r_2^4\Big)\\
&+H^3\beta^3 r_1 r_2^4\alpha^2M_{\rm p}^2r_2^2\left(1+k^2\eta^2\right)\left(1+k^4\eta^4\right) + M_{\rm p}^5\left(A_1+A_2+A_3\right) + A_4 + A_5 + A_6\Bigg)\Bigg)\Bigg]\,,
\end{aligned} \tag{A.11}$$

where the denominator at the exponent and the A_i are given by

$$
\begin{aligned}
D =& H^2\beta^2\eta\left(1+k^2\eta^2\right)(r_1-r_2)\Big[-H^4\beta^4 r_1\left(3+2k^2\eta^2\right)-H^4\beta^4 r_2\left(3+2k^2\eta^2\right)\\
&+\alpha^3\left(1+k^2\eta^2\right)M_{\rm p}^3 r_1^4\left(H\beta k^3\eta^3+\alpha M_{\rm p} r_2\left(1+k^2\eta^2\right)\right)\\
&+H\alpha^2\beta k^3\eta^3 M_{\rm p}^2 r_1^2 r_2\left(H\beta k^3\eta^3+\alpha M_{\rm p} r_2\left(1+k^2\eta^2\right)\right)+\alpha^2 M_{\rm p}^2 r_1^3\Big(H^2\beta^2 k^6\eta^6\\
&+3H\alpha\beta k^3\eta^3 M_{\rm p} r_2\left(1+k^2\eta^2\right)+\alpha^2 M_{\rm p}^2 r_2^2\left(1+k^2\eta^2\right)^2\Big)\Big]\\
A_1 =& \alpha^3 r_1^8\left(1+k^2\eta^2\right)\left(\frac{H^2\beta^2k^3\eta^3}{M_{\rm p}^2}-\frac{H\alpha\beta r_2\left(2+k^2\eta^2+k^4\eta^4\right)}{M_{\rm p}}-\alpha^2 r_2^2 k\eta\left(1+k^2\eta^2\right)\right)\\
A_2 =& -\alpha^2 r_1^7\Bigg(-\frac{H^3\beta^3k^6\eta^6}{M_{\rm p}^3}-\frac{H^2\alpha\beta^2k^3\eta^3 r_2\left(12+17k^2\eta^2+3k^4\eta^4\right)}{M_{\rm p}^2}\\
&+\frac{H\alpha^2\beta r_2^2\left(8+20k^2\eta^2+21k^4\eta^4+9k^6\eta^6\right)}{M_{\rm p}}+\alpha^3 k\eta r_2^3\left(1+k^2\eta^2\right)^2\Bigg)\\
A_3 =& r_1^5\Bigg(\frac{H^5\beta^5\left(6+4k^2\eta^2-3k^4\eta^4\right)}{M_{\rm p}^5}-\frac{H^4\alpha\beta^4k\eta r_2\left(3+2k^2\eta^2-2k^4\eta^4+k^6\eta^6\right)}{M_{\rm p}^4}\\
&-\frac{H^3\alpha^2\beta^3 r_2^2\left(+1+k^2\eta^2+2k^4\eta^4-10k^6\eta^6-4k^8\eta^8\right)}{M_{\rm p}^3}+\alpha^5 k\eta r_2^5\left(1+k^2\eta^2\right)^2\\
&+\frac{H^2\alpha^3\beta^2k\eta r_2^3\left(18+39k^2\eta^2+26k^4\eta^4+5k^6\eta^6\right)}{M_{\rm p}^2}+\frac{2H\alpha^4\beta k^4\eta^4 r_2^4\left(1+k^2\eta^2\right)}{M_{\rm p}}\Bigg)\\
A_4 =& H^2\beta^2 r_1^2 r_2^3\Big(H^3\beta^3\left(18+9k^2\eta^2+k^4\eta^4\right)+H\alpha^2\beta M_{\rm p}^2 r_2^2\left(1+k^2\eta^2+5k^4\eta^4+2k^6\eta^6\right)\\
&+H^2\alpha\beta^2k\eta M_{\rm p} r_2\left(18+29k^2\eta^2+10k^4\eta^4+k^6\eta^6\right)+\alpha^3 k\eta M_{\rm p}^3 r_2^3\left(4+5k^2\eta^2+k^4\eta^4\right)\Big)\\
A_5 =& -HM_{\rm p}\alpha\beta r_1^6\Big(3H^3\beta^3k\eta\left(1+k^2\eta^2\right)+\alpha^3M_{\rm p}^3 r_2^3\left(6+17k^2\eta^2+18k^4\eta^4+7k^6\eta^6\right)\\
&+H^2\alpha\beta^2 M_{\rm p} r_2\left(1+k^2\eta^2+k^4\eta^4-12k^6\eta^6-4k^8\eta^8\right)\\
&-H\alpha^2\beta k\eta M_{\rm p}^2 r_2^2\left(-1+17k^2\eta^2+21k^4\eta^4+k^6\eta^6\right)\Big)\\
A_6 =& r_1^4 r_2\Big(H^5\beta^5\left(24+49k^2\eta^2+16k^4\eta^4\right)+H^4\alpha\beta^4k^3\eta^3M_{\rm p} r_2\left(16+27k^2\eta^2+9k^4\eta^4\right)\\
&+H^3\alpha^2\beta^3k^4\eta^4M_{\rm p}^2 r_2^2\left(18+9k^2\eta^2+k^4\eta^4\right)+H\alpha^4\beta M_{\rm p}^4 r_2^4\left(1+k^2\eta^2\right)\left(1+4k^4\eta^4\right)\\
&+H^2\alpha^3\beta^2k\eta M_{\rm p}^3 r_2^3\left(39+57k^2\eta^2+20k^4\eta^4+4k^6\eta^6\right)+\alpha^5 k\eta M_{\rm p}^5 r_2^5\left(1+k^2\eta^2\right)^2\Big)\,,
\end{aligned}
$$

and where r_1 and r_2 are defined as

$$
r_{1,2}=\sqrt{1\pm\sqrt{1-\frac{\beta^2}{2\alpha^4}}}\,. \tag{A.12}
$$

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Biographical Information

NAME OF AUTHOR: Michele Fontanini

PLACE OF BIRTH: Udine, UD, Italy

DATE OF BIRTH: March 9, 1979

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

Università degli Studi di Trieste, Trieste, Italy

Syracuse University, Syracuse, New York

DEGREES AWARDED:

Bachelor of Science in Physics, 2001, Università degli Studi di Trieste

Masters of Science in Theoretical Physics, 2005, Università degli Studi di Trieste

PROFESSIONAL EXPERIENCE:

Teaching Assistant, Department of Physics, Syracuse University, 2006-2008

Instructor, Department of Physics, Syracuse University, Summer 2007