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Nonlinear Hydrodynamics of *Disentangled* Flux-Line Liquids

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In this paper we use non-Gaussian hydrodynamics to study the magnetic response of a flux-line liquid in the mixed state of a type-II superconductor. Both the derivation of our model, which goes beyond conventional Gaussian flux liquid hydrodynamics, and its relationship to other approaches used in the literature are discussed. We focus on the response to a transverse tilting field which is controlled by the tilt modulus, c_{44}, of the flux array. We show that interaction effects can enhance c_{44} even in infinitely thick clean materials. This enhancement can be interpreted as the appearance of a disentangled flux-liquid fraction. In contrast to earlier work, our theory incorporates the nonlocality of the intervortex interaction in the field direction. This nonlocality is crucial for obtaining a nonvanishing renormalization of the tilt modulus in the thermodynamic limit of thick samples.

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I. INTRODUCTION

The static and dynamical properties of magnetic flux lattices in type-II superconductors have been the focus of much theoretical and experimental work over the last ten years[1,2]. Interest in this field was revived by the discovery of the high-T_c materials, where thermal fluctuations melt the Abrikosov flux lattice at temperatures and fields well below the mean field transition at $H_{c2}(T)$[3,4]. The flux lattice melting is a first order transition in clean samples[5], with an associated jump in the bulk magnetization, and it has been observed experimentally[6–13]. In conventional low-temperature type-II superconductors, the region of the phase diagram where thermal fluctuations are important is extremely small and mean field theory provides a good description of the physics of the flux-line array. In the high-T_c materials, in contrast, the melted flux liquid replaces the Abrikosov lattice over a large region of the phase diagram. Understanding the properties of the flux liquid is therefore crucial for controlling the magnetic response of these materials.

The conventional Abrikosov flux lattice is characterized by two broken symmetries. First, the translational symmetry is broken by the ordering of the magnetic flux lines in a triangular lattice in the plane perpendicular to the external field. Secondly, the gauge symmetry along the field is broken by the alignment of the vortices with the external field. A natural question then arises of whether these two symmetries are recovered simultaneously upon melting, or rather they are recovered in succession at two different temperatures. The latter scenario would allow for the appearance of a disentangled flux liquid phase where translational symmetry is recovered, but the longitudinal gauge symmetry is still broken. At a second transition temperature the disentangled flux liquid would then be replaced by an entangled flux liquid where the longitudinal gauge symmetry is also recovered. Alternatively, if both symmetries are recovered simultaneously, the Abrikosov lattice would melt directly into an entangled flux liquid. The precise nature of such an entangled liquid remains an open question[14]. The existence of a disentangled liquid phase, exhibiting longitudinal superconductivity – the ability to support currents flowing without dissipation in the direction parallel to the flux lines – in clean samples has been proposed some time ago by Feigel'man and collaborators[15]. Early simulations provided support for Feigel'man's ideas[16–18], but more recent numerical work indicates that the two transitions observed in earlier work may have been the consequence of finite size effects[19,20]. Recent numerical results support the scenario that the Abrikosov lattice melts directly into an entangled liquid and no disentangled liquid phase exists in infinitely thick samples[19–21]. Open questions, however, remain concerning the role of various approximations used in the different numerical models, particularly the range of the intervortex interaction.

A closely related property of the vortex array that provides a direct measure of longitudinal vortex correlations is the tilt modulus, c_{44}. It can be probed by measuring the response of the flux array to a small additional magnetic field $\delta\mathbf{H}_\perp$, applied perpendicular to the external field $\hat{\mathbf{z}}H_0$ responsible for the onset of the vortex state. Such a transverse field tilts the lines away from the direction of alignment with H_0. Correlated disorder induced, for instance, by aligned damage tracks in the material can drive $1/c_{44}$ to zero, yielding a transverse Meissner effect, which has been proposed as the signature of the Bose glass phase[22,23]. The role of correlated disorder in enhancing c_{44} in the liquid phase has also been observed experimentally in materials with a single family of twin planes by using the dc flux transformer configuration[24]. These materials contain practically no small-scale disorder, so that the macroscopic flux liquid regions in the channels between twin planes are very clean. The experiments suggest that the enhancement of c_{44}, interpreted as the onset of disentangled liquid phase, be a finite-size effect, that decreases with increasing sample thickness[25]. In thick samples the experiments indicate that the vortex-lattice melting and the loss of longitudinal superconductivity coincide in clean materials. Even though a true Meissner effect with vanishing $1/c_{44}$ is not expected in infinitely thick, clean samples, it is clear that interactions can enhance the tilt modulus of clean flux liquids and suppress the transverse response of the superconductor.

In this paper we employ hydrodynamics to evaluate the renormalization of the tilt modulus of a clean flux liquid due to interactions. Our starting point is a long-wavelength hydrodynamic free energy that includes *non-Gaussian* couplings in the hydrodynamic fields. It therefore goes beyond the Gaussian flux-line liquid hydrodynamic free energy discussed before in the literature[26,27]. We show that such a non-Gaussian hydrodynamic free energy can either be written down phenomenologically or it can be derived by using the mapping of the classical statistical mechanics of vortex lines with *nonlocal* interactions onto the quantum statistical mechanics of two-dimensional *charged* bosons, introduced some time ago by Feigel'man and collaborators[15]. Our central result is the expression for the renormalized *wave vector-dependent* tilt modulus given in Eq. 1.7 below. This is a perturbative result that extends earlier results by other authors[28,29] in two important ways. First, it incorporates both the finite range and the nonlocality of the intervortex interaction in the field direction. This nonlocality plays a crucial role in controlling the tilt response. It is only when the nonlocality is properly accounted for that a finite renormalization of c_{44} is obtained in a clean flux-line liquids of infinite thickness. In addition, our formalism allows us to evaluate the full wave vector dependence of the renormalized tilt modulus - a result that was not discussed before in the literature.

Before discussing our result in more detail, it is useful to make contact with already existing work. The tilt modulus of the Abrikosov lattice is easily calculated from the Ginzburg-Landau free energy for a superconductor in a field. It is dispersive both in the longitudinal and in the in-plane directions due to the non-local character of the intervortex interaction and it has a rather complicated expression, particularly for layered material. It naturally separates in the sum of two contributions,

$$c_{44}(q_\perp, q_z) = c_{44}^v(q_z) + c_{44}^c(q_\perp, q_z), \tag{1.1}$$

with $q_\perp$ and q_z wave vectors perpendicular and parallel to the external field, respectively. The first term on the right hand side of Eq. (1.1) is the single vortex contribution, arising from the self-energy part of the tilt energy. Neglecting its weak logarithmic dependence on q_z, it is given by[31–34]

$$c_{44}^v \approx n_0 \tilde{\epsilon}_1, \tag{1.2}$$

where $n_0 = B_{0z}/\phi_0$ is the average areal density of vortices, with B_{0z} the mean induction along the external field direction and $\phi_0 = hc/2e$ the flux quantum, and $\tilde{\epsilon}_1$ is the single-vortex tilt energy defined below. The second term in Eq. (1.1) is the compressional contribution from intervortex interactions. It is strongly dispersive and in layered materials it is given by[32–34]

$$c_{44}^c(q_\perp, q_z) = \frac{B_{0z}^2}{4\pi} \frac{1}{1 + q_z^2 \tilde{\lambda}_\perp^2 + q_\perp^2 p^2 \tilde{\lambda}_\perp^2}, \tag{1.3}$$

where $\tilde{\lambda}_\perp = \lambda_\perp/(1 - H/H_{c2})^{1/2}$ is the effective penetration length in the ab plane (the field is applied along the $\hat{c}$ axis) and p is the anisotropy ratio. It is important to stress that the long wavelength tilt modulus,

$$c_{44} = c_{44}(q_\perp = 0, q_z = 0) = \frac{B_{0z}^2}{4\pi}\Big(1 + \frac{1}{4\pi\tilde{\lambda}_\perp^2 p^2 n_0}\Big) \tag{1.4}$$

is generally dominated by the large compressional contribution ($B_{0z}^2/4\pi$). The second term inside the brackets in Eq. (1.4), arising from the single-vortex contribution, is important only at very low vortex densities.

The tilt modulus of a *flux-line liquid* cannot be evaluated directly. It is, however, expected that the bare flux-liquid tilt modulus, denoted here by $c_{44}^0(q_\perp, q_z)$, does not differ considerably from that of the lattice given in Eq. (1.1)[35]. In fact, a direct coarse-graining of the microscopic intervortex interaction yields a Gaussian long-wavelength free energy of an entangled flux-line liquid with a tilt modulus given precisely by Eq. (1.1) above[36]. Interactions responsible for nonlinearities in the long-wavelength free energy will, however, renormalize c_{44}^0.

The renormalization of c_{44} in flux-line liquids has been studied before by employing the analogy between the directed vortex lines induced in a three dimensional superconductor by the external field $\hat{\mathbf{z}}H_0$ and the imaginary-time world lines of two dimensional bosons[37,3,4]. The most severe approximation made in the form of this boson mapping introduced by Nelson[3,4], is that the pairwise interaction between flux lines is approximated as local in the field direction (z), i.e., only the interaction between vortex segments at equal height z is considered. This corresponds to an instantaneous pairwise interaction between the bosons. One of the consequences of this approximation is that it completely neglects the compressional part of the tilt modulus. Hence in this model c_{44} is given by the single vortex part, which is inversely proportional to the boson superfluid density, n_s,

$$c_{44}^v = \frac{B_{0z}^2}{4\pi} \frac{1}{4\pi\lambda_\perp^2 p^2 n_s}. \tag{1.5}$$

The superfluid phase of bosons ($n_s = n_0$) corresponds to an entangled liquid of magnetic flux lines with c_{44}^v given by Eq. (1.2). A finite normal-fluid fraction of bosons of density $n_n = n_0 - n_s$ corresponds to a disentangled fraction of flux liquid and enhances the tilt modulus. A normal-fluid phase of bosons with $n_s = 0$ corresponds to a disentangled flux liquid with infinite tilt modulus and transverse Meissner effect. Täuber and Nelson (TN) recently employed this boson mapping to evaluate the renormalization of c_{44}^v due to sample thickness, different boundary conditions and various type of disorder[28]. They found that for finite sample thickness (corresponding to a nonzero boson temperature) there is a nonvanishing normal-fluid component which suppresses c_{44}^v. On the other hand, the normal-fluid density always vanishes for infinitely thick samples (or vanishing boson temperature), so that the flux liquid is always entangled in this limit.

Feigel'man and coworkers[15] incorporated the nonlocality of the intervortex interaction in the field direction in the boson formalism. They showed that the statistical mechanics of vortex lines with *nonlocal* interactions maps onto that of two-dimensional *charged* bosons. This nonlocal mapping incorporates the compressional part of the vortex tilt modulus. Larkin and Vinokur[29] and later Geshkenbein[30] used this nonlocal boson mapping to generalize the expression (1.5) obtained by TN. These authors proposed that the long-wavelength renormalized tilt modulus can be written in terms of the superfluid density n_s of two-dimensional bosons interacting with a gauge field as

$$c_{44}^{LV} = \frac{B_{0z}^2}{4\pi}\Big(1 + \frac{1}{4\pi\tilde{\lambda}_\perp^2 p^2 n_s}\Big). \tag{1.6}$$

The superfluid density was evaluated perturbatively by Feigel'man and coworkers[15] for the case where the repulsive interaction among the bosons is infinitely long-ranged, corresponding to a vortex liquid with $\lambda_\perp \to \infty$. These authors argued that in this limit a distinct disentangled flux liquid phase with diverging c_{44} exists in infinitely thick superconducting samples.

The calculation of the interaction-renormalization of the flux liquid tilt modulus via hydrodynamics described here has the advantage that it naturally incorporates the nonlocality of the intervortex interaction and it allows us to easily treat the case of finite λ. The non-Gaussian hydrodynamics used as the starting point contains bare elastic constants that are determined by the intervortex interaction. In particular, the bare tilt modulus is given by Eq. (1.1). The corrections to c_{44} due to the nonlinearities are evaluated perturbatively. Our main result is an expression for the wave vector-dependent renormalized tilt modulus, given by

$$\frac{1}{c_{44}^R(q_\perp, q_z)} = \frac{1}{c_{44}^0(q_\perp, q_z)}\left[1 - \frac{n_0\tilde{\epsilon}_1}{c_{44}^0(q_\perp, q_z)}\frac{n_n(q_\perp, q_z)}{n_0}\right], \tag{1.7}$$

where $n_n(q_\perp, q_z)$ has the rather complicated integral expression given in Eq. (6.7) below. The corrections to the tilt modulus incorporated in n_n can be interpreted in terms of a disentangled fraction of the flux liquid - hence a "normal-fluid component". When the nonlocality of the intervortex interaction in the field direction is neglected, Eq. (1.7) becomes identical to the result obtained by Täuber and Nelson (see Eq. (3.33) of Ref. 28). In this case the long-wavelength c_{44} is not renormalized in infinitely thick samples.

Our result, Eq. (1.7), is also simply related to the Larkin-Vinokur formula given in Eq. (1.6). This is immediately seen by introducing a normal fluid fraction in Eq. (1.6) as $n_n = n_0 - n_s$, and then expanding for small values of the normal fluid fraction, $n_n/n_0 << 1$, to obtain

$$\frac{1}{c_{44}^{LV}} \approx \frac{1}{c_{44}^0}\left[1 - \frac{n_0\tilde{\epsilon}_1}{c_{44}^0}\frac{n_n}{n_0}\right], \tag{1.8}$$

with c_{44}^0 given by Eq. (1.4). This expression is formally identical to the long-wavelength ($q_\perp = 0, q_z = 0$) limit of our result.

We find that interaction effects in a clean flux liquid do lead to a nonvanishing renormalization of the tilt modulus in the thermodynamic limit of thick samples. This correction is present only if the nonlocality of the intervortex interaction is properly incorporated. The correction remains, however, small at all but very low ($B < 1$Tesla) fields. Our results are perturbative and cannot be used to infer quantitative conclusions about the existence of a true disentangled flux liquid phase. One of the main outcomes of our work is the development of a transparent hydrodynamic framework that can be used to study the role of the nonlocality of the intervortex interaction on the tilt response, both in clean materials and in the presence of disorder of various geometries. Note that in conventional, Gaussian hydrodynamics the effect of disorder on c_{44} cannot be detected.

In section II we discuss the general form of the London free energy used as the starting point to study the magnetic properties of superconductors in the mixed state. The various response functions of interest are also defined there. After discussing the response to a tilt field in section III, we review and contrast in sections IV and V, respectively, the results obtained by conventional Gaussian hydrodynamics and by the local boson mapping. After showing how hydrodynamics can be derived from the boson model in section VI, we introduce our non-Gaussian hydrodynamic model and discuss its relationship to previous work. Our results are discussed in section VII. Finally, a rigorous derivation of the nonlocal, non-Gaussian hydrodynamics from the charged boson analogy is displayed in Appendix A, and the perturbative evaluation of the renormalization of c_{44} from interactions is displayed in Appendix B.

II. MAGNETIC RESPONSE OF THE VORTEX ARRAY

High-T_c superconductors are uniaxial, strongly type-II materials with very large values of the Ginzburg-Landau parameter $\kappa = \lambda/\xi$. For applied fields $H_{c1} << H <<$

H_{c2}, their mixed state can be described in the London limit with a Ginzburg-Landau Hamiltonian given by

$$\mathcal{H}[\theta, \mathbf{A}] = \frac{1}{2}\int_{\mathbf{r}}\Big\{\frac{c^2}{4\pi\tilde{\lambda}_\mu^2}\Big(\frac{\phi_0}{2\pi}\partial_\mu\theta - A_\mu\Big)^2 + \\ +\frac{1}{4\pi}(\boldsymbol{\nabla}\times\mathbf{A})^2\Big\}. \qquad (2.1)$$

Here the z direction has been chosen along the anisotropy (c) axis of the superconductor. Greek indices $\mu, \nu, ...$ run over all Cartesian components ($\mu = x, y, z$) and summation is intended in Eq. (2.1). Latin indices $i, j, k, ...$ run only over x and y. The integral $\int_{\mathbf{r}} ... \equiv \int_0^L dz \int d\mathbf{r}_\perp ...$ is over the volume $\Omega = LA$ of the superconductor, with L the thickness in the direction of the c axis and A the area in the ab plane. Also, $\tilde{\lambda}_\mu = \lambda_\mu/(1 - H/H_{c2})^{1/2}$, where $\lambda_x = \lambda_y = \lambda_\perp$ are the penetration depths from supercurrents in the ab plane, while $\lambda_z = p\lambda_\perp$ is the penetration depth from supercurrents along the c axis, with p the anisotropy ratio arising from an effective mass tensor for the superconducting electrons ($p = (m_z/m_\perp)^{1/2}$). Finally, $\mathbf{A}$ is the total vector potential, with $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ the internal field in the material, and $\phi_0 = hc/2e$ is the flux quantum. The corresponding Gibbs free energy functional is

$$\mathcal{G}[\theta, \mathbf{H}] = \mathcal{H}[\theta, \mathbf{A}] - \frac{1}{4\pi}\int_{\mathbf{r}}\mathbf{B}\cdot\mathbf{H}, \qquad (2.2)$$

where $\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{A}^{\rm ext}$ is the applied external field.

The London free energy functional can be rewritten in terms of interacting vortex lines by introducing a "vortex line density" vector defined as

$$\hat{\mathbf{T}}(\mathbf{r}) = \frac{1}{2\pi}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\theta). \qquad (2.3)$$

Here and below a hat ($\hat{}$) is used, when needed, to distinguish microscopic fluctuating quantities from average ones. We will specifically consider situations where the magnetic field responsible for the onset of the vortex state is applied along the z direction. Vortex line configurations are then conveniently characterized by a set of N single-valued functions $\mathbf{r}_n(z)$, which specify the position of the n-th vortex line in the xy plane as it wanders along the z axis. The three-dimensional position of each flux line is parametrized as $\mathbf{R}_n(z) = [\mathbf{r}_n(z), z]$ and the vortex density vector can be written as

$$\hat{\mathbf{T}}(\mathbf{r}) = \sum_{n=1}^{N}\frac{d\mathbf{R}_n(z)}{dz}\delta^{(2)}(\mathbf{r}_\perp - \mathbf{r}_n(z)), \qquad (2.4)$$

where $\mathbf{r} = (\mathbf{r}_\perp, z)$. The vortex density vector can be written as $\hat{\mathbf{T}}(\mathbf{r}) = (\hat{\mathbf{t}}, \hat{n})$, where $\hat{\mathbf{t}}$ is a two-dimensional vector describing the local tilt of flux lines away from the direction of the external field and $\hat{n}$ is the areal density of vortices,

$$\hat{n}(\mathbf{r}) = \sum_{n=1}^{N}\delta^{(2)}(\mathbf{r}_\perp - \mathbf{r}_n(z)), \qquad (2.5)$$

$$\hat{\mathbf{t}}(\mathbf{r}) = \sum_{n=1}^{N}\frac{d\mathbf{r}_n(z)}{dz}\delta^{(2)}(\mathbf{r}_\perp - \mathbf{r}_n(z)) \; . \qquad (2.6)$$

The vortex density vector is also directly related to the superfluid velocity of the electrons in the superconductor, $\mathbf{v}^s = (\phi_0/2\pi)\boldsymbol{\nabla}\theta - \mathbf{A}$, by

$$\phi_0\hat{\mathbf{T}} - \hat{\mathbf{B}} = \boldsymbol{\nabla}\times\mathbf{v}^s. \qquad (2.7)$$

The Cartesian components of the local supercurrent are $j^s_\mu = (c/4\pi\tilde{\lambda}^2_\mu)v^s_\mu$ (no summation over μ intended here). After some manipulations (see, for instance, Ref. 17 for the details) and neglecting spin wave fluctuations, one obtains

$$\mathcal{G}[\hat{\mathbf{T}}, \mathbf{H}] = \frac{1}{8\pi\Omega}\sum_{\mathbf{q}}\Big\{\big[\phi_0\hat{T}_\mu(\mathbf{q}) - \hat{B}_\mu(\mathbf{q})\big]U_{\mu\nu}(\mathbf{q})\big[\phi_0\hat{T}_\nu(-\mathbf{q}) - \hat{B}_\nu(-\mathbf{q})\big] + |\hat{\mathbf{B}}(\mathbf{q})|^2 - 2\mathbf{H}(\mathbf{q})\cdot\hat{\mathbf{B}}(-\mathbf{q})\Big\}, \qquad (2.8)$$

with

$$U_{\mu\nu}(\mathbf{q}) = \frac{1}{\tilde{\lambda}_\perp q^2}\Big[\delta_{\mu\nu} - \delta_{\mu i}\delta_{\nu j}\frac{(\tilde{\lambda}_z^2 - \tilde{\lambda}_\perp^2)q_\perp^2}{\tilde{\lambda}_z^2 q_\perp^2 + \tilde{\lambda}_\perp^2 q_z^2}P^T_{ij}(\mathbf{q}_\perp)\Big]. \qquad (2.9)$$

Here, $\mathbf{q} = (\mathbf{q}_\perp, q_z)$ and $P^T_{ij}(\mathbf{q}_\perp) = \delta_{ij} - \hat{q}_{\perp i}\hat{q}_{\perp j}$ is the two-dimensional transverse projection operator, with $\hat{\mathbf{q}}_\perp = \mathbf{q}_\perp/q_\perp$. The corresponding longitudinal projection operator is $P^L_{ij}(\mathbf{q}_\perp) = \delta_{ij} - P^T_{ij}(\mathbf{q}_\perp)$.

In this paper we will only consider magnetic field fluctuations due to fluctuations in the vortices' degrees of freedom. This London part of the field fluctuations is obtained by minimizing the Ginzburg-Landau free energy (2.8) for fixed vortex configurations $\hat{T}(\mathbf{q})$ and it is given by

$$\hat{\mathbf{B}}(\mathbf{q}) = \hat{\mathbf{B}}^V(\mathbf{q}) + \hat{\mathbf{B}}^M(\mathbf{q}), \qquad (2.10)$$

where $\hat{\mathbf{B}}^V(\mathbf{q})$ is the part of the internal field due to the vortices,

$$\hat{B}^V_\mu(\mathbf{q}) = \big(\mathbf{1} + \mathbf{U}(\mathbf{q})\big)^{-1}_{\mu\sigma}U_{\sigma\nu}(\mathbf{q})\phi_0\hat{T}_\nu(\mathbf{q}) \qquad (2.11)$$
$$= \frac{1}{1 + \tilde{\lambda}_\perp^2 q^2}\Big[\delta_{\mu\nu} - \delta_{\mu i}\delta_{\nu j}\frac{(\tilde{\lambda}_z^2 - \tilde{\lambda}_\perp^2)q_\perp^2}{1 + \tilde{\lambda}_\perp^2 q_z^2 + \tilde{\lambda}_z^2 q_\perp^2}P^T_{ij}(\mathbf{q}_\perp)\Big]\phi_0\hat{T}_\nu(\mathbf{q}),$$

and $\hat{\mathbf{B}}^M(\mathbf{q})$ is the Meissner response of the material to a spatially inhomogeneous external field,

$$\hat{B}^M_\mu(\mathbf{q}) = \big(\mathbf{1} + \mathbf{U}(\mathbf{q})\big)^{-1}_{\mu\nu} H_\nu(\mathbf{q}) \\ = \frac{1}{1+\tilde{\lambda}_\perp^2 q^2}\left[\tilde{\lambda}_\perp^2 q^2 \delta_{\mu\nu} + \delta_{\mu i}\delta_{\nu j}\frac{(\tilde{\lambda}_z^2 - \tilde{\lambda}_\perp^2)q_\perp^2}{1+\tilde{\lambda}_\perp^2 q_z^2 + \tilde{\lambda}_z^2 q_\perp^2} P^T_{ij}(\mathbf{q}_\perp)\right] H_\nu(\mathbf{q}). \tag{2.12}$$

In addition to the contributions given in Eq. (2.10), there are field fluctuations representing thermal deviations from the solution of the London equation, which are neglected here. By inserting Eqs. (2.11) and (2.12) into Eq. (2.8), we obtain the vortex free energy functional expressed entirely in terms of vortex degrees of freedom,

$$\mathcal{G}[\hat{\mathbf{T}}, \mathbf{H}] = \frac{1}{2\Omega}\sum_{\mathbf{q}} \left\{ \hat{T}_\mu(\mathbf{q}) V_{\mu\nu}(\mathbf{q}) \hat{T}_\nu(-\mathbf{q}) - \frac{1}{\phi_0} H_\mu(\mathbf{q}) V_{\mu\nu}(\mathbf{q}) \hat{T}_\nu(-\mathbf{q}) - \frac{1}{4\pi} H_\mu(\mathbf{q}) \big(\mathbf{1} + \mathbf{U}(\mathbf{q})\big)^{-1}_{\mu\nu} H_\nu(-\mathbf{q}) \right\}, \tag{2.13}$$

where

$$V_{\mu\nu}(\mathbf{q}) = V_0 \big(\mathbf{1} + \mathbf{U}(\mathbf{q})\big)^{-1}_{\mu\sigma} U_{\sigma\nu}(\mathbf{q}) \\ = \frac{V_0}{1+\tilde{\lambda}_\perp^2 q^2}\left[\delta_{\mu\nu} - \delta_{\mu i}\delta_{\nu j}\frac{(\tilde{\lambda}_z^2 - \tilde{\lambda}_\perp^2)q_\perp^2}{1+\tilde{\lambda}_\perp^2 q_z^2 + \tilde{\lambda}_z^2 q_\perp^2} P^T_{ij}(\mathbf{q}_\perp)\right], \tag{2.14}$$

are the Fourier components of the anisotropic intervortex interaction, with $V_0 = \phi_0^2/4\pi$. One important property of the intervortex interaction is its nonlocality. In particular, the nonlocality in the z direction, reflecting that flux-line elements at different z heights repel each other via a Yukawa-like potential, will play a very important role in the discussion below.

The Gibbs free energy of the vortex system is given by

$$G(\mathbf{H}, T) = -k_B T \ln \mathcal{Z}(\mathbf{H}, T), \tag{2.15}$$

where

$$\mathcal{Z}(\mathbf{H}, T) = \int' \mathcal{D}\hat{\mathbf{T}}(\mathbf{r}) e^{-\mathcal{G}/k_B T} \tag{2.16}$$

is the canonical partition function. The prime over the integral sign indicates that the integration must be performed with the constraint $\boldsymbol{\nabla}\cdot\hat{\mathbf{B}} = 0$. The average local field in the superconductor is then given by

$$\mathbf{B}(\mathbf{r}) = <\hat{\mathbf{B}}(\mathbf{r})> = -4\pi \frac{\delta G}{\delta \mathbf{H}(\mathbf{r})}, \tag{2.17}$$

where the brackets denote a statistical average with Boltzmann weight $\sim \exp[-\mathcal{G}/k_B T]$.

For a spatially homogeneous external field applied along the z direction, $\mathbf{H}(\mathbf{r}) = \hat{\mathbf{z}} H_0$, we obtain the familiar form[1],

$$\mathcal{G}_0(\hat{\mathbf{T}}, H_0) = -NL\frac{H_0\phi_0}{4\pi} \\ + \frac{1}{2\Omega}\sum_{\mathbf{q}} \hat{T}_\mu(\mathbf{q}) V_{\mu\nu}(\mathbf{q}) \hat{T}_\nu(-\mathbf{q}). \tag{2.18}$$

For a uniform applied field $\mathbf{H} = \hat{\mathbf{z}} H_0$, the Meissner part of the transverse local field given in Eq. (2.12) vanishes. The local field in the superconductor is entirely due to the vortices and it given by Eq. (2.11). From here on we will always refer to the vortex system created by the homogeneous field $\mathbf{H} = \hat{\mathbf{z}} H_0$ and the local field is to be understood as the field given by Eq. (2.11).

The focus of this paper is on the response of the vortex array created by the external field $\hat{\mathbf{z}} H_0$ to a small additional spatially inhomogeneous external field $\delta\mathbf{H}(\mathbf{r})$. The Gibbs free energy functional in the presence of this perturbation can be written as

$$\mathcal{G}(\hat{\mathbf{T}}, \hat{\mathbf{z}} H_0 + \delta\mathbf{H}) = \mathcal{G}_0(\hat{\mathbf{T}}, H_0) + \delta\mathcal{G}(\hat{\mathbf{T}}, \delta\mathbf{H}), \tag{2.19}$$

where $\mathcal{G}_0$ is given by Eq. (2.18) and the perturbation is

$$\delta\mathcal{G}(\hat{\mathbf{T}}, \delta\mathbf{H}) = -\frac{1}{4\pi}\int_{\mathbf{r}} \hat{\mathbf{B}}^V \cdot \delta\mathbf{H} \tag{2.20}$$

$$= -\frac{1}{c}\int_{\mathbf{r}} \hat{\mathbf{j}}^s \cdot \delta\mathbf{A}^{\rm ext}. \tag{2.21}$$

The local field $\hat{\mathbf{B}}^V$ in Eq. (2.20) is the field in the absence of the perturbation $\delta\mathbf{H}$ and is related to the vortex degrees of freedom via Eq. (2.11). It does not include the Meissner response to the perturbation $\delta\mathbf{H}$. The supercurrent is defined as $\hat{\mathbf{j}}^s = (c/4\pi)\boldsymbol{\nabla}\times\hat{\mathbf{B}}^V$.

Below we will use $\langle ...\rangle_0$ to denote a statistical average over the unperturbed ensemble described by $\mathcal{G}_0$, while $\langle ...\rangle_H$ will denote the average over the perturbed ensemble, with free energy given by Eq. (2.19). The mean local field $\mathbf{B}^H$ in the material in the presence of the perturbation $\delta\mathbf{H}$ can be written as the sum of vortex and Meissner parts as

$$\mathbf{B}^H(\mathbf{q}) = \langle \hat{\mathbf{B}}^V(\mathbf{q})\rangle_H + \delta\mathbf{B}^M(\mathbf{q}), \tag{2.22}$$

where $\delta\mathbf{B}^M(\mathbf{q})$ is the Meissner response to the perturbation, given by Eq. (2.12) with $\mathbf{H}(\mathbf{q}) = \delta\mathbf{H}(\mathbf{q})$. To linear order in the perturbing field, the vortex contribution can be expressed in terms of correlation functions in the unperturbed ensemble as,

$$\langle \hat{B}^V_\mu(\mathbf{q})\rangle_H = \langle \hat{B}^V_\mu(\mathbf{q})\rangle_0 + \frac{\beta}{4\pi}\langle \hat{B}^V_\mu(\mathbf{q})\hat{B}^V_\nu(-\mathbf{q})\rangle^c_0 \delta H_\nu(\mathbf{q}), \tag{2.23}$$

where $\langle ...\rangle^c$ is the connected part of the correlator, i.e., $\langle AB\rangle^c = \langle AB\rangle - \langle A\rangle\langle B\rangle$. Finally, the corresponding linear response function defines the magnetic susceptibility $\chi_{ij}(\mathbf{q})$ of the material according to

$$B^H_\mu(\mathbf{q}) - \langle \hat{B}^V_\mu(\mathbf{q})\rangle_0 = \left[4\pi\chi_{\mu\nu}(\mathbf{q}) + \delta_{\mu\nu}\right]\delta H_\nu(\mathbf{q}). \tag{2.24}$$

The components of the susceptibility tensor can also be expressed in terms of vortex density correlations,

$$4\pi\chi_{\mu\nu}(\mathbf{q}) = -\frac{V_{\mu\nu}}{V_0} + \frac{\phi_0^2}{k_B T V_0^2} V_{\mu\sigma}(\mathbf{q})V_{\nu\lambda}(-\mathbf{q})T_{\sigma\lambda}(\mathbf{q}), \tag{2.25}$$

where $T_{\mu\nu}(\mathbf{q})$ is the correlation function of the vortex density vector,

$$T_{\mu\nu}(\mathbf{q}) = \langle \hat{T}_\mu(\mathbf{q})\hat{T}_\nu(-\mathbf{q})\rangle^c_0. \tag{2.26}$$

The density-vector correlation function can be expressed in terms of derivatives of the partition function of the perturbed system as

$$\langle \hat{T}_\mu(\mathbf{q})\hat{T}_\nu(\mathbf{q}')\rangle^c_0 = (\phi_0 k_B T)^2 (V^{-1})_{\mu\kappa}(V^{-1})_{\nu\lambda} \times \Big[\frac{\delta^2 \ln \mathcal{Z}(H_0\hat{\mathbf{z}} + \delta\mathbf{H}, T)}{\delta H_\kappa(\mathbf{q})\delta H_\lambda(\mathbf{q}')}\Big]_{\delta\mathbf{H}=0}, \tag{2.27}$$

where $(V^{-1})_{\mu\nu}$ are the components of the inverse of the interaction tensor (2.14).

The tensor $T_{\mu\nu}$ is block diagonal, with $T_{\mu\nu} = (T_{ij}, T_{zz})$. The component T_{zz} is the density-density correlation function or structure function of the vortices,

$$T_{zz}(\mathbf{q}) = S(\mathbf{q}) = \langle \delta\hat{n}(\mathbf{q})\delta\hat{n}(-\mathbf{q})\rangle_0, \tag{2.28}$$

where $\delta\hat{n}(\mathbf{q}) = \hat{n}(\mathbf{q}) - n_0\Omega\delta_{\mathbf{q},\mathbf{0}}$ describes the fluctuation of the local density field from its mean value $n_0 = B_{0z}/\phi_0$, with $B_{0z} \approx H_0$ the equilibrium value of the z component of the internal field. The in-plane part T_{ij} is the tilt-tilt autocorrelator and it is the central quantity of interest here. It can be written in terms of transverse and longitudinal components as

$$T_{ij}(\mathbf{q}) = T_L(\mathbf{q})P^L_{ij}(\mathbf{q}_\perp) + T_T(\mathbf{q})P^T_{ij}(\mathbf{q}_\perp). \tag{2.29}$$

The transverse part of the tilt autocorrelator determines the tilt modulus of the vortex array. The wave-vector-dependent tilt modulus is defined by

$$T_T(\mathbf{q}) = \frac{n_0^2 k_B T}{c_{44}(q_\perp, q_z)}. \tag{2.30}$$

Finally, in order to make contact with the literature, it is useful to write the perturbing field in terms of a vector potential, $\delta\mathbf{H} = \boldsymbol{\nabla} \times \delta\mathbf{A}^{\rm ext}$. The linear response to the vector potential $\delta\mathbf{A}^{\rm ext}$ is then characterized by the helicity tensor $\Upsilon_{\mu\nu}$, which relates the induced current to $\delta\mathbf{A}^{\rm ext}$,

$$j^H_\mu(\mathbf{q}) = -c\Upsilon_{\mu\nu}(\mathbf{q})\delta A^{\rm ext}_\nu(\mathbf{q}), \tag{2.31}$$

where $\mathbf{j}^H$ is the total screening current induced in the material by the perturbing vector potential, comprising of both the vortex and Meissner contributions. The helicity tensor can be immediately related to the components of the susceptibility tensor,

$$\Upsilon_{\mu\nu}(\mathbf{q}_\perp) = -\epsilon_{\mu\sigma\xi}\epsilon_{\nu\alpha\beta}q_\sigma q_\alpha \chi_{\xi\beta}(\mathbf{q}). \tag{2.32}$$

Using Eq. (2.25), it can also be expressed in term of the correlations of the vortex density tensor.

III. TILTING FIELD

In the remainder of this paper we focus on the response of the vortex array to a spatially inhomogeneous field $\delta\mathbf{H}_\perp(\mathbf{q})$ applied normal to the direction of H_0 and that tilts the flux lines away from the z direction. As discussed by Chen and Teitel[17], we distinguish two types of perturbations. The first is a tilt perturbation, corresponding to a tilting field which is spatially homogeneous in the xy plane and may be modulated in the z direction. The long wavelength response to this tilt perturbation is determined by the long wavelength tilt modulus, c_{44}, defined as

$$\frac{n_0^2 k_B T}{c_{44}} = \lim_{q_z\to 0}\lim_{q_\perp\to 0} T_T(q_\perp, q_z). \tag{3.1}$$

The order of the limits ($q_\perp \to 0$ first, followed by $q_z \to 0$) is important here and reflects the physical situation of the relevant experiment. The vanishing of the long wavelength tilt modulus signals the onset of a transverse Meissner effect, where the perturbing field is completely expelled from the material (as seen from Eq. (2.13), the corresponding static susceptibility equals $-1/4\pi$). This occurs, for instance, in vortex arrays pinned by columnar defects.

The second physical experiment of interest here is the response to a tilting field $\delta\mathbf{H}_\perp(\mathbf{q}_\perp)$ which is spatially homogeneous in the z direction (i.e., independent of q_z) and generates a shear perturbation of the vortex array. Such a field can be obtained from a vector potential $\delta\mathbf{A}^{\rm ext} = \hat{\mathbf{z}}\delta A^{\rm ext}_z(\mathbf{r}_\perp)$, which induces screening currents along the z direction. In the literature the response of the superconductor to such a shear perturbation is often characterized by the corresponding component of the helicity modulus ($\Upsilon_{zz}(\mathbf{q}_\perp)$) defined in Eq. (2.31), which in turn is related to the transverse part of the tilt-tilt correlator by

$$\Upsilon_{zz}(q_\perp) = \frac{1}{4\pi}\frac{q_\perp^2}{1+q_\perp^2\tilde{\lambda}_z^2}\left[1 - \frac{V_0}{k_BT}\frac{T_T(q_\perp, q_z=0)}{1+q_\perp^2\tilde{\lambda}_z^2}\right], \tag{3.2}$$

where the first term arises from the Meissner part of the response. The long wavelength limit of the helicity modulus is

$$\lim_{q_\perp\to 0}\Upsilon_{zz}(q_\perp) = \frac{q_\perp^2}{4\pi}\left[1 - \frac{V_0}{k_BT}\lim_{q_\perp\to 0}T_T(q_\perp, q_z=0)\right]. \tag{3.3}$$

The vanishing of $\lim_{q_\perp\to 0}T_T(q_\perp, q_z = 0)$ yields $\lim_{q_\perp\to 0}4\pi\Upsilon_{zz}(q_\perp)/q_\perp^2 = 1$, which corresponds to a perfect Meissner response in the z direction and signals longitudinal superconductivity.

We emphasize, however, that both the perturbations just described simply probe the magnetic response of the superconductor, which is the true equilibrium test of superconductivity. In fact the relevant response function in each case (tilt or helicity modulus) is simply the transverse part of the susceptibility tensor,

$$\chi_T(\mathbf{q}) = P_{ij}^T(\mathbf{q}_\perp)\chi_{ij}(\mathbf{q}). \tag{3.4}$$

The long wavelength tilt modulus is given by

$$\frac{n_0^2V_0}{c_{44}} = 1 + 4\pi\lim_{q_z\to 0}\chi_T(q_\perp = 0, q_z), \tag{3.5}$$

and the component of the helicity modulus that controls longitudinal superconductivity is

$$\lim_{q_\perp\to 0}\Upsilon_{zz}(q_\perp) = -\lim_{q_\perp\to 0}{q_\perp}^2\chi_T(q_\perp, q_z = 0). \tag{3.6}$$

In a flux-line *lattice* the transverse part of the tilt-tilt correlator is non-analytic at small wave-vectors and the different order of limits of the two perturbations discussed above is important. This is because the vortex array has a nonzero long wavelength shear modulus, c_{66}. As a result, the flux lattice exhibits longitudinal superconductivity, with $\lim_{q_\perp\to 0}T_T(q_\perp, q_z = 0) = 0$, and

$$\lim_{q_\perp\to 0}\chi_T^{\rm lattice}(q_\perp, q_z = 0) = -\frac{1}{4\pi}, \tag{3.7}$$

but no transverse Meissner effect, as $\lim_{q_z\to 0}T_T(\mathbf{q}_\perp = 0, q_z) \neq 0$ and

$$\lim_{q_z\to 0}\chi_T^{\rm lattice}(q_\perp = 0, q_z) = -\frac{1}{4\pi} + \frac{V_0n_0^2}{c_{44}}. \tag{3.8}$$

In a flux-line *liquid*, in contrast, we find that the order of limits is not important and the flux array in general exhibits neither longitudinal superconductivity, nor perfect Meissner effect, as

$$\lim_{q_z\to 0}\chi_T^{\rm liquid}(q_\perp = 0, q_z) = \lim_{q_\perp\to 0}\chi_T^{\rm liquid}(q_\perp, q_z = 0) = -\frac{1}{4\pi} + \frac{V_0n_0^2}{c_{44}^R}, \tag{3.9}$$

where c_{44}^R is the flux liquid tilt modulus, renormalized by interaction effects. We will see below, however, that interactions can yield a strong upward renormalization of c_{44} even in clean flux liquids.

IV. GAUSSIAN HYDRODYNAMICS

A useful framework for discussing the long wavelength properties of flux-line liquids that naturally incorporates all nonlocalities of the intervortex interaction is hydrodynamics, where vortex fluctuations are described in terms of a few coarse-grained fields. By long wavelengths, we mean wavelengths large compared to the spacing between CuO_2 planes in the $\hat{\mathbf{z}}$ direction, and large compared to the intervortex spacing in the ab plane normal to $\hat{\mathbf{z}}$.

The coarse-grained hydrodynamic fields for a flux-line liquid are the fluctuating areal density,

$$\hat{n}^H(\mathbf{r}) = \sum_{n=1}^{N}\delta_{BZ}^{(2)}(\mathbf{r}_\perp - \mathbf{r}_n(z)), \tag{4.1}$$

and a tilt field,

$$\hat{\mathbf{t}}^H(\mathbf{r}) = \sum_{n=1}^{N}\frac{d\mathbf{r}_n}{dz}\delta_{BZ}^{(2)}(\mathbf{r}_\perp - \mathbf{r}_n(z)) . \tag{4.2}$$

Here $\delta_{BZ}^{(2)}(\mathbf{r}_\perp)$ is a smeared-out two-dimensional δ-function with a finite spatial extent of the order of the inverse of the Brillouin zone boundary $k_{BZ} = \sqrt{4\pi n_0}$. It is defined as

$$\delta_{BZ}^{(2)}(\mathbf{r}_\perp) = \frac{1}{A}\sum_{q_\perp\leq k_{BZ}} e^{-i\mathbf{q}_\perp\cdot\mathbf{r}_\perp}. \tag{4.3}$$

We stress that these hydrodynamic fields differ from the microscopic fields defined in Eq. (2.5) and (2.6) as they are coarse-grained quantities obtained by averaging out the more microscopic and rapidly varying degrees of freedom.

A *Gaussian* hydrodynamic free energy containing terms quadratic in the deviations of the fields from their equilibrium values can be obtained by coarse-graining the microscopic energy of interacting vortices given in Eq. (2.19), with the result[36],

$$F_G = \frac{1}{2n_0^2}\int_{\mathbf{r}}\int_{\mathbf{r}'}\Big[B(\mathbf{r}-\mathbf{r}')\delta\hat{n}^H(\mathbf{r})\delta\hat{n}^H(\mathbf{r}') + K(\mathbf{r}-\mathbf{r}')\hat{\mathbf{t}}^H(\mathbf{r})\cdot\hat{\mathbf{t}}^H(\mathbf{r}')\Big], \tag{4.4}$$

where $\delta\hat{n}^H(\mathbf{r}) = \hat{n}^H(\mathbf{r}) - n_0$ and $B(\mathbf{r})$ and $K(\mathbf{r})$ are nonlocal liquid elastic constants. The density and tilt fields are not independent quantities, but are related by a "continuity" equation expressing the constraint that vortex lines cannot start or stop inside the sample,

$$\partial_z \delta\hat{n}^H + \boldsymbol{\nabla}_\perp \cdot \hat{\mathbf{t}}^H = 0. \tag{4.5}$$

The Gaussian hydrodynamic free energy is rewritten in a more familiar form by passing to Fourier space,

$$F_G = \frac{1}{2n_0^2\Omega}\sum_{\mathbf{q}}\Big[c_{11}^0(\mathbf{q})|\delta\hat{n}^H(\mathbf{q})|^2 + c_{44}^0(\mathbf{q})|\hat{\mathbf{t}}^H(\mathbf{q})|^2\Big], \tag{4.6}$$

where $c_{11}^0(\mathbf{q})$ and $c_{44}^0(\mathbf{q})$ are the bare compressional and tilt moduli of the flux liquid. The compressional modulus is given by

$$c_{11}^0(\mathbf{q}) = \frac{B_{0z}^2}{4\pi}\frac{1+q^2\tilde{\lambda}_\perp^2 p^2}{(1+q^2\tilde{\lambda}_\perp^2)(1+q_z^2\tilde{\lambda}_\perp^2+q_\perp^2p^2\tilde{\lambda}_\perp^2)} \ . \tag{4.7}$$

The bare tilt modulus is found to be to a good approximation identical to the flux lattice tilt modulus given in Eqs. (1.1-1.3)[35,36].

In this Gaussian approximation, the probability of a fluctuation is proportional to $\exp(-F_G/k_BT)$ and averages must be carried out subject to the continuity constraint, Eq. (4.5). The correlation functions of the hydrodynamic fields are then immediately calculated and are given by

$$\langle\delta\hat{n}^H(-\mathbf{q})\delta\hat{n}^H(\mathbf{q})\rangle_G = \frac{n_0^2k_BTq_\perp^2}{c_{44}^0(\mathbf{q})q_z^2+c_{11}^0(\mathbf{q})q_\perp^2} \ , \tag{4.8}$$

$$\langle\hat{t}_i^H(-\mathbf{q})\delta\hat{n}^H(\mathbf{q})\rangle_G = \frac{n_0^2k_BTq_{\perp i}q_z}{c_{44}^0(\mathbf{q})q_z^2+c_{11}^0(\mathbf{q})q_\perp^2} \ , \tag{4.9}$$

$$\langle\hat{t}_i^H(-\mathbf{q})\hat{t}_j^H(\mathbf{q})\rangle_G = T_T^0(\mathbf{q})P_{ij}^T(\mathbf{q}_\perp) + T_L^0(\mathbf{q})P_{ij}^L(\mathbf{q}_\perp) \ , \tag{4.10}$$

with

$$T_T^0(\mathbf{q}) = \frac{n_0^2k_BT}{c_{44}^0(\mathbf{q})} \tag{4.11}$$

and

$$T_L^0(\mathbf{q}) = \frac{n_0^2k_BTq_z^2}{c_{44}^0(\mathbf{q})q_z^2+c_{11}^0(\mathbf{q})q_\perp^2}. \tag{4.12}$$

The long wavelength tilt modulus is determined by the transverse part of the tilt autocorrelator, according to Eq. (4.11). To this Gaussian order it is then identically given by its bare value, c_{44}^0, given in Eq. (1.4). Gaussian hydrodynamics does not allow for any renormalization of the tilt modulus, even in the presence of disorder. This is because a disorder potential couples to the flux-line areal density that, within a Gaussian theory, is in turn decoupled from the transverse part of the tilt field. In particular, this naive hydrodynamic theory does not describe the possibility of a disentangled flux-line liquid, with a tilt modulus enhanced by interaction or disorder. In other words, Gaussian hydrodynamics is by definition a theory of *entangled* flux-line liquids.

V. 2D BOSON MODEL

Considerable progress in understanding the properties of vortex-line arrays has been made by employing the formal analogy between the classical statistical mechanics of directed lines in three dimensions and the quantum statistical mechanics of two-dimensional bosons. The advantage of this approach is that it can incorporate interaction effects accounting for localization or disentanglement of the vortices. The drawback is that this model, at least in its simplest implementation employed by Nelson and coworkers[3,4,38,28], neglects the nonlocality of the intervortex interaction. We will show below that the nonlocality of the interaction in the field (z) direction plays a crucial role in controlling the tilt modulus.

In this section we briefly review the local version of the boson mapping employed by Nelson and coworkers[3,4,38] and the results obtained recently for the tilt modulus by Täuber and Nelson[28] using this model.

Neglecting the nonlocality of the intervortex interaction, the free energy of interacting vortex lines in a field $\mathbf{H} = H_0\hat{\mathbf{z}} + \delta\mathbf{H}_\perp$ given in Eq. (2.19) is approximated as

$$\mathcal{G}(\{\mathbf{r}_n\},\mathbf{H}) = NL\big(H_0\frac{\phi_0}{4\pi}-\epsilon_1\big) + \int_z\Big\{\sum_{n=1}^N\frac{\tilde{\epsilon}_1}{2}\Big[\frac{d\mathbf{r}_n}{dz}\Big]^2 + \frac{1}{2}\sum_{m\neq n}V_\perp(|\mathbf{r}_n(z)-\mathbf{r}_m(z)|)\Big\} - \frac{\phi_0}{4\pi}\int_z\sum_{n=1}^N\delta\mathbf{H}_\perp(\mathbf{r}_n(z),z)\cdot\frac{d\mathbf{r}_n}{dz}, \tag{5.1}$$

where $\tilde{\epsilon}_1 = \epsilon_1/p^2$, with $\epsilon_1 = \epsilon_0\ln\kappa$ the effective line tension and $\epsilon_0 = (\phi_0/4\pi\tilde{\lambda}_\perp)^2$ a characteristic energy scale. The nonlocality relating fields and vortex variables has been neglected also in the last term of Eq. (5.1). Two crucial approximations have been made in rewriting the general intervortex energy given in (2.8) in the form (5.1). First, the leading elastic term in the self-energy part of Eq. (2.8) has been linearized, according to

$\sqrt{1+\frac{1}{p^2}\left(\frac{d\mathbf{r}_n}{dz}\right)^2} \approx 1+\frac{1}{2p^2}\left(\frac{d\mathbf{r}_n}{dz}\right)^2$. Secondly, the pair interaction among different flux lines has been replaced by an interaction $V_\perp(r_\perp)$ acting locally in each constant-z plane, given by

$$V_\perp(r_\perp) = \frac{\phi_0^2}{8\pi^2\tilde{\lambda}_\perp^2} K_0(r_\perp/\tilde{\lambda}_\perp) \ , \tag{5.2}$$

with $K_0(x)$ a modified Bessel function. Of these approximations the latter is the most severe, since it amounts to neglecting the q_z dependence of the elastic constants - an approximation that strongly affects the tilt modulus, as we will see below. Letting $\mathcal{G}(\{\mathbf{r}_n(z)\},\mathbf{H}) = \mu N L + \mathcal{F}_N(\{\mathbf{r}_n(z)\},\mathbf{H})$, with $\mu = H_0\frac{\phi_0}{4\pi} - \epsilon_1 = \phi_0(H_0 - H_{c1})/4\pi$ a chemical potential, the grand canonical partition function of the vortex liquid can be written as

$$\mathcal{Z}_{\rm gr}(\mathbf{H}) = \sum_{N=0}^{\infty} \frac{1}{N!} e^{\beta L\mu N} \prod_{n=0}^{N} \int \mathcal{D}\mathbf{r}_n(z) e^{-\mathcal{F}_N(\mathbf{H})/k_BT}. \tag{5.3}$$

The integral in Eq. (5.3) is over all vortex line configurations. It has the form of a quantum-mechanical partition function in the path integral representation for the world lines of N particles of mass $\tilde{\epsilon}_1$, moving through imaginary time z and interacting with the repulsive pair potential $V_\perp(r_\perp)$. The vortex model with this simplified interaction can therefore be mapped into a model of $2D$ massive bosons with instantaneous pairwise interaction. The mapping results in the following correspondences:

$$\begin{aligned} & z \leftrightarrow \tau \\ & L \leftrightarrow \hbar\beta_{\rm boson} \\ & \tilde{\epsilon}_1 \leftrightarrow m \\ & k_BT \leftrightarrow \hbar \\ & H_0\frac{\phi_0}{4\pi} - \epsilon_1 \leftrightarrow \mu, \end{aligned} \tag{5.4}$$

where $\beta_{\rm boson} = 1/k_BT_{\rm boson}$ is the inverse temperature of the bosons. The precise mapping of the grand canonical vortex line partition function onto the Feynman path integral in imaginary time τ of a gas of two-dimensional bosons requires the introduction of a second quantized Hamiltonian corresponding to Eq. (5.1) and is described in the literature[4,38–40]. Some care must be taken in dealing with the tilting field $\delta\mathbf{H}_\perp$ which introduces velocity-dependent terms into the fictitious boson Lagrangian. One important difference between the flux-line array and the boson system is in the boundary conditions in the fictitious time variable z. The mapping of the free energy (5.1) of vortex lines onto the "action" of two-dimensional bosons is exact only when one imposes periodic boundary conditions for the flux lines in the z direction, i.e., $\mathbf{r}_n(L) = \mathbf{r}_n(0)$. In contrast the natural boundary condition for flux line would be free boundary conditions, corresponding to $\left(\frac{d\mathbf{r}_n}{dz}\right)_{z=L} = \left(\frac{d\mathbf{r}_n}{dz}\right)_{z=0} = 0$. As shown by Täuber and Nelson[28], the choice of the boundary conditions does affect the tilt modulus of a finite-thickness sample. We will not, however, discuss this here as we are ultimately interested in infinitely thick samples.

To complete the mapping, the grand canonical partition function (5.1) is first rewritten in a coherent-state path integral representation as

$$\mathcal{Z}_{\rm gr}(\mathbf{H}) = \int \mathcal{D}\psi(\mathbf{r}_\perp,z) \int \mathcal{D}\psi^*(\mathbf{r}_\perp,z) e^{-\mathcal{S}[\psi,\psi^*;\mathbf{h}]/k_BT}. \tag{5.5}$$

The boson "action" in the imaginary-time path integral is

$$\begin{aligned} \mathcal{S}[\psi,\psi^*;\mathbf{h}] = \int_{\mathbf{r}} \Big[& \psi^*\Big(k_BT\partial_z - \frac{(k_BT)^2}{2\tilde{\epsilon}_1}\nabla_\perp^2\Big)\psi \\ & - \frac{k_BT}{2\tilde{\epsilon}_1}\mathbf{h}\cdot(\psi^*\boldsymbol{\nabla}_\perp\psi - \psi\boldsymbol{\nabla}_\perp\psi^*) - \frac{1}{2\tilde{\epsilon}_1}h^2|\psi|^2 \\ & + \int d\mathbf{r'}_\perp V_\perp(\mathbf{r}_\perp - \mathbf{r'}_\perp)|\psi(\mathbf{r}_\perp,z)|^2|\psi(\mathbf{r'}_\perp,z)|^2 \Big], \end{aligned} \tag{5.6}$$

and $\mathbf{h}(\mathbf{r}) = (\phi_0/4\pi)\delta\mathbf{H}_\perp(\mathbf{r})$. The complex fields ψ and ψ^* correspond to boson annihilation and creation operators in the second quantized Hamiltonian. It is convenient to rewrite these fields in terms of an amplitude and a phase as

$$\psi(\mathbf{r}_\perp,z) = \sqrt{\hat{n}(\mathbf{r}_\perp,z)}e^{i\theta(\mathbf{r}_\perp,z)}. \tag{5.7}$$

The magnitude $\hat{n}(\mathbf{r}_\perp,z)$ of the field ψ corresponds to the fluctuating local boson density. The phase field θ determines the boson momentum density,

$$\mathbf{g}(\mathbf{r}_\perp,z) = k_BT\hat{n}\boldsymbol{\nabla}_\perp\theta \ . \tag{5.8}$$

Upon inserting Eq. (5.7) into Eq. (5.6), the action can be written in terms of density and phase variables as

$$\begin{aligned} \mathcal{S}[\psi,\psi^*;\mathbf{h}] = \int_{\mathbf{r}} \Big\{ & ik_BT\hat{n}\partial_z\theta \\ & + \frac{(k_BT)^2}{8\tilde{\epsilon}_1}\frac{(\boldsymbol{\nabla}_\perp\hat{n})^2}{\hat{n}} + \frac{(k_BT)^2}{2\tilde{\epsilon}_1}\hat{n}(\boldsymbol{\nabla}_\perp\theta)^2 \\ & - \frac{k_BT}{\tilde{\epsilon}_1}i\hat{n}\mathbf{h}\cdot\boldsymbol{\nabla}_\perp\theta - \frac{h^2}{2\tilde{\epsilon}_1}\hat{n} \\ & + \int d\mathbf{r'}_\perp V_\perp(\mathbf{r}_\perp - \mathbf{r'}_\perp)\hat{n}(\mathbf{r}_\perp,z)\hat{n}(\mathbf{r'}_\perp,z) \Big\}, \end{aligned} \tag{5.9}$$

where we have dropped surface terms that vanish for periodic boundary conditions.

The tilt-tilt correlator $T_{ij}(\mathbf{q})$ can be calculated using Eq. (2.27), with the result,

$$T_{ij}(\mathbf{q}) = \frac{k_BT}{\tilde{\epsilon}_1}\delta_{ij}\langle\hat{n}(\mathbf{q})\rangle_{\mathbf{h}=0} - \left(\frac{k_BT}{\Omega\tilde{\epsilon}_1}\right)^2\sum_{\mathbf{p},\mathbf{p}'} p_{\perp i}p'_{\perp j}\langle\hat{n}(\mathbf{q}-\mathbf{p})\theta(\mathbf{p})\hat{n}(-\mathbf{q}-\mathbf{p}')\theta(\mathbf{p}')\rangle_{\mathbf{h}=0} , \tag{5.10}$$

where the brackets denote an average over the full nonlinear action (5.9), evaluated at $\mathbf{h} = 0$.

To proceed, a standard approximation is to consider only small fluctuations of the fields from their mean values. Letting

$$\hat{n}(\mathbf{r}_\perp, z) = n_0 + \delta\hat{n}(\mathbf{r}_\perp, z), \tag{5.11}$$

and retaining only terms quadratic in the fields in the action, the corresponding Gaussian action in zero tilting field is given by

$$\mathcal{S}_G[\psi,\psi^*;\mathbf{0}] = \int_{\mathbf{r}}\Big\{ik_BT\delta\hat{n}\partial_z\theta + \frac{(k_BT)^2}{8\tilde{\epsilon}_1}\frac{(\boldsymbol{\nabla}_\perp\delta\hat{n})^2}{n} + \frac{(k_BT)^2}{2\tilde{\epsilon}_1}n_0(\boldsymbol{\nabla}_\perp\theta)^2 + \int_{\mathbf{r}'_\perp} V_\perp(\mathbf{r}_\perp - \mathbf{r}'_\perp)\delta\hat{n}(\mathbf{r}_\perp, z)\delta\hat{n}(\mathbf{r}'_\perp, z)\Big\}. \tag{5.12}$$

To Gaussian order the tilt autocorrelator is given by

$$T^0_{ij}(\mathbf{q}) = \frac{n_0k_BT}{\tilde{\epsilon}_1}\delta_{ij} + \left(\frac{n_0k_BT}{\tilde{\epsilon}_1}\right)^2 q_{\perp i}q_{\perp j}\langle\theta(\mathbf{q}_\perp)\theta(-\mathbf{q}_\perp)\rangle_G , \tag{5.13}$$

where $\langle ...\rangle_G$ denotes an average over the Gaussian action (5.12). The correlation functions of the fluctuating fields are easily calculated within the Gaussian approximation, with the result,

$$\langle\delta\hat{n}(-\mathbf{q})\delta\hat{n}(\mathbf{q})\rangle_G = \frac{n_0k_BTq^2/\tilde{\epsilon}_1}{q_z^2 + \epsilon_B(q)^2/(k_BT)^2} , \tag{5.14}$$

$$\langle\theta(-\mathbf{q})\delta\hat{n}(\mathbf{q})\rangle_G = \frac{q_z}{q_z^2 + \epsilon_B(q)^2/(k_BT)^2} , \tag{5.15}$$

$$\langle\theta(-\mathbf{q})\theta(\mathbf{q})\rangle_G = \frac{\tilde{\epsilon_1}\epsilon_B(q)^2/(n_0q^2(k_BT)^2)}{q_z^2 + \epsilon_B(q)^2/(k_BT)^2} , \tag{5.16}$$

where

$$\frac{\epsilon_B(q_\perp)}{k_BT} = \Big[\ \frac{n_0k_BTq_\perp{}^2V_\perp(q_\perp)}{\tilde{\epsilon}_1} + (\frac{k_BTq_\perp{}^2}{2\tilde{\epsilon}_1})^2\ \Big]^{1/2} \tag{5.17}$$

corresponds to the Bogoliubov spectrum of the two-dimensional boson superfluid. The quartic term in the Bogoliubov spectrum arises from the $|\boldsymbol{\nabla}_\perp\hat{n}|^2$ "kinetic" term in the action. To this Gaussian order of approximation the tilt modulus is dispersionless and simply the bare part of the single-vortex contribution to c_{44}, given by

$$c^0_{44} = c^{v0}_{44} = n_0\tilde{\epsilon}_1, \tag{5.18}$$

as given in Eq. (1.2). By comparing the correlation functions given in Eqs. (5.14-5.16) to those of the hydrodynamic fields given in Eqs. (4.8-4.10), we see that the results obtained by these two methods agree with each other provided we drop the term of $\mathcal{O}(q_\perp^4)$ in the Bogoliubov spectrum (which is of higher order in the wave vector and therefore is consistently neglected in a long wavelength theory) and make the identifications $c^0_{44}(q_\perp, q_z) = n_0\tilde{\epsilon}_1$ and $c^0_{11}(q_\perp, q_z) = n_0^2V_\perp(q_\perp)$. The quantity that replaces the "Bogoliubov spectrum" in hydrodynamics is a characteristic inverse length scale ξ_z^{-1} that controls the decay of correlations along the z direction, given by

$$\left[\frac{\epsilon(q_\perp)}{k_BT}\right]^{1/2} \to \xi_z^{-1}(q_\perp, q_z) = q_\perp\sqrt{\frac{c^0_{11}(q_\perp, q_z)}{c^0_{44}(q_\perp, q_z)}}. \tag{5.19}$$

Notice, however, that, in contrast to the boson spectrum, the correlation length ξ_z depends on q_z, not just on $q_\perp$. This dependence arises from the nonlocality of the intervortex interaction in the field direction and will have important consequences on the renormalization of c_{44}. Finally, we stress that the hydrodynamic tilt field does *not* simply map onto the momentum density of two-dimensional bosons, which in turn is related to the boson phase variable by Eq. (5.8). The boson momentum density is to lowest order purely longitudinal while the tilt field always has a transverse part.

Täuber and Nelson evaluated perturbatively the corrections to c^v_{44} arising from terms beyond Gaussian in the free energy[28]. These corrections can be obtained by factorizing the fourth order correlator on the right hand side of Eq. (5.10) as a product of Gaussian correlators using Wick's theorem[41]. For the long wavelength tilt modulus, these authors obtained

$$\frac{1}{c^{vR}_{44}} = \frac{1}{n_0\tilde{\epsilon}_1}\Big[1 - \frac{n^B_n}{n_0}\Big], \tag{5.20}$$

where

$$n^B_n = \frac{Lk_BT}{8\tilde{\epsilon}_1}\int\frac{d^2\mathbf{q}_\perp}{(2\pi)^2}\left[\frac{q_\perp}{\sinh\frac{L\epsilon_B(q_\perp)}{2k_BT}}\right]^2 \tag{5.21}$$

is the normal-fluid density of the two-dimensional boson liquid. The long-wavelength tilt modulus can also be written as

$$\frac{1}{c^{vR}_{44}} = \frac{n^B_s}{n_0^2\tilde{\epsilon}_1} , \tag{5.22}$$

where $n_s^B = n_0 - n_n^B$ is the boson superfluid density. As easily seen from Eq. (5.21) and discussed in TN[28], the normal-fluid density is finite only for samples of finite thickness L, corresponding to a nonzero boson temperature. In this case one obtains a renormalization of the tilt modulus due to finite-size effects. The sign of this correction is sensitive to the choice of boundary conditions (the result for periodic boundary conditions is displayed here). The normal fluid density vanishes, however, for $L \to \infty$. The local boson model therefore predicts that the tilt modulus of an infinitely thick, clean superconductor is unrenormalized and equals its bare value $n_0\tilde{\epsilon}_1$. In other words, the flux-line liquid is always entangled in the thermodynamic limit.

VI. NON-GAUSSIAN HYDRODYNAMICS AND DISENTANGLED FLUX LIQUIDS

Our goal in the remainder of this paper is to construct a *non-Gaussian* fully *nonlocal* hydrodynamic theory and use it to evaluate the renormalization of the tilt modulus. As a first step in this direction, in this section we derive a non-Gaussian hydrodynamic free energy from the *local* boson action given in Eq. (5.9). Of course such a hydrodynamic theory neglects interactions that are nonlocal in z and will mainly be used as a guide for constructing a more general non-Gaussian nonlocal hydrodynamics in the next section. The non-Gaussian terms in the free energy renormalize the tilt modulus. When these corrections are evaluated perturbatively, the resulting c_{44}^R is identical to that obtained by by Täuber and Nelson using the boson formalism[28]. The main goal of this section is to emphasize the relationship between the boson formalism and hydrodynamics and to stress that equivalent results can be obtained by either method.

To derive the hydrodynamic free energy from the boson action, we employ the method used by Kamien and collaborators[42] for the formally analog problem of directed polymers in a nematic solvent. We begin by eliminating the term $\frac{(k_BT)^2}{2\tilde{\epsilon}_1}\hat{n}(\boldsymbol{\nabla}_\perp\theta)^2$ in Eq. (5.10) in favor of a new vector field $\hat{\mathbf{P}}$, via a Hubbard-Stratonovich transformation, with the result

$$\mathcal{Z}_{\rm gr}(\mathbf{H}) = \int \mathcal{D}\hat{\mathbf{P}}\mathcal{D}\hat{n}\mathcal{D}\theta e^{-\mathcal{S}'[\hat{\mathbf{P}},\hat{n},\theta;\mathbf{h}]/k_BT}, \tag{6.1}$$

where

$$\mathcal{S}'[\hat{\mathbf{P}},\hat{n},\theta;\mathbf{h}] = \int_{\mathbf{r}}\Big\{ik_BT\hat{n}\partial_z\theta + \frac{(k_BT)^2}{8\tilde{\epsilon}_1}\frac{(\boldsymbol{\nabla}_\perp\hat{n})^2}{\hat{n}} + \frac{(k_BT)}{\tilde{\epsilon}_1}i\hat{n}\boldsymbol{\nabla}_\perp\theta\cdot\big[k_BT\hat{\mathbf{P}} - \mathbf{h}\big] \\ + \frac{\hat{n}}{2\tilde{\epsilon}_1}\big[(k_BT\hat{\mathbf{P}})^2 - h^2\big] + \int_{\mathbf{r'}_\perp} V_\perp(\mathbf{r}_\perp - \mathbf{r'}_\perp)\hat{n}(\mathbf{r}_\perp,z)\hat{n}(\mathbf{r'}_\perp,z)\Big\}. \tag{6.2}$$

If we integrate over $\hat{\mathbf{P}}$ in Eq. (6.3), we return to the original nonlinear action. Instead we integrate over θ which only appears linearly in the new action. This integration results in a δ-functional, yielding

$$\tilde{\mathcal{Z}}_{\rm gr}(\mathbf{H}) = \int \mathcal{D}\hat{n}\mathcal{D}\hat{\mathbf{P}}\exp\Big[\frac{2k_BTn_0^2}{\tilde{\epsilon}_1}\int_{\mathbf{r}}\ln(\hat{n}(\mathbf{r})/n_0)\Big]e^{-\tilde{\mathcal{S}}_H[\hat{n},\hat{\mathbf{P}};\mathbf{h}]/k_BT}\delta\big(\partial_z\hat{n} + \boldsymbol{\nabla}_\perp\cdot\frac{n}{\tilde{\epsilon}_1}(k_BT\hat{\mathbf{P}} + \mathbf{h})\big), \tag{6.3}$$

with

$$\tilde{\mathcal{S}}_H[\hat{n},\hat{\mathbf{P}};\mathbf{h}] = \frac{1}{2}\int_{\mathbf{r}}\Big\{\frac{(k_BT)^2}{\tilde{\epsilon}_1}\hat{n}\hat{\mathbf{P}}^2 + \frac{(k_BT)^2}{4\tilde{\epsilon}_1}\frac{(\boldsymbol{\nabla}_\perp\hat{n})^2}{\hat{n}} - \frac{\hat{n}}{2\tilde{\epsilon}_1}h^2 + \int_{\mathbf{r'}_\perp} V_\perp(\mathbf{r}_\perp - \mathbf{r'}_\perp)\hat{n}(\mathbf{r}_\perp,z)\hat{n}(\mathbf{r'}_\perp,z)\Big\}. \tag{6.4}$$

In obtaining Eq. (6.4) we have discretized the nonlinear action (5.10) in real space, according to

$$\int_{\mathbf{r}} f(\mathbf{r}) \to v_0\sum_i f_i, \tag{6.5}$$

with v_0 an elementary volume, $v_0 = \tilde{\epsilon_1}/(2k_BTn_0^2)$. This is the volume of a box with base area equal to $1/n_0$ and height equal to the single-vortex entanglement length,

$$l_z = \frac{\tilde{\epsilon_1}}{2k_BTn_0}. \tag{6.6}$$

The term containing the logarithm of the fluctuating density arises from the Jacobian of the functional integration over the full nonlinear action. It represents the nonlinear "ideal gas" part of the flux liquid free energy.

Statistical averages have to be performed by integrating over the fields $\hat{n}(\mathbf{r})$ and $\hat{\mathbf{P}}(\mathbf{r})$ with the constraint provided by the δ-functional in Eq. (6.3). Comparison of Eq. (6.3) to the hydrodynamic free energy (4.6) of a flux-line liquid with the constraint (4.5) suggests a physical interpretation for the auxiliary vector field $\hat{\mathbf{P}}$. The quantity $\hat{n}(k_BT\hat{\mathbf{P}} + \mathbf{h})\tilde{\epsilon}_1$ takes the place of the hydrodynamic tilt field $\hat{\mathbf{t}}^H$ introduced in the previous section. The difference between the vector field $\hat{\mathbf{P}}$ and the tilt field can be understood by noting that, as pointed out by Nelson and Le Doussal[38], the canonically conjugate momentum of the fictitious particle that corresponds to the n-th flux-line is $\mathbf{p}_n = i\big(\tilde{\epsilon}_1\frac{d\mathbf{r}_n}{dz} + \mathbf{h}\big)$. The vector field $\hat{\mathbf{P}}$ can then be interpreted as a sort of "velocity" field, while the tilt field $\hat{\mathbf{t}}^H$ represents the canonically conjugate momentum density. The two differ in the presence of an applied transverse field $\mathbf{h}$ that contributes to the single-vortex "canonical momentum".

The relationship between the effective action $\tilde{\mathcal{S}}_H$ and the hydrodynamic free energy of a tilted flux-line liquid is made more transparent by performing and additional change of variable that replaces the field $\hat{\mathbf{P}}$ by a tilt field defined as

$$\hat{\mathbf{t}}(\mathbf{r}) = \frac{\hat{n}(\mathbf{r})}{\tilde{\epsilon}_1}\big[k_BT\hat{\mathbf{P}}(\mathbf{r}) + \mathbf{h}(\mathbf{r})\big]. \tag{6.7}$$

The Jacobian of this transformation cancels the Jacobian of the Hubbard-Stratonovich transformation used earlier and we obtain,

$$\mathcal{Z}_{gr}(\mathbf{H}) = \int \mathcal{D}\hat{n}\mathcal{D}\hat{\mathbf{t}} e^{-\mathcal{S}_H[\hat{n},\hat{\mathbf{t}};\mathbf{h}]/k_BT}\delta\big(\partial_z\hat{n} + \boldsymbol{\nabla}_\perp\cdot\hat{\mathbf{t}}\big)\ , \tag{6.8}$$

with

$$\mathcal{S}_H[\hat{n},\hat{\mathbf{t}};\mathbf{h}] = \frac{1}{2k_BT}\int_{\mathbf{r}}\Big[\tilde{\epsilon}_1\frac{\hat{\mathbf{t}}^2}{\hat{n}} + \frac{(k_BT)^2}{4\tilde{\epsilon}_1}\frac{(\boldsymbol{\nabla}_\perp\hat{n})^2}{\hat{n}} - \mathbf{h}\cdot\mathbf{t} + \int_{\mathbf{r}'_\perp} V_\perp(\mathbf{r}_\perp - \mathbf{r}'_\perp)\hat{n}\big(\mathbf{r}_\perp, z)\hat{n}(\mathbf{r}'_\perp, z)\Big]. \tag{6.9}$$

The effective action of a tilted flux-line liquid given in Equation (6.9) becomes formally identical to the corresponding nonlinear hydrodynamic free energy, provided we make the identifications,

$$\begin{aligned} &\hat{n}(\mathbf{r}) \leftrightarrow \hat{n}^H(\mathbf{r}), \\ &\hat{\mathbf{t}}(\mathbf{r}) \leftrightarrow \hat{\mathbf{t}}^H(\mathbf{r}), \\ &n_0\tilde{\epsilon}_1 \leftrightarrow c_{44}^0(\mathbf{q}), \\ &{n_0}^2V_\perp(q_\perp) \leftrightarrow c_{11}^0(\mathbf{q}). \end{aligned} \tag{6.10}$$

The corresponding hydrodynamic free energy is nonlinear, but local in z, and it given by

$$F^\ell[\hat{n}^H, \hat{\mathbf{t}}^H;\mathbf{h}] = k_BT\mathcal{S}_H[\hat{n},\hat{\mathbf{t}};\mathbf{h}]. \tag{6.11}$$

The subscript "ℓ" indicates that only local interaction among the vortices has been retained in this hydrodynamic free energy. The free energy F^l contains the term quadratic in the density gradient that is neglected in conventional hydrodynamics. We will retain this term here to make our comparison with the results of the boson theory more transparent. Also this term will be needed below to provide a large wave vector cutoff to the integrals determining the renormalized tilt modulus.

The long wavelength part of the tilt-tilt autocorrelator can now be evaluated using the definition, Eq. (2.27). The non-Gaussian terms in the local hydrodynamic free energy (6.11) are separated out by writing

$$F^\ell = F^\ell_G + \delta F^\ell, \tag{6.12}$$

where F^ℓ_G is given by Eq. (4.6), but with the values specified in Eqs. (6.10) for the elastic constants, and

$$\delta F^\ell = -\frac{\tilde{\epsilon}_1}{2n_0}\int_{\mathbf{r}}\hat{\mathbf{t}}^2\frac{\delta\hat{n}}{\hat{n}}. \tag{6.13}$$

The tilt autocorrelator is then evaluated perturbatively in the non-Gaussian part δF^ℓ of the free energy. The perturbation expansion is outlined in Appendix B. To leading order, we obtain

$$T_{ij}(\mathbf{q}) = T^0_{ij}(\mathbf{q}) + \delta T_{ij}(\mathbf{q}), \tag{6.14}$$

where $T^0_{ij}(\mathbf{q})$ is the bare part of the correlator, given by Eq. (4.10-4.12). The hydrodynamic limit of the correction $\delta T_{ij}(\mathbf{q})$ is given by

$$\lim_{q_z\to 0}\delta T_{ij}(0,q_z) = \frac{n_0k_BT}{\tilde{\epsilon_1}}\delta_{ij} - \frac{(k_BT)^2}{\tilde{\epsilon_1}^2LA}\sum_{\mathbf{q}'_\perp,q'_z} q'_iq'_j\frac{(\epsilon_B(q'_\perp)/(k_BT))^2 - {q'}_z^2}{[(\epsilon_B(q'_\perp)/(k_BT))^2 + {q'}_z^2]^2}. \tag{6.15}$$

This result is identical to that obtained obtained by Täuber and Nelson via the boson formalism. In particular, the long wavelength tilt modulus defined according to Eq. (3.1) is found to be given by Eq. (5.20), with

$$n_n^B = \frac{n_0k_BT}{2LA}\sum_{\mathbf{q}_\perp,q_z} q_\perp^2\frac{(\epsilon_B(q_\perp)/(k_BT))^2 - q_z^2}{[(\epsilon_B(q_\perp)/(k_BT))^2 + q_z^2]^2}, \tag{6.16}$$

which becomes identical to Eq. (5.21) in the thermodynamic limit of large sample size.

VII. TILT MODULUS FROM NONLOCAL, NON-GAUSSIAN HYDRODYNAMICS

As discussed in the Introduction, neglecting the interaction among vortex segments at different "heights" z

has severe effects on the flux liquid tilt modulus, namely it completely neglects its compressional part, which is the largest contribution over a wide part of the (H, T) phase diagram. Hence our desire to develop a simple formalism for the calculation of the tilt modulus of a flux-line liquid that incorporates such nonlocalities.

A generalization of the boson mapping that incorporates the z-nonlocality of the vortex interaction was proposed some time ago by Feigel'man and collaborators[15]. The z-nonlocality yields a retarded interaction among the bosons that can be handled by the introduction of a Chern-Simons gauge field. In the limit of infinite penetration depth, $\lambda_\perp$, considered by these authors, the flux-line array then maps onto a *charged* superfluid. These authors argued that the charged boson system possesses a normal-fluid phase at zero temperature, corresponding to a thermodynamically distinct disentangled flux liquid phase, with infinite tilt modulus and longitudinal superconductivity.

Nonlocality is incorporated in a natural way in hydrodynamics. A *nonlinear* hydrodynamic free energy that incorporates all nonlocalities of the intervortex interaction can be obtained phenomenologically by coarse-graining of the microscopic energy of the vortex liquid, following the methods described in Ref. 36. Care must be taken in handling the self-interaction between segments of the same flux-line at different z heights, which is responsible for the non-Gaussian terms in the hydrodynamic free energy. Such non-Gaussian terms are neglected in the linearized theory, but as seen in the previous section they control the renormalization of the tilt modulus. The nonlinear hydrodynamic free energy obtained by such a procedure is given by

$$F = \frac{1}{2n_0^2}\int_{\mathbf{r}}\int_{\mathbf{r}'}\left\{\Big[\frac{n_0^2\tilde{\epsilon}_1}{\hat{n}^H(\mathbf{r})}\delta(\mathbf{r}-\mathbf{r}') + K_c(\mathbf{r}-\mathbf{r}')\Big]\hat{\mathbf{t}}^H(\mathbf{r})\hat{\mathbf{t}}^H(\mathbf{r}') + B(\mathbf{r}-\mathbf{r}')\delta\hat{n}^H(\mathbf{r})\delta\hat{n}^H(\mathbf{r}')\right\}, \tag{7.1}$$

where $B(\mathbf{r})$ is the real space compressional modulus and $K_c(\mathbf{r})$ is the interaction part of the real space tilt modulus. The first term in Eq. (7.1) arises from the self-energy part of the interaction and it represents a sort of nonlinear "kinetic" contribution to the total energy of the flux-line array. To make contact with conventional notation, it is convenient to rewrite the interaction part of the free energy in wave-vector space,

$$F = \frac{1}{2}\int_{\mathbf{r}}\tilde{\epsilon}_1\frac{[\hat{\mathbf{t}}^H(\mathbf{r})]^2}{\hat{n}^H(\mathbf{r})} + \frac{1}{2n_0^2\Omega}\sum_{\mathbf{q}}\left\{c_{44}^{c0}(\mathbf{q})|\hat{\mathbf{t}}^H(\mathbf{q})|^2 + c_{11}^0(\mathbf{q})|\delta\hat{n}^H(\mathbf{q})|^2\right\} \tag{7.2}$$

where the bare compressional modulus, $c_{11}^0(\mathbf{q})$, and the interaction part of the bare tilt modulus, $c_{44}^{c0}(\mathbf{q})$, are given in Eqs. (4.7) and (1.3), respectively.

The non-Gaussian hydrodynamic free energy can also be derived from the action of two-dimensional bosons with retarded interaction written down by Feigel'man and collaborators by successively eliminating nonhydrodynamic fields in favor of hydrodynamic fields via formal manipulations analogous to those described in the previous section. This derivation is outlined in Appendix A. The resulting free energy differs from the phenomenological one given in Eq. (7.2) only in that it contains an additional term proportional to density gradients (see Appendix A). This term is usually neglected in hydrodynamics because it is of higher order in the gradients. We will, however, retain it here as it provides an intrinsic large-wave-vector cutoff to the integrals determining the renormalized tilt modulus. It can be incorporated in the free energy of Eq. (7.2) by the replacement

$$c_{11}^0(\mathbf{q}) \rightarrow c_{11}^0(\mathbf{q}) + (k_BT)^2 n_0 q_\perp^2/(4\tilde{\epsilon}_1). \tag{7.3}$$

It is convenient for the following to separate out the non-Gaussian part of the hydrodynamic free energy of Eq. (7.2) by letting

$$F = F_G + \delta F, \tag{7.4}$$

where F_G is given by Eq. (4.6), and

$$\delta F = -\frac{1}{2}\int_{\mathbf{r}}\frac{\tilde{\epsilon}_1[\hat{\mathbf{t}}^H(\mathbf{r})]^2}{n_0}\frac{\delta\hat{n}^H(\mathbf{r})}{\hat{n}^H(\mathbf{r})}. \tag{7.5}$$

The tilt autocorrelator can be evaluated by treating the non-Gaussian part of the free energy (7.5) perturbatively. Some details are given in Appendix B. The dimensionless parameter that controls the expansion in $\delta F/k_BT$ is proportional to $(\tilde{\epsilon}_1/2k_BT\sqrt{n_0})^2 = (l_z/a_0)^2$, with l_z the entanglement length given in Eq. (6.6). Small values of l_z/a_0 correspond to an entangled flux-line liquid. The "kinetic" nonlinearities that are incorporated perturbatively stiffen the tilt modulus of the line liquid, making it therefore less entangled.

The nonlinearities embodied in δF yield corrections to all the correlation functions. Here, we only display the result for the transverse part of the tilt-tilt correlator, that determines the wave vector-dependent tilt modulus. Using Eq. (2.30), the wave vector-dependent tilt modulus is given by

$$\frac{1}{c_{44}^R(q_\perp, q_z)} = \frac{1}{c_{44}^0(q_\perp, q_z)}\left[1 - \frac{n_0\tilde{\epsilon}_1}{c_{44}^0(q_\perp, q_z)}\frac{n_n(q_\perp, q_z)}{n_0}\right], \tag{7.6}$$

with

$$n_n(q_\perp,q_z)=\frac{k_BT}{LA}\sum_{\mathbf{q}'_\perp,q'_z}\left\{\frac{{q'}_\perp^2}{c_{44}^0(\mathbf{q}')}\frac{1}{{q'}_z^2+[\xi_z(\mathbf{q}')]^{-2}}-\frac{n_0\tilde{\epsilon}_1(\mathbf{q}_\perp-\mathbf{q}'_\perp)^2}{c_{44}^0(\mathbf{q}')c_{44}^0(\mathbf{q}-\mathbf{q}')}\frac{1}{(q_z-q'_z)^2+[\xi_z(\mathbf{q}-\mathbf{q}')]^{-2}}\right\} \tag{7.7}$$

$$+\frac{n_0\tilde{\epsilon}_1k_BT}{LA}\sum_{\mathbf{q}'_\perp,q'_z}\frac{(\hat{\mathbf{q}}_\perp\cdot\hat{\mathbf{q}}'_\perp)^2(\mathbf{q}_\perp-\mathbf{q}'_\perp)^2[\xi_z(\mathbf{q}')]^{-2}-\left[1-(\hat{\mathbf{q}}_\perp\cdot\hat{\mathbf{q}}'_\perp)^2\right]{q'}_\perp^2{q'}_z({q'}_z-q_z)}{c_{44}^0(\mathbf{q}')c_{44}^0(\mathbf{q}-\mathbf{q}')\left[{q'}_z^2+[\xi_z(\mathbf{q}')]^{-2}\right]\left[(q_z-q'_z)^2+[\xi_z(\mathbf{q}-\mathbf{q}')]^{-2}\right]},$$

and

$$[\xi_z(\mathbf{q})]^{-2}=\frac{q_\perp^2}{c_{44}^0(\mathbf{q})}\left[c_{11}^0(\mathbf{q})+\frac{(k_BT)^2n_0q_\perp^2}{4\tilde{\epsilon}_1}\right]. \tag{7.8}$$

The length scale $\xi_z(\mathbf{q})$ differs from the one defined in Eq. (5.19) in that it contains an additional term arising from the coupling to the density gradient contained in our free energy and usually neglected in hydrodynamics. For simplicity, we use, however, the same notation as in Eq. (5.19).

The long-wavelength tilt modulus is determined by $n_n=\lim_{q_z\to 0}\lim_{q_\perp\to 0}n_n(q_\perp,q_z)$, given by

$$n_n=\frac{k_BT}{LA}\sum_{\mathbf{q}_\perp,q_z}\frac{q_\perp^2}{c_{44}^0(\mathbf{q})}\left[1-\frac{n_0\tilde{\epsilon}_1}{c_{44}^0(\mathbf{q})}\right]\frac{1}{q_z^2+[\xi_z(\mathbf{q})]^{-2}}+\frac{n_0\tilde{\epsilon}_1k_BT}{2LA}\sum_{\mathbf{q}_\perp,q_z}\frac{q_\perp^2}{[c_{44}^0(\mathbf{q})]^2}\frac{[\xi_z(\mathbf{q})]^{-2}-q_z^2}{\left[q_z^2+[\xi_z(\mathbf{q})]^{-2}\right]^2}. \tag{7.9}$$

Equations (7.6-7.9) are the central result of this paper. If the z-nonlocality of the intervortex interaction is neglected in Eq. (7.7) by replacing the elastic constants on the right-hand side with the corresponding values used in the the local boson formalism, according to Eq. (6.10), then Eq. (7.7) becomes identical to the result obtained by TN. In particular, the first term on the right hand side of Eq. (7.9) is absent in the local boson model of TN, where $c_{44}^0=n_0\tilde{\epsilon}_1$. The long-wavelength normal fluid density is then given by Eq. (5.21) and vanishes for $L\to\infty$.

The normal fluid density given in Eq. (7.9) can be evaluated explicitly for the case of an isotropic superconductor ($p=1$) in the limit of infinite thickness ($L\to\infty$). After inserting in Eq. (7.9) the expression for the nonlocal bare elastic constants given in Eqs. (4.7) and (1.1-1.3), the q_z integral in Eq. (7.9) can be evaluated. The resulting normal-fluid fraction depends on the three length scales that characterize the system. These are the average intervortex spacing, $a_0=1/\sqrt{n_0}$, the the ab plane London penetration depth, $\tilde{\lambda}_\perp$, and the single-vortex entanglement length, l_z. We have introduced two dimensionless parameters,

$$u=\frac{2l_z}{\sqrt{\pi}a_0}=\frac{2\tilde{\epsilon}_1}{k_BT\sqrt{4\pi n_0}}, \tag{7.10}$$

and a dimensionless volume fraction of vortex lines,

$$v^*=\frac{1}{4\pi n_0\tilde{\lambda}_\perp^2}, \tag{7.11}$$

The renormalized long-wavelength tilt modulus is written in terms of our dimensionless parameters as

$$\frac{1}{c_{44}^R}=\frac{1}{c_{44}^0}\left[1-\frac{v^*}{1+v^*}\frac{n_n}{n_0}\right] \tag{7.12}$$

and the normal fluid fraction is given by

$$\frac{n_n}{n_0}=\frac{1}{2u}\int_0^\infty dx\{K(x|u,v^*)+L(x|u,v^*)\}\ , \tag{7.13}$$

where

$$K(x|u,v^*)=\frac{x^2[1+(x+v^*)(1+x/u^2)]+2z_1z_2x(x+v^*)}{\sqrt{x+v^*}z_1z_2(z_1+z_2)[\sqrt{1+x+v^*}(x+z_1z_2)+z_1z_2(z_1+z_2)]}\ , \tag{7.14}$$

$$L(x|u,v^*)=v^*\frac{x(z_1^2+z_2^2)}{z_1z_2(z_1+z_2)(z_1^2-z_2^2)}\ , \tag{7.15}$$

with

$$z_{1,2}=\frac{1}{\sqrt{2}}\{1+x+(x/u)^2+v^*\pm[(1+(x/u)^2-x-v^*)^2+4v^*]^{1/2}\}^{1/2}\ . \tag{7.16}$$

These integrals have been evaluated numerically. The resulting normal fluid fraction is shown in Fig. 1 as a function of u for several values of the volume fraction v^*. We note that the dependence on v^* is rather weak, particularly for small values of u.

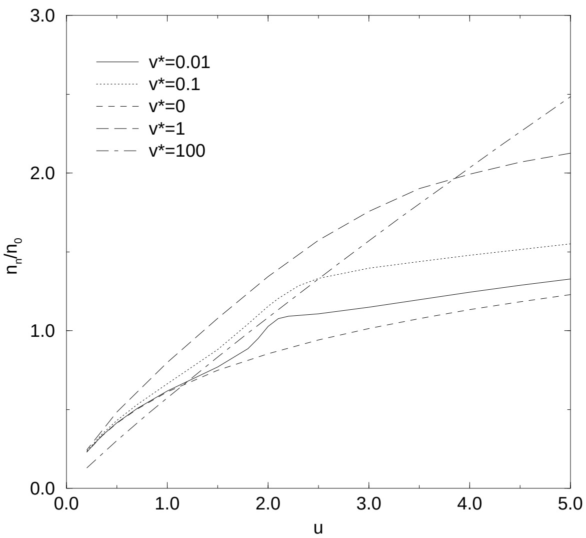

Fig.1.The normal-fluid fraction given by Eq. (7.13) as a function of u for five different values of v^*. Notice the weak dependence of n_n/n_0 on v^* for small values of u.

For $v^* = 0$ (which can be interpreted as either the high density limit or the infinite $\lambda_\perp$ case treated by Feigel'man and collaborators[15]), the normal-fluid density given in Eq. (7.13) reduces – up to an overall factor of 2 – to the result obtained by Feigel'man et al.[15,43]. Our Eq. (7.9) generalizes the result obtained by Feigel'man and coworkers to the case of finite penetration depth.

We stress that our calculation is perturbative and we have only evaluated the leading correction in perturbation theory. As discussed above, the small parameter in the perturbation theory is proportional to $u^2 \sim (l_z/a_0)^2$. In other words, the unperturbed state is an entangled flux liquid, with a very small value of the z-axis coherence length l_z, and interactions stiffen the vortices, enhancing the tilt modulus. We can estimate the values of magnetic field and temperature where our perturbation theory breaks down as determined by the root of the equation

$$\frac{v^*}{1+v^*}\frac{n_n}{n_0} = 1. \tag{7.17}$$

The solution $u_0(v^*)$ of Eq. (7.17) defines a line $B_{D0}(T)$ in the (H,T) phase diagram that can be interpreted as an estimate of the phase boundary between entangled and disentangled liquid regions. For $B > B_{D0}(T)$ the liquid is entangled, while for $B < B_{D0}(T)$ the perturbation theory breaks down, signaling the appearance of a disentangled flux-line liquid. Of course, in order to interpret the region $B < B_{D0}(T)$ as a disentangled flux liquid the $B_{D0}(T)$ line must lie in the molten region of the (H,T) phase diagram. At high density, $v^* << 1$ and Eq. (7.17) can be approximated as $n_n/n_0 \sim 1/v^* >> 1$. It is clear from Fig. 1 that the roots of this equation occur at large values of u, where $n_n/n_0 \sim (1/2)ln(u)$. We then estimate that our perturbation theory breaks down for $u_0(v^*) \sim \exp(2/v^*)$. Converting to field and temperature, this corresponds to $B_{D0}(T) \sim (H_{c1}/2\ln\kappa)\ln(H_{c1}\phi_0/\pi k_B T 4p^2\sqrt{\ln\kappa})$, with $H_{c1} = \phi_0/4\pi\tilde{\lambda}_\perp^2 \ln\kappa$. Below this line, c_{44} is strongly renormalized upward by interactions and a large disentangled flux-line liquid fraction may appear. Conversely, at low density, $v^* >> 1$ and Eq. (7.17) becomes $n_n/n_0 \sim 1$. The solution of this equation depends weakly on v^*, as seen from Fig. 1, and is approximately $u_0 \sim 2$, corresponding to $B_{D0}(T) \sim (\phi_0/4\pi)(\tilde{\epsilon}_1/k_BT)^2$. This result coincides with the estimate obtained by Feigel'man et al[15], but it applies in a different field regime. The solution $u_0(v^*)$ of Eq. (7.17) for general values of v^* has been obtained numerically and is shown in Fig. 2 as a solid line. For small v^* (high vortex-line density) Eq. (7.17) predicts that the perturbation theory breaks down at very large values of u, in a region that is well beyond its range of applicability.

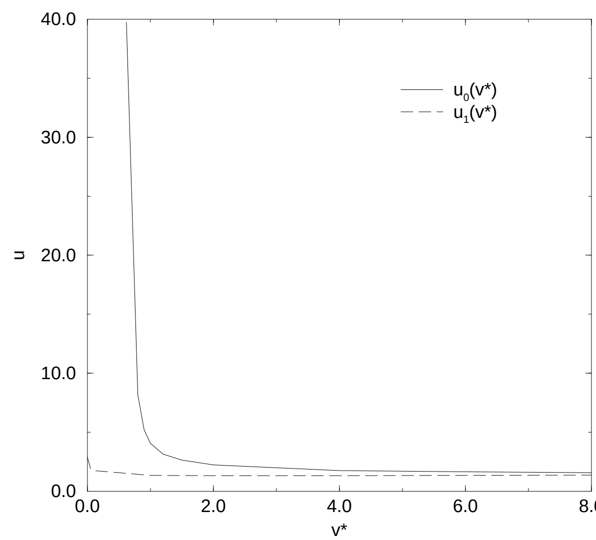

Fig.2.The solid line is the numerical solution of Eq. (7.17). It defines the line $u_0(v^*)$ in the (u, v^*) parameter space where the perturbation expansion of the tilt autocorrelator breaks down. The dashed line is $u_1(v^*)$, where $n_n/n_0 = 1$.

We now wish to compare our perturbative result to the nonperturbative expression for c_{44} proposed by Larkin and Vinokur and given in Eq. (1.6). As discussed in the Introduction, if the Larkin-Vinokur formula is expanded for small values of the normal fluid fraction n_n/n_0, the leading term has the form given in Eq. (1.8), which is identical to the long wavelength limit of our result (7.6), provided we identify n_n in Eq. (1.8) with our perturbative expression for the normal fluid density given in Eq. (7.9). It is then tempting to conjecture that a nonperturbative generalization of our calculation may indeed yield the expression (1.6) proposed by Larkin and Vinokur for the renormalized long wavelength tilt modulus, but with a normal fluid fraction given by Eq. (7.9), corresponding to

$$\frac{1}{c_{44}^R} = \frac{1}{c_{44}^c + \frac{n_0 \tilde{\epsilon}_1}{1 - n_n/n_0}}, \tag{7.18}$$

with n_n given by Eq. (7.9). We stress that Eq. (7.18), which is simply a rewriting of the Larkin-Vinokur result, is purely a conjecture in the context of our work. It is, however, interesting to explore its consequences. According to Eq. (7.18), the condition for the vanishing of $1/c_{44}^R$, corresponding to the onset of a macroscopic disentangled fluid fraction, would read

$$\frac{n_n}{n_0} = 1 \, . \tag{7.19}$$

The numerical solution of this equation, denoted by $u_1(v^*)$, is shown in Fig. 2 as a dashed line. We note that the line $u_0(v^*)$, where the perturbation theory breaks down, and the line $u_1(v^*)$, where the conjectured nonperturbative form of $1/c_{44}^R$ vanishes, coincide at large v^*, but diverge at small v^*. In this high density region it appears that the perturbation theory strongly underestimates the stiffening of c_{44} from interactions. The line $u_1(v^*)$ defines a second "disentanglement line", $B_{D1}(T)$, in the (H, T) phase diagram. Assuming $u_1(v^*) \sim 2 \sim$ constant over the range of v^* values of interest, we estimate $B_{D1}(T) \sim (\phi_0/4\pi)(\tilde{\epsilon}_1/k_B T)^2$. Notice that the field $B_{D1}(T)$ (which coincides with $B_{D0}(T)$ at low vortex density) is of the order of the melting field $B_{\rm m}(T)$ of the vortex lattice. Using a Lindeman criterion for melting, this is found to be $B_{\rm m}(T) = (16 c_L^4 \phi_0 p^2/(\ln\kappa)^2 (\tilde{\epsilon}_1/k_B T)^2$, where c_L is the Lindeman parameter[44].

Before discussing the location of the disentanglement lines $B_{D0}(T)$ and $B_{D1}(T)$ in the (H, T) phase diagram, we recall that the explicit evaluation of the integrals determining the normal fluid density has been carried out for isotropic superconductors ($p = 1$). To estimate the relevance of our result to the anisotropic CuO_2 materials, we have used the above estimate for the boundary between disentangled and entangled liquid regions and inserted parameter values typical of these materials. To justify this approximation, we note that for $p >> 1$ the compressional part of the tilt modulus arising from the nonlocality of the vortex interaction in the z direction becomes less important relative to the vortex part. As it is precisely this nonlocality that is responsible for a nonvanishing renormalization of c_{44} in infinitely thick samples, we expect that the results that we have obtained for the isotropic case will provide an upper bound for the size of the renormalization in anisotropic materials. A sketch of a phase diagram showing the location of the disentanglement lines $B_{D0}(T)$ (dashed line) and $B_{D1}(T)$ (dotted line) is shown in Fig. 3. It is not drawn to scale.

Using parameter values of YBCO and BSCCO we have estimated that in both these materials at high fields ($B > 1$Tesla) the $B_{D0}(T)$ boundary defining the breaking down of our perturbation theory lies well within the flux lattice phase. At low fields there is a possibility for a disentangled phase in the reentrant liquid region. This region is, however, rather narrow, particularly in YBCO where it is expected to have a width of the order of 1 Gauss[45]. For this reason, while we have drwan in Fig. 3 the "horizontal" part of the $B_{D0}(T)$ curve as passing through this reentrant liquid phase, it could very well be that this line is located either above (in the lattice) or below (in the Meissner phase) the sketched position. The disentanglement line $B_{D1}(T)$ is shown as dotted in Fig. 3 and it is estimated to lie in the liquid phase. The existence of this line is, however, just a conjecture in the context of our work, as our results are strictly perturbative. In general we expect the actual disentanglement line to lie between our perturbative estimate $B_{D0}(T)$ and the conjectured $B_{D1}(T)$. It could therefore lie almost entirely in the solid phase, indicating that a true thermodynamic disentangled liquid phase does not exist. This conclusion would appear to agree with the latest results from simulations[19–21]. Further work beyond the naive lowest order perturbation expansion discussed here is needed, however, to settle this point.

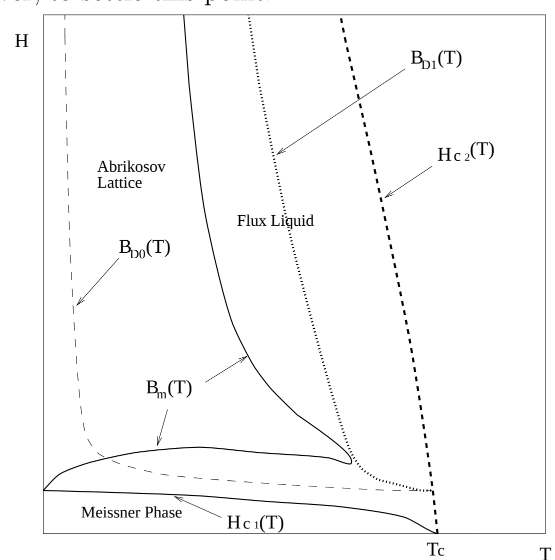

Fig.3. A sketch (not to scale) of the phase diagram showing the location of the "disentanglement" lines discussed in the text. The dashed line, $B_{D0}(T)$, marks the breaking down of the perturbation expansion for the inverse tilt modulus; the dotted line, $B_{D1}(T)$, corresponds to $n_n/n_0 = 1$ and signals the divergence of the conjectured form of c_{44}^R, given in Eq. (7.18). The width of the reentrant liquid phase is in reality much smaller than shown here and the line $B_{D0}(T)$ may or may not pass through it. $B_m(T)$ is the melting line. $H_{c2}(T)$ marks the onset of a Meissner effect and is not a sharp phase transition.

One important outcome of our work is that the nonlocality of the intervortex interaction in the field direction has important qualitative effects on the tilt modulus. In particular, it always yields a finite – although often small – upward renormalization of c_{44} even in infinitely thick samples. This renormalization is absent in calculations based on the local boson mapping[28]. In fact, in the work of TN an important role is played by the invariance of the flux-line interaction under an affine transformation or uniform tilt (corresponding to Galilean invariance of a pure boson system). L.D. Landau[46] has shown that the Galilean invariance implies that the superfluid density at the ground state ($T = 0$) of a superfluid equals the total density. The affine transformation invariance is not present in the more general intervortex free energy that allows for pairwise interaction among vortex segments at different heights, z. This nonlocality breaks the "Galilean invariance" and yields a tilt-tilt interaction which penalizes any misalignment of the flux-lines, therefore favoring disentanglement.

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VIII. APPENDIX A - DERIVATION OF NONLOCAL HYDRODYNAMICS FROM THE PARTITION FUNCTION OF $2D$ CHARGED BOSONS

In this appendix we show that the nonlocal, non-Gaussian hydrodynamic free energy given in Eq. (7.2) can be derived by formal manipulations of the partition function of a two-dimensional charged boson fluid. Feigel'man and collaborators[15] have shown that the partition function of an array of flux-lines described in the London approximation by the Ginzburg-Landau free energy of Eq. (2.18) can be mapped onto that of a two-dimensional system of bosons interacting via a massive vector potential. The nonlocality of the intervortex interaction is incorporated via a gauge field that mediates a retarded interaction among the bosons. The coherent-state formulation of the boson problem yields the imaginary-time action:

$$\mathcal{S}_c[\psi,\psi^*,\mathbf{a},\mathbf{A}] = \int_0^{\beta\hbar} d\tau \int d\mathbf{r}_\perp \{ \; \psi^*[\; \hbar\partial_\tau + ia_0 - \frac{1}{2m}(\hbar\boldsymbol{\nabla}_\perp + i\mathbf{a}_\perp)^2 - \mu \;]\psi + V_{sr}(\psi\psi^*) + \frac{p^2}{2g^2}(\boldsymbol{\nabla}_\perp \times \mathbf{a}_\perp)^2 + \frac{1}{2g^2}\big[\hat{\mathbf{z}} \times (\partial_\tau \mathbf{a}_\perp - \boldsymbol{\nabla} a_0)\big]^2 + \frac{i}{2\sqrt{\pi}\tilde{\lambda}_\perp g}(\boldsymbol{\nabla}\times\mathbf{a})\cdot\mathbf{A} + \frac{1}{8\pi}(\boldsymbol{\nabla}\times\mathbf{A})^2 \; \} \; . \tag{8.1}$$

The correspondence between vortex and boson variables is summarized in Eq. (5.4). The coupling constant g corresponds to the strength of the vortex interaction, according to $g^2 \leftrightarrow \phi_0^2/(4\pi\tilde{\lambda}_\perp^2)$ and p is the anisotropy parameter that here allows for different scalar an transverse interaction among the bosons. $\mathbf{A}$ is the vector potential of the real magnetic field ($\boldsymbol{\nabla}\times\mathbf{A} = \mathbf{B}$), and $\mathbf{a} = (a_0, \mathbf{a}_\perp)$ is a gauge field that mediates the non-instantaneous interaction among the bosons. The boson chemical potential μ has to be determined so that the equilibrium boson density n_B equals the vortex density, $n_B = n_0 = B/\phi_0$. Finally, V_{sr} is a short range repulsion (on scale ξ) between the bosons. This action is based on the gauge $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ and $\boldsymbol{\nabla}_\perp\cdot\mathbf{a}_\perp = 0$. The choice of $\boldsymbol{\nabla}_\perp\cdot\mathbf{a}_\perp = 0$ instead of $\boldsymbol{\nabla}\cdot\mathbf{a} = 0$ reflects the assumption of nonrelativistic velocities for the bosons, corresponding to small tilt of the flux lines away from the z direction[47]. By rewriting the boson fields in terms of an amplitude and a phase, as defined in Eq. (5.7), we obtain

$$\mathcal{S}_c[\hat{n},\theta,\mathbf{a},\mathbf{A}] = \int_0^{\beta\hbar} d\tau \int d\mathbf{r}_\perp \{ \, i\hbar\hat{n}\partial_\tau\theta + i\hat{n}a_0 + \frac{\hbar^2}{8m}\frac{(\boldsymbol{\nabla}_\perp\hat{n})^2}{\hat{n}} + V_{sr}(\hat{n}) + \frac{\hat{n}}{2m}\mathbf{a}_\perp^2 + \frac{\hbar}{m}\hat{n}(\boldsymbol{\nabla}\theta)\cdot\mathbf{a}_\perp - \mu\hat{n} + \frac{\hbar^2}{2m}n(\boldsymbol{\nabla}_\perp\theta)^2 + \frac{p^2}{2g^2}(\boldsymbol{\nabla}_\perp\times\mathbf{a}_\perp)^2 + \frac{1}{2g^2}\big[\hat{\mathbf{z}}\times(\partial_\tau\mathbf{a}_\perp - \boldsymbol{\nabla}a_0)\big]^2 + \frac{i}{2\sqrt{\pi}\tilde{\lambda}_\perp g}(\boldsymbol{\nabla}\times\mathbf{a})\cdot\mathbf{A} + \frac{1}{8\pi}(\boldsymbol{\nabla}\times\mathbf{A})^2 \; \} \; . \tag{8.2}$$

The assumption of small fluctuations allows us to extend the range of θ from $[-\pi,\pi]$ to $[-\infty,+\infty]$. As described in section V, we now eliminate the phase θ in favor of a vector field $\hat{\mathbf{P}}$ via a Hubbard-Stratonovich transformation, to obtain

$$\tilde{\mathcal{S}}_c'[\hat{n},\hat{\mathbf{P}},\mathbf{a},\mathbf{A}] = \int_0^{\beta\hbar} d\tau \int d\mathbf{r}_\perp \{ \, \frac{\hbar^2}{2m}\hat{n}\hat{\mathbf{P}}^2 + \frac{\hat{n}}{2m}\mathbf{a}_\perp^2 + i\hat{n}a_0 + \frac{\hbar^2}{2m}\frac{(\boldsymbol{\nabla}_\perp\hat{n})^2}{\hat{n}} - \mu\hat{n} + V_{sr}(\hat{n}) + \frac{p^2}{2g^2}(\boldsymbol{\nabla}_\perp\times\mathbf{a}_\perp)^2 + \frac{1}{2g^2}\big[\hat{\mathbf{z}}\times(\partial_\tau\mathbf{a}_\perp - \boldsymbol{\nabla}a_0)\big]^2 + \frac{i}{2\sqrt{\pi}\tilde{\lambda}_\perp g}(\boldsymbol{\nabla}\times\mathbf{a})\cdot\mathbf{A} + \frac{1}{8\pi}(\boldsymbol{\nabla}\times\mathbf{A})^2 + \frac{n_0\hbar^2}{m}\ln(\frac{\hat{n}}{n_0}) \; \}, \tag{8.3}$$

with the constraint

$$\partial_\tau \hat{n} + \boldsymbol{\nabla}_\perp \cdot \frac{\hat{n}}{m}(\hbar\hat{\mathbf{P}} + i\mathbf{a}_\perp) = 0 \; . \tag{8.4}$$

The last term in the action in Eq. (8.3), logarithmic in the density, is the Jacobian of the transformation. We then make a change of variables,

$$\hat{\mathbf{t}} = \frac{\hat{n}}{m}(\hbar\hat{\mathbf{P}} + i\mathbf{a}_\perp) \; , \tag{8.5}$$

and obtain

$$\mathcal{S}'_c[\hat{n}, \hat{\mathbf{t}}, \mathbf{a}, \mathbf{A}] = \int_0^{\beta\hbar} d\tau \int d\mathbf{r}_\perp \Big\{ \frac{m\hat{\mathbf{t}}^2}{2\hat{n}} - i\mathbf{a}_\perp \cdot \hat{\mathbf{t}} + i\hat{n}a_0 - \mu\hat{n} + V_{sr}(\hat{n}) + \frac{\hbar^2}{8m}\frac{(\boldsymbol{\nabla}_\perp\hat{n})^2}{\hat{n}} + \frac{p^2}{2g^2}(\boldsymbol{\nabla}_\perp \times \mathbf{a}_\perp)^2 + \frac{1}{2g^2}\big[\hat{\mathbf{z}} \times (\partial_\tau \mathbf{a}_\perp - \boldsymbol{\nabla} a_0)\big]^2 + \frac{i}{2\sqrt{\pi}\tilde{\lambda}_\perp g}(\boldsymbol{\nabla} \times \mathbf{a}) \cdot \mathbf{A} + \frac{1}{8\pi}(\boldsymbol{\nabla} \times \mathbf{A})^2 \Big\}, \tag{8.6}$$

with the constraint

$$\partial_\tau \hat{n} + \boldsymbol{\nabla}_\perp \cdot \hat{\mathbf{t}} = 0 \; . \tag{8.7}$$

The Jacobian of this transformation cancels that of the previous one.

Finally, we define an effective action $\mathcal{S}_c^{\rm eff}$ for the bosons by integrating out both the vector potential $\mathbf{A}(\mathbf{r})$ and the gauge field $\mathbf{a}(\mathbf{r})$,

$$\int' \mathcal{D}\hat{n}\mathcal{D}\hat{\mathbf{t}}\mathcal{D}\mathbf{A}\mathcal{D}\mathbf{a}\; e^{-\mathcal{S}'_c[\hat{n},\hat{\mathbf{t}},\mathbf{a},\mathbf{A}]}\delta(\partial_\tau\hat{n} + \boldsymbol{\nabla}_\perp \cdot \hat{\mathbf{t}}) = \int \mathcal{D}\hat{n}\mathcal{D}\hat{\mathbf{t}} e^{-\mathcal{S}_c^{\rm eff}[\hat{n},\hat{\mathbf{t}}]}\delta(\partial_\tau\hat{n} + \boldsymbol{\nabla}_\perp \cdot \hat{\mathbf{t}}). \tag{8.8}$$

The prime over the integral sign on the left hand side of the equation indicates that the integration over $\mathbf{A}$ and $\mathbf{a}$ has to be performed by taking into account the constraints imposed by our choice of gauge. The vector potential and gauge field are most easily integrated out by rewriting the field part of the action (8.6) in Fourier space, with the result,

$$\mathcal{S}_c^{\rm eff}[\hat{n}, \hat{\mathbf{t}}] = \int_0^{\beta\hbar} d\tau \int d\mathbf{r}_\perp \Big\{ \frac{m\hat{\mathbf{t}}^2}{2\hat{n}} - \mu\hat{n} + V_{sr}(\hat{n}) + \frac{\hbar^2}{8m}\frac{(\boldsymbol{\nabla}_\perp\hat{n})^2}{\hat{n}} \Big\} + \frac{1}{2\Omega}\sum_{\mathbf{q}} \Big\{ \frac{g^2\tilde{\lambda}_\perp^2}{1 + q_z^2\tilde{\lambda}_\perp^2 + q_\perp^2 p^2\tilde{\lambda}_\perp^2}|\hat{\mathbf{t}}_T(\mathbf{q})|^2 + \frac{q^2}{q_\perp^2}\frac{g^2\tilde{\lambda}_\perp^2}{1 + q^2\tilde{\lambda}_\perp^2}|\hat{n}(\mathbf{q})|^2 \Big\}, \tag{8.9}$$

where $\hat{\mathbf{t}}_T(\mathbf{q}) = \hat{\mathbf{q}}_\perp \times \hat{\mathbf{t}}(\mathbf{q})$. By making use of the continuity constraint given in Eq. (8.7), we can write

$$\frac{g^2\tilde{\lambda}_\perp^2}{1 + q_z^2\tilde{\lambda}_\perp^2 + q_\perp^2 p^2\tilde{\lambda}_\perp^2}|\hat{\mathbf{t}}_T(\mathbf{q})|^2 + \frac{q^2}{q_\perp^2}\frac{g^2\tilde{\lambda}_\perp^2}{1 + q^2\tilde{\lambda}_\perp^2}|\hat{n}(\mathbf{q})|^2 = \frac{g^2\tilde{\lambda}_\perp^2}{1 + q_z^2 + q_\perp^2 p^2\tilde{\lambda}_\perp^2}|\hat{\mathbf{t}}(\mathbf{q})|^2 + \frac{g^2\lambda^2(1 + q^2p^2\tilde{\lambda}_\perp^2)}{(1 + q^2\tilde{\lambda}_\perp^2)(1 + q_z^2\tilde{\lambda}_\perp^2 + q_\perp^2 p^2\tilde{\lambda}_\perp^2)}|\hat{n}(\mathbf{q})|^2. \tag{8.10}$$

Finally, if we replace the short range repulsion $V_{sr}(\hat{n})$ by a short-wavelength cutoff and identify the boson density $\hat{n}$ and momentum field $\hat{\mathbf{t}}$ with the corresponding hydrodynamic quantities for the vortices, we see that Eq. (8.9) yields precisely the nonlocal non-Gaussian hydrodynamic free energy discussed in section VI.

IX. APPENDIX B - PERTURBATIVE CORRECTIONS TO THE TILT MODULUS FROM NONLINEAR HYDRODYNAMICS

The wave-vector dependent tilt modulus is defined in terms of the transverse part of the tilt-tilt correlator as in Eq. (3.1). In the hydrodynamic approximation, the tilt-tilt correlator can be written as

$$T_{ij}(\mathbf{r}, \mathbf{r}') = \frac{\int \mathcal{D}\hat{n}(\mathbf{r})\mathcal{D}\hat{\mathbf{t}}(\mathbf{r})\hat{t}_i(\mathbf{r})\hat{t}_j(\mathbf{r}')e^{-F/k_BT}\delta(\partial_z\hat{n} + \boldsymbol{\nabla}_\perp \cdot \hat{\mathbf{t}})}{\int \mathcal{D}\hat{n}(\mathbf{r})\mathcal{D}\hat{\mathbf{t}}(\mathbf{r})\mathbf{e}^{-\mathbf{F}/\mathbf{k_B T}}\delta(\partial_\mathbf{z}\hat{\mathbf{n}} + \boldsymbol{\nabla}_\perp \cdot \hat{\mathbf{t}})} \; , \tag{9.1}$$

Where F is the hydrodynamic free energy given in Eq. (7.2). The free energy can be written as the sum of a Gaussian part, F_G, and non-Gaussian corrections, δF, as in Eq. (7.4). We want to calculate up to lowest-order in the small parameter, u^2, nonlinear corrections to the tilt autocorrelator. By keeping only terms up to fourth order in the fluctuations of the hydrodynamic fields, the non-Gaussian part of the free energy is given by,

$$\delta F \approx -\frac{\tilde{\epsilon}_1}{2n_0^2\Omega^2}\sum_{\mathbf{q}_1,\mathbf{q}_2}\hat{t}_i(\mathbf{q}_1)\hat{t}_i(\mathbf{q}_2)\delta\hat{n}(-\mathbf{q}_1-\mathbf{q}_2)+\frac{\tilde{\epsilon}_1}{2n_0^3\Omega^3}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}\hat{t}_i(\mathbf{q}_1)\hat{t}_i(\mathbf{q}_2)\delta\hat{n}(\mathbf{q}_3)\delta\hat{n}(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\,. \tag{9.2}$$

The tilt-tilt correlator is then evaluated in Fourier space perturbatively in the non-Gaussian part of the free energy, with the result,

$$T_{ij}(\mathbf{q},\mathbf{q}')=\Omega\delta_{\mathbf{q}+\mathbf{q}',\mathbf{0}}T^0_{ij}(\mathbf{q})-\frac{1}{k_BT}\langle\hat{t}_i(\mathbf{q})\hat{t}_j(\mathbf{q}')\delta F\rangle^c_G+\frac{1}{2(k_BT)^2}\langle\hat{t}_i(\mathbf{q})\hat{t}_j(\mathbf{q}')(\delta F)^2\rangle^c_G\,, \tag{9.3}$$

where $\langle ...\rangle^c_G$ denotes a cumulant average over the Gaussian ensemble with weight $\sim\exp(-F_G/k_BT)$. The first term on the right hand side of Eq. (9.3) is the Gaussian result given in Eqs. (4.10-4.12).

Using Wick's theorem, the corrections arising from the non-Gaussian part of the free energy are easily expressed in terms of the correlations in the Gaussian ensemble given in Eq. (4.8-4.12), with the result

$$P^T_{ij}(\mathbf{q}_\perp)\langle\hat{t}_i(\mathbf{q})\hat{t}_j(\mathbf{q}')\delta F\rangle^c_G=\Omega\delta_{\mathbf{q}+\mathbf{q}',\mathbf{0}}[T^0_T(\mathbf{q})]^2\frac{\tilde{\epsilon}_1}{n_0^3}\frac{1}{\Omega}\sum_{\mathbf{q}_1}\langle|\delta\hat{n}(\mathbf{q}_1)|^2\rangle_G \tag{9.4}$$

and

$$P^T_{ij}(\mathbf{q}_\perp)\langle\hat{t}_i(\mathbf{q})\hat{t}_j(\mathbf{q}')(\delta F)^2\rangle^c_G=2\Omega\delta_{\mathbf{q}+\mathbf{q}',\mathbf{0}}[T^0_T(\mathbf{q})]^2P^T_{ij}(\mathbf{q}_\perp)\frac{\tilde{\epsilon}_1^2}{n_0^4}\frac{1}{\Omega}\sum_{\mathbf{q}_1}\Big\{\langle\hat{t}^H_i(\mathbf{q}_1)\hat{t}^H_j(-\mathbf{q}_1)\rangle_G\langle|\delta\hat{n}^H(\mathbf{q}-\mathbf{q}_1)|^2\rangle_G+$$
$$\langle\hat{t}^H_i(\mathbf{q}_1)\delta\hat{n}^H(-\mathbf{q}_1)\rangle_G\langle\hat{t}^H_j(\mathbf{q}_1-\mathbf{q})\delta\hat{n}^H(\mathbf{q}-\mathbf{q}_1)\rangle_G\Big\}. \tag{9.5}$$

By substituting the expressions for the Gaussian correlators given in Eqs. (4.8-4.12), we obtain the following expression for the transverse part of the tilt autocorrelator to lowest order in the non-Gaussian terms,

$$\begin{aligned}T_T(\mathbf{q})=&\frac{n_0^2k_BT}{c^0_{44}(\mathbf{q})}\\&-\frac{n_0^3\tilde{\epsilon}_1(k_BT)^2}{[c^0_{44}(\mathbf{q})]^2}\frac{1}{LA}\sum_{\mathbf{q}'_\perp,q'_z}\left\{\frac{{q'}^2_\perp}{c^0_{44}(\mathbf{q}')}\frac{1}{{q'}^2_z+[\xi_z(\mathbf{q}')]^{-2}}-\frac{n_0\tilde{\epsilon}_1(\mathbf{q}_\perp-\mathbf{q}'_\perp)^2}{c^0_{44}(\mathbf{q}')c^0_{44}(\mathbf{q}-\mathbf{q}')}\frac{1}{(q_z-q'_z)^2+[\xi_z(\mathbf{q}-\mathbf{q}')]^{-2}}\right\}\\&-\frac{n_0^4\tilde{\epsilon}_1^2(k_BT)^2}{[c^0_{44}(\mathbf{q})]^2}\frac{1}{LA}\sum_{\mathbf{q}'_\perp,q'_z}\frac{(\hat{\mathbf{q}}_\perp\cdot\hat{\mathbf{q}}'_\perp)^2(\mathbf{q}_\perp-\mathbf{q}'_\perp)^2[\xi_z(\mathbf{q}')]^{-2}-\big[1-(\hat{\mathbf{q}}_\perp\cdot\hat{\mathbf{q}}'_\perp)^2\big]{q'}^2_\perp q'_z(q'_z-q_z)}{c^0_{44}(\mathbf{q}')c^0_{44}(\mathbf{q}-\mathbf{q}')\big[{q'}^2_z+[\xi_z(\mathbf{q}')]^{-2}\big]\big[(q_z-q'_z)^2+[\xi_z(\mathbf{q}-\mathbf{q}')]^{-2}\big]}\,.\end{aligned} \tag{9.6}$$

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