3-1991

A Study of Approximating the Moments of the Job completion Time in PERT Networks

Kishan Mehrotra
Syracuse University, mehrtra@syr.edu

John Chai

Sharma Pillutla

SU-CIS-91-19

A Study of Approximating the Moments of the Job completion Time in PERT Networks

Kishan Mehrotra, John Chai, and Sharma Pillutla

March 1991

School of Computer and Information Science
Syracuse University
Suite 4-116, Center for Science and Technology
Syracuse, New York 13244-4100

A Study of Approximating the Moments of the Job Completion Time in PERT Networks

Kishan Mehrotra
School of Computer and Information Science
John Chai[1] and Sharma Pillutla[2]
Quantitative Methods Department
School of Management

Syracuse University
Syracuse, New York 13244-2130

March 28, 1991

[1]Funded by a Summer Grant from the Brethen Institute of Manufacturing Sysytems
[2]Partly funded by a fellowship from the Graduate School.

Abstract

The importance of proper management of projects has not gone unrecognized in industry and academia. Consequently tools like Critical Path Method (CPM) and Program Evaluation Review Technique (PERT) for project planning have been the focus of attention of both practitioners and researchers. Determination of the *T*ime to *C*omplete the *J*ob (**TCJ**) in PERT networks is important for planning and bidding purposes. The complexity involved in accurately determining the TCJ has led to the development of many approximating procedures. Most of them ignore the dependence between paths in the network. We propose an approximation to determine the TCJ which *explicitly* recognizes this dependency. Experimental results which demonstrate the accuracy of our approximation for a wide variety of networks are presented.

1 Introduction

Program Evaluation and Review Technique (PERT) was developed in the 1950's. An early application of PERT was made by the U.S. government in planning and scheduling the research project for developing the Polaris Ballistic Missile. Soon PERT became the primary tool for planning and scheduling of projects, especially those which were funded by the U.S. government. PERT networks have been used to represent large projects in the industry and hence have a lot of applicability in the business world [see Elmagrabhy (1977)]. Analysis of PERT networks, also known as stochastic activity networks, has received considerable attention in the literature.

PERT is based on the concept that a project is divided into a number of activities which are arranged in some order according to the job requirements. A PERT network is graphically represented using a set of nodes and arcs where a node represents the beginning or completion of one or more activities and an activity is represented by an arc (arrow) connecting two nodes. The project starts at the initial node and ends at the terminal node. A path is a set of nodes connected by arrows which begin at the initial node and end at the terminal node. This collection of arcs, nodes and paths is collectively called an activity network. A project is deemed complete if work along all paths is complete.

If activity times are deterministic, the duration of the project completion time is determined by the length of the longest path in the network. However, things become complicated when activity times are stochastic in nature. For a stochastic activity network, Kulkarni and Adlakha (1986) have identified three important measures of performance.

(a) Distribution of the project completion time

(b) The probability that a given path is critical

(c) The probability that a given activity belongs to a critical path.

Performance measures derived from (a) are the most commonly used measures and most of the work has concentrated on the properties of the Time to Completion of the Job (TCJ).

Determination of the exact distribution of TCJ is complicated by the fact that different paths are correlated and also because of the need to find the maximum of a set of random variables, as we shall see later. Hence one cannot easily determine the exact distribution of the TCJ. The research has primarily branched off in three directions:

(i) Exact methods: Martin (1965), Dodin (1985), Fisher at el (1985), and Hagstrom (1990) are some of the papers that deal with these methods. Most of their results are limited in that they make quite restrictive assumptions. For example Martin (1965) assumes that the arc duration density functions are polynomial. Hagstrom (1990) assumes task durations have discrete distributions.

(ii) Approximating and bounding approaches: These have been the most prolific in the literature. Malcolm et al. (1959), Sculli (1983), Golenko-Ginzburg (1989), Dodin (1985b), Sculli and Wong (1985), and Dodin and Sirvanci (1986) determine approximations for the distribution and moments of the TCJ. Kamburowski (1985), Shogan (1977), Kleindorfer (1971), and Robillard and Trahan (1977), on the other hand, try to find upper and/or lower bounds for the distributions and moments of the TCJ.

(iii) Simulation methods: These methods have been discussed in the literature by Van Slyke (1963), Burt and Garman (1971), and Sigal et al. (1979)

We adopt approach (ii) above and present a simple and practical method to determine close approximations for the first two moments of the TCJ. We do not undertake the task of determination of the bounds for these moments. Though it is informative to know the best and worst completion times for a project, a single approximation for the TCJ is more useful for bidding purposes as compared to a range. In general researchers are more interested in the moments of the TCJ rather than completely specifying the exact distribution. In fact, the distribution is merely a first step towards obtaining the moments.

Dodin and Sirvanci (1986) propose the extreme value distribution as an approximation to the TCJ. They claim that the distribution of the TCJ varies from a normal to an extreme value distribution depending on factors like the size of the network, the dependence between paths and the number of dominating paths. We *explicitly* take into account this dependence between paths which occurs due to common activities on various paths. We show, using simulation results as a benchmark, that the distribution of the TCJ is better approximated by a mixture of distributions. In addition, we use the critical path concept which is easier to comprehend and extremely simple to operationalize, as opposed to a dominating path concept (Dodin and Sirvanci, 1986). Section 2 presents the theoretical underpinnings of our approach and illustrates its use by an example. Section 3 compares the simulation results and those obtained using our approximation for a wide variety of networks appearing in the literature. Section 4 presents the conclusions and additional mathematical details are presented in the appendices.

2 Development of the Proposed Approximation

In this section we lay down the theoretical arguments underlying our approach. We then explicate the concepts using a widely cited network in the literature — Kleindorfer's network, as an illustrative example.

2.1 Theoretical Concepts

Let T be a random variable that stands for the time to complete the job; let X_{ij} be the time required to finish the *j-th* activity in the *i-th* path, where n_i represents the number of activities in the *i-th* path, and N represents the total number of paths in the network; and define $Y_i = \sum_{j=1}^{n_i} X_{ij}$. Then we can write $T = \max_{1 \leq i \leq N} Y_i$. We make use of the critical path concept, as opposed to the dominating path concept used by Dodin and Sirvanci (1986), in trying to determine the distribution of T. The traditional definition of the critical path is that path which takes the longest expected time [see Elmagrabhy (1977)]. This is obtained by summing the expected times of the activities on that path. As stated earlier this is a much simpler concept and less cumbersome from an analytical point of view.

Now consider the situation where there is more than one critical path. In this case, the time to complete the job will depend heavily upon that critical path which is completed *last*. In fact, the TCJ will be determined by *any* path which takes the longest time. To complicate matters, it may be possible that several activities of two critical paths are identical. Therefore, it becomes necessary to treat the common and non-common activities separately. Consider an "ideal" situation as shown in Figure 1. Now consider the

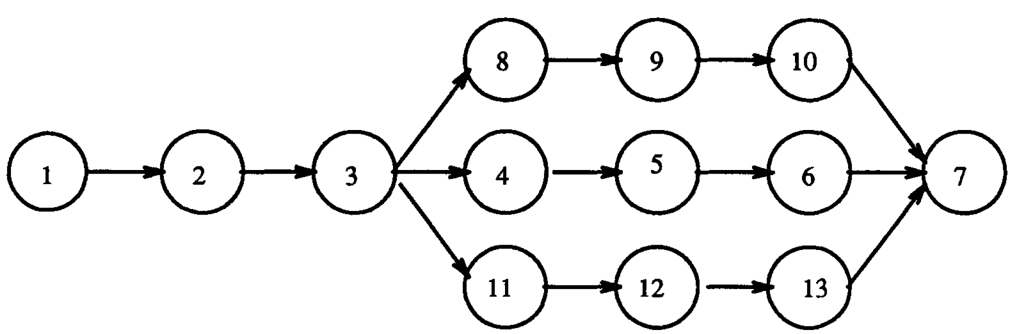

Figure 1: The "Ideal" Setting of Several Critical Paths

set of K critical paths of the given network. Let U_i be the the sum of the "non-common" activities in the *i-th* critical path and V be the sum of the "common" activities for the K critical paths. Then we can approximate $T = \max_{1 \leq i \leq N} Y_i$ where N is the total number of paths in the network by $T \approx \max_{1 \leq i \leq K}(U_i) + V$ where K is the number of critical paths in a network. So far we have discussed only the ideal condition. In practice however, the critical paths do not have exactly the same activities common to all of them. Typically observed critical paths are as shown in Figure 2.

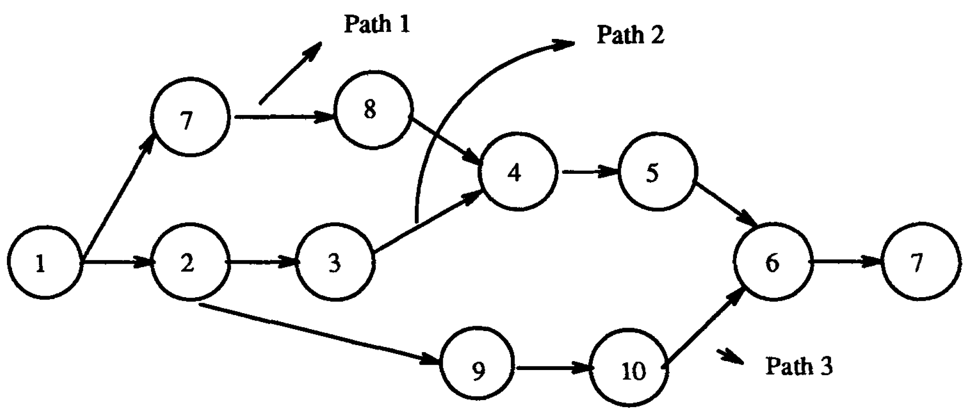

Figure 2: Typically Observed Critical Paths

Here it is observed that all paths do not have exactly the same number of common activities. For example paths P_1 and P_2 have only three common activities, whereas P_2, and P_3 have two common activities. Also, all common activities are not exactly the same — paths P_1 and P_2 have activities $4-5, 5-6$ and $6-7$ common whereas paths P_2 and P_3 have $1-2$ and $6-7$ as common activities. In such cases a *subjective* assessment can be made and then the results of the ideal situation can be used. For example, for the network whose critical paths are represented in Figure 2, it would be reasonable to argue that among three paths comprising six activities each, there are three common activities and three non-common activities. Although this is a subjective assessment, however, in section 3 we observe that it provides a close approximation for the first two moments of T. We will shortly discuss an example which will provide some guideline on choosing the number of common activities.

The beta distribution has been traditionally suggested to model the durations of the stochastic activities comprising the PERT network. However, there is a preponderant usage of the normal distribution in the literature. Sculli (1983) states that

...this can be justified by the fact that most large networks can be reduced to a guide network where a completely independent path becomes one activity. The central limit theorem justifies the normality assumption for the duration of activities in the guide network.

Moreover, as observed in Golenko-Ginzberg (1989), the beta distribution is not stable with respect to convolution and maximization. Therefore, for the purposes of our analysis, we assume that the activity durations are *iid* normal random variables. The assumption of *iid* distributed activities is not overly restrictive. It was made only for purposes of computational ease in illustrating our approach. The proposed approximation can be used with *non-iid* distributed activities with equal facility. Subsequently we also consider the setting of *iid* exponential activities. We summarize the following theoretical properties about the distribution of $U = \max_{1 \leq i \leq K}(U_i)$, V, and T.

Properties of V: The distribution of V is, in general, given by the distribution of the sum of the X_{ij}s that are common to the critical paths. Therefore, we know that the distribution of V is (a) normal if each X_{ij} is normal, and (b) gamma if each X_{ij} is exponential, and (c) approximately normal, by the Central Limit Theorem, if the number of common activities is large. The expected value and variance of V are obtained by adding the expected values and variances of the common activities.

Properties of U: Properties of U_i's, for each value of i, are the same as properties of V. The distribution of $U = \max_{1 \leq i \leq K} U_i$, is given by some appropriate distribution obtained from the theory of order statistics. For example, if each U_i is a normal random variable; i. e. $P(U_i \leq x) = \mathcal{N}(x; \mu, \sigma^2)$, then the distribution of U is given by

$$P(U \leq x) = \{\mathcal{N}(x; \mu, \sigma^2)\}^K \equiv \mathcal{N}^K(x; \mu, \sigma^2).$$

More generally, if $P(U_i \leq x) = \mathcal{F}(x)$ for $i = 1, 2, \ldots, K$; then

$$P(U \leq x) = \{\mathcal{F}(x)\}^K \equiv \mathcal{F}^K(x).$$

For large values of K, the distribution of U can be approximated by the extreme value distribution.

Properties of T: The distribution of $T = U + V$ is therefore represented by the convolution of distribution of U and V. The exact form of the distribution of T is not easy to assess, because the convolution distributions are, in general, not of any well known standard family of distributions or of closed forms. However, the moments of the distribution, particularly the first two moments, can be evaluated relatively easily because

$$E(T) = E(U) + E(V), \qquad \text{and} \qquad \mathrm{Var}(T) = \mathrm{Var}(U) + \mathrm{Var}(V).$$

Calculation of $E(U)$ and $\mathrm{Var}(U)$ may cause difficulties for larger values of K because expected values of the largest observation in a sample are not available for all distributions. In these cases a reasonably accurate approximation can be used as suggested in appendix A.3.

2.2 Illustrative Example

We now present an example of the theoretical distribution of T using a widely cited network, Kleindorfer's network (See Figure 3). Figure 4 shows all possible paths in this network.

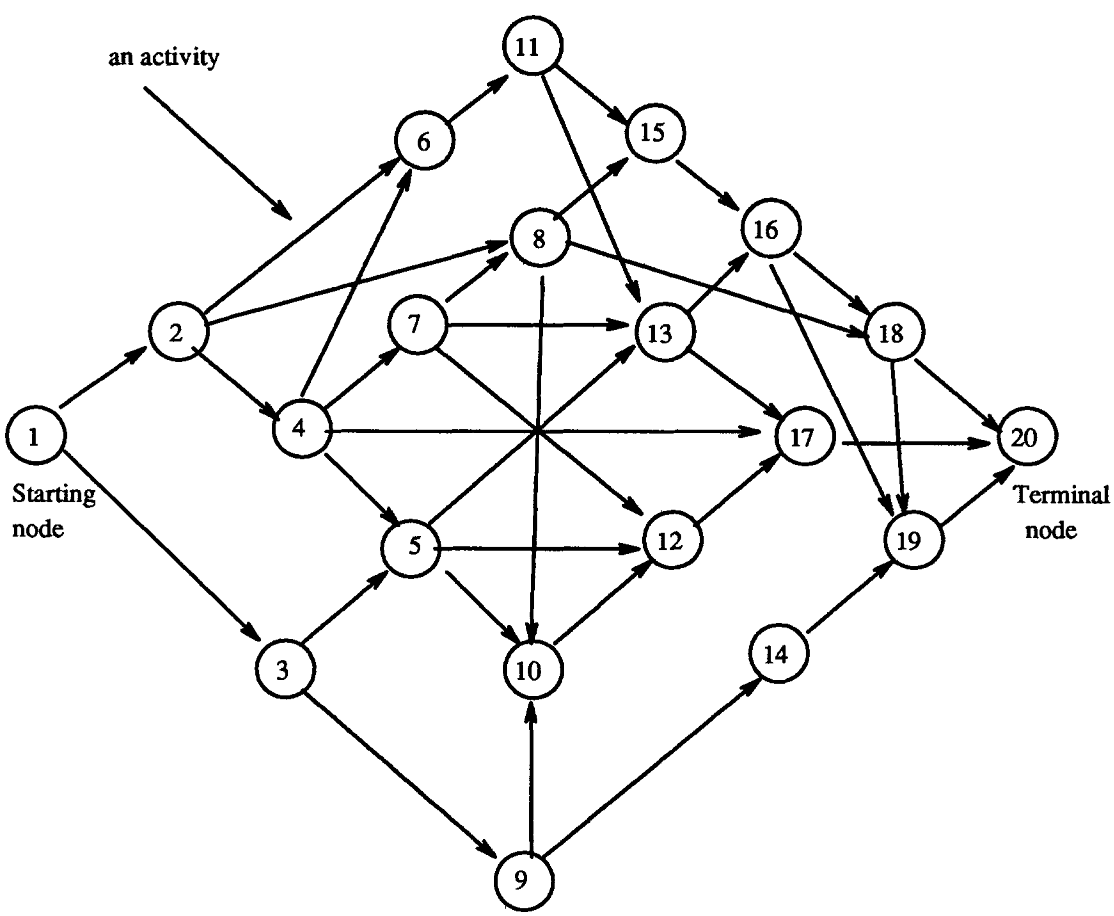

Figure 3: Kleindorfer's Network

Figure 4: All Possible Paths on Kleindorfer Network

PATH # 1: 1 2 4 5 10 12 17 20
PATH # 2: 1 2 4 5 12 17 20
PATH # 3: 1 2 4 5 13 16 18 19 20
PATH # 4: 1 2 4 5 13 16 18 20
PATH # 5: 1 2 4 5 13 16 19 20
PATH # 6: 1 2 4 5 13 17 20
PATH # 7: 1 2 4 6 11 13 16 18 19 20
PATH # 8: 1 2 4 6 11 13 16 18 20
PATH # 9: 1 2 4 6 11 13 16 19 20
PATH # 10: 1 2 4 6 11 13 17 20
PATH # 11: 1 2 4 6 11 15 16 18 19 20
PATH # 12: 1 2 4 6 11 15 16 18 20
PATH # 13: 1 2 4 6 11 15 16 19 20
PATH # 14: 1 2 4 7 8 10 12 17 20
PATH # 15: 1 2 4 7 8 15 16 18 19 20
PATH # 16: 1 2 4 7 8 15 16 18 20
PATH # 17: 1 2 4 7 8 15 16 19 20
PATH # 18: 1 2 4 7 8 18 19 20
PATH # 19: 1 2 4 7 8 18 20
PATH # 20: 1 2 4 7 12 17 20
PATH # 21: 1 2 4 7 13 16 18 19 20
PATH # 22: 1 2 4 7 13 16 18 20
PATH # 23: 1 2 4 7 13 16 19 20
PATH # 24: 1 2 4 7 13 17 20
PATH # 25: 1 2 4 17 20
PATH # 26: 1 2 6 11 13 16 18 19 20
PATH # 27: 1 2 6 11 13 16 18 20
PATH # 28: 1 2 6 11 13 16 19 20
PATH # 29: 1 2 6 11 13 17 20
PATH # 30: 1 2 6 11 15 16 18 19 20
PATH # 31: 1 2 6 11 15 16 18 20
PATH # 32: 1 2 6 11 15 16 19 20
PATH # 33: 1 2 8 10 12 17 20
PATH # 34: 1 2 8 15 16 18 19 20
PATH # 35: 1 2 8 15 16 18 20
PATH # 36: 1 2 8 15 16 19 20
PATH # 37: 1 2 8 18 19 20
PATH # 38: 1 2 8 18 20
PATH # 39: 1 3 5 10 12 17 20
PATH # 40: 1 3 5 12 17 20
PATH # 41: 1 3 5 13 16 18 19 20
PATH # 42: 1 3 5 13 16 18 20
PATH # 43: 1 3 5 13 16 19 20
PATH # 44: 1 3 5 13 17 20
PATH # 45: 1 3 9 10 12 17 20
PATH # 46: 1 3 9 14 19 20

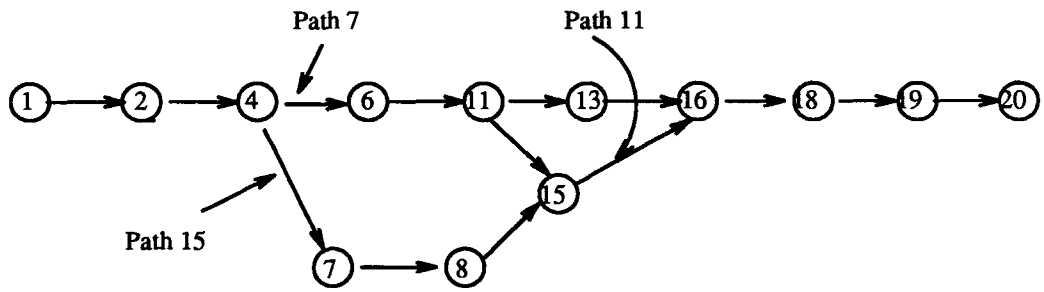

Figure 5: Three Critical Paths of the Kleindorfer's Network

It has three critical paths, P_7, P_{11}, and P_{15}. There are five activities that are common to all three critical paths. The remaining four activities are not common to all three critical paths. Figure 5 shows the subgraph of the three critical paths. Now, from the above, we know that $T \approx V + \max(U_i) = V + U$.

Case I: Let us consider the case where each activity has the normal distribution with mean 4 and variance 1. Here V is the sum of five normal random variables and therefore is itself a normal random variable with mean 20 and variance 5. In a similar manner U_7, U_{11}, and U_{15} are also normal random variables each with mean 16 and variance 4. Finally,

$$P(U \leq x) = \mathcal{N}^3(x; 16, 4).$$

The mean and variance of T can be easily evaluated from the above representation of the distribution of T. One can obtain the mean and variance of $\mathcal{N}^3$ for the standardized normal random variable from the statistical tables by Owen (1962). Using these properties, $E(T) \approx 37.692$ and $Var(T) \approx 7.238$. (For details, see Appendix A.1.)

Case II: In this case, where each activity follows an exponential distribution with mean 4, the procedure for deriving the distribution is the same as in Case I. The only exception is that V is the sum of five exponential distributions, each with mean 4, and therefore the distribution of this convolution is given by a gamma distribution (Γ) with mean 20 and shape parameter 5. Similarly, the distribution of each U_i is given by a gamma with mean 16 and shape parameter 4 and, finally, $P(U \leq u) = \Gamma^3(u; 20, 4)$. To find the expected value and variance of U we need to know the expected value and variance of the largest observation in a sample of size 3 from a gamma distribution with shape parameter 5. Expected values of the order statistics for the gamma distribution are tabulated [see Sarhan and Greenberg (1962)]. Using these results it is observed that $E(T) \approx 42.924$ and $Var(T) \approx 140.064$. (For more detail, see the Appendix A.2.).

Table 1: Structural Descriptions Of Different Networks Being Evaluated

	Number of				
Name of Network	Nodes	'Critical' Activities	Critical Paths	Total Paths	Common Activities
Kleindorfer	20	9	3	46	5
Large Network	43	12	19	617	8
Shogan (1977)	6	3	4	4	1
Kamburowski (1985)	8	3	5	5	0 – 1
Fulkerson (1962)	10	5	16	16	2
Ringer (1971)	7	4	2	5	1
Martin (1965)	9	6	2	3	4
Dodin (1985)	7	4	4	4	2
Pritsker & Kiviat (1969)	9	5	3	6	3
Provan & Ball (1984)	9	3	9	9	0 – 1

In our evaluation above it could be argued that V should be approximated as a sum of 6 independent random variables because paths P_7 and P_{11} have 7 activities common while P_7 and P_{15} have 5 common activities and P_{11} and P_{15} have 6 activities in common, giving an average of 6. If this is taken into account then the first two moments of T will change to 37.466 and 7.678 for the normal case and 41.984 and 145.152 for the exponential case. These difference in the moments are small when compared with either the normal or extreme value approximations.

3 Empirical Study

To the best of our knowledge, the exact distribution of T has not been derived for any reasonable size network. We therefore use Monte Carlo simulation to obtain the "true" moments of the distribution of the TCJ for a variety of networks cited in PERT-related literature. Table 1 elaborates on the structural characteristics of these networks based on the assumption of *iid* activities. The dimensionality and complexity of these networks varies considerably. For example, the total number of paths in the network ranges from three (Martin, 1965) to 617 for the "large network" that appeared in Dodin and Sirvanci (1986).

The simulation program was coded in Pascal and run on an IBM 3090 machine. The simulation of each network comprised a sample size of 20,000 runs. We use a simulation run length of 20,000 to obtain values as close to the "exact" mean and the "exact" variance as possible. With this run-length the standard error in the mean of a simulation study is of the order of $\sqrt{1/20000} = \pm 0.007$. For the normal $\mathcal{N}(4,1)$ distribution of each arc and for the Kleindorfer's network the standard error of mean from the simulation study is 0.0196, and this implies that the true value of $E(T) \in (37.377, 37.495)$ with confidence coefficient 99%.

The first two moments of the TCJ for different activity time distributions were obtained from these simulation runs. Table 2 presents, inter alia, the simulation results for a normally distributed activity time and Table 3 presents the corresponding results when the activity times are exponentially distributed. Tables 2 and 3 also present the first two moments obtained using (i) our approximation discussed above, (ii) the Malcolm et al.'s normal approximation and, (iii) the extreme value approximation. Appendix B discusses the procedure for obtaining the moments assuming that the TCJ follows extreme value distribution. From Table 2 it is clear that the normal approximation underestimates the mean and overestimates the variance. On the other hand the extreme value approximation, in general, overestimates the mean and underestimates the variance. In comparison to these two approaches, the suggested approximation gives more accurate moments. These results agree with the theoretical arguments put forth in section 2, that the distribution of the TCJ is neither a normal nor an extreme value but a mixture of some distributions. The chi-square values show that for an underlying exponential activity distribution, we can reject the hypothesis that the distribution of the TCJ is either normal or extreme value at at 0.001 significance level for all ten networks. The chi-square values using our approximation tend to be close to those using the simulation mean and variance. This similarity further reinforces our hypothesis about the distribution of the TCJ. With a normal activity distribution we can conclude at a 0.001 significance level that the distribution of the TCJ is not an extreme value.

4 Conclusions

We conclude from the above that explicit recognition of dependence between paths enhances the accuracy of estimates of the first two moments of the distribution of the TCJ. Furthermore, incorporation of this approximation in standard PERT software is facilitated, given the simplicity of the approach and the availability of published tables. Though we

Table 2: Comparative Evaluation of Different Approximations

Normally Distributed Activity Durations

Mean Activity Time = 4

Variance of Activity Time = 1

Name of the Network	Simulation Mean/ Variance	Our approach Mean/ Variance	Normal distn. Mean/ Variance	Extreme Value dist. Mean/ Variance	χ^2 test for normality using Simulation results	Our results	χ^2 test for extreme value using Simulation results	Our results
Kleindorfer	37.430 7.710	37.692 7.238	36 9	40.127 6.738	24.83	250.05	5197.49	7902.03
Large Network	52.407 7.673	51.689 9.119	48 12	55.4770 3.3520	35.14	5119.02	5034.12	2943.99
Shogan (1977)	13.544 1.862	13.456 1.983	12 3	14.598 1.778	48.61	154.66	4989.09	3689.84
Kamburowski (1985)	13.812 1.697	14.159 2.118	12 3	14.771 1.533	94.99	1603.72	4894.88	10455.72
Fulkerson (1962)	23.012 2.915	23.058 2.885	20 5	24.676 1.483	50.55	73.59	5117.68	5730.39
Ringer(1971)	17.0338 3.192	16.987 3.040	16 4	18.477 4.746	47.24	82.92	5024.31	4560.09
Martin (1965)	24.788 5.615	24.798 5.363	24 6	27.034 7.119	11.81	31.18	5125.80	5577.55
Dodin (1985)	17.561 2.901	17.456 2.983	16 4	19.001 2.373	60.90	142.50	5349.06	4105.01
Pritsker & Kiviat (1969)	21.272 4.062	21.197 4.119	20 5	23.076 3.743	17.99	40.03	5279.15	4516.37
Provan & Ball (1984)	14.337 1.375	14.336 1.394	12 3	15.214 1.123	92.10	100.69	4970.84	4877.02

Table 3: Comparative Evaluation of Different Approximations

Exponentially Distributed Activity Durations

Mean Activity Time = 4

Name of the Network	Simulation Mean/ Variance	Our approach Mean/ Variance	Normal distn. Mean/ Variance	Extreme Value dist. Mean/ Variance	χ^2 test for normality using Simulation results	χ^2 test for normality using Our results	χ^2 test for extreme value using Simulation results	χ^2 test for extreme value using Our results
Kleindorfer	41.703 146.633	42.924 140.064	36 144	43.427 165.593	824.17	1155.23	4844.91	8050.98
Large Network	65.791 173.6138	65.482 173.20	48 192	77.631 132.372	526.80	516.90	4921.59	4358.09
Shogan (1977)	18.139 53.963	18.188 51.997	12 48	19.510 71.347	1541.13	1518.47	4891.41	5257.08
Kamburowski (1985)	19.726 51.791	19.233 52.041	12 48	20.643 64.112	1508.14	1434.35	4904.61	3514.92
Fulkerson (1962)	32.646 92.227	34.997 75.343	20 80	39.031 144.945	998.93	3098.38	4851.16	18534.34
Ringer(1971)	19.830 67.890	19.750 64.937	16 64	20.190 149.469	1373.25	1675.00	4953.89	4914.57
Martin (1965)	27.031 100.031	27.000 99.000	24 96	29.536 232.081	1108.86	1098.94	4979.93	4952.35
Dodin (1985)	22.054 67.381	22.189 67.998	16 64	24.734 91.212	1102.32	1126.92	5060.31	5369.72
Pritsker & Kiviat (1969)	24.895 81.056	24.852 83.768	20 80	27.797 131.710	1142.79	1162.67	5004.79	4720.11
Provan & Ball (1984)	22.356 52.422	22.980 48.659	12 48	24.045 56.188	1329.13	1576.05	4931.86	8003.37

have presented the approach for only normal and exponentially distributed activity durations, the approach can be extended to any underlying activity distribution. Obviously, the facility with which the approximation can be applied would vary with the distribution.

In a stochastic network it is possible (i.e. may occur with positive probability) that a path with M *iid* activities takes less time to complete than another path with $(M-1)$ activities. In a network that has a critical path of M activities we define a path with $(M-1)$ activities as a "sub-critical" path. Then, our above argument suggests that the role of a sub-critical path may be important in further improving the approximations for the moments of T. Hence, another extension that is immediately perceivable is the development of a procedure that accounts for the contribution of the sub-critical paths in a given network.

References

[1] Burt, J.M., Jr., and M.B. Garman, "Conditional Monte Carlo: A Simulation Technique for Stochastic Network Analysis", *Management Science*, Vol. 18, 1971, pp. 207–217.

[2] David, H.A. *Order Statistics*, John Wiley and Sons, NY, 1970.

[3] Dodin, Bajis and Mete Sirvanci, "Stochastic Activity Networks and The Extreme Value Distribution", *Technical Paper*, November 1986.

[4] Dodin, B.M., "Bounding the Project Completion Time Distribution in PERT Networks", *Operations Research*, Vol. 24, No. 4, 1985a, pp. 862–881.

[5] Dodin, B.M, "Approximating the Distribution Functions in Stochastic Networks", *Computers and Operations Research*, Vol. 12, No. 3, 1985b, pp. 251–264.

[6] Dodin, B.M, "Reducibility of Stochastic Networks", *Omega*, Vol. 13, No. 3, 1985c, pp. 223–232.

[7] Elmaghraby, S.E., *Activity Networks: Project Planning and Control by Network Models*, J.Wiley & Sons, New York, 1977.

[8] Fisher, D.L., D. Saisi and W.M. Goldstein, "Stochastic PERT Networks: OP Diagrams, Critical Paths and the Project Completion Time", *Computers and Operations Research*, Vol. 12, No. 5, 1985, pp. 471–482.

[9] Fulkerson, D.R., "Expected Critical Path Lengths in PERT Networks", *Operations Research*, Vol. 10, 1967, pp. 808–817.

[10] Golenko-Ginzburg, Dimitri, "A New Approach to the Activity-Time Distribution in PERT", *Journal of Operational Research Society*, Vol. 40, No. 4, pp. 389–393.

[11] Hagstrom, J.N., "Computing the Probability Distribution of Project Duration in a PERT Network", *Networks*, Vol. 20, 1990, pp. 231–244.

[12] Kamburowski, Jerzy, "An Upper Bound on the Expected Completion Time of PERT Networks", *European Journal of Operational Research*, Vol. 21, 1985a, pp. 206–212.

[13] Kamburowski, Jerzy, "Normally Distributed Activity Durations in PERT Networks", *Journal of Operational Research Society*, Vol. 36, No. 11, 1985b, pp. 1051–1057.

[14] Kleindorfer, G.B., "Bounding Distributions for a Stochastic Acyclic Network", *Operations Research*, Vol. 19, 1971, pp. 1586–1601.

[15] Kulkarni, V.G. and V.G. Adhlaka, "Markov and Markov-Regenerative PERT Networks", *Operations Research*, Vol. 34, No. 5, Sep-Oct 1986, pp. 769–781.

[16] Malcolm, D.G., J.H. Roseboom, C.E. Clark, and W. Fazar, "Application of a Technique for Research and Development Program Evaluation", *Operations Research*, Vol. 7, 1959, pp. 646–669.

[17] Martin, J.J., "Distribution of the Time Through a Directed Acyclic Network", *Operations Research*, Vol. 13, 1965, pp. 46–66.

[18] Owen, D.B., *Handbook of Statistical Tables*, Addison-Wesley, Reading, MA, 1962.

[19] Ringer, L.J., "Numerical Operators for Statistical PERT Critical Path Analysis", *Management Science*, Vol. 16, No. 2, 1969, pp. B-136–B-143.

[20] Robillard, P. and M. Trahan, "The Completion Time of PERT Networks", *Operations Research*, Vol. 25, No. 1, Jan-Feb 1977, pp. 15–29.

[21] Sarhan, A.E. and Greenberg, B.G. *Contributions to Order Statistics*, John Wiley and Sons, NY, 1962.

[22] Sculli, D. and K.L. Wong, "The Maximum and Sum of Two Beta Variables and the Analysis of PERT Networks", *Omega*, Vol. 13, No. 3, 1985, pp. 233–240.

[23] Sculli, D., "The Completion Time of PERT Networks", *Journal of Operational Research Society*, Vol. 34, 1983, pp. 155–158.

[24] Shogan, A.W., "Bounding Distributions for a Stochastic PERT Network", *Networks*, Vol. 7, 1977, pp. 359–381.

[25] Sigal, C.E., A.A.B. Pritsker and J.J. Solberg, "The Stochastic Shortest Route Problem", *Operations Research*, Vol. 28, No. 5, Sep-Oct 1980, pp. 1122–1129.

[26] Van Slyke, R.M., "Monte Carlo Methods and the PERT Problem", *Operations Research*, Vol. 11, 1963, pp. 839–860.

A Appendix: Derivation of Moments of TCJ

Let there be K critical paths in the network. Let $M-m$ be the number of common activities out of a total of M activities on the critical path. We present below the derivation of the first two moments of the TCJ and associated approximations.

A.1

We know that $T = \max_{1\leq i\leq K}(U_i) + V = U + V$. Let each X_{ij} be a *iid* normal random variable, i.e. $X_{ij} \sim \mathcal{N}(\mu, \sigma^2)$. Then it follows that

$$\begin{aligned} U_i &\sim \mathcal{N}(m\mu, m\sigma^2) \\ V &\sim \mathcal{N}((M-m)\mu, (M-m)\sigma^2) \end{aligned}$$

Thus $U = \max_{1\leq i\leq K}(U_i)$ represents the maximum of K normal random variables and its distribution is given by $\mathcal{N}^K(m\mu, m\sigma^2)$. Suppose that Z_K denotes the largest observation in a sample of size K from standard normal distribution i.e. $\mathcal{N}$(0,1). Then, it is easy to verify that

$$\begin{aligned} E(U) &= \sqrt{m}\sigma\ E(Z_K) \\ \text{Var}(U) &= m\sigma^2\ \text{Var}(Z_K) \end{aligned}$$

For small values of K the mean and variance of Z_K are tabulated e.g. see Sarhan and Greenberg (1962). For large values of K one can use the approximations discussed in Case A.3 below. In summary,

$$\begin{aligned} E(T) &= M\mu + \sqrt{m}\sigma\ E(Z_K) \\ \text{Var}(T) &= (M-m)\sigma^2 + m\sigma^2\ \text{Var}(Z_K). \end{aligned}$$

A.2

Assuming now that the activity distributions follow an exponential distribution with mean λ. As discussed earlier in section 2 of this paper, the distribution of each U_i is given by a $\Gamma(\lambda, m)$, where λ is the mean parameter and m is the shape parameter. The distribution of V is also a gamma disribution, $\Gamma(\lambda, M-m)$. As in the case A.1 above, suppose that now

Z_K denotes the largest among K observations drawn from the gamma distribution $\Gamma(\lambda, m)$ then

$$\begin{aligned} E(U) &= m\lambda\, E(Z_K) \\ \text{var}(U) &= m^2\lambda\, \text{Var}(Z_K) \end{aligned}$$

As above we can refer to published tables to obtain moments of Z_K for for small values of K and A.3 for large values.

A.3

If the number of critical paths K is very large or the distribution of U_i is not of the form for which the moments of the largest observation are tabulated, then recourse can be taken to the approximation suggested below. This approximation is based on the probability integral transformation and where the Taylor series expansion is carried only upto one term.

Suppose that the distribution of each U_i is given by $\mathcal{F}(.)$ and $\mathcal{Q}$ satisfies the relation: whenever $\mathcal{F}(x) = y$ then $\mathcal{Q}(y) = x$, i.e. $\mathcal{Q}$ is the inverse function of $\mathcal{F}$, then

$$\begin{aligned} E(U) &\approx \mathcal{Q}\left(\frac{K}{K+1}\right) \\ \text{Var}(U) &\approx \frac{K}{2(K+1)^2(K+2)}\mathcal{Q}'^2\left(\frac{K}{K+1}\right) \end{aligned}$$

where $\mathcal{Q}'$ denotes the first derivative of $\mathcal{Q}$.

Better approximations, using more terms of the Taylor expansion, are provided in David (1970).

B Appendix: Method for Calculating the Extreme-Value Approximation

Consider *iid* random variables X_i's, with distribution function $\mathcal{F}(x)$ and the density function $f(x)$. Set $Y_n = \max_{1 \le t \le n} X_i$. Then for large values of n the distribution of Y_n can be approximated by the extreme-value distribution. A precise statement is:

Theorem .1 *Suppose $\mathcal{F}(x) < 1$ for all values of $x < \infty$; $\mathcal{F}(x)$ is twice differentiable with respect to x for $x > x'$ where x' is some fixed real number; and*

$$\lim_{x \longrightarrow \infty} \frac{d}{dx} \left[\frac{1 - \mathcal{F}(x)}{f(x)} \right] = 0.$$

Then

$$\lim_{n \longrightarrow \infty} P\{b_n(Y_n - a_n) \le x\} = \exp(-\exp(-x)),$$

holds uniformly for $x \in (-\infty, \infty)$. The constants a_n and b_n satisfy

$$\mathcal{F}(a_n) = \frac{n-1}{n}, \quad b_n = n \; f(a_n). \qquad (a.1)$$

The first two moments of Y_n can be approximated by

$$E(Y_n) \approx a_n + \frac{.577722}{b_n}, \quad \mathrm{Var}(Y_n) \approx \frac{\pi^2}{6b_n^2}.$$

Application of the above theorem to specific distributions:

To apply the theorem to special cases requires solution of the two equations in (a.1). Typically, b_n is easy to obtain but the constant a_n, given by

$$a_n = \mathcal{F}^{-1}\left(\frac{n-1}{n}\right),$$

is difficult whenever the inverse of $\mathcal{F}$ is not available in a closed form.

Case 1: If X_i's are normally distributed, $\mathcal{N}(\mu, \sigma^2)$, then it can be seen that

$$a_n = \mu + \sigma \left[\sqrt{2 \log n} \; - \frac{1}{2} \; \frac{(\log \log n + \log 4\pi)}{\sqrt{2 \log n}} \right]$$

and

$$b_n = \frac{\sqrt{2 \log n}}{\sigma}.$$

Case 2: If each X_i is distributed as $\mathcal{F} = \Gamma(\lambda, m)$, then we solve the equation $\mathcal{F}(a_n) = n^{-1}(n-1)$ by making use of the relation between $\mathcal{F}$ and the Poisson distribution function. We then obtain a_n such that it satisfies

$$\sum_{j=0}^{m-1} \exp(-a_n/\lambda) \left(\frac{a_n}{\lambda}\right)^j \frac{1}{j!} = \frac{1}{n},$$

and use this value of a_n to get

$$b_n = \frac{n}{\Gamma(m)\lambda^m} \exp(-a_n/\lambda)\, a_n^{m-1}.$$