2-1-2008

Hydrodynamics of Liquids of Arbitrarily Curved Flux-Lines and Vortex Loops

Panayotis Benetatos
Syracuse University

M. Cristina Marchetti
Syracuse University

Recommended Citation
arXiv:cond-mat/0101222v1

Hydrodynamics of liquids of arbitrarily curved flux-lines and vortex loops

Panayotis Benetatos* and M. Cristina Marchetti
Physics Department, Syracuse University, Syracuse, NY 13244
(February 1, 2008)

We derive a hydrodynamic model for a liquid of arbitrarily curved flux-lines and vortex loops using the mapping of the vortex liquid onto a liquid of *relativistic* charged quantum bosons in 2+1 dimensions recently suggested by Tešanović and by Sudbø and collaborators. The loops in the flux-line system correspond to particle-antiparticle fluctuations in the bosons. We explicitly incorporate the externally applied magnetic field which in the boson model corresponds to a chemical potential associated with the conserved *charge* density of the bosons. We propose this model as a convenient and physically appealing starting point for studying the properties of the vortex liquid.

I. INTRODUCTION

The physics of vortex matter has been a very active field of research since the discovery of high-temperature superconductivity [1,2]. The melting of the Abrikosov vortex lattice at a field well below the mean field upper critical line, $H_{c2}(T)$, is now well established both theoretically and experimentally. The nature of the resulting vortex liquid phase remains, however, unclear. Various approaches have been used to study the properties of the liquid, ranging from the mapping of the statistical mechanics of vortex lines onto that of (nonrelativistic) two-dimensional quantum bosons [3–6], to continuum hydrodynamic models [7–9]. Recently, it has been proposed that the vortex liquid phase should be viewed not as a collection of well-defined directed vortex lines induced by the externally applied field, but rather as a system where closed vortex loops proliferate, with components in all directions [10,11]. This picture has been substantiated by numerical work by Sudbø and collaborators [12–18]. Such a vortex liquid has been described by Kiometzis et al. [19] and by Tešanović [10,11] by mapping it onto a system of *relativistic* two-dimensional quantum bosons, where the proliferation of vortex loops and overhangs in the flux-line system corresponds to particle-antiparticle creation and annihilation events in the bosons. This mapping has been developed in some detail for the case of zero external field.

In this paper we consider a vortex-line liquid in an arbitrary homogeneous external field, $\mathbf{H}$. The only restriction is that H be well below the mean field upper critical line, H_{c2}, so that the London approximation applies. The liquid in general consists of both field-induced vortices (on the average aligned with the external field, but where large fluctuations leading to overhangs are not excluded) and closed vortex loops generated spontaneously by thermal fluctuations. On the basis of general considerations, the statistical mechanics of such a vortex liquid maps onto that of *relativistic* charged quantum bosons in $2+1$ dimensions coupled to a screened electromagnetic field. Our model generalizes that of Tešanović [11] in that it explicitly incorporates the external field $\mathbf{H}$ which in the boson model corresponds to a chemical potential, μ_r, associated with the conserved *charge* density of the bosons. By manipulating the relativistic boson action, we then derive a hydrodynamic model for a liquid of arbitrarily curved flux-lines and closed vortex loops. We propose this model as a convenient and physically appealing starting point for studying the properties of the vortex liquid.

We consider for simplicity a clean, isotropic type-II superconductor in the mixed state. The anisotropic case is discussed in the Appendix. In the London limit, the familiar Ginzburg-Landau free energy functional can be rewritten in terms of discrete vortex degrees of freedom by parametrizing the i-th line by its position $\mathbf{r}_i(s_i)$, with s_i the arclength along its curve, as [20–22]

$$\mathcal{G}\big[\{\mathbf{r}_i(s_i)\},\mathbf{H}\big] = U_{self} + U_{int} - \frac{\phi_0}{4\pi}\sum_{i=1}^{N}\int d\mathbf{r}_i\cdot\mathbf{H}\,, \qquad (1.1)$$

where

$$U_{self} \approx \epsilon_1 \sum_{i=1}^{N}\int ds_i \qquad (1.2)$$

is an approximate form for the self-energy of a single vortex line, and

$$U_{int} = \frac{1}{2}\sum_{i\neq j}\int d\mathbf{r}_i\cdot\int' d\mathbf{r}_j V(|\mathbf{r}_i-\mathbf{r}_j|)\,, \qquad (1.3)$$

with $V(r) = (\epsilon_0/r)e^{-r/\tilde{\lambda}}$, describes the screened interaction between vortex segments on different lines. Here $\epsilon_0 = \phi_0^2/(4\pi\tilde{\lambda})^2$ and $\epsilon_1 = \epsilon_0\ln\kappa$. Also, $\tilde{\lambda}$ is the effective London penetration depth and $\kappa \equiv \tilde{\lambda}/\xi$, with ξ the

*Present address: Physics Department, Harvard University, Cambridge, MA 02138

coherence length. The prime on the integral sign indicates that the interaction must be cutoff at intervortex separations of the order of ξ. The last term in Eq. (1.1) represents the interaction with the homogeneous external field, $\mathbf{H}$. A short distance cutoff $\approx \xi$ has been assumed to regularize the integrals at short scales. The canonical partition function of the system is

$$\mathcal{Z}(\mathbf{H},T)=\int' \mathcal{D}\{\mathbf{r}_i(s_i)\}e^{-\mathcal{G}/k_BT} \ . \qquad (1.4)$$

The prime over the integral sign indicates that the functional integration must be performed over all flux-line configurations $\{\mathbf{r}_i(s_i)\}$ subject to the constraint $\nabla\cdot\mathbf{B}=0$ (i.e. no flux-line can start or stop within the sample). The expression for the vortex energy given in Eqs. (1.1–1.3) is the starting point of much theoretical work on vortex arrays. As required, the free energy is a scalar and it is therefore rotationally invariant.

A familiar approximation to the general free energy given in Eq. (1.1) can be obtained at low fields and temperatures, where field-induced vortex lines interact weakly and fluctuations away from the external field direction are small. In this case it is convenient to choose the z axis in the direction of the applied field, $\mathbf{H}=\hat{\mathbf{z}}H_0$ and to parametrize the vortex positions as $\mathbf{r}_i=\big[\mathbf{r}_{\perp i}(z),z\big]$. Asuming small transverse fluctuations of the vortices and retaining only the interaction among vortex segments at the same "height" z (to be referred to as the *local* approximation), one obtains

$$\mathcal{G}\big[\{\mathbf{r}_i(s_i)\},H_0\big] \approx \sum_{i=1}^{N}\int dz\Big\{\frac{\epsilon_1}{2}\Big[\frac{d\mathbf{r}_{\perp i}(z)}{dz}\Big]^2-\mu\Big\} +\frac{1}{2}\sum_{i\neq j}\int dz V(|\mathbf{r}_{\perp i}(z)-\mathbf{r}_{\perp j}(z)|) \ . \qquad (1.5)$$

with $\mu=H_0\phi_0/(4\pi)-\epsilon_1$ [3,5]. Eq. (1.5) is clearly only appropriate to describe field-induced *directed* lines and it neglects the possibility of thermally-induced closed vortex loops. It is well known that the partition function of the array of directed vortex lines described by Eq. (1.5) maps onto the quantum mechanical partition function in the path integral formulation for a gas of two-dimensional bosons [3,5,23–25]. In this mapping, the vortex lines correspond to the boson world lines, ϵ_1 is the boson mass, and z represents the imaginary time. This boson mapping has been exploited by Nelson and collaborators to study the properties of vortex liquids and is particularly useful for the description of vortex arrays pinned by correlated disorder [6]. The approximate vortex free energy given in Eq. (1.5) is invariant under a uniform tilt of the external field away from the z direction. In the boson analogy, this translates into invariance with respect to Galileian boosts [26].

One of the limitations of this boson model is that it neglects the nonlocality of the intervortex interaction in the external field direction. Feigel'man and collaborators argued that this nonlocality can be incorporated in the boson formalism by mapping the vortex partition function onto the path integral of two-dimensional *charged* bosons [27] coupled to a screened gauge field that mediates the non-instantaneous interaction among the bosons [28,29]. The boson action proposed by these authors suffers, however, from one crucial problem — it does not incorporate the symmetry of the original vortex free energy given in Eq. (1.1). More precisely, the boson analogue of a rotation of the three-dimensional space where the vortices are embedded is a Lorentz transformation of the 2 + 1-dimensional space-time where the bosons are embedded. Since the original vortex energy is rotationally invariant, its dual boson action should be Lorentz invariant. But the model implemented by Feigel'man and collaborators lacks a definite symmetry as the field and the interaction part of the Lagrangian density are Lorentz invariant, but the approximate form used for the free-particle part is only Galileian invariant. We note that the non-Gaussian hydrodynamic model proposed recently by us [8] suffers from the same problem, as it was actually *derived* from the nonrelativistic charged boson model. This problem is in retrospect immediately apparent in the hydrodynamic formulation, where the free energy can be written in terms of the coarse-grained fields $\big[\hat{n}^H(\mathbf{r}),\hat{\mathbf{t}}^H(\mathbf{r})\big]$ as the sum of three terms, as in Eq. (1.1),

$$\mathcal{G}^H\big[\hat{n}^H,\hat{\mathbf{t}}^H,H_0\big] \;=\; U^H_{self} \;+\; U^H_{int} \;-\; NLH_0\frac{\phi_0}{4\pi} \ , \qquad (1.6)$$

with

$$U^H_{self}=NL\epsilon_1+\frac{1}{2}\int_{\mathbf{r}}\epsilon_1\frac{|\hat{\mathbf{t}}^H(\mathbf{r})|^2}{\hat{n}^H(\mathbf{r})} \qquad (1.7)$$

and

$$U^H_{int}=\frac{1}{2n_0^2}\int_{\mathbf{r}}\int_{\mathbf{r}'}\Big[K_t(\mathbf{r}-\mathbf{r}')\hat{\mathbf{t}}^H(\mathbf{r})\cdot\hat{\mathbf{t}}^H(\mathbf{r}') + K_c(\mathbf{r}-\mathbf{r}')\hat{n}^H(\mathbf{r})\hat{n}^H(\mathbf{r}')\Big] \ . \qquad (1.8)$$

Here $K_c(\mathbf{r})$ is the real space compressional modulus and $K_t(\mathbf{r})$ is the compressional part of the real space tilt modulus. For an isotropic superconductor, $K_c(\mathbf{r})=K_t(\mathbf{r})$. Since $[\hat{n}^H(\mathbf{r}),\hat{\mathbf{t}}^H(\mathbf{r})]$ is a vector under rotations (proportial to the local induction $\mathbf{B}$ in the superconductor, with $\hat{n}^H(\mathbf{r})\sim B_z(\mathbf{r})$ and $\hat{\mathbf{t}}^H(\mathbf{r})\sim\mathbf{B}_\perp(\mathbf{r})$), we see that, for $L\to\infty$, U^H_{int} is a scalar and therefore rotationally invariant, while U^H_{self} is not a scalar and therefore is not rotationally invariant. Note that, in contrast, both the uncharged boson model à la Nelson [25] and conventional Gaussian hydrodynamics [30,7] (where the field $\hat{n}^H(\mathbf{r})$ in the denominator of Eq. (1.7) is replaced by its equilibrium value) are consistent approximations to the free energy of directed vortex lines and respect the restricted symmetry under global rotations about the z axis. These observations indicate that, as recently pointed out in Refs. [19,10–15,18], the proper two-dimensional boson model dual to the original three-dimensional vortex problem is a gas of *relativistic* charged bosons in $2+1$-dimensional space-time, interacting via a screened electromagnetic field.

In this paper, starting from the action of relativistic charged quantum bosons, we derive a nonlocal, nonlinear hydrodynamic model of liquids of arbitrarly curved vortex lines that properly incorporates the full rotational symmetry of the vortex energy. As discussed in Refs. [19,10–18], this leads to some interesting new concepts, namely, the appearence of thermally excited *antivortex* lines and *vortex loops*. It is well known that a loop gas is dual to a gas of relativistic bosons in $2+1$ space-time dimensions. This duality has been exploited recently by a few authors to study the properties of vortex liquids in type-II superconductors. Kleinert and collaborators [19,31] have developed a field theory of a gas of fluctuating closed vortex loops and have applied renormalization group ideas to study its phases. Tešanović has proposed a theory [11] which separates "field-induced" from "thermally-generated" degrees of freedom. The former correspond to directed flux-lines, while the latter correspond to closed vortex loops. A prediction of this work is the existence of a phase transition, dubbed the Φ-transition, within the flux liquid phase from a low temperature liquid of lines with finite tension to a liquid phase that lacks superconducting coherence in all directions and where vortex loops proliferate yielding the vanishing of the flux-line tension. Numerical evidence for this Φ-transition has recently come from the numerical simulations of Sudbø and collaborators [18].

In this paper, we propose a long wavelength description of a liquid of interacting field-induced vortex lines and vortex loops which treats both *on equal footing*. The external magnetic field is incorporated explicitly and controls the density of field-induced lines. In the dual description it enters as the chemical potential of the bosons, in analogy to the nonrelativistic boson model. The important difference here is that the chemical potential couples to the local *charge* density of bosons, which is the relevant conserved variable as contrasted to the particle number density. The hydrodynamic model proposed here is physically appealing model and naturally generalizes the familiar Gaussian hydrodynamics. Its ramifications and particularly its standing with respect to the aforementioned Φ-transition are interesting directions for future work.

II. DERIVATION OF THE HYDRODYNAMICS OF FLUX-LINES AND FLUX-LOOPS FROM A RELATIVISTIC BOSON ANALOGY

In this section, we use the methods discussed in Ref. [8] to derive the hydrodynamic free energy of arbitrarly curved vortex lines. As usual, the vortex lines are viewed as imaginary-time world lines of $2D$ bosons. The need for a relativistic description is immediately apparent by considering the self-energy part of the vortex energy, given in Eq. (1.2), where the arclength of a curve segment can be parametrized as $ds = \sqrt{(dz)^2 + (d\mathbf{r}_\perp)^2}$. Eq. (1.2) is formally identical to the imaginary-time action of a relativistic free particle of rest mass $m \leftrightarrow \epsilon_1$ [32],

$$S = -mc \int_{s_1}^{s_2} ds = -mc \int_{s_1}^{s_2} \sqrt{c^2(dt)^2 - (d\mathbf{r}_\perp)^2}\ , \quad (2.1)$$

where s_1 and s_2 are two space-time events and $c\tau \leftrightarrow z$, with $\tau \equiv it$ the imaginary time. The speed of light c does not have a counterpart in the flux-line free energy, but is explicitly kept in this section to enable us to obtain the nonrelativistic limit. The correspondence between boson and vortex variables is summarized in Table 1. One important diference between our mapping and the mapping to $2D$ *nonrelativistic* bosons discussed extensively in the literature is that here the boson mass, m_b, corresponds to the vortex line energy rather than to the titl energy per unit length. This distinction becomes important in the case of anisotropic materials, as discussed in the Appendix.

TABLE I. Correspondence between boson and vortex variables.

bosons	vortices
$c\tau$	z
$\beta\hbar c$	L
$S/\hbar$	$\mathcal{F}/k_BT$
$m_bc^2/\hbar c$	ϵ_1/k_BT
$e_b^2/\hbar c$	$4\pi\epsilon_0/k_BT$
$\mu/\hbar c$	$H_0\phi_0/(4\pi k_BT)$
λ_b	$\tilde{\lambda}$
$\mathbf{t}_b/c$	$\mathbf{t} \approx \mathbf{B}/\phi_0$

As shown below, the vortex lines can be interpreted as the world lines of *relativistic* spin-0, charged bosons in $2+1$ space-time dimensions, interacting via a screened electromagnetic interaction [33]. By applying standard methods [34,35] of rewriting the boson Hamiltonian in the language of second quantization and transforming to coherent states, the imaginary-time path integral representation for the grand-canonical partition function of the bosons is written as

$$Z_G^{\rm bos} = \int_{\substack{\Phi^*(\mathbf{r}_\perp,\beta\hbar)=\Phi^*(\mathbf{r}_\perp,0)\\ \Phi(\mathbf{r}_\perp,\beta\hbar)=\Phi(\mathbf{r}_\perp,0)}} \mathcal{D}\Phi\mathcal{D}\Phi^*\mathcal{D}\mathbf{a}\mathcal{D}\mathbf{A}e^{-\mathcal{S}_r[\Phi,\Phi^*,\mathbf{a},\mathbf{A}]/\hbar}, \tag{2.2}$$

with the action

$$\mathcal{S}_r[\Phi,\Phi^*,\mathbf{a},\mathbf{A}] = \int_0^{\beta\hbar} d\tau \int d^2\mathbf{r}_\perp \Big\{ \frac{m_b c^2}{2}\Phi\Phi^* + \frac{1}{2m_b c^2}\big[(-i\hbar\partial_\tau + e_b a_0 + i\mu_r)\Phi\big]\big[(i\hbar\partial_\tau + e_b a_0 + i\mu_r)\Phi^*\big] + \frac{1}{2m_b}|(-i\hbar\nabla_\perp + \frac{e_b}{c}\mathbf{a}_\perp)\Phi|^2 + \mathcal{L}_F[\mathbf{a},\mathbf{A}] \Big\} , \tag{2.3}$$

where

$$\mathcal{L}_F[\mathbf{a},\mathbf{A}] = \frac{1}{8\pi}\Big\{(\nabla\times\mathbf{a})^2 + \frac{2i}{\lambda_b}(\nabla\times\mathbf{a})\cdot\mathbf{A} + (\nabla\times\mathbf{A})^2\Big\} \tag{2.4}$$

is the free field Lagrangian density and $\nabla \equiv \big(\frac{1}{c}\partial_\tau, \nabla_\perp\big)$. Here, $\Phi(\mathbf{r}_\perp,t)$ is a complex scalar matter field that describes bosons with positive and negative charge, with e_b the boson charge. The world lines of bosons of opposite charge, which are *antiparticles* to each other, correspond to flux-lines with opposite vorticity. The constraint on the fields Φ and Φ^* comes from the permutation symmetry requirement and reflects boson statistics. The boson field Φ is coupled to a *massive* electromagnetic field $\mathbf{a} = \big(a_0, \mathbf{a}_\perp\big)$ and the gauge field $\mathbf{A} = \big(A_0, \mathbf{A}_\perp\big)$ provides the screening. Integrating out the field $\mathbf{A}$ in Eqs. (2.3) and (2.4) under the gauge $\nabla\cdot\mathbf{a} = 0$ gives a mass term $|\mathbf{a}|^2/(8\pi\lambda_b^2)$. Under Lorentz transformations, the field Φ is a scalar and the boson action has full Lorentz invariance. The action also respects a global $U(1)$ symmetry, $\Phi \to \Phi' = \Phi e^{i\alpha}$, which leads to the conserved current, $\mathbf{j} = (j_0, \mathbf{j}_\perp)$, given by

$$\mathbf{j} = \frac{e_b}{2m_b}\big[\Phi^*\big(-i\hbar\nabla + \frac{e_b}{c}\mathbf{a}\big)\Phi + \Phi\big(i\hbar\nabla + \frac{e_b}{c}\mathbf{a}\big)\Phi^*\big], \tag{2.5}$$

with $\nabla\cdot\mathbf{j} = 0$. The temporal component j_0/c of the 2+1-current is the charge density, which is the appropriate conserved quantity in a relativistic theory. Finally, we have introduced a chemical potential, μ_r, which couples to the conserved *charge* density. This should be contrasted to the nonrelativistic boson model which maps onto a liquid of *directed* vortex lines and where the relevant conserved quantity is particle number. The quantum relativistic model naturally incorporates spontaneous creation and annihilation of particle-antiparticle pairs. The flux-line analogue of this is the creation of oriented *vortex loops*. It will become apparent below that the chemical potential corresponds to the external field $\mathbf{H}$.

In order to highlight the connection with the nonrelativistic boson model used by Nelson and collaborators, we rewrite the boson field in terms of an amplitude and a phase, $\Phi(\mathbf{r}_\perp,\tau) = \sqrt{\rho(\mathbf{r}_\perp,\tau)}\exp[i\theta(\mathbf{r}_\perp,\tau)]$, with the result

$$\mathcal{S}_r[\rho,\theta,\mathbf{a},\mathbf{A}] = \int_0^{\beta\hbar} d\tau \int d^2\mathbf{r}_\perp \Big\{ \frac{m_b c^2}{2}\rho + \frac{\hbar^2}{8m_b\rho}(\nabla\rho)^2 + \frac{\rho}{2m_b}(\hbar\nabla_\perp\theta + \frac{e_b}{c}\mathbf{a}_\perp)^2 + \frac{\rho}{2m_b c^2}(\hbar\partial_\tau\theta + e_b a_0 + i\mu_r)^2 + \mathcal{L}_F[\mathbf{a},\mathbf{A}] \Big\} . \tag{2.6}$$

A boson condensate phase is signalled by a macroscopic occupation ρ_c of the $\mathbf{p} = 0$ momentum state, defined in terms of the average order parameter as

$$\rho_c = |\langle\Phi(\mathbf{r}_\perp,\tau)\rangle|^2 . \tag{2.7}$$

Notice that in general $\rho_c \neq \rho_0 \equiv \langle\rho(\mathbf{r}_\perp,\tau)\rangle = \langle|\Phi(\mathbf{r}_\perp,\tau)|^2\rangle$, although the two quantities are equal within the mean field approximation described below.

In the spirit of Landau-Ginzburg mean field theory, we evaluate the partition function by the method of steepest descent. The stationarity condition gives nonlinear differential equations for the various fields. Restricting ourselves to solutions which are stationary and spatially homogeneous, the extrema conditions yield the equations,

$$\frac{\delta S_r}{\delta\rho} = \frac{m_b c^2}{2} + \frac{e_b^2|\mathbf{a}_\perp|^2}{2m_b c^2} + \frac{1}{2mc^2}(e_b a_0 + i\mu_r)^2 = 0 , \tag{2.8}$$

$$\frac{\delta S_r}{\delta\mathbf{a}_\perp} = \frac{e_b^2}{m_b c^2}\rho\mathbf{a}_\perp + \frac{\mathbf{a}_\perp}{4\pi\lambda_b^2} = 0 , \tag{2.9}$$

$$\frac{\delta S_r}{\delta a_0} = \frac{e_b}{m_b c^2}\rho(e_b a_0 + i\mu_r) + \frac{a_0}{4\pi\lambda_b^2} = 0. \tag{2.10}$$

One finds of course $\mathbf{a}_\perp = 0$, corresponding to the absence of charge current in the homogeneous state. A spatially homogeneous saddle point solution with $|\langle\Phi\rangle|^2 = \rho_0 \neq 0$ exists only for $|\mu_r| > m_b c^2$ [36]. The solution is given by

$$\rho_0 = \frac{|\mu_r| - m_b c^2}{4\pi\lambda_b^2 e_b^2}\ , \quad \text{for } |\mu_r| > m_b c^2\ ,$$
$$\rho_0 = 0\ , \quad \text{for } |\mu_r| \leq m_b c^2\ . \tag{2.11}$$

Within this mean field approximation, the occupation of the zero momentum state is $\rho_c = \rho_0$. The free energy in the condensate phase of bosons is given by

$$F_G^c(T,\Omega,\mu_r) = -k_B T \ln \mathcal{Z}_G^c = -\hbar\Omega\frac{(m_b c^2 - \mu_r)^2}{8\pi\lambda_b^2 e_b^2}\ , \tag{2.12}$$

where Ω is the area of the $2D$ boson gas. The mean charge density, $\langle j_0/c\rangle$, in this mean-field approximation is

$$\langle j_0/c\rangle = -\frac{1}{\Omega}\Big(\frac{\partial F_G^c}{\partial\mu_r}\Big)_{T,\Omega} = \pm i e_b\rho_0\ , \tag{2.13}$$

where the positive (negative) sign should be chosen for $\mu_r > 0$ ($\mu_r < 0$). For charged bosons the chemical potential is associated with the electric charge and allows for a positive (or negative) average charge density. Antiparticles have a charge and a chemical potential opposite to that of particles.

Within mean field theory, all the charge is in the condensate phase. When fluctuations are incorporated in the theory, the mean charge density, $\langle\rho\rangle$, differs from the condensate fraction, ρ_c, even at zero temperature.

The nonrelativistic limit for the action can be obtained by shifting the the chemical potential by $m_b c^2$,

$$\mu_r = \mu + m_b c^2\ , \tag{2.14}$$

where μ is the nonrelativistic chemical potential [36]. The action becomes

$$\mathcal{S}_r[\rho,\theta,\mathbf{a},\mathbf{A}] = \int_0^{\beta\hbar} d\tau\int d^2\mathbf{r}_\perp\Big\{\frac{\hbar^2}{8m_b c^2\rho}(\partial_\tau\rho)^2 + \frac{\hbar^2}{8m_b\rho}(\nabla_\perp\rho)^2 + i\rho(\hbar\partial_\tau\theta + e_b a_0 + i\mu)$$
$$+\frac{\rho}{2m_b}(\hbar\nabla_\perp\theta + \frac{e_b}{c}\mathbf{a}_\perp)^2 + \frac{\rho}{2m_b c^2}(\hbar\partial_\tau\theta + e_b a_0 + i\mu)^2 + \mathcal{L}_F[\mathbf{a},\mathbf{A}]\Big\}\ . \tag{2.15}$$

The nonrelativistic boson model of Täuber and Nelson [25] is then recoverd by letting $c\to\infty$. In this limit the 2D bosons only interact with the scalar field a_0, and there is no magnetic interaction,

$$\mathcal{S}_{nr}[\rho,\theta] = \int_0^{\beta\hbar} d\tau\int d^2\mathbf{r}_\perp\Big\{i\hbar\rho\partial_\tau\theta + i\rho e_b a_0 - \rho\mu + \frac{\hbar^2}{8m_b\rho}(\nabla_\perp\rho)^2 + \frac{\rho}{2m_b}(\hbar\nabla_\perp\theta)^2$$
$$+\frac{1}{8\pi}\big[(\hat{\mathbf{z}}\times\nabla_\perp)a_0\big]^2 + \frac{a_0^2}{8\pi\lambda_b^2}\Big\}\ . \tag{2.16}$$

The scalar potential a_0 mediates the instantaneous screened Coulomb interaction. By integrating it out we recover the familiar *instantaneous* interaction term with Fourier components $|\rho(\mathbf{q})|^2 4\pi\lambda_b^2 e_b^2/2(1+q_\perp^2\lambda_b^2)$.

In order to obtain the hydrodynamic free energy, we proceed as in Ref. [8] and perform a Hubbard-Stratonovich transformation to eliminate the boson amplitude and phase fields in favor of familiar hydrodynamic fields. First we eliminate the phase θ in Eq. (2.6) in favor of a new vector field $\mathbf{P}$, with the result

$$\mathcal{S}'_r[\rho,\mathbf{P},\mathbf{a},\mathbf{A}] = \int_0^{\beta\hbar} d\tau\int d^2\mathbf{r}_\perp\Big\{\frac{m_b c^2}{2}\rho + \frac{\hbar^2}{8m_b c^2\rho}(\partial_\tau\rho)^2 + \frac{\hbar^2}{8m_b\rho}(\nabla_\perp\rho)^2 + \frac{\rho}{2m_b}\Big[\frac{P_z^2}{c^2} + \mathbf{P}_\perp^2\Big]$$
$$+i\frac{\rho P_z}{m_b c^2}\big(e_b a_0 + i\mu_r\big) + \frac{ie_b\rho}{m_b c}\mathbf{P}_\perp\cdot\mathbf{a}_\perp + \mathcal{L}_F[\mathbf{a},\mathbf{A}] + \frac{\rho_0\hbar^2}{m_b}\ln\big(\frac{\rho}{\rho_0}\big)\Big\} \tag{2.17}$$

with the constraint

$$\frac{1}{c^2}\partial_\tau(\rho P_z) + \nabla_\perp\cdot(\rho\mathbf{P}_\perp) = 0\ . \tag{2.18}$$

Notice that the field $\mathbf{P}$ has *three* components. The temporal component P_z is related to the conserved charge density of the theory, which in turn is related to the temporal variations of the phase. The last term in Eq. (2.17) arises from the Jacobian of the functional integration and represents the nonlinear "ideal gas" part of the free energy. Finally, to make contact with the hydrodynamic fields used in our earlier work, we let

$$t_{bz} = \frac{\rho P_z}{m_b c} \ , \tag{2.19}$$

$$\mathbf{t}_{b\perp} = \frac{\rho \mathbf{P}_\perp}{m_b} \ , \tag{2.20}$$

and integrate out the fields $\mathbf{a}$ and $\mathbf{A}$ to obtain an effective action

$$\mathcal{S}'_r[\rho, \mathbf{t}_b] = \int_0^{\beta\hbar} d\tau \int d^2\mathbf{r}_\perp \Big\{ \frac{m_b c^2}{2}\rho + \frac{\hbar^2}{8 m_b c^2 \rho}(\partial_\tau \rho)^2 + \frac{\hbar^2}{8 m_b \rho}(\nabla_\perp \rho)^2 + \frac{m_b}{2\rho}\mathbf{t}_b^2 - \frac{1}{c}\mu_r t_{bz} \Big\} + \frac{1}{2\Omega\beta\hbar}\sum_{\mathbf{q}} \frac{4\pi\lambda_b^2 e_b^2/c^2}{1+q^2\lambda_b^2}|\mathbf{t}(\mathbf{q})|^2 \tag{2.21}$$

with the constraint

$$\nabla \cdot \mathbf{t}_b = 0 \ , \tag{2.22}$$

where $\mathbf{q} = (q_z/c, \mathbf{q}_\perp)$.

Since we are interested in the fluctuating vortices rather than in the relativistic bosons, we shall rewrite Eq. (2.21) in the vortex language using the "translation" Table 1. We are interested in the thermodynamic limit where periodic or free boundary conditions give the same result and $\int_0^{\beta\hbar} c d\tau \int d^2\mathbf{r}_\perp \to \int_{\mathbf{r}} = \int_0^L \int d\mathbf{r}_\perp$, where L is the thickness of the sample. The resulting vortex free energy is given by

$$\mathcal{F}_r[\mathbf{t}, \rho] = \int_{\mathbf{r}} \Big\{ \frac{\epsilon_1}{2}\rho + \frac{(k_B T)^2}{8\epsilon_1 \rho}(\nabla\rho)^2 + \frac{\epsilon_1}{2\rho}\mathbf{t}^2 - \frac{H_0\phi_0}{4\pi}t_z \Big\} + \frac{1}{2\Omega L}\sum_{\mathbf{q}} \frac{4\pi\epsilon_0\tilde{\lambda}^2}{1+q^2\tilde{\lambda}^2}|\mathbf{t}(\mathbf{q})|^2 \tag{2.23}$$

with the constraint given by Eq. (2.22) above. The fluctuating field $\mathbf{t}$ corresponds to the coarse-grained vortex density which determines the magnetic induction $\mathbf{B}$ in the material. The chemical potential, μ_r, of the relativistic bosons corresponds to the external applied field which gives rise to a net density of field-induced lines. The physical interpretation becomes clear by discussing the mean-field saddle point solution, obtained by applying the stationarity condition to the free energy, with the result

$$\left(\frac{\delta \mathcal{F}_r}{\delta \mathbf{t}_\perp}\right)_{\mathbf{t}=\mathbf{t}_0, \rho=\rho_0} = 0 \Longrightarrow \mathbf{t}_{\perp 0} = 0 \ , \tag{2.24}$$

$$\left(\frac{\delta \mathcal{F}_r}{\delta \rho}\right)_{\mathbf{t}=\mathbf{t}_0, \rho=\rho_0} = 0 \Longrightarrow t_{z0} = \pm c\rho_0 \ , \tag{2.25}$$

$$\left(\frac{\delta \mathcal{F}_r}{\delta t_z}\right)_{\mathbf{t}=\mathbf{t}_0, \rho=\rho_0} = 0 \Longrightarrow H_0 = \pm\Big(H_{c1} + \phi_0\rho_0\Big) \ . \tag{2.26}$$

Conversely, the equilibrium solution is given by

$$\rho_0 = \frac{|H_0| - H_{c1}}{\phi_0} \ , \quad \text{for} \ |H_0| > H_{c1}, \qquad \rho_0 = 0 \ , \quad \text{for} \quad |H_0| \le H_{c1}, \tag{2.27}$$

where $H_{c1}\epsilon_1 4\pi/\phi_0 \leftrightarrow mc^2$. In other words, ρ_0 is a measure of the number density of *directed* field-induced vortices. The field ρ is a scalar and it is defined to be always positive (for simplicity, in the following we will consider the case of $H_0 > 0$, corresponding to the + signs in Eqs. (2.25) and (2.26). The field t_z is proportional to the z component of the magnetic induction. Its equilibrium value in a spatially homogeneous system is simply proportional to ρ_0 because on large scales the contribution from vortices precisely cancels that of antivortices, and $t_{z0} = c\rho_0 = cB_{z0}/\phi_0$. Locally, the fields t_z and ρ can, however, fluctuate independently, allowing for spontaneous vortex loop fluctuations, independent of the externally applied field. As will be seen more explicitely in the next section, the field ρ mediates the renormalization of the single-vortex stiffness due to such spontaneous loop fluctuations.

The hydrodynamic free energy given in Eq. (2.23) provides a starting point for describing the long wavelength properties of a liquid of interacting (*directed*) field-induced vortex lines and oriented vortex loops. It should be stressed that the distinction between directed lines and loops, while physically appealing, is strictly a single-line notion and loses much of its meaning in a continuum theory. At the level of hydrodynamics, spontaneous vortex loop fluctuations are incorporated via the field $t_z^L = t_z - \rho$. As we will see below, one can construct an effective theory where loops are integrated out. Their role then enters as a renormalization of the vortex-line tension.

III. CORRELATIONS

It is useful to evaluate the two-point correlation function of the hydrodynamic fields appearing in the free energy of Eq. (2.23). This is easily done within a Gaussian approximation for the free energy, obtained by introducing fluctuations of the fields about their equilibrium values, $\delta\mathbf{t} = \mathbf{t} - \mathbf{t}_0$ and $\delta\rho = \rho - \rho_0$, and expanding the free energy to quadratic order in these fluctuations. For $\rho_0 > 0$, the Gaussian free energy is given by

$$\mathcal{F}_r^G[\delta\mathbf{t},\delta\rho] = F_0 + \frac{1}{2\Omega L}\sum_{\mathbf{q}}\Big\{\Big[\frac{\epsilon_1}{\rho_0}+\frac{(k_BT)^2}{4\epsilon_1\rho_0}q^2\Big]|\delta\rho(\mathbf{q})|^2 + \Big[\frac{\epsilon_1}{\rho_0}+V(q)\Big]|\delta\mathbf{t}(\mathbf{q})|^2 \\ -\frac{\epsilon_1}{\rho_0}\Big[\delta t_z(\mathbf{q})\delta\rho(-\mathbf{q})+\delta t_z(-\mathbf{q})\delta\rho(\mathbf{q})\Big]\Big\}\;, \tag{3.1}$$

where averages over the free energy $\mathcal{F}_r^G$ are to be evaluated with the constraint

$$i\mathbf{q}\cdot\delta\mathbf{t}(\mathbf{q}) = 0\;. \tag{3.2}$$

In Eq. (3.1), F_0 is the equilibrium value, $V(q) = 4\pi\epsilon_0\tilde{\lambda}^2/(1+q^2\tilde{\lambda}^2)$, and $q^2 = q_\perp^2 + q_z^2$. It is instructive to integrate out $\delta\rho$ to obtain the effective free energy,

$$\mathcal{F}_r^{\rm eff}[\delta t_z,\mathbf{t}_\perp] = \frac{1}{2\Omega L}\sum_{\mathbf{q}}\Big\{\Big[\frac{(k_BT)^2q^2/4\rho_0}{\epsilon_1+(k_BT)^2q^2/4\epsilon_1}+V(q)\Big]|\delta t_z(\mathbf{q})|^2 + \Big[\frac{\epsilon_1}{\rho_0}+V(q)\Big]|\mathbf{t}_\perp(\mathbf{q})|^2\Big\}\;, \tag{3.3}$$

which is written entirely in terms of fluctuations in the local induction, as at long wavelengths,

$$\delta\mathbf{B}\approx\phi_0\Big(\mathbf{t}_\perp,\delta t_z\Big)\;. \tag{3.4}$$

Finally, by separating $\mathbf{t}_\perp$ in its components longitudinal and transverse to $\hat{\mathbf{q}}_\perp = \mathbf{q}_\perp/q_\perp$ according to

$$\mathbf{t}_\perp(\mathbf{q}) = \hat{\mathbf{q}}_\perp t_\perp^{\rm L}(\mathbf{q}) + (\hat{z}\times\hat{\mathbf{q}}_\perp)t_\perp^{\rm T}(\mathbf{q})\;, \tag{3.5}$$

and using the constraint (3.2) to eliminate $t_\perp^{\rm L}$ in favor of δt_z we obtain,

$$\mathcal{F}_r^{\rm eff}[\delta t_z,\mathbf{t}_\perp] = \frac{1}{2\Omega L}\sum_{\mathbf{q}}\Big\{\Big[\frac{\epsilon_1 q_z^2}{\rho_0 q_\perp^2}+\frac{\epsilon_1}{\rho_0}\frac{(k_BT)^2q^2/4\epsilon_1^2}{1+(k_BT)^2q^2/4\epsilon_1^2}+V(q)\frac{q_z^2}{q_\perp^2}\Big]|\delta t_z(\mathbf{q})|^2 + \Big[\frac{\epsilon_1}{\rho_0}+V(q)\Big]|t_\perp^{\rm T}(\mathbf{q})|^2\Big\}\;. \tag{3.6}$$

Comparing the effective free energy of Eq. (3.6) to the conventional Gaussian hydrodynamic free energy [30,7] obtained neglecting relativistic effects and given by

$$\mathcal{F}^G[\delta t_z,\mathbf{t}_\perp] = \frac{1}{2\Omega L}\sum_{\mathbf{q}}\Big\{\Big[\frac{\epsilon_1 q_z^2}{\rho_0 q_\perp^2}+V(q)\frac{q_z^2}{q_\perp^2}\Big]|\delta t_z(\mathbf{q})|^2 + \Big[\frac{\epsilon_1}{\rho_0}+V(q)\Big]|t_\perp^{\rm T}(\mathbf{q})|^2\Big\}\;, \tag{3.7}$$

it is evident that "relativistic" effects yield short wavelength corrections to the single-vortex effective tension. Similarly, in the boson system they are responsible for corrections to the quasi-particle spectrum due to spontaneous particle-antiparticle pair creation, Again, these effects are important only at finite wavevector.

It is now straightforward to evaluate the Gaussian two-point correlation functions, with the result

$$\langle\delta t_z(\mathbf{q})\delta t_z(-\mathbf{q})\rangle_G = \frac{\rho_0 k_BT q_\perp^2}{\rho_0 V(q)q^2+\epsilon_1 q_z^2+\epsilon_1 q_\perp^2\frac{(k_BT)^2q^2/4\epsilon_1^2}{1+(k_BT)^2q^2/4\epsilon_1^2}}\;, \tag{3.8}$$

$$\langle\delta\rho(\mathbf{q})\delta\rho(-\mathbf{q})\rangle_G = \frac{\rho_0 k_BT q^2}{\rho_0 V(q)q^2+\epsilon_1 q_z^2+\epsilon_1 q_\perp^2\frac{(k_BT)^2q^2/4\epsilon_1^2}{1+(k_BT)^2q^2/4\epsilon_1^2}}\;\frac{1+\rho_0V(q)/\epsilon_1}{1+(k_BT)^2q^2/4\epsilon_1^2}\;, \tag{3.9}$$

$$\langle\delta t_z(\mathbf{q})\delta\rho(-\mathbf{q})\rangle_G = \frac{\rho_0 k_BT q_\perp^2}{\rho_0 V(q)q^2+\epsilon_1 q_z^2+\epsilon_1 q_\perp^2\frac{(k_BT)^2q^2/4\epsilon_1^2}{1+(k_BT)^2q^2/4\epsilon_1^2}}\;\frac{1}{1+(k_BT)^2q^2/4\epsilon_1^2}\;. \tag{3.10}$$

The correlations of the in-plane part $\mathbf{t}_\perp$ of the tilt field are given by

$$\langle t_i(-\mathbf{q})t_j(\mathbf{q})\rangle_G = T_T^0(\mathbf{q})P_{ij}^T(\mathbf{q}_\perp) + T_L^0(\mathbf{q})P_{ij}^L(\mathbf{q}_\perp) \;, \tag{3.11}$$

with

$$T_T^0(\mathbf{q}) = \frac{\rho_0 k_B T}{\epsilon_1 + \rho_0 V(q)} \;, \tag{3.12}$$

and

$$T_L^0(\mathbf{q}) = \frac{q_z^2}{q_\perp^2}\langle t_z(-\mathbf{q})t_z(\mathbf{q})\rangle_G \;. \tag{3.13}$$

$P_{ij}^L(\mathbf{q}_\perp) = \hat{q}_{\perp i}\hat{q}_{\perp j}$ and $P_{ij}^T(\mathbf{q}_\perp) = \delta_{ij} - P_{ij}^L(\mathbf{q}_\perp)$ are the in-plane longitudinal and transverse projection operators respectively. To Gaussian order, the transverse part of the tilt field autocorrelator is the same as that obtained from the familiar hydrodynamics of directed lines.

The structure function, given by the autocorrelator of fluctuations in the vortex line density, δt_z, differs from its "non-relativistic" counterpart only at finite wave-vectors. In the hydrodynamic limit it simply reduces to the familiar result obtained using the ("non-relativistic") hydrodynamics of directed lines,

$$\langle \delta t_z(\mathbf{q})\delta t_z(-\mathbf{q})\rangle_G \approx \frac{\rho_0 k_B T q_\perp^2}{\rho_0 V(q) q^2 + \epsilon_1 q_z^2} \;. \tag{3.14}$$

When "relativistic effects" are included the two fields ρ and t_z are no longer independent. It is then instructive to also consider fluctuations in a new field defined as their difference as $t_z^L = t_z - \rho$. This field fluctuates about a zero equilibrium value and vanishes identically when relativistic effects are neglected. Fluctuations in t_z^L may be interpreted as a measure of fluctuations due to the spontaneous excitation of vortex loops. Its correlation function is given by

$$\langle \delta t_z^L(\mathbf{q})\delta t_z^L(-\mathbf{q})\rangle_G = \frac{\rho_0 k_B T}{\rho_0 V(q) q^2 + \epsilon_1 q_z^2 + \epsilon_1 q_\perp^2 \frac{(k_BT)^2 q^2/4\epsilon_1^2}{1+(k_BT)^2q^2/4\epsilon_1^2}} \, \frac{q_z^2 + \rho_0 V(q) q^2/\epsilon_1}{1 + (k_BT)^2 q^2/4\epsilon_1^2} \;. \tag{3.15}$$

The long wavelength limit of this correlation function is given by

$$\lim_{q_z\to 0}\lim_{q_\perp\to 0}\langle \delta t_z^L(-\mathbf{q})\delta t_z^L(\mathbf{q})\rangle_G = \frac{k_B T}{\epsilon_1} \;, \tag{3.16}$$

irrespective of the order of the limits ($q_z \to 0$ first or $q_\perp \to 0$ first). In other words we have identified a correlation function that at long wavelengths yields the inverse single-line tension, ϵ_1. Calculating perturbative corrections to this Gaussian correlator will be a way to probe the renormalization of the single-line tension.

IV. DISCUSSION

We have presented a long wavelength (hydrodynamic) description of a liquid of arbitrarly curved vortex line and loops that describes on the same footing both field-induced and spontaneously generated vortices. The long-wavelength model was obtained by exploiting the mapping of such a vortex liquid onto a gas of relativistic charged bosons in $2D$ that was recently discussed by Tešanović [10,11] and by Sudbø and collaborators [18,16,17]. Although our model and that of Tešanović [11] yield free energies that are formally very similar, the two models differ in one important respect. In Ref. [11], the matter field Φ that describes the relativistic quantum bosons corresponds to a *fictitious* vortex system of *zero total vorticity.* The field-induced vortices are separated out of the field Φ. In our model, in contrast, the boson field Φ corresponds to the *actual* flux-line system, including both field-induced and spontaneously generated vortices. The externally applied magnetic field $\mathbf{H}$ which yields a non-zero vorticity enters explicitely in the free energy and controls the net mean flux threading the superconductor. It corresponds to a chemical potential for the bosons which yields a non-zero average charge. Furthermore, Tešanović's fictitious vortex loops have a short-range scalar steric repulsion which enters as a Φ^4 coupling in the action. This is in addition to the usual screened magnetic interaction and results from the singular gauge transformation which gives rise to the fictitious vortex loops. Such a steric repulsion does not exist among the *actual* vortex loops [1,22] and therefore it is absent in our model. In our action, vortices only interact through the long range magnetic interaction (screened by the Chern-Simons term) which is cutoff at distances $\leq \xi$ to ensure a finite repulsive energy barrier at short distances.

An important property of the flux-line array which provides a direct measure of vortex correlations along the direction of the applied external field is the tilt modulus. A drawback of the hydrodynamics of directed flux-lines familiar from the literature [8] is that it has been impossible to separate the renormalization of the single-vortex part of the tilt modulus (related to the single-line tension) from that of the compressional part. The hydrodynamic model that we present here allows us to probe the renormalization of the single-line tension through the autocorrelator of the t_z^L field. This is an interesting direction for future work as the vanishing of the effective line tension is considered to be a signature of the Φ-transition suggested by Tešanović [11] and by Sudbø and collaborators [18,16,17].

Finally, the hydrodynamic model of *directed* flux-line liquids has successfully been used to evaluate the renormalization of the tilt modulus from columnar defects parallel to the external magnetic field [37]. This type of quenched disorder suppresses the wandering of the flux-lines away from their average direction and it renders the directed line approximation appropriate. In contrast, other types of quenched disorder, namely point defects or splayed columnar defects with large splay, enhance vortex wandering, so that the resulting vortex liquid can no longer be described as a liquid of directed lines. The hydrodynamic model introduced here is particularly appropriate for studying the effect of this kind of disorder, which can be modelled by a random potential coupled to the field ρ. This is another interesting direction for future work.

This work was supported by the National Science Foundation through Grant No. DMR97-30678. It is a great pleasure to acknowledge valuable discussions with Steve Teitel. PB acknowledges the hospitality of the Physics Department of the Technical University of Munich where part of this work was completed.

APPENDIX: ANISOTROPIC SUPERCONDUCTORS

Our discussion has so far been limited to isotropic materials. High-T_c superconductors are, however, layered materials where the anisotropy can play an important role and change substantially vortex behavior. For this reason in this section we generalize our model to include finite anisotropy. As usual, anisotropy is incorporated in the Ginzburg-Landau free energy via an anisotropic effective mass tensor which leads to different values for the penetration and coherence lengths in the ab plane and in the c direction [1].

We first consider a uniaxial superconductor in an external field $\mathbf{H}$ applied along the c axis chosen as the z direction. The energy of a single vortex line with position parametrized as $\mathbf{r} = \left[\mathbf{r}_\perp(z), z\right]$ ($\mathbf{r}_\perp(z)$ can be a multivalued function of z to allow for the possibility of overhangs) wandering between points a and b is given by

$$\tilde{U}_{self} \approx \epsilon_1 \int_a^b \sqrt{(dz)^2 + \frac{1}{p^2}(d\mathbf{r}_\perp)^2}\ , \tag{4.1}$$

where $p = \lambda_c/\lambda_{ab}$ is the familiar anisotropy parameter. In copper-oxide high-T_c materials, $p >> 1$. By comparing Eq. (4.1) to the action of a free relativistic particle given in Eq. (2.1), we see that a vortex line in a uniaxial material can be interpreted as the world line of a relativistic particle which moves in an “anisotropic space-time” with rescaled spatial coordinates, $\mathbf{r}_\perp \to \mathbf{r}_\perp/p$. At the single-vortex level, anisotropy effectively *enhances* the “relativistic” behavior, as it reduces the energy per unit length associated with transverse vortex fluctuations, as expected for a layered material. In this context, it may be tempting to interpret anisotropy as responsible for an effective reduction of the speed of light in the relativistic $2D$ boson problem with $c \to c/p$. This interpretation is, however, misleading as it does not carry through when interactions are included. In the interaction part of the vortex-line free energy, anisotropy does not simply lead to a rescaling of the transverse coordinates. This is apparent from the fact that the collective part of the wavevector-dependent elastic constants of a vortex lattice in a uniaxial material is not simply obtained from the corresponding elastic constants of an isotropic material with the replacement $q_\perp \to q_\perp p$.

In the interaction and the free field part of the boson model, the role of anisotropy is simply that of allowing for different scalar and transverse interactions among the bosons, precisely as originally proposed by Feigel'man and collaborators [29]. Note that the aforementioned parts in our relativistic boson action are formally the same as the corresponding parts in Feigel'man's boson action - our model differs in the free particle part only. The relativistic boson action that maps onto the free energy of interacting vortex lines and loops in a uniaxial superconductor with an external field applied along the c axis is given by

$$\tilde{\mathcal{S}}_r[\Phi,\Phi^*,\mathbf{a},\mathbf{A}] = \int_0^{\beta\hbar} d\tau \int d^2\mathbf{r}_\perp \Big\{ \frac{mc^2}{2}\Phi\Phi^* + \frac{1}{2m_b c^2}\left[(-i\hbar\partial_\tau + e_b a_0 + i\mu_r)\Phi\right]\left[(i\hbar\partial_\tau + e_b a_0 + i\mu_r)\Phi^*\right] + \frac{p^2}{2m_b}|(-i\hbar\nabla_\perp + \frac{e_b}{c}\mathbf{a}_\perp)\Phi|^2 + \tilde{\mathcal{L}}_F[\mathbf{a},\mathbf{A}] \Big\}\ , \tag{4.2}$$

where

$$\tilde{\mathcal{L}}_F[\mathbf{a},\mathbf{A}] = \frac{1}{8\pi}\Big\{ p^2(\nabla_\perp\times\mathbf{a}_\perp)^2 + \left[\hat{\mathbf{z}}\times(\frac{1}{c}\partial_\tau\mathbf{a}_\perp - \nabla a_0)\right]^2 + \frac{2i}{\lambda_b}(\nabla\times\mathbf{a})\cdot\mathbf{A} + (\nabla\times\mathbf{A})^2 \Big\}\ . \tag{4.3}$$

We stress that in this mapping the boson mass, m_b, corresponds to the vortex *line energy*, ϵ_1, precisely as indicated in Table 1. This is indeed the appropriate interpretation as it is made apparent by comparing the single-vortex energy given in Eq. (4.1) to the action of a relativistic boson, given in Eq. (2.1). In contrast, in all the previous literature, and particularly in the work by Nelson and coworkers, it is the tilt energy per unit length, $\tilde{\epsilon}_1 = \epsilon_1/p^2$, that is interpreted as the boson mass.

Finally, by using the methods described in Section III and the translation Table 1, one can immediately obtain the hydrodynamic free energy of a liquid of arbitrarly fluctuating vortex lines and loops in a uniaxial superconductor with $\mathbf{H} \parallel c$. It is given by

$$\tilde{\mathcal{F}}_r[\mathbf{t},\rho] = \int_{\mathbf{r}} \left\{ \frac{(k_BT)^2}{8\tilde{\epsilon}_1\rho}(\nabla_\perp\rho)^2 + \frac{(k_BT)^2}{8\epsilon_1\rho}(\partial_z\rho)^2 + \frac{\tilde{\epsilon}_1}{2\rho}(\mathbf{t}_\perp)^2 + \frac{\epsilon_1}{2\rho}(t_z)^2 + \frac{\epsilon_1}{2}\rho - \mu_r t_z \right\} + \frac{1}{2\Omega}\sum_{\mathbf{q}} \left\{ \frac{4\pi\epsilon_0\tilde{\lambda}_\perp^2}{1+q_z^2\tilde{\lambda}_\perp^2+q_\perp^2p^2\tilde{\lambda}_\perp^2}|\mathbf{t}_\perp(\mathbf{q})|^2 + \frac{4\pi\epsilon_0\tilde{\lambda}_\perp^2(1+q^2p^2\tilde{\lambda}_\perp^2)}{(1+q^2\tilde{\lambda}_\perp^2)(1+q_z^2\tilde{\lambda}_\perp^2+q_\perp^2p^2\tilde{\lambda}_\perp^2)}|t_z(\mathbf{q})|^2 \right\} \tag{4.4}$$

with $\tilde{\epsilon}_1 = \epsilon_1/p^2$ and the familiar constraint $\nabla\cdot\mathbf{t} = 0$.

---

[1] G. Blatter *et al.*, Rev. Mod. Phys. **66**, 1125 (1994).
[2] G. W. Crabtree and D. R. Nelson, Phys. Today **50(4)**, 38 (1997).
[3] D. R. Nelson, Phys. Rev. Lett. **60**, 1973 (1988).
[4] M. Fisher, Phys. Rev. Lett. **62**, 1415 (1989).
[5] D. R. Nelson and H. S. Seung, Phys. Rev. B **39**, 9153 (1989).
[6] D. R. Nelson and V. N. Vinokur, Phys. Rev. B **48**, 13 060 (1993).
[7] M. C. Marchetti and D. R. Nelson, Physica C **174**, 40 (1991).
[8] P. Benetatos and M. C. Marchetti, Phys. Rev. B **59(9)**, 6499 (1999).
[9] M. C. Marchetti and D. R. Nelson, Physica C **330**, 105 (2000).
[10] Z. Tešanović, Phys. Rev. B **51**, 16204 (1995).
[11] Z. Tešanović, Phys. Rev. B **59**, 6449 (1999).
[12] A. K. Nguyen, A. Sudbø, and R. E. Hetzel, Phys. Rev. Lett. **77**, 1592 (1996).
[13] A. K. Nguyen and A. Sudbø, Phys. Rev. B **57**, 3123 (1998).
[14] A. K. Nguyen and A. Sudbø, Phys. Rev. B **58**, 2802 (1998).
[15] S. Chin, A. K. Nguyen, and A. Sudbø, Phys. Rev. B **59**, 14 017 (1999).
[16] J. Hove and A. Sudbø, Phys. Rev. Lett. **84**, 3426 (2000).
[17] J. Hove, S. Mo, and A. Sudbø, Phys. Rev. Lett. **85**, 2368 (2000).
[18] A. K. Nguyen and A. Sudbø, Phys. Rev. B **60**, 15 307 (1999).
[19] M. Kiometzis, H. Kleinert, and A. M. J. Schakel, Prog. of Phys. **43**, 697 (1995).
[20] W. Barford and J. Gunn, Physica C **156**, 515 (1988).
[21] M. C. Marchetti, Phys. Rev. B **43**, 8012 (1991).
[22] E. H. Brandt, Int. J. Mod. Phys. B **5(5)**, 751 (1991).
[23] M. P. A. Fisher and D. H. Lee, Phys. Rev. B **39**, 2756 (1989).
[24] D. R. Nelson, J. Stat. Phys. **57**, 511 (1989).
[25] U. C. Täuber and D. R. Nelson, Phys. Rep. **289**, 157 (1997).
[26] For a discussion of the symmetry properties of the nonrelativistic boson model, see Ref. [25].
[27] Here all bosons have the same charge, either positive or negative. In contrast, the relativistic boson system discussed below consists of a mixture of positively and negatively charged bosons.
[28] M. V. Feigel'man, V. B. Geshkenbein, and V. M. Vinokur, Sov. Phys. - JETP Lett. **52**, 546 (1990).
[29] M. V. Feigel'man, V. B. Geshkenbein, L. B. Ioffe, and A. I. Larkin, Phys. Rev. B **48**, 16 641 (1993).
[30] M. C. Marchetti and D. R. Nelson, Phys. Rev. B **42**, 9938 (1990).
[31] H. Kleinert, *Gauge Fields in Condensed Matter, Vol. 1* (World Scientific, Singapore, 1989).
[32] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, Oxford, 1975).
[33] For a continuum proof of the equivalence of the quantum-mechanical path integral of charged spin-0 particles coupled to an external electromagnetic field to the the field theory of a complex massive scalar field minimally coupled to the electromagnetic potential see, for instance, K. Bardakci and S. Samuel, Phys. Rev. D **18**, 2849 (1978).
[34] J. I. Kapusta, *Finite-Temperature Field Theory* (Cambridge University Press, Cambridge, 1989).

[35] J. I. Kapusta, Phys. Rev. D **24**, 426 (1981).
[36] The chemical potential, μ, customarily defined in non-relativistic many-body theory, is related to our μ_r by $\mu = \mu_r - mc^2$ (for $\mu_r \geq 0$, chosen here to correspond to particles). The equilibrium solution is then given by $\rho_0 = 0$ for $\mu \leq 0$, and $\rho_0 = \mu/(4\pi\lambda_b^2 e_b^2)$ for $\mu > 0$.
[37] P. Benetatos and M. C. Marchetti, Physica C **332**, 237 (2000).