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March 1991

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A Study of Approximating the Moments of the Job Completion Time in PERT Networks

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Abstract

The importance of proper management of projects has not gone unrecognized in industry and academia. Consequently tools like Critical Path Method (CPM) and Program Evaluation Review Technique (PERT) for project planning have been the focus of attention of both practitioners and researchers. Determination of the Time to Complete the Job (TCJ) in PERT networks is important for planning and bidding purposes. The complexity involved in accurately determining the TCJ has led to the development of many approximating procedures. Most of them ignore the dependence between paths in the network. We propose an approximation to determine the TCJ which explicitly recognizes this dependency. Experimental results which demonstrate the accuracy of our approximation for a wide variety of networks are presented.
1 Introduction

Program Evaluation and Review Technique (PERT) was developed in the 1950's. An early application of PERT was made by the U.S. government in planning and scheduling the research project for developing the Polaris Ballistic Missile. Soon PERT became the primary tool for planning and scheduling of projects, especially those which were funded by the U.S. government. PERT networks have been used to represent large projects in the industry and hence have a lot of applicability in the business world [see Elmagrabhy (1977)]. Analysis of PERT networks, also known as stochastic activity networks, has received considerable attention in the literature.

PERT is based on the concept that a project is divided into a number of activities which are arranged in some order according to the job requirements. A PERT network is graphically represented using a set of nodes and arcs where a node represents the beginning or completion of one or more activities and an activity is represented by an arc (arrow) connecting two nodes. The project starts at the initial node and ends at the terminal node. A path is a set of nodes connected by arrows which begin at the initial node and end at the terminal node. This collection of arcs, nodes and paths is collectively called an activity network. A project is deemed complete if work along all paths is complete.

If activity times are deterministic, the duration of the project completion time is determined by the length of the longest path in the network. However, things become complicated when activity times are stochastic in nature. For a stochastic activity network, Kulkarni and Adlakha (1986) have identified three important measures of performance.

(a) Distribution of the project completion time
(b) The probability that a given path is critical
(c) The probability that a given activity belongs to a critical path.

Performance measures derived from (a) are the most commonly used measures and most of the work has concentrated on the properties of the Time to Completion of the Job (TCJ).

Determination of the exact distribution of TCJ is complicated by the fact that different paths are correlated and also because of the need to find the maximum of a set of random variables, as we shall see later. Hence one cannot easily determine the exact distribution of the TCJ. The research has primarily branched off in three directions:
(i) Exact methods: Martin (1965), Dodin (1985), Fisher et al. (1985), and Hagstrom (1990) are some of the papers that deal with these methods. Most of their results are limited in that they make quite restrictive assumptions. For example Martin (1965) assumes that the arc duration density functions are polynomial. Hagstrom (1990) assumes task durations have discrete distributions.

(ii) Approximating and bounding approaches: These have been the most prolific in the literature. Malcolm et al. (1959), Sculli (1983), Golenko-Ginzburg (1989), Dodin (1985b), Sculli and Wong (1985), and Dodin and Sirvanci (1986) determine approximations for the distribution and moments of the TCJ. Kamburowski (1985), Shogan (1977), Kleindorfer (1971), and Robillard and Trahan (1977), on the other hand, try to find upper and/or lower bounds for the distributions and moments of the TCJ.

(iii) Simulation methods: These methods have been discussed in the literature by Van Slyke (1963), Burt and Garman (1971), and Sigal et al. (1979)

We adopt approach (ii) above and present a simple and practical method to determine close approximations for the first two moments of the TCJ. We do not undertake the task of determination of the bounds for these moments. Though it is informative to know the best and worst completion times for a project, a single approximation for the TCJ is more useful for bidding purposes as compared to a range. In general researchers are more interested in the moments of the TCJ rather than completely specifying the exact distribution. In fact, the distribution is merely a first step towards obtaining the moments.

Dodin and Sirvanci (1986) propose the extreme value distribution as an approximation to the TCJ. They claim that the distribution of the TCJ varies from a normal to an extreme value distribution depending on factors like the size of the network, the dependence between paths and the number of dominating paths. We explicitly take into account this dependence between paths which occurs due to common activities on various paths. We show, using simulation results as a benchmark, that the distribution of the TCJ is better approximated by a mixture of distributions. In addition, we use the critical path concept which is easier to comprehend and extremely simple to operationalize, as opposed to a dominating path concept (Dodin and Sirvanci, 1986). Section 2 presents the theoretical underpinnings of our approach and illustrates its use by an example. Section 3 compares the simulation results and those obtained using our approximation for a wide variety of networks appearing in the literature. Section 4 presents the conclusions and additional mathematical details are presented in the appendices.
2 Development of the Proposed Approximation

In this section we lay down the theoretical arguments underlying our approach. We then explicate the concepts using a widely cited network in the literature — Kleindorfer's network, as an illustrative example.

2.1 Theoretical Concepts

Let $T$ be a random variable that stands for the time to complete the job; let $X_{ij}$ be the time required to finish the $j$-th activity in the $i$-th path, where $n_i$ represents the number of activities in the $i$-th path, and $N$ represents the total number of paths in the network; and define $Y_i = \sum_{j=1}^{n_i} X_{ij}$. Then we can write $T = \max_{1 \leq i \leq N} Y_i$. We make use of the critical path concept, as opposed to the dominating path concept used by Dodin and Sirvanci (1986), in trying to determine the distribution of $T$. The traditional definition of the critical path is that path which takes the longest expected time [see Elmagrabhy (1977)]. This is obtained by summing the expected times of the activities on that path. As stated earlier this is a much simpler concept and less cumbersome from an analytical point of view.

Now consider the situation where there is more than one critical path. In this case, the time to complete the job will depend heavily upon that critical path which is completed last. In fact, the TCJ will be determined by any path which takes the longest time. To complicate matters, it may be possible that several activities of two critical paths are identical. Therefore, it becomes necessary to treat the common and non-common activities separately. Consider an "ideal" situation as shown in Figure 1. Now consider the

![Figure 1: The "Ideal" Setting of Several Critical Paths](image)

set of $K$ critical paths of the given network. Let $U_i$ be the the sum of the "non-common"
activities in the $i$-th critical path and $V$ be the sum of the "common" activities for the $K$ critical paths. Then we can approximate $T = \max_{1 \leq i \leq N} Y_i$ where $N$ is the total number of paths in the network by $T \approx \max_{1 \leq i \leq K} (U_i) + V$ where $K$ is the number of critical paths in a network. So far we have discussed only the ideal condition. In practice however, the critical paths do not have exactly the same activities common to all of them. Typically observed critical paths are as shown in Figure 2.

![Typically Observed Critical Paths](image)

Figure 2: Typically Observed Critical Paths

Here it is observed that all paths do not have exactly the same number of common activities. For example paths $P_1$ and $P_2$ have only three common activities, whereas $P_2$, and $P_3$ have two common activities. Also, all common activities are not exactly the same — paths $P_1$ and $P_2$ have activities $4 - 5, 5 - 6$ and $6 - 7$ common whereas paths $P_2$ and $P_3$ have $1 - 2$ and $6 - 7$ as common activities. In such cases a subjective assessment can be made and then the results of the ideal situation can be used. For example, for the network whose critical paths are represented in Figure 2, it would be reasonable to argue that among three paths comprising six activities each, there are three common activities and three non-common activities. Although this is a subjective assessment, however, in section 3 we observe that it provides a close approximation for the first two moments of $T$. We will shortly discuss an example which will provide some guideline on choosing the number of common activities.

The beta distribution has been traditionally suggested to model the durations of the stochastic activities comprising the PERT network. However, there is a preponderant usage of the normal distribution in the literature. Sculli (1983) states that

\[ \ldots \text{this can be justified by the fact that most large networks can be reduced} \]
to a guide network where a completely independent path becomes one activity. The central limit theorem justifies the normality assumption for the duration of activities in the guide network.

Moreover, as observed in Golenko-Ginzberg (1989), the beta distribution is not stable with respect to convolution and maximization. Therefore, for the purposes of our analysis, we assume that the activity durations are iid normal random variables. The assumption of iid distributed activities is not overly restrictive. It was made only for purposes of computational ease in illustrating our approach. The proposed approximation can be used with non-iid distributed activities with equal facility. Subsequently we also consider the setting of iid exponential activities. We summarize the following theoretical properties about the distribution of \( U = \max_{1 \leq i \leq K}(U_i), \) \( V, \) and \( T. \)

Properties of \( V: \) The distribution of \( V \) is, in general, given by the distribution of the sum of the \( X_{ij} \)'s that are common to the critical paths. Therefore, we know that the distribution of \( V \) is (a) normal if each \( X_{ij} \) is normal, and (b) gamma if each \( X_{ij} \) is exponential, and (c) approximately normal, by the Central Limit Theorem, if the number of common activities is large. The expected value and variance of \( V \) are obtained by adding the expected values and variances of the common activities.

Properties of \( U: \) Properties of \( U_i \)'s, for each value of \( i, \) are the same as properties of \( V. \) The distribution of \( U = \max_{1 \leq i \leq K} U_i, \) is given by some appropriate distribution obtained from the theory of order statistics. For example, if each \( U_i \) is a normal random variable; i.e. \( P(U_i \leq x) = \mathcal{N}(x; \mu, \sigma^2), \) then the distribution of \( U \) is given by

\[
P(U \leq x) = \{\mathcal{N}(x; \mu, \sigma^2)\}^K = \mathcal{N}^K(x; \mu, \sigma^2).
\]

More generally, if \( P(U_i \leq x) = \mathcal{F}(x) \) for \( i = 1, 2, \ldots, K; \) then

\[
P(U \leq x) = \{\mathcal{F}(x)\}^K = \mathcal{F}^K(x).
\]

For large values of \( K, \) the distribution of \( U \) can be approximated by the extreme value distribution.

Properties of \( T: \) The distribution of \( T = U + V \) is therefore represented by the convolution of distribution of \( U \) and \( V. \) The exact form of the distribution of \( T \) is not easy to assess, because the convolution distributions are, in general, not of any well known standard family of distributions or of closed forms. However, the moments of the distribution, particularly the first two moments, can be evaluated relatively easily.
because

\[ E(T) = E(U) + E(V), \quad \text{and} \quad \text{Var}(T) = \text{Var}(U) + \text{Var}(V). \]

Calculation of \( E(U) \) and \( \text{Var}(U) \) may cause difficulties for larger values of \( K \) because expected values of the largest observation in a sample are not available for all distributions. In these cases a reasonably accurate approximation can be used as suggested in appendix A.3.

### 2.2 Illustrative Example

We now present an example of the theoretical distribution of \( T \) using a widely cited network, Kleindorfer’s network (See Figure 3). Figure 4 shows all possible paths in this network.

![Figure 3: Kleindorfer’s Network](image-url)
Figure 4: All Possible Paths on Kleindorfer Network

PATH # 1: 1 2 4 5 10 12 17 20
PATH # 2: 1 2 4 5 12 17 20
PATH # 3: 1 2 4 5 13 16 18 19 20
PATH # 4: 1 2 4 5 13 16 18 20
PATH # 5: 1 2 4 5 13 16 19 20
PATH # 6: 1 2 4 5 13 17 20
PATH # 7: 1 2 4 6 11 13 16 18 19 20
PATH # 8: 1 2 4 6 11 13 16 18 20
PATH # 9: 1 2 4 6 11 13 16 19 20
PATH # 10: 1 2 4 6 11 13 17 20
PATH # 11: 1 2 4 6 11 15 16 18 19 20
PATH # 12: 1 2 4 6 11 15 16 18 20
PATH # 13: 1 2 4 6 11 15 16 19 20
PATH # 14: 1 2 4 7 8 10 12 17 20
PATH # 15: 1 2 4 7 8 10 15 16 18 19 20
PATH # 16: 1 2 4 7 8 15 16 18 20
PATH # 17: 1 2 4 7 8 15 16 19 20
PATH # 18: 1 2 4 7 8 18 19 20
PATH # 19: 1 2 4 7 8 18 20
PATH # 20: 1 2 4 7 12 17 20
PATH # 21: 1 2 4 7 13 16 18 19 20
PATH # 22: 1 2 4 7 13 16 18 20
PATH # 23: 1 2 4 7 13 16 19 20
PATH # 24: 1 2 4 7 13 17 20
PATH # 25: 1 2 4 17 20
PATH # 26: 1 2 6 11 13 16 18 19 20
PATH # 27: 1 2 6 11 13 16 18 20
PATH # 28: 1 2 6 11 13 16 19 20
PATH # 29: 1 2 6 11 13 17 20
PATH # 30: 1 2 6 11 15 16 18 19 20
PATH # 31: 1 2 6 11 15 16 18 20
PATH # 32: 1 2 6 11 15 16 19 20
PATH # 33: 1 2 8 10 12 17 20
PATH # 34: 1 2 8 15 16 18 19 20
PATH # 35: 1 2 8 15 16 18 20
PATH # 36: 1 2 8 15 16 19 20
PATH # 37: 1 2 8 18 19 20
PATH # 38: 1 2 8 18 20
PATH # 39: 1 3 5 10 12 17 20
PATH # 40: 1 3 5 12 17 20
PATH # 41: 1 3 5 13 16 18 19 20
PATH # 42: 1 3 5 13 16 18 20
PATH # 43: 1 3 5 13 16 19 20
PATH # 44: 1 3 5 13 17 20
PATH # 45: 1 3 9 10 12 17 20
PATH # 46: 1 3 9 14 19 20
It has three critical paths, $P_7$, $P_{11}$, and $P_{15}$. There are five activities that are common to all three critical paths. The remaining four activities are not common to all three critical paths. Figure 5 shows the subgraph of the three critical paths. Now, from the above, we know that $T \approx V + \max(U_i) = V + U$.

Case I: Let us consider the case where each activity has the normal distribution with mean 4 and variance 1. Here $V$ is the sum of five normal random variables and therefore is itself a normal random variable with mean 20 and variance 5. In a similar manner $U_7$, $U_{11}$, and $U_{15}$ are also normal random variables each with mean 16 and variance 4. Finally,

$$P(U \leq x) = N^3(x; 16, 4).$$

The mean and variance of $T$ can be easily evaluated from the above representation of the distribution of $T$. One can obtain the mean and variance of $N^3$ for the standardized normal random variable from the statistical tables by Owen (1962). Using these properties, $E(T) \approx 37.692$ and $Var(T) \approx 7.238$. (For details, see Appendix A.1.)

Case II: In this case, where each activity follows an exponential distribution with mean 4, the procedure for deriving the distribution is the same as in Case I. The only exception is that $V$ is the sum of five exponential distributions, each with mean 4, and therefore the distribution of this convolution is given by a gamma distribution ($\Gamma$) with mean 20 and shape parameter 5. Similarly, the distribution of each $U_i$ is given by a gamma with mean 16 and shape parameter 4 and, finally, $P(U \leq u) = \Gamma^3(u; 20, 4)$. To find the expected value and variance of $U$ we need to know the expected value and variance of the largest observation in a sample of size 3 from a gamma distribution with shape parameter 5. Expected values of the order statistics for the gamma distribution are tabulated [see Sarhan and Greenberg (1962)]. Using these results it is observed that $E(T) \approx 42.924$ and $Var(T) \approx 140.064$. (For more detail, see the Appendix A.2.)
Table 1: Structural Descriptions Of Different Networks Being Evaluated

<table>
<thead>
<tr>
<th>Name of Network</th>
<th>Number of</th>
<th>‘Critical’</th>
<th>Critical Activities</th>
<th>Total Paths</th>
<th>Common Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nodes</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kleindorfer</td>
<td>20</td>
<td>9</td>
<td>3</td>
<td>46</td>
<td>5</td>
</tr>
<tr>
<td>Large Network</td>
<td>43</td>
<td>12</td>
<td>19</td>
<td>617</td>
<td>8</td>
</tr>
<tr>
<td>Shogan (1977)</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Kamburowski (1985)</td>
<td>8</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>0 - 1</td>
</tr>
<tr>
<td>Fulkerson (1962)</td>
<td>10</td>
<td>5</td>
<td>16</td>
<td>16</td>
<td>2</td>
</tr>
<tr>
<td>Ringer (1971)</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Martin (1965)</td>
<td>9</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>Dodin (1985)</td>
<td>7</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Pritsker &amp; Kiviat (1969)</td>
<td>9</td>
<td>5</td>
<td>3</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Provan &amp; Ball (1984)</td>
<td>9</td>
<td>3</td>
<td>9</td>
<td>9</td>
<td>0 - 1</td>
</tr>
</tbody>
</table>

In our evaluation above it could be argued that $V$ should be approximated as a sum of 6 independent random variables because paths $P_7$ and $P_{11}$ have 7 activities common while $P_7$ and $P_{15}$ have 5 common activities and $P_{11}$ and $P_{15}$ have 6 activities in common, giving an average of 6. If this is taken into account then the first two moments of $T$ will change to 37.466 and 7.678 for the normal case and 41.984 and 145.152 for the exponential case. These difference in the moments are small when compared with either the normal or extreme value approximations.

3 Empirical Study

To the best of our knowledge, the exact distribution of $T$ has not been derived for any reasonable size network. We therefore use Monte Carlo simulation to obtain the “true” moments of the distribution of the TCJ for a variety of networks cited in PERT-related literature. Table 1 elaborates on the structural characteristics of these networks based on the assumption of iid activities. The dimensionality and complexity of these networks varies considerably. For example, the total number of paths in the network ranges from three (Martin, 1965) to 617 for the “large network” that appeared in Dodin and Sirvanci (1986).
The simulation program was coded in Pascal and run on an IBM 3090 machine. The simulation of each network comprised a sample size of 20,000 runs. We use a simulation run length of 20,000 to obtain values as close to the "exact" mean and the "exact" variance as possible. With this run-length the standard error in the mean of a simulation study is of the order of $\sqrt{1/20000} = \pm 0.007$. For the normal $N(4,1)$ distribution of each arc and for the Kleindorfer's network the standard error of mean from the simulation study is 0.0196, and this implies that the true value of $E(T) \in (37.377, 37.495)$ with confidence coefficient 99%.

The first two moments of the TCJ for different activity time distributions were obtained from these simulation runs. Table 2 presents, inter alia, the simulation results for a normally distributed activity time and Table 3 presents the corresponding results when the activity times are exponentially distributed. Tables 2 and 3 also present the first two moments obtained using (i) our approximation discussed above, (ii) the Malcolm et al.'s normal approximation and, (iii) the extreme value approximation. Appendix B discusses the procedure for obtaining the moments assuming that the TCJ follows extreme value distribution. From Table 2 it is clear that the normal approximation underestimates the mean and overestimates the variance. On the other hand the extreme value approximation, in general, overestimates the mean and underestimates the variance. In comparison to these two approaches, the suggested approximation gives more accurate moments. These results agree with the theoretical arguments put forth in section 2, that the distribution of the TCJ is neither a normal nor an extreme value but a mixture of some distributions. The chi-square values show that for an underlying exponential activity distribution, we can reject the hypothesis that the distribution of the TCJ is either normal or extreme value at at 0.001 significance level for all ten networks. The chi-square values using our approximation tend to be close to those using the simulation mean and variance. This similarity further reinforces our hypothesis about the distribution of the TCJ. With a normal activity distribution we can conclude at a 0.001 significance level that the distribution of the TCJ is not an extreme value.

4 Conclusions

We conclude from the above that explicit recognition of dependence between paths enhances the accuracy of estimates of the first two moments of the distribution of the TCJ. Furthermore, incorporation of this approximation in standard PERT software is facilitated, given the simplicity of the approach and the availability of published tables. Though we
Table 2: Comparative Evaluation of Different Approximations

<table>
<thead>
<tr>
<th>Name of the Network</th>
<th>Simulation Mean/ Variance</th>
<th>Our approach Mean/ Variance</th>
<th>Normal distr. Mean/ Variance</th>
<th>Extreme Value dist. Mean/ Variance</th>
<th>( \chi^2 ) test for normality using Simulation results</th>
<th>Our results</th>
<th>( \chi^2 ) test for extreme value using Simulation results</th>
<th>Our results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kleindorfer</td>
<td>37.430 7.710</td>
<td>37.692 7.238</td>
<td>36 9</td>
<td>40.127 6.738</td>
<td>24.83 250.05</td>
<td>5197.49</td>
<td>7902.03</td>
<td></td>
</tr>
<tr>
<td>Large Network</td>
<td>52.407 7.673</td>
<td>51.689 9.119</td>
<td>48 12</td>
<td>55.4770 3.3520</td>
<td>35.14 5119.02</td>
<td>5034.12</td>
<td>2943.99</td>
<td></td>
</tr>
<tr>
<td>Fulkerson (1962)</td>
<td>23.012 2.915</td>
<td>23.058 2.885</td>
<td>20 5</td>
<td>24.676 1.483</td>
<td>50.55 73.59</td>
<td>5117.68</td>
<td>5730.39</td>
<td></td>
</tr>
<tr>
<td>Ringer (1971)</td>
<td>17.0338 3.192</td>
<td>16.987 3.040</td>
<td>16 4</td>
<td>18.477 4.746</td>
<td>47.24 82.92</td>
<td>5024.31</td>
<td>4560.09</td>
<td></td>
</tr>
<tr>
<td>Martin (1965)</td>
<td>24.788 5.615</td>
<td>24.798 5.363</td>
<td>24 6</td>
<td>27.034 7.119</td>
<td>11.81 31.18</td>
<td>5125.80</td>
<td>5577.55</td>
<td></td>
</tr>
<tr>
<td>Dodin (1985)</td>
<td>17.561 2.901</td>
<td>17.456 2.983</td>
<td>16 4</td>
<td>19.001 2.373</td>
<td>60.90 142.50</td>
<td>5349.06</td>
<td>4105.01</td>
<td></td>
</tr>
<tr>
<td>Provan &amp; Ball (1984)</td>
<td>14.337 1.375</td>
<td>14.336 1.394</td>
<td>12 3</td>
<td>15.214 1.123</td>
<td>92.10 100.69</td>
<td>4970.84</td>
<td>4877.02</td>
<td></td>
</tr>
</tbody>
</table>
Table 3: Comparative Evaluation of Different Approximations

Exponentially Distributed Activity Durations

<table>
<thead>
<tr>
<th>Name of the Network</th>
<th>Simulation Mean / Variance</th>
<th>Our approach Mean / Variance</th>
<th>Normal distr. Mean / Variance</th>
<th>Extreme Value distr. Mean / Variance</th>
<th>$\chi^2$ test for normality using Simulation results</th>
<th>$\chi^2$ test for extreme value using Simulation results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kleindorfer</td>
<td>41.703 146.633</td>
<td>42.924 140.064</td>
<td>36</td>
<td>43.427 165.593</td>
<td>824.17 1155.23</td>
<td>4844.91 8050.98</td>
</tr>
<tr>
<td>Large Network</td>
<td>65.791 173.6138</td>
<td>65.482 173.20</td>
<td>48</td>
<td>77.631 132.372</td>
<td>526.80 516.90</td>
<td>4921.59 4358.09</td>
</tr>
<tr>
<td>Shogan (1977)</td>
<td>18.139 53.963</td>
<td>18.188 51.997</td>
<td>12</td>
<td>20.643 71.347</td>
<td>1508.14 1434.35</td>
<td>4904.61 3514.92</td>
</tr>
<tr>
<td>Fulkerson (1962)</td>
<td>32.646 92.227</td>
<td>34.997 75.343</td>
<td>20</td>
<td>39.031 144.945</td>
<td>998.93 3098.38</td>
<td>4851.16 18534.34</td>
</tr>
<tr>
<td>Ringer (1971)</td>
<td>19.830 67.890</td>
<td>19.750 64.937</td>
<td>16</td>
<td>20.190 149.469</td>
<td>1373.25 1675.00</td>
<td>4953.89 4914.57</td>
</tr>
<tr>
<td>Martin (1965)</td>
<td>27.031 100.031</td>
<td>27.000 99.000</td>
<td>24</td>
<td>29.536 232.081</td>
<td>1108.86 1098.94</td>
<td>4979.93 4952.35</td>
</tr>
<tr>
<td>Dodin (1985)</td>
<td>22.054 67.381</td>
<td>22.189 67.998</td>
<td>16</td>
<td>24.734 91.212</td>
<td>1102.32 1126.92</td>
<td>5060.31 5369.72</td>
</tr>
<tr>
<td>Pritsker &amp; Kiviat (1969)</td>
<td>24.895 81.056</td>
<td>24.852 83.768</td>
<td>20</td>
<td>27.797 131.710</td>
<td>1142.79 1162.67</td>
<td>5004.79 4720.11</td>
</tr>
<tr>
<td>Provan &amp; Ball (1984)</td>
<td>22.356 52.422</td>
<td>22.980 48.659</td>
<td>12</td>
<td>24.045 56.188</td>
<td>1329.13 1576.05</td>
<td>4931.86 8003.37</td>
</tr>
</tbody>
</table>
have presented the approach for only normal and exponentially distributed activity durations, the approach can be extended to any underlying activity distribution. Obviously, the facility with which the approximation can be applied would vary with the distribution.

In a stochastic network it is possible (i.e. may occur with positive probability) that a path with $M$ iid activities takes less time to complete than another path with $(M-1)$ activities. In a network that has a critical path of $M$ activities we define a path with $(M-1)$ activities as a "sub-critical" path. Then, our above argument suggests that the role of a sub-critical path may be important in further improving the approximations for the moments of $T$. Hence, another extension that is immediately perceivable is the development of a procedure that accounts for the contribution of the sub-critical paths in a given network.
References


A Appendix: Derivation of Moments of TCJ

Let there be $K$ critical paths in the network. Let $M-m$ be the number of common activities out of a total of $M$ activities on the critical path. We present below the derivation of the first two moments of the TCJ and associated approximations.

A.1

We know that $T = \max_{1 \leq i \leq K} (U_i) + V = U + V$. Let each $X_{ij}$ be a iid normal random variable, i.e. $X_{ij} \sim \mathcal{N}(\mu, \sigma^2)$. Then it follows that

$$U_i \sim \mathcal{N}(m\mu, m\sigma^2)$$
$$V \sim \mathcal{N}((M-m)\mu, (M-m)\sigma^2)$$

Thus $U = \max_{1 \leq i \leq K} (U_i)$ represents the maximum of $K$ normal random variables and its distribution is given by $\mathcal{N}^K(m\mu, m\sigma^2)$. Suppose that $Z_K$ denotes the largest observation in a sample of size $K$ from standard normal distribution i.e. $\mathcal{N}(0,1)$. Then, it is easy to verify that

$$E(U) = \sqrt{m}\sigma \ E(Z_K)$$
$$\text{Var}(U) = m\sigma^2 \ Var(Z_K)$$

For small values of $K$ the mean and variance of $Z_K$ are tabulated e.g. see Sarhan and Greenberg (1962). For large values of $K$ one can use the approximations discussed in Case A.3 below. In summary,

$$E(T) = M\mu + \sqrt{m}\sigma \ E(Z_K)$$
$$\text{Var}(T) = (M-m)\sigma^2 + m\sigma^2 \ Var(Z_K).$$

A.2

Assuming now that the activity distributions follow an exponential distribution with mean $\lambda$. As discussed earlier in section 2 of this paper, the distribution of each $U_i$ is given by a $\Gamma(\lambda, m)$, where $\lambda$ is the mean parameter and $m$ is the shape parameter. The distribution of $V$ is also a gamma distribution, $\Gamma(\lambda, M-m)$. As in the case A.1 above, suppose that now
\( Z_K \) denotes the largest among \( K \) observations drawn from the gamma distribution \( \Gamma(\lambda, m) \) then

\[
E(U) = m\lambda E(Z_K) \\
\text{var}(U) = m^2\lambda \text{Var}(Z_K)
\]

As above we can refer to published tables to obtain moments of \( Z_K \) for for small values of \( K \) and A.3 for large values.

**A.3**

If the number of critical paths \( K \) is very large or the distribution of \( U_i \) is not of the form for which the moments of the largest observation are tabulated, then recourse can be taken to the approximation suggested below. This approximation is based on the probability integral transformation and where the Taylor series expansion is carried only upto one term.

Suppose that the distribution of each \( U_i \) is given by \( \mathcal{F}(\cdot) \) and \( Q \) satisfies the relation: whenever \( \mathcal{F}(x) = y \) then \( Q(y) = x \), i.e. \( Q \) is the inverse function of \( \mathcal{F} \), then

\[
E(U) \approx Q \left( \frac{K}{K+1} \right) \\
\text{Var}(U) \approx \frac{K}{2(K+1)^2(K+2)} Q'^2 \left( \frac{K}{K+1} \right)
\]

where \( Q' \) denotes the first derivative of \( Q \).

Better approximations, using more terms of the Taylor expansion, are provided in David (1970).
B Appendix: Method for Calculating the Extreme-Value Approximation

Consider iid random variables $X_i$'s, with distribution function $F(x)$ and the density function $f(x)$. Set $Y_n = \max_{1 \leq i \leq n} X_i$. Then for large values of $n$ the distribution of $Y_n$ can be approximated by the extreme-value distribution. A precise statement is:

**Theorem 1** Suppose $F(x) < 1$ for all values of $x < \infty$; $F(x)$ is twice differentiable with respect to $x$ for $x > x'$ where $x'$ is some fixed real number; and

$$\lim_{x \to \infty} \frac{d}{dx} \left[ \frac{1-F(x)}{f(x)} \right] = 0.$$  

Then

$$\lim_{n \to \infty} P\{b_n(Y_n - a_n) \leq x\} = \exp(-\exp(-x)),$$

holds uniformly for $x \in (-\infty, \infty)$. The constants $a_n$ and $b_n$ satisfy

$$F(a_n) = \frac{n-1}{n}, \quad b_n = n f(a_n). \quad \text{(a.1)}$$

The first two moments of $Y_n$ can be approximated by

$$E(Y_n) \approx a_n + \frac{.577722}{b_n}, \quad \text{Var}(Y_n) \approx \frac{\pi^2}{6 b_n^2}.$$  

Application of the above theorem to specific distributions:

To apply the theorem to special cases requires solution of the two equations in (a.1). Typically, $b_n$ is easy to obtain but the constant $a_n$, given by

$$a_n = F^{-1}\left(\frac{n-1}{n}\right),$$

is difficult whenever the inverse of $F$ is not available in a closed form.

**Case 1:** If $X_i$'s are normally distributed, $N(\mu, \sigma^2)$, then it can be seen that

$$a_n = \mu + \sigma \left[ \sqrt{2\log n} - \frac{1}{2} \frac{(\log \log n + \log 4\pi)}{\sqrt{2\log n}} \right]$$

and

$$b_n = \frac{\sqrt{2\log n}}{\sigma}.$$
Case 2: If each $X_i$ is distributed as $\mathcal{F} = \Gamma(\lambda, m)$, then we solve the equation $\mathcal{F}(a_n) = n^{-1}(n - 1)$ by making use of the relation between $\mathcal{F}$ and the Poisson distribution function. We then obtain $a_n$ such that it satisfies

$$\sum_{j=0}^{m-1} \exp\left(-\frac{a_n}{\lambda}\right) \left(\frac{a_n}{\lambda}\right)^j \frac{1}{j!} = \frac{1}{n},$$

and use this value of $a_n$ to get

$$b_n = \frac{n}{\Gamma(m)\lambda^m} \exp\left(-\frac{a_n}{\lambda}\right) a_n^{m-1}.$$