A Remark on the Topology of (n,n) Springer Varieties

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A REMARK ON THE TOPOLOGY OF \((n,n)\) SPRINGER VARIETIES

STEPHAN M. WEHRLI

Abstract. We prove a conjecture of Khovanov [Kho04] which identifies the topological space underlying the Springer variety of complete flags in \(\mathbb{C}^{2n}\) stabilized by a fixed nilpotent operator with two Jordan blocks of size \(n\).

1. Introduction

Let \(E_n\) be a complex vector space of dimension \(2n\) and \(z_n: E_n \to E_n\) a nilpotent linear endomorphism with two nilpotent Jordan blocks, each of them of size \(n\). A complete flag in \(E_n\) is an ascending sequence of linear subspaces \(0 \subsetneq L_1 \subsetneq L_2 \subsetneq \ldots \subsetneq L_{2n} = E_n\). The \((n,n)\) Springer variety is the set

\[ B_{n,n} := \{\text{complete flags in } E_n \text{ stabilized by } z_n\}, \]

where a complete flag is said to be stabilized by \(z_n\) if each of the subspaces \(L_j\) is stable under \(z_n\), i.e. if \(z_n L_j \subseteq L_j\) for all \(j \in \{1, \ldots, 2n\}\).

It is known that \(B_{n,n}\) is a complex projective variety of (complex) dimension \(n\), and that the irreducible components of \(B_{n,n}\) are topologically trivial (but algebraically non-trivial) iterated \(\mathbb{P}^1\)-bundles over a point (where \(\mathbb{P}^1\) is the complex projective line, i.e., topologically, \(\mathbb{P}^1 \cong S^2\)). Moreover, a result of Fung [Fun02] (going back to earlier work of Spaltenstein [Spa76] and Vargas [Var79]), describes the irreducible components of \(B_{n,n}\) explicitly in terms of crossingless matchings of \(2n\) points:

Proposition 1.1 (Fung). The irreducible components of \(B_{n,n}\) are parametrized by crossingless matchings of \(2n\) points. Furthermore, the irreducible component \(K_a\) associated to \(a \in B^n\) can be described explicitly, as follows:

\[ K_a = \{(L_1, \ldots, L_{2n}) \in B_{n,n}: L_{s_a(j)} = z_n^{-d_a(j)} L_{j-1} \forall j \in O_a\} \]

Here, \(B^n\) is the set of all crossingless matchings of \(2n\) points. Elements of \(B^n\) can be thought of as diagrams consisting of \(n\) disjoint, nested cups, as in Figure 1. Equivalently, elements of \(B^n\) are partitions of the set \(\{1, 2, \ldots, 2n\}\) into pairs, such that there is no quadruple \(i < j < k < l\) with \((i,k)\) and \((j,l)\) paired. For an element \(a \in B^n\), we denote by \(O_a\) the set of all \(i\) appearing in a pair \((i,j) \in a\) with \(i < j\); and if \((i,j) \in a\) is a pair with \(i < j\), then we define \(s_a(i) := j\) and \(d_a(i) := (s_a(i) - i + 1)/2\). Note that \(d_a(i)\) is always an integer because \(s_a(i) - i - 1\) is twice the number of cups that are contained strictly inside the cup with endpoints \(i\) and \(s_a(i)\).
In [Kho04], Khovanov proved that the integer cohomology ring of $\mathcal{B}_{n,n}$ is isomorphic to the center of the ring $H^n = \bigoplus_{a,b \in B^n} b(H^n)_a$, defined in [Kho02]. To show this, Khovanov first proved that $\mathcal{B}_{n,n}$ has the same integer cohomology ring as a topological space $\tilde{S} \subset (\mathbb{P}^1)^{2n} = \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ ($2n$ factors), defined by $\tilde{S} := \bigcup_{a \in B^n} S_a \subset (\mathbb{P}^1)^{2n}$, where

$$S_a := \{(l_1, \ldots, l_{2n}) \in (\mathbb{P}^1)^{2n} : l_j = l_{s_a(j)} \forall j \in O_a\}.$$ 

The goal of this paper is to show the following stronger statement, which was also conjectured by Khovanov ([Kho04, Conjecture 1]):

**Theorem 1.2.** $\mathcal{B}_{n,n}$ and $\tilde{S}$ are homeomorphic.

Our proof of Theorem 1.2 is based on Proposition 1.1 and on the observation of Cautis and Kamnitzer [CK07] that $\mathcal{B}_{n,n}$ can be embedded into a (smooth) complex projective variety $Y_{2n}$ diffeomorphic to $(\mathbb{P}^1)^{2n}$. Besides the diffeomorphism

$$\phi_{2n} : Y_{2n} \rightarrow (\mathbb{P}^1)^{2n}$$

of Cautis and Kamnitzer, whose definition we review in Section 2, we will need an involutive diffeomorphism

$$I_{2n} : (\mathbb{P}^1)^{2n} \rightarrow (\mathbb{P}^1)^{2n}$$

defined by $I_{2n}(l_1, \ldots, l_{2n}) := (l'_1, \ldots, l'_{2n})$ with

$$l'_j := \begin{cases} l_j & \text{if } j \text{ is odd,} \\ l_j^\perp & \text{if } j \text{ is even,} \end{cases}$$

where $l_j^\perp \subset \mathbb{C}^2$ is the orthogonal complement (w.r.t. the standard hermitian product on $\mathbb{C}^2$) of the complex line $l_j \subset \mathbb{C}^2$ (or, equivalently, the antipode of the point $l_j \in \mathbb{P}^1 \cong S^2$). In Section 3 we prove the following result, which implies Theorem 1.2.

**Proposition 1.3.** The diffeomorphism $I_{2n} \circ \phi_{2n}$ maps $K_a \subset Y_{2n}$ to $S_a \subset (\mathbb{P}^1)^{2n}$ for all $a \in B^n$, and hence $\mathcal{B}_{n,n}$ to $\tilde{S}$.

The author had the main idea for this article in Spring 2007 while he was preparing a talk for an informal seminar on link homology and coherent sheaves organized by Mikhail Khovanov at Columbia University. In a recent article [RT08], Russell and Tymoczko studied an action of the symmetric group $S_{2n}$ on the cohomology ring of $\mathcal{B}_{n,n}$. In this context, they also proved Theorem 1.2. Although our proof is similar to theirs, our work is completely independent.

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2. Diffeomorphism $\phi_m$

In the following, $E$ is the complex vector space $E := \mathbb{C}^N \oplus \mathbb{C}^N$ (for some $N > 0$), and $z: E \to E$ is the nilpotent linear endomorphism given by $ze_j := e_{j-1}$ and $zf_j := f_{j-1}$ for all $j \in \{2, \ldots, N\}$, and $ze_1 := zf_1 := 0$, where $\{e_1, \ldots, e_N\}$ is the standard basis for the first $\mathbb{C}^N$ summand in $E$, and $\{f_1, \ldots, f_N\}$ is the standard basis of the second $\mathbb{C}^N$ summand in $E$. For $n \leq N$, we denote by $E_n \subset E$ the subspace $E_n := \mathbb{C}^n \oplus \mathbb{C}^n = \text{span}(e_1, \ldots, e_n) \oplus \text{span}(f_1, \ldots, f_n)$, or equivalently, $E_n = z^{-n}(0) = \ker(z^n) = \text{im}(z^{N-n})$, and we denote by $\langle \cdot, \cdot \rangle_E$ the standard hermitian product on $E$, satisfying
\[
\langle e_i, e_j \rangle_E := \langle f_i, f_j \rangle_E := \delta_{i,j}, \quad \langle e_i, f_j \rangle_E := 0,
\]
for all $i, j \in \{1, \ldots, N\}$, and by $\langle \cdot, \cdot \rangle$ the standard hermitian product on $\mathbb{C}^2$, satisfying
\[
\langle e, e \rangle := \langle f, f \rangle := 1, \quad \langle e, f \rangle := 0,
\]
where $\{e, f\}$ is the standard basis of $\mathbb{C}^2$.

2.1. Stable subspaces. A subspace $W \subset E$ is called stable under $z$ if it satisfies $zW \subset W$. Note that this condition also implies $z^2W \subset zW$ and $W \subset z^{-1}W$, so if $W$ is stable under $z$, then so are its images and preimages under $z$. Moreover, if a stable subspace $W$ satisfies $W \subset \text{im}(z)$, then $z: z^{-1}W \to W$ is surjective and therefore
\[
\dim((z^{-1}W) \cap W^\perp) = \dim(z^{-1}W/W) = \dim(z^{-1}W) - \dim(W) = \dim(E_1) = 2
\]
where we have used that $z^{-1}W \cap z^{-1}(0) = \ker(z) = E_1$. Let $C: E \to \mathbb{C}^2$ be the linear map defined by $C(e_j) := e$ and $C(f_j) := f$ for all $j \in \{1, \ldots, N\}$. The following lemma is taken from [CK07] Lemma 2.2:

**Lemma 2.1.** If $W \subset E$ is stable under $z$ and contained in $\text{im}(z)$, then the restriction $C|_{(z^{-1}W) \cap W^\perp}: (z^{-1}W) \cap W^\perp \to \mathbb{C}^2$ is an isometric isomorphism.

For the convenience of the reader, we recall the proof given in [CK07].

**Proof.** Since $(z^{-1}W) \cap W^\perp$ is two-dimensional, it suffices to show that the restriction of $C$ to $(z^{-1}W) \cap W^\perp$ is an isometry. For this, let $v, w \in (z^{-1}W) \cap W^\perp$ with $v = v_1 + \ldots + v_N$ and $w = w_1 + \ldots + w_N$ where $v_j, w_j \in \text{span}(e_j, f_j)$. Then we have
\[
\langle v, w \rangle_E = \sum_i \langle v_i, w_i \rangle_E = \sum_i \langle C(v_i), C(w_i) \rangle
\]
and
\[
\langle C(v), C(w) \rangle = \langle \sum_i C(v_i), \sum_j C(w_j) \rangle = \sum_{i,j} \langle C(v_i), C(w_j) \rangle.
\]
To prove that the restriction of $C$ to $(zW) \cap W^\perp$ is an isometry, i.e. that $\langle v, w \rangle_E = \langle C(v), C(w) \rangle$, we must therefore show $\sum_{i \neq j} \langle C(v_i), C(w_j) \rangle = 0$. We will actually prove a stronger statement, namely that $\sum_{i=j+k} \langle C(v_i), C(w_j) \rangle = 0$ for each fixed $k \neq 0$. Assuming $k > 0$ (the case $k < 0$ being similar), we can write
\[
\sum_{i=j+k} \langle C(v_i), C(w_j) \rangle = \sum_{i=j+k} \langle v_i, w_j \rangle_E = \langle v, z^k w \rangle_E,
\]
and since \( v, w \in (z^{-1}W) \cap W^\perp \), we have \( v \in W^\perp \) and \( z^k w \in z^k(z^{-1}W) \subset z^{k-1}W \subset W \), whence \( \langle v, z^k w \rangle_E = 0 \), as desired. 

\[ \square \]

**Lemma 2.2.** Let \( W \subset E \) be a stable subspace such that \( \ker(z) \subset W \subset \im(z) \). Then \( z \) maps \( W^\perp \cap z^{-1}W \) isomorphically to \( (zW)^\perp \cap W \), and the following diagram commutes:

\[
\begin{array}{ccc}
(z^{-1}W) \cap W^\perp & \xrightarrow{z} & W \cap (zW)^\perp \\
\downarrow c & & \downarrow c \\
\mathbb{C}^2 & & \mathbb{C}^2
\end{array}
\]

**Proof.** It is apparent that \( W \cap (zW)^\perp \cong W/(zW) \) is two-dimensional, and, by the previous lemma, \( C \) restricts to an isomorphism on \( (z^{-1}W) \cap W^\perp \), so we only need to prove that \( z \) maps elements of \( (z^{-1}W) \cap W^\perp \) to elements of \( W \cap (zW)^\perp \), and that the above diagram commutes. Thus, let \( v \in (z^{-1}W) \cap W^\perp \), and write \( v \) as

\[
v = v_1 + \ldots + v_N
\]

for \( v_j \in \text{span}(e_j, f_j) \). Since \( v \in W^\perp \) and \( W \supset \ker(z) = E_1 = \text{span}(e_1, f_1) \), we have \( v_1 = 0 \), and since \( C(zv_j) = C(v_j) \) for all \( j \geq 2 \), this implies \( C(zv) = C(v) \). We clearly have \( zv \in W \) (because \( v \in z^{-1}W \)), so the only thing that remains to be shown is that \( zv \in (zW)^\perp \). For this, consider any \( w \in W \) and write \( w \) as \( w = w_1 + \ldots + w_N \) for \( w_j \in \text{span}(e_j, f_j) \). Since \( (zv_j, zw_j)_E = \langle v_j, w_j \rangle_E \) for all \( j \geq 2 \), and since \( v_1 = 0 \) and \( v \in W^\perp \), we see that \( \langle zv, zw \rangle_E = \langle v, w \rangle_E = 0 \), and thus \( zv \in (zW)^\perp \). 

\[ \square \]

### 2.2. \( Y_m \) and \( \phi_m \)

For \( m \leq N \), Cautis and Kamnitzer [CK07] Section 2] define a complex projective variety \( Y_m \),

\[
Y_m := \{(L_1, \ldots, L_m) \in F_m : \dim(L_j) = j \text{ and } zL_j \subset L_j \forall j\},
\]

where \( F_m \) is the set of all partial flags \( 0 \not\subset L_1 \not\subset L_2 \not\subset \ldots \not\subset L_m \subset E \). Note that the conditions \( zL_j \subset L_j \) and \( zL_{j-1} \subset L_{j-1} \) imply that \( z \) descends to an endomorphism of \( L_j/L_{j-1} \), and since \( L_j/L_{j-1} \) is one-dimensional and \( z \) nilpotent, this endomorphism must be the zero-map, so the spaces \( L_j \in (L_1, \ldots, L_m) \in Y_m \) actually satisfy the seemingly stronger condition \( zL_j \subset L_{j-1} \). In particular, \( L_m \subset z^{-1}L_{m-1} \subset z^{-2}L_{m-2} \subset \ldots \subset z^{-m}(0) = \ker(z^m) = E_m \), so as far as the definition of \( Y_m \) is concerned, we could restrict ourselves to the space \( E_m = \mathbb{C}^m \oplus \mathbb{C}^m \) instead of working with the bigger space \( E = \mathbb{C}^N \oplus \mathbb{C}^N \). In particular, \( Y_m \) is independent of the choice of \( N \) (as long as \( N \geq m \)).

Note also that the assignment \( (L_1, \ldots, L_{m-1}, L_m) \mapsto (L_1, \ldots, L_{m-1}) \) defines a \( \mathbb{P}^1 \)-bundle \( Y_m \to Y_{m-1} \). Indeed, a point in the fiber above \( (L_1, \ldots, L_{m-1}) \in Y_{m-1} \) is obtained from \( (L_1, \ldots, L_{m-1}) \) by choosing an \( L_m \) such that \( L_m \subset L_{m-1} \subset z^{-1}L_{m-1} \), and since \( z^{-1}L_{m-1}/L_{m-1} \) is two-dimensional, we have a \( \mathbb{P}^1 \) worth of choices. Denoting by \( L_{j-1}^\perp \) the orthogonal complement of \( L_{j-1} \) w.r.t. \( \langle ., . \rangle_E \), we can identify \( z^{-1}L_{m-1}/L_{m-1} \) with \( (z^{-1}L_{m-1}) \cap L_{m-1}^\perp \), and by Lemma 2.1 the map \( C : E \to \mathbb{C}^2 \) identifies \( (z^{-1}L_{m-1}) \cap L_{m-1}^\perp \) with \( \mathbb{C}^2 \). Therefore, the \( \mathbb{P}^1 \)-bundle \( Y_m \to Y_{m-1} \) is topologically trivial (i.e., topologically, \( Y_m \cong \mathbb{P}^1 \times Y_{m-1} \)), and Cautis and Kamnitzer use
this to define a diffeomorphism
\[ \phi_m : Y_m \to (\mathbb{P}^1)^m \]
by \( \phi_m(L_1, \ldots, L_m) := (C(L_1), C(L_2 \cap L_1^\perp), C(L_3 \cap L_2^\perp), \ldots, C(L_m \cap L_{m-1}^\perp)) \).

2.3. Subvarieties \( X_{m,i} \subset Y_m \). For each \( i \in \{1, \ldots, m-1\} \), Cautis and Kamnitzer define a subvariety \( X_{m,i} \subset Y_m \),
\[ X_{m,i} := \{(L_1, \ldots, L_m) \in Y_m : L_{i+1} = z^{-1}(L_{i-1})\}, \]
together with a surjection
\[ q_{m,i} : X_{m,i} \to Y_{m-2}, \]
given by \( q_{m,i}(L_1, \ldots, L_m) := (L_1, \ldots, L_{i-1}, zL_{i+2}, \ldots, zL_m) \in Y_{m-2} \). The following (easy) Lemma was shown in [CK07, Theorem 2.1].

**Lemma 2.3.** The map \( \phi_m : Y_m \to (\mathbb{P}^1)^m \) takes \( X_{i,m} \) diffeomorphically to
\[ A_{m,i} := \{(l_1, \ldots, l_m) \in (\mathbb{P}^1)^m : l_{i+1} = l_i^\perp\}, \]
where \( l_i^\perp \) denotes the orthogonal complement of the line \( l_i \subset \mathbb{C}^2 \) w.r.t. \( \langle \cdot, \cdot \rangle \).

Let \( f_{m,i} : (\mathbb{P}^1)^m \to (\mathbb{P}^1)^{m-2} \) be the forgetful map sending \((l_1, \ldots, l_m) \in (\mathbb{P}^1)^m\) to \((l_1, \ldots, l_{i-1}, l_{i+2}, \ldots, l_m) \in (\mathbb{P}^1)^{m-2}\), and let
\[ g_{m,i} : A_{m,i} \to (\mathbb{P}^1)^{m-2} \]
be the restriction of \( f_{m,i} \) to \( A_{m,i} \).

**Lemma 2.4.** Let \( \psi_{m,i} : X_{m,i} \to A_{m,i} \) be the restriction of \( \phi_m \) to \( X_{m,i} \subset Y_m \). Then the following diagram commutes:
\[ \begin{array}{ccc}
X_{m,i} & \xrightarrow{g_{m,i}} & Y_{m-2} \\
\downarrow{\psi_{m,i}} & & \downarrow{\phi_{m-2}} \\
A_{m,i} & \xrightarrow{g_{m,i}} & (\mathbb{P}^1)^{m-2}
\end{array} \]

**Proof.** It is straightforward to check that \( g_{m,i} \circ \psi_m \) maps \((L_1, \ldots, L_m) \in X_{m,i}\) to the tuple \((l'_1, \ldots, l'_{m-2}) \in (\mathbb{P}^1)^{m-2}\), where
\[ l'_j = \begin{cases} 
C(L_j \cap L_{j-1}^\perp) & \text{if } j < i, \\
C(L_{j+2} \cap L_{j+1}^\perp) & \text{if } j \geq i,
\end{cases} \]
and \( \phi_{m-2} \circ g_{m,i} \) maps \((L_1, \ldots, L_m) \in X_{m,i}\) to the tuple \((l''_1, \ldots, l''_{m-2}) \in (\mathbb{P}^1)^{m-2}\), where
\[ l''_j = \begin{cases} 
C(L_j \cap L_{j-1}^\perp) & \text{if } j < i, \\
C(zL_{j+2} \cap (zL_{j+1})^\perp) & \text{if } j \geq i.
\end{cases} \]
To prove \( g_{m,i} \circ \psi_m = \phi_{m-2} \circ g_{m,i} \), we must therefore show that
\[ C(L_{j+2} \cap L_{j+1}^\perp) = C(zL_{j+2} \cap (zL_{j+1})^\perp) \]
holds for all \( j \geq i \). But if \( j \geq i \), then \( L_{j+1} \supset L_{i+1} = z^{-1}(L_{i-1}) \supset z^{-1}(0) = \ker(z) \), and (by increasing \( N \) if necessary) we can also assume that \( L_{j+1} \subset \im(z) \). Thus,
Lemma 2.2 applied to \( W := L_{j+1} \) tells us that \( z \) maps \( (z^{-1}W) \cap W^\perp \) to \( W \cap (zW)^\perp \), and that \( C(v) = C(xv) \) for all \( v \in (z^{-1}W) \cap W^\perp \). Now the equality \( C(L_{j+2} \cap L_{j+1}^\perp) = C(zL_{j+2} \cap (zL_{j+1})^\perp) \) follows because \( z \) maps \( L_{j+2} \cap L_{j+1}^\perp \subseteq (z^{-1}W) \cap W^\perp \) to \( zL_{j+2} \cap (zL_{j+1})^\perp \subseteq W \cap (zW)^\perp \). \( \square \)

3. PROOF OF PROPOSITION 1.3

In this section, we use the same notations as before, except that we now assume \( m = 2n \) (and hence \( N \geq 2n \)). Then the Springer variety \( \mathcal{B}_{n,n} \) is naturally contained in \( Y_{2n} \) as
\[
\mathcal{B}_{n,n} := \{ (L_1, \ldots, L_{2n}) \in Y_{2n} : L_{2n} = E_n \},
\]
where \( E_n := \text{span}(e_1, \ldots, e_n) \oplus \text{span}(f_1, \ldots, f_n) \), and Proposition 1.1 tells us that the irreducible component \( K_a \subset \mathcal{B}_{n,n} \subset Y_{2n} \) associated to the crossingless matching \( a \in B^n \) is equal to the set of all \( (L_1, \ldots, L_{2n}) \in Y_{2n} \) satisfying
\[
L_{s_a(j)} = z^{-d_a(j)}L_{j-1}
\]
for all \( j \in O_a \), where \( z_a \colon E_n \to E_n \) is the restriction of \( z \) to \( E_n \). A priori, \( z^{-d_a(j)}L_{j-1} \) could a priori be a proper subspace of \( z^{-d_a(j)}L_{j-1} \) (because \( z^{-d_a(j)}L_{j-1} \) might not be contained in \( E_n \)), but it turns out that \( z_n^{-d_a(j)}L_{j-1} \) is equal to \( z^{-d_a(j)}L_{j-1} \) whenever \( (L_1, \ldots, L_{2n}) \in K_a \). In fact, we have:

**Lemma 3.1.** \( K_a = \{ (L_1, \ldots, L_{2n}) \in Y_{2n} : L_{s_a(j)} = z^{-d_a(j)}L_{j-1} \forall j \in O_a \} \).

*Proof.* Suppose \( (L_1, \ldots, L_{2n}) \) is contained in \( K_a \). Then the condition \( z_a^{-d_a(j)}L_{j-1} = L_{s_a(j)} \), combined with \( \dim(L_{j-1}) = j-1 \), \( \dim(L_{s_a(j)}) = s_a(j) \), and \( \dim(\ker(z)) = 2 \), implies
\[
\dim(z^{-d_a(j)}L_{j-1}) = 2d_a(j) + \dim(L_{j-1}) = 2d_a(j) + j - 1 = s_a(j)
\]
and thus \( z^{-d_a(j)}L_{j-1} = z_n^{-d_a(j)}L_{j-1} \). Conversely, suppose \( (L_1, \ldots, L_{2n}) \in Y_{2n} \) satisfies \( z^{-d_a(j)}L_{j-1} = L_{s_a(j)} \) for all \( j \in O_a \). Then we must show that \( L_{2n} = E_n \). To prove this, let us call a pair \( (k, l) \in a \) *outermost* if there is no pair \( (k', l') \in a \) such that \( k' < k < l < l' \). Then the outermost pairs in \( a \) form a sequence \( (k_1, l_1), (k_2, l_2), \ldots, (k_r, l_r) \in a \) such that \( k_1 = 1 \), \( l_r = 2n \), and \( k_{s+1} = l_s + 1 \) for all \( s < r \), and \( d_a(k_1) + \ldots + d_a(k_r) = n \). Using \( z^{-d_a(j)}L_{j-1} = L_{s_a(j)} \) successively for \( j \in \{ k_r, k_{r-1}, \ldots, k_1 \} \subset O_a \), we obtain
\[
L_{2n} = z^{-d_a(k_r)}L_{l_{r-1}} = z^{-d_a(k_{r-1})}L_{l_{r-2}} = \ldots = z^{-n(0)} = E_n,
\]
as desired. \( \square \)

From now on, \( a \in B^n \) is a fixed crossingless matching of \( 2n \) points, and \( i \) is an index such that \( s_a(i) = i + 1 \), i.e., such that \( (i, i + 1) \) is a pair in \( a \). We denote by \( a' \in B^{n-1} \) the crossingless matching obtained from \( a \) by removing the pair \( (i, i + 1) \) (and renumbering indices \( j \geq i + 2 \) such that \( j \in \{ i + 2, \ldots, 2n \} \) becomes \( j - 2 \in \{ i, \ldots, 2n - 2 \} \)), and by \( q \) the map \( q_{2n,i} \colon X_{2n,i} \to Y_{2n-2} \), defined as in the previous section.
Lemma 3.2. \( K_a = q^{-1}(K_{a'}) \).

Proof. Since \( s_a(i) = i + 1 \) and \( d_a(i) = (s_a(i) - i + 1)/2 = 1 \), the equality \( L_{i+1} = z^{-1}L_{i-1} \) holds for each \((L_1, \ldots, L_{2n}) \in K_a\), and thus \( K_a \subset Y_{2n} \) is contained in \( X_{2n,i} \).

It remains to show that an element \((L_1, \ldots, L_{2n}) \in X_{2n,i}\) satisfies the conditions \( L_{s_{a,j}} = z^{-d_a(j)}L_{j-1} \) for all \( j \in O_a \setminus \{i\} \) if and only if the element \((L', \ldots, L'_{2n-2}) := q(L_1, \ldots, L_{2n}) = (L_1, \ldots, L_{i-1}, zL_{i+2}, \ldots, zL_{2n}) \in Y_{2n-2} \) satisfies the conditions \( L'_{s_{a,j}'} = z^{-d_{a'}(j)}L'_{j-1} \) for all \( j \in O_{a'} \). We divide the proof into three cases.

Case 1. If \( j < s_a(j) < i \), then the equivalence

\[
L_{s_a(j)} = z^{-d_a(j)}L_{j-1} \iff L'_{s_{a'}(j)} = z^{-d_{a'}(j)}L'_{j-1}
\]

is obvious because the quantities appearing on either side of \( \iff \) are identical.

Case 2. If \( j < i < i + 1 < s_a(j) \), then \( L'_{j-1} = L_{j-1}, L'_{s_{a'}(j)} = zL_{s_a(j)}, \) and \( d_{a'}(j) = d_a(j) - 1 \), so we must show:

\[
L_{s_a(j)} = z^{-d_a(j)}L_{j-1} \iff zL_{s_a(j)} = z^{-d_a(j) + 1}L_{j-1}
\]

But this follows simply by applying \( z \) (resp., \( z^{-1} \)) to the equalities on either side of \( \iff \), and observing that \( z^{-1}(zL_{s_a(j)}) = L_{s_a(j)} \) (because \( L_{s_a(j)} \supset L_{i+1} = z^{-1}L_{i-1} \supset z^{-1}(0) = \ker(z) \)), and that \( z(z^{-d_a(j)}L_{j-1}) = z^{-(d_a(j)+1)}L_{j-1} \) (because, by increasing \( N \) if necessary, we may assume \( z^{-d_a(j)+1}L_{j-1} \subset \text{im}(z) \)).

Case 3. If \( i + 1 < j < s_a(j) \), then \( L'_{j-3} = zL_{j-1}, L_{s_{a'}(j-2)} = zL_{s_a(j)}, \) and \( d_{a'}(j-2) = d_a(j) \), so we must show:

\[
L_{s_a(j)} = z^{-d_a(j)}L_{j-1} \iff zL_{s_a(j)} = z^{-d_a(j)}zL_{j-1}
\]

As in Case 2, this follows by applying \( z \) (resp., \( z^{-1} \)) to the equalities on either side of \( \iff \). \( \square \)

Note that (since \( s_a(j) - j \) is odd for all \( j \in O_a \)) the involutive diffeomorphism \( I_{2n} : (\mathbb{P}^1)^{2n} \to (\mathbb{P}^1)^{2n} \) defined in the introduction exchanges the subset \( S_a \subset (\mathbb{P}^1)^{2n} \) with the subset

\[
T_a := \{(l_1, \ldots, l_{2n}) \in (\mathbb{P}^1)^{2n} : l_{s_a(j)} = l_j^+ \forall j \in O_a\} \subset (\mathbb{P}^1)^{2n}
\]

To prove Proposition 3.3, we must therefore show that \( \phi_{2n} \) maps \( K_a \) to \( T_a \) for all \( a \in B^n \). We will need the following lemma, in which \( a, i \) and \( a' \) are as in the previous lemma, and \( g \) denotes the map \( g_{2n,i} : A_{2n,i} \to (\mathbb{P}^1)^{2n-2} \), defined as in the previous section.

Lemma 3.3. \( T_a = g^{-1}(T_{a'}) \).

Proof. This follows directly from the definitions of \( g, A_{2n,i}, T_a \) and \( T_{a'} \). \( \square \)

We are now ready to prove Proposition 3.3.

Proof of Proposition 3.3. Induction on \( n \). The case \( n = 1 \) is trivial because the only crossingless matching of 2 points is \( a_1 := \{(1, 2)\} \), and \( \phi_2 : Y_2 \to \mathbb{P}^1 \times \mathbb{P}^1 \) maps \( \Phi_{1,1} = K_{a_1} = X_{2,1} \subset Y_2 \) diffeomorphically to \( T_{a_1} = A_{2,1} \subset \mathbb{P}^1 \times \mathbb{P}^1 \).

Thus, let \( n > 1 \), and suppose we have already proven the proposition for \( n - 1 \). Let \( a \in B^n \). Then there is an \( i \in \{1, \ldots, 2n - 1\} \) such that \( s_a(i) = i + 1 \), i.e., such
that \((i, i+1) \in a\). As above, we denote by \(a' \in B^{n-1}\) the crossingless matching obtained from \(a\) by removing the pair \((i, i+1)\) (and renumbering all \(j \geq i+2\)), and by \(q\) (resp., \(g\)) the map \(q_{2n,i}\) (resp., \(g_{2n,i}\)). By induction, we know that \(\phi_{2n-2}\) maps \(K_{a'}\) to \(T_{a'}\), so Lemma 2.4 gives us the following commutative diagram:

\[
\begin{array}{c}
q^{-1}(K_{a'}) \quad \psi_{2n,i} \quad X_{2n,i} \quad q \quad Y_{2n-2} \quad K_{a'} \\
\downarrow \psi_{2n,i} \quad \downarrow \phi_{2n-2} \\
g^{-1}(T_{a'}) \quad A_{2n,i} \quad g \quad (\mathbb{P}^1)^{2n-2} \quad T_{a'}
\end{array}
\]

Hence we get \(\psi_{2n,i}(q^{-1}(K_{a'})) = g^{-1}(T_{a'})\), and by Lemmas 3.2 and 3.3, this implies

\[
\psi_{2n,i}(K_a) = T_a,
\]

thus completing the inductive step. □

References


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