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# ON A CONJECTURE OF AUSLANDER AND REITEN

CRAIG HUNEKE AND GRAHAM J. LEUSCHKE

ABSTRACT. In studying Nakayama's 1958 conjecture on rings of infinite dominant dimension, Auslander and Reiten proposed the following generalization: Let  $\Lambda$  be an Artin algebra and  $M$  a  $\Lambda$ -generator such that  $\text{Ext}_{\Lambda}^i(M, M) = 0$  for all  $i \geq 1$ ; then  $M$  is projective. This conjecture makes sense for any ring. We establish Auslander and Reiten's conjecture for excellent Cohen–Macaulay normal domains containing the rational numbers, and slightly more generally.

## 0. INTRODUCTION

The generalized Nakayama conjecture of M. Auslander and I. Reiten is as follows [4]: For an Artin algebra  $\Lambda$ , every indecomposable injective  $\Lambda$ -module appears as a direct summand in the minimal injective resolution of  $\Lambda$ . Equivalently, if  $M$  is a finitely generated  $\Lambda$ -generator such that  $\text{Ext}_{\Lambda}^i(M, M) = 0$  for all  $i \geq 1$ , then  $M$  is projective. This latter formulation makes sense for any ring, and Auslander, S. Ding, and Ø. Solberg [3] widened the context to algebras over commutative local rings.

**Conjecture (AR).** *Let  $\Lambda$  be a Noetherian ring finite over its center and  $M$  a finitely generated left  $\Lambda$ -module such that  $\text{Ext}_{\Lambda}^i(M, \Lambda) = \text{Ext}_{\Lambda}^i(M, M) = 0$  for all  $i > 0$ . Then  $M$  is projective.*

In the same paper, Auslander and Reiten proved AR for modules  $M$  that are *ultimately closed*, that is, there is some syzygy  $N$  of  $M$  all of whose indecomposable direct summands already appear in some previous syzygy of  $M$ . This includes all modules over rings of finite representation type, all rings  $\Lambda$  such that for some integer  $n$ ,  $\Lambda$  has only a finite number of indecomposable summands of  $n^{\text{th}}$  syzygies, and all rings of radical square zero.

Auslander, Ding, and Solberg [3, Proposition 1.9] established AR in case  $\Lambda$  is a quotient of a ring  $\Gamma$  of finite global dimension by a regular sequence. In fact, in this case they prove something much

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stronger: If  $\text{Ext}_\Lambda^2(M, M) = 0$ , then  $\text{pd}_\Lambda M < \infty$  [3, Proposition 1.8]. This in turn was generalized by L. Avramov and R.-O. Buchweitz [5, Theorem 4.2]: A finite module  $M$  over a (commutative) complete intersection ring  $R$  has finite projective dimension if and only if  $\text{Ext}_R^{2i}(M, M) = 0$  for some  $i > 0$ .

M. Hoshino [9] proved that if  $R$  is a symmetric Artin algebra with radical cube zero, then  $\text{Ext}_R^1(M, M) = 0$  implies that  $M$  is free. Huneke, L.M. Şega, and A.N. Vraciu have recently extended this to prove that if  $R$  is Gorenstein local with  $\mathfrak{m}^3 = 0$ , and if  $\text{Ext}_R^i(M, M) = 0$  for some  $i \geq 1$ , then  $M$  is free, and have further verified the Auslander-Reiten conjecture for all finitely generated modules  $M$  over Artinian commutative local rings  $(R, \mathfrak{m})$  such that  $\mathfrak{m}^2 M = 0$  [11]. In particular, this verifies the Auslander-Reiten conjecture for commutative local rings with  $\mathfrak{m}^3 = 0$ .

The assumption that  $\Lambda$  be finite over its center is essential, given a counterexample due to R. Schultz [15].

Our main theorem establishes the AR conjecture for a class of commutative Cohen–Macaulay rings and well-behaved modules. Moreover, our result is effective; we can specify how many Ext are needed to vanish to give the conclusion of AR.

**Main Theorem.** *Let  $R$  be a Cohen–Macaulay ring which is a quotient of a locally excellent ring  $S$  of dimension  $d$  by a locally regular sequence. Assume that  $S$  is locally a complete intersection ring in codimension one, and further assume either that  $S$  is Gorenstein, or that  $S$  contains the field of rational numbers. Let  $M$  be a finitely generated  $R$ -module of constant rank such that*

$$(1) \quad \begin{aligned} \text{Ext}_R^i(M, M) &= 0 \quad \text{for } i = 1, \dots, d, \text{ and} \\ \text{Ext}_R^i(M, R) &= 0 \quad \text{for } i = 1, \dots, 2d + 1. \end{aligned}$$

*Then  $M$  is projective.*

The restriction imposed on  $R$  by assuming that  $S$  be locally complete intersection in codimension one is equivalent to assuming that  $R$  is a quotient by a regular sequence of some normal domain  $T$ , by [10, Theorem 3.1]. However, replacing  $S$  by  $T$  according to the construction in [10] would increase  $d$ , the number of Ext required to vanish. In any case, this observation gives the following corollary.

**Theorem 0.1.** *Let  $R$  be a Cohen–Macaulay ring which is a quotient of a locally excellent ring  $S$  of dimension  $d$  by a locally regular sequence. Assume that  $S$  is locally a complete intersection ring*

in codimension one, and further assume either that  $S$  is Gorenstein, or that  $S$  contains the field of rational numbers. Then the AR conjecture holds for all finitely generated  $R$ -modules, that is, if  $\text{Ext}_R^i(M, R) = \text{Ext}_R^i(M, M) = 0$  for all  $i > 0$ , then  $M$  is projective.

Not every zero-dimensional ring  $R$  is a factor of a ring  $S$  as in the theorem, since not all Artinian local rings can be smoothed. For example, Anthony Iarrobino has pointed out that the easiest such example is a polynomial ring in four variables modulo an ideal generated by seven general quadrics (note, however, that the cube of the maximal ideal of such a ring is zero, so this case is covered by [11]). For other examples of non-smoothable rings, see Mumford [14].

In the next section we prove some preliminary lemmas, and then prove the main result. This requires extra work regarding the trace of a module. Since we could not find a satisfactory reference for what we needed, we include basic facts concerning the trace in an appendix.

Throughout the following, all rings are Noetherian and all modules finitely generated. For an  $R$ -module  $M$ , we define the *dual* of  $M$  by  $M^* = \text{Hom}_R(M, R)$ . There is a natural homomorphism  $\theta_M : M \rightarrow M^{**}$  defined by sending  $x \in M$  to “evaluation at  $x$ ”. We say that  $M$  is *torsion-free* if  $\theta_M$  is injective, and *reflexive* if  $\theta_M$  is an isomorphism. It is known (cf. [2, Theorem 2.17], for example) that  $M$  is torsion-free if and only if  $M$  is a first syzygy, and reflexive if and only if  $M$  is a second syzygy. We will say that a torsion-free  $R$ -module  $M$  *has constant rank* if  $M$  is locally free of constant rank at the minimal primes of  $R$ . This is equivalent to  $K \otimes_R M$  being a free  $K$ -module, where  $K$  is the total quotient ring of  $R$  obtained by inverting all nonzerodivisors.

## 1. PROOF OF THE MAIN THEOREM

We begin by observing that the vanishing of  $\text{Ext}$  and the projectivity of  $M$  are both local questions, so that in proving our main theorem we may assume that both  $S$  and  $R$  are local. Furthermore, since  $S$  is assumed to be excellent we can (and do) complete  $S$  at its maximal ideal without loss of generality.

Next we point out the following consequence of the lifting criterion of Auslander, Ding, and Solberg [3, Proposition 1.6].

**Lemma 1.1.** *Let  $S$  be a complete local ring,  $x \in S$  a nonunit nonzerodivisor, and  $R = S/(x)$ . Assume that there exists  $t \geq 2$  such that for any  $S$ -module  $N$ ,  $\text{Ext}_S^t(N, N) = \text{Ext}_S^t(N, S) = 0$  for*

$i = 1, \dots, t$  implies that  $N$  is free. Then for any  $R$ -module  $M$ ,  $\text{Ext}_R^i(M, M) = \text{Ext}_R^i(M, R) = 0$  for  $i = 1, \dots, t$  implies that  $M$  is free. Furthermore, if  $AR$  holds for  $S$ -modules then it holds for  $R$ -modules.

*Proof.* Let  $M$  be an  $R$ -module such that  $\text{Ext}_R^i(M, M) = \text{Ext}_R^i(M, R) = 0$  for  $i = 1, \dots, t$ . Then in particular  $\text{Ext}_R^2(M, M) = 0$ , and so by [3, Proposition 1.6] there exists an  $S$ -module  $N$  on which  $x$  is a nonzerodivisor and such that  $R \otimes_S N \cong M$ . Apply  $\text{Hom}_S(-, N)$  to the short exact sequence  $0 \rightarrow N \rightarrow N \rightarrow M \rightarrow 0$  and use the fact that  $\text{Ext}_S^{i+1}(M, N) \cong \text{Ext}_R^i(M, M) = 0$  for  $i = 1, \dots, t$  to see that multiplication by  $x$  is surjective on  $\text{Ext}_S^i(N, N)$  for  $i = 1, \dots, t$ . Then Nakayama's Lemma implies that  $\text{Ext}_S^i(N, N) = 0$  for  $i = 1, \dots, t$ . The same argument, applying  $\text{Hom}_S(-, S)$  and observing that  $\text{Ext}_S^{i+1}(M, S) \cong \text{Ext}_R^i(M, R) = 0$  for  $i = 1, \dots, t$ , shows that  $\text{Ext}_S^i(N, S) = 0$  for  $i = 1, \dots, t$  as well. Since this forces  $N$  to be  $S$ -free,  $M$  is  $R$ -free.

Finally, repeating the argument with “all  $i \geq 1$ ” in place of “ $i = 1, \dots, t$ ” gives the last statement.  $\square$

With Lemma 1.1 in mind, we now focus on the case  $R = S$  in our main theorem. Indeed, if  $\dim(S) \leq 1$ , then  $S$  is locally a complete intersection ring by hypothesis, and hence  $R$  is as well. By [3, Proposition 1.9], then,  $AR$  holds for  $R$ -modules. So we may assume that  $R = S$ , and in particular we take  $d = \dim R$ . Our next goal is to modify the module  $M$ .

**Lemma 1.2.** [4, Lemma 1.4] *In proving the Main Theorem, we may replace  $M$  by  $\text{syz}_R^n(M)$ , where  $n = \max\{2, d + 1\}$ , and assume that  $M$  is reflexive and that  $\text{Ext}_R^i(M^*, R) = 0$  for  $i = 1, \dots, d$ . In proving  $AR$ , we may replace  $M$  by any syzygy module  $\text{syz}_R^t(M)$ .*

*Proof.* Put  $N = \text{syz}_R^n(M)$ . It is a straightforward computation with the long exact sequences of  $\text{Ext}$  to show that if  $\text{Ext}_R^i(M, M) = 0$  for  $i = 1, \dots, d$  and  $\text{Ext}_R^i(M, R) = 0$  for  $i = 1, \dots, 2d + 1$ , then  $\text{Ext}_R^i(N, N) = 0$  for  $i = 1, \dots, d$  and  $\text{Ext}_R^i(N, R) = 0$  for  $i = 1, \dots, d$ . Assume, then, that we have shown that  $N$  is free. Then since  $\text{Ext}_R^n(M, R) = 0$ , the  $n$ -fold extension of  $M$  by  $N$  consisting of the free modules in the resolution of  $M$  must split, so  $M$  is free as well. This proves the last statement.

To prove that  $N$  is reflexive and  $\text{Ext}_R^i(N^*, R) = 0$  for  $i = 1, \dots, d$ , one shows by induction on  $t$  that  $\text{Ext}_R^i((\text{syz}_R^t(M))^*, R) = 0$  for  $i = 1, \dots, t$ . For the base case  $t = 2$ , observe that since

$\text{Ext}_R^i(M, R) = 0$  for  $i = 1, 2$ , the dual of the exact sequence

$$(*) \quad 0 \longrightarrow \text{syz}_R^2(M) \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

where  $F_1$  and  $F_0$  are free modules, is still exact. Dualizing again gives  $(*)$  back, so  $N = \text{syz}_R^2(M)$  is reflexive and satisfies  $\text{Ext}_R^i(N^*, R) = 0$  for  $i = 1, 2$ . For the inductive step, dimension-shifting shows that if  $\text{Ext}_R^i(M^*, R) = 0$  for  $i = 1, \dots, t-1$ , then  $\text{Ext}_R^i((\text{syz}_R^1(M))^*, R) = 0$  for  $i = 2, \dots, t$ , and the same argument as above shows that  $\text{Ext}_R^1((\text{syz}_R^1(M))^*, R) = 0$ .  $\square$

It is worth noting that if  $R$  is a Cohen–Macaulay (CM) ring, then  $\text{syz}_R^d(M)$  is a maximal Cohen–Macaulay (MCM) module for any  $M$ . Also, the replacement in Lemma 1.2 has consequences for the assumptions (1) in the main theorem: If  $\text{Ext}_R^i(M, R) = 0$  for  $i = 1, \dots, t$ , then  $\text{Ext}_R^i(\text{syz}_R^1(M), R) = 0$  for  $i = 1, \dots, t-1$ . This observation combines with Lemmas 1.1 and 1.2 to reduce the proof of our main theorem to the following:

**Theorem 1.3.** *Let  $(R, \mathfrak{m})$  be a complete local CM ring of dimension  $d$  which is a complete intersection in codimension one. Assume either that  $R$  is Gorenstein, or that  $R$  contains  $\mathbb{Q}$ . Let  $M$  be a MCM  $R$ -module of constant rank such that for  $i = 1, \dots, d$ ,*

$$(2) \quad \begin{aligned} \text{Ext}_R^i(M, M) &= 0, \\ \text{Ext}_R^i(M, R) &= 0, \quad \text{and} \\ \text{Ext}_R^i(M^*, R) &= 0. \end{aligned}$$

*Then  $M$  is free.*

We postpone the proof of Theorem 1.3 to the end of this section, and establish some preparatory results.

By Cohen’s structure theorem, the complete local ring  $R$  is a homomorphic image of a regular local ring, and so has a canonical module  $\omega$ . Since  $R$  is complete intersection in codimension one, it is in particular Gorenstein at the associated primes, and so  $\omega$  has constant rank. Hence  $\omega$  is isomorphic to an ideal of  $R$ . For a MCM  $R$ -module  $N$ , we write  $N^\vee$  for the canonical dual  $\text{Hom}_R(N, \omega)$ .

We next apply a result found in [6, Corollary B4] (see also [8, Lemma 2.1]):

**Proposition 1.4.** *Let  $R$  be a CM local ring with a canonical module  $\omega$  and let  $N$  be a MCM  $R$ -module. If  $\text{Ext}_R^i(N, R) = 0$  for  $i = 1, \dots, \dim R$  then  $\omega \otimes_R N \cong (N^*)^\vee$  is a MCM  $R$ -module.*

Applied to our current context, this gives the following fact.

**Corollary 1.5.** *Under our assumptions (2) in Theorem 1.3, both  $\omega \otimes_R M$  and  $\omega \otimes_R M^*$  are MCM  $R$ -modules.*

We will also show that the triple tensor product  $\omega \otimes_R M^* \otimes_R M$  is MCM, but for this we use the following lemma. It requires that we add one further assumption to (2): that the module  $M$  in question has constant rank.

**Lemma 1.6.** *Let  $(R, \mathfrak{m}, k)$  be a CM local ring with canonical ideal  $\omega$ , and let  $N$  be a MCM  $R$ -module of constant rank. Assume that  $\text{Hom}_R(N, N)$  is also a MCM  $R$ -module, and that for some maximal regular sequence  $\underline{x}$ , we have*

$$\text{Hom}_R(N, N) \otimes_R R/(\underline{x}) \cong \text{Hom}_{R/(\underline{x})}(N/\underline{x}N, N/\underline{x}N).$$

*Then  $\underline{x}$  is a regular sequence on  $N \otimes_R N^\vee$ . In particular,  $N \otimes_R N^\vee$  is MCM.*

*Proof.* We indicate reduction modulo  $\underline{x}$  by an overline, and use  $\lambda(-)$  for the length of a module. We also continue to use  $-\vee$  for  $\text{Hom}_{\overline{R}}(-, \overline{\omega})$  without fear of confusion. Since  $\overline{\omega} \cong E_{\overline{R}}(k)$ , the injective hull of the residue field of  $\overline{R}$ , we have  $\lambda(M^\vee) = \lambda(M)$  for all  $\overline{R}$ -modules  $M$ .

First, a short computation using Hom-Tensor adjointness:

$$\begin{aligned} (\overline{N} \otimes_{\overline{R}} \overline{N}^\vee)^\vee &= \text{Hom}(\overline{N} \otimes_{\overline{R}} \overline{N}^\vee, \overline{\omega}) \\ &\cong \text{Hom}_{\overline{R}}(\overline{N}, \overline{N}^{\vee\vee}) \\ &\cong \text{Hom}_{\overline{R}}(\overline{N}, \overline{N}) \end{aligned}$$

In particular, this implies that  $\lambda(\overline{N} \otimes_{\overline{R}} \overline{N}^\vee) = \lambda((\overline{N} \otimes_{\overline{R}} \overline{N}^\vee)^\vee) = \lambda(\text{Hom}_{\overline{R}}(\overline{N}, \overline{N}))$ . Since  $\overline{N} \otimes_{\overline{R}} \overline{N}^\vee = \overline{N} \otimes_{\overline{R}} \overline{N}^{\vee\vee}$ , our hypothesis yields  $\lambda(\overline{N} \otimes_{\overline{R}} \overline{N}^\vee) = \lambda(\overline{\text{Hom}_R(N, N)})$ . Finally, we compute, using the

fact that  $N$ ,  $N \otimes_R N^\vee$ , and  $\text{Hom}_R(N, N)$  all have constant rank:

$$\begin{aligned}
\lambda(\overline{N \otimes_R N^\vee}) &= \lambda(\overline{\text{Hom}_R(N, N)}) \\
&= e(\underline{x}, \text{Hom}_R(N, N)) \\
&= \text{rank}(\text{Hom}_R(N, N))e(\underline{x}, R) \\
&= \text{rank}(N)^2 e(\underline{x}, R) \\
&= \text{rank}(N \otimes_R N^\vee) e(\underline{x}, R) \\
&= e(\underline{x}, N \otimes_R N^\vee)
\end{aligned}$$

Here  $e(\underline{x}, \ )$  denotes the multiplicity of the ideal  $(\underline{x})$  on the module. The second equality follows since we have assumed that  $\text{Hom}_R(N, N)$  is also a MCM  $R$ -module. The equality of the first and last items implies that  $N \otimes_R N^\vee$  is MCM by [7, 4.6.11].  $\square$

**Proposition 1.7.** *Let  $(R, \mathfrak{m})$  be a CM local ring with canonical ideal  $\omega$  and let  $M$  be a reflexive  $R$ -module of constant rank such that  $\text{Ext}_R^i(M, M) = \text{Ext}_R^i(M^*, R) = 0$  for  $i = 1, \dots, d = \dim R$ . Then  $\omega \otimes_R M^* \otimes_R M$  is a MCM  $R$ -module.*

*Proof.* We will take  $N = M$  in Lemma 1.6. By Proposition 1.4,  $M^\vee \cong \omega \otimes_R M^*$ , so we need only show that  $\text{Hom}_R(M, M)$  cuts down correctly. Induction on the length of a regular sequence  $\underline{x}$ , using the vanishing of  $\text{Ext}_R^i(M, M)$ , then proves that  $\underline{x}$  is also regular on  $\text{Hom}_R(M, M)$  and that  $\text{Hom}_R(M, M) \otimes_R R/(\underline{x}) \cong \text{Hom}_{R/(\underline{x})}(M/\underline{x}M, M/\underline{x}M)$ , finishing the proof.  $\square$

**Proposition 1.8.** *In addition to the assumptions (2) of Theorem 1.3, suppose also that  $M$  has constant rank. Then  $\omega \otimes_R M^* \otimes_R M$  is a MCM  $R$ -module. Furthermore, the natural homomorphism*

$$1 \otimes \alpha : \omega \otimes_R M^* \otimes_R M \longrightarrow \omega \otimes_R \text{Hom}_R(M, M),$$

where  $\alpha$  is defined by  $\alpha(f \otimes x)(y) = f(y) \cdot x$ , is injective.

*Proof.* The first statement follows immediately from Proposition 1.7. For the second, pass to the total quotient ring  $K$  of  $R$ . Since  $R$  is generically Gorenstein,  $\omega \otimes_R K \cong K$ , and since  $M$  has a rank,  $M \otimes_R K$  is a free  $K$ -module. Since  $\alpha$  is an isomorphism when  $M$  is free, the kernel of  $1 \otimes \alpha$  must be torsion. But  $\omega \otimes_R M^* \otimes_R M$  is MCM, and so torsion-free. Hence the kernel of  $1 \otimes \alpha$  is zero.  $\square$



We return to the assumptions of Theorem 1.3:  $(R, \mathfrak{m}, k)$  is a complete local CM ring with a canonical ideal  $\omega$ , and  $M$  is a torsion-free  $R$ -module of constant rank, satisfying

$$(3) \quad \begin{aligned} \text{Ext}_R^i(M, M) &= 0, \\ \text{Ext}_R^i(M, R) &= 0, \quad \text{and} \\ \text{Ext}_R^i(M^*, R) &= 0, \quad \text{for } i = 1, \dots, d = \dim R. \end{aligned}$$

We also assume that  $R$  is locally a complete intersection ring in codimension one. As we observed above, this implies by the work of Auslander, Ding, and Solberg that  $M$  is locally free in codimension one. We therefore assume  $d \geq 2$ . The following lemma is standard. (See [13, Theorems 16.6, 16.7].)

**Lemma 1.9.** *Let  $(R, \mathfrak{m}, k)$  be a CM local ring of dimension at least 2. Let  $X$  be a MCM  $R$ -module and  $L$  a module of finite length over  $R$ . Then  $\text{Ext}_R^1(L, X) = 0$ .*

Recall from Proposition 1.8 that under the assumptions (3), the homomorphism  $1 \otimes \alpha : \omega \otimes_R M^* \otimes_R M \rightarrow \omega \otimes_R \text{Hom}_R(M, M)$  is injective.

**Lemma 1.10.** *If  $M$  is locally free on the punctured spectrum, then the homomorphism  $1 \otimes \alpha$  is a split monomorphism with cokernel of finite length.*

*Proof.* We have the following exact sequence:

$$(4) \quad 0 \longrightarrow \omega \otimes_R M^* \otimes_R M \xrightarrow{1 \otimes \alpha} \omega \otimes_R \text{Hom}_R(M, M) \longrightarrow C \longrightarrow 0.$$

Since  $M$  is locally free on the punctured spectrum,  $1 \otimes \alpha$  is an isomorphism when localized at any nonmaximal prime of  $R$ , which forces  $C$  to have finite length. Since  $\omega \otimes_R M^* \otimes_R M$  is MCM by Proposition 1.8,  $\text{Ext}_R^1(C, \omega \otimes_R M^* \otimes_R M) = 0$ , and so (4) splits.  $\square$

*Proof of Theorem 1.3.* We will proceed by induction on  $d = \dim R$ . As mentioned above, the case  $d = 1$  follows from [3, Proposition 1.9], so we may assume  $d \geq 2$ , and that the statement is true for all modules over CM local rings matching our hypotheses (3) and having dimension less than that of  $R$ . In particular, we may assume that  $M$  is locally free on the punctured spectrum. Also, we may assume that  $M$  is indecomposable.

First assume that  $R$  is Gorenstein. Then  $\alpha : M^* \otimes_R M \rightarrow \text{Hom}_R(M, M)$  must be a split monomorphism with cokernel of finite length, by Lemma 1.10. Since  $\text{Hom}_R(M, M)$  is torsion-free, this implies  $\alpha$  is an isomorphism, and hence that  $M$  is free.

Next assume that  $R$  is not necessarily Gorenstein, but contains the rationals. Consider the following diagram involving the trace homomorphism (see Appendix A).

$$\begin{array}{ccc} \omega \otimes_R M^* \otimes_R M & \xrightarrow{1 \otimes \alpha} & \omega \otimes_R \operatorname{Hom}_R(M, M) \\ & \searrow 1 \otimes \operatorname{ev} & \downarrow 1 \otimes \operatorname{tr} \\ & & \omega \otimes_R R \end{array}$$

By Lemma A.6, the diagram commutes. Furthermore, by Lemma 1.10,  $1 \otimes \alpha$  is a split monomorphism with finite-length cokernel  $C$ , so  $\omega \otimes_R \operatorname{Hom}_R(M, M)$  has  $C$  as a direct summand and  $1 \otimes \alpha$  is surjective onto the complement. Since  $\omega$  is torsion-free,  $1 \otimes \operatorname{tr}$  must kill  $C$ .

As  $R$  contains  $\mathbb{Q}$ ,  $\operatorname{rank} M$  is invertible and so  $\operatorname{tr}$  is surjective by Corollary A.5. It follows that the composition  $1 \otimes \operatorname{tr} \alpha$  is surjective, so that  $1 \otimes \operatorname{ev}$  is as well. In other words, the evaluation map  $M^* \otimes_R M \rightarrow R$  induces a surjection when tensored with  $\omega$ . By Nakayama's Lemma, then, the evaluation map is surjective, and it follows that  $M$  has a free direct summand. Since  $M$  is indecomposable,  $M$  is free.  $\square$

#### APPENDIX A. THE TRACE OF A MODULE

In this section we give a general description of the trace of a module. Our treatment is intrinsic to the module, and it satisfies the usual properties of a trace defined for torsion-free modules over a normal domain. We include full proofs for convenience.

Throughout this section, let  $R$  be a Noetherian ring with total quotient ring  $K$ ; that is,  $K$  is obtained from  $R$  by inverting all nonzerodivisors. Let  $M$  be a torsion-free  $R$ -module. The *trace of  $M$*  will be a certain homomorphism  $\operatorname{tr} : \operatorname{Hom}_R(M, M) \rightarrow R$ . To define the trace, let

$$\alpha : M^* \otimes_R M \rightarrow \operatorname{Hom}_R(M, M)$$

be the natural homomorphism defined by  $\alpha(f \otimes x)(y) = f(y) \cdot x$ . Note that dualizing  $\alpha$  gives a homomorphism  $\alpha^*$  from  $\operatorname{Hom}_R(M, M)^* = \operatorname{Hom}_R(\operatorname{Hom}_R(M, M), R)$  to  $(M^* \otimes_R M)^* \cong \operatorname{Hom}_R(M^*, M^*)$ . It is known (see [12], for example) that  $\alpha$  is an isomorphism if and only if  $M$  is free.

**Definition A.1.** *Assume that  $\alpha^* : \operatorname{Hom}_R(M, M)^* \rightarrow \operatorname{Hom}_R(M^*, M^*)$  is an isomorphism. The trace of  $M$  is defined by  $\operatorname{tr} = (\alpha^*)^{-1}(1_{M^*})$ . We say in this case that  $M$  has a trace.*

Observe that the target of  $\alpha^*$  is  $(M^* \otimes_R M)^*$ , which we have used Hom-Tensor adjointness to identify with  $\text{Hom}_R(M^*, M^*)$ . Under this identification, the identity map  $M^* \rightarrow M^*$  corresponds to the evaluation map  $\text{ev} : M^* \otimes_R M \rightarrow R$  defined by  $\text{ev}(f \otimes x) = f(x)$ . To see this, recall that the Hom-Tensor morphism  $\Phi_{ABC} : \text{Hom}(A \otimes B, C) \rightarrow \text{Hom}(A, \text{Hom}(B, C))$  is defined by  $[\Phi_{ABC}(f)(a)](b) = f(a \otimes b)$  for  $a \in A$ ,  $b \in B$ . Taking  $A = M^*$ ,  $B = M$ ,  $C = R$ , we see that for  $x \in M$  and  $f \in M^*$ ,  $[\Phi_{M^*MR}(\text{ev})(f)](x) = \text{ev}(f \otimes x) = f(x)$ . So  $\Phi_{M^*MR}(\text{ev})$  is the map  $M^* \rightarrow M^*$  taking  $f$  to  $f$ . In particular, we could also define the trace by  $\text{tr} = (\alpha^*)^{-1}(\text{ev})$ .

Our first proposition generalizes the standard fact that a torsion-free module over a normal domain has a trace.

**Proposition A.2.** *If  $M_{\mathfrak{p}}$  is a free  $R_{\mathfrak{p}}$ -module for all primes  $\mathfrak{p}$  of height one in  $R$ , and  $R$  satisfies Serre's condition  $(S_2)$ , then  $M$  has a trace.*

*Proof.* We must show that  $\alpha^* : \text{Hom}_R(M, M)^* \rightarrow \text{Hom}_R(M^*, M^*)$  is an isomorphism. Let  $L = \ker(\alpha)$ ,  $I = \text{im}(\alpha)$ ,  $C = \text{coker}(\alpha)$ . Then dualizing  $\alpha$  gives two exact sequences:

$$\begin{aligned} 0 &\rightarrow I^* \rightarrow \text{Hom}_R(M^*, M^*) \rightarrow L^* \\ 0 &\rightarrow C^* \rightarrow (\text{Hom}_R(M, M))^* \rightarrow I^* \rightarrow \text{Ext}_R^1(C, R) \end{aligned}$$

Since  $\alpha$  is an isomorphism at all minimal primes of  $R$ , the annihilator of  $L$  is not contained in any minimal prime. Hence  $L$  is a torsion module, and so  $L^* = 0$ .

Since, further,  $\alpha$  is an isomorphism at all primes of height one in  $R$ , the annihilator of  $C$  is not contained in any height-one prime. By the assumption that  $R$  satisfies condition  $(S_2)$ , then,  $\text{grade}(\text{Ann } C) \geq 2$ , so  $C^* = \text{Ext}_R^1(C, R) = 0$ . This shows that  $\alpha^*$  is an isomorphism.  $\square$

**Lemma A.3.** *For  $f \in \text{Hom}_R(R^n, R^n)$ ,  $\text{tr}(f)$  is the sum of the diagonal entries of a matrix representing  $f$ .*

*Proof.* Since  $R^n$  is free,  $\alpha$  is an isomorphism already, and of course  $\alpha^*$  is as well. Write  $f = \alpha(\sum_{i=1}^n a_{ij} g_j \otimes e_i)$ , where  $e_i$  and  $g_i$  are the canonical bases for  $R^n$  and its dual, respectively. Then since  $g_j(e_i) = \delta_{ij}$ , we see that

$$\text{tr}(f) = \text{ev}\left(\sum_{i=1}^n a_{ij} g_j \otimes e_i\right) = \sum_{1 \leq i, j \leq n} a_{ij} g_j(e_i) = \sum_{1 \leq j \leq n} a_{jj},$$

as desired.  $\square$

Recall that the torsion-free  $R$ -module  $M$  is said to have constant rank  $n$  if  $K \otimes_R M$  is a free  $K$ -module of rank  $n$ . If this is the case, we fix a basis  $\{e_1, \dots, e_n\}$  for  $K \otimes_R M$ , and let  $\{g_1, \dots, g_n\}$  be the dual basis, so that  $g_i(e_j) = \delta_{ij}$ .

**Lemma A.4.** *Assume that  $M$  is a torsion-free  $R$ -module of constant rank and that  $M$  has a trace. Then for any  $f \in M^*$  and  $x \in M$ , we have  $x = \sum_{i=1}^n g_i(x)e_i$  and  $\text{tr}(f) = \sum_{i=1}^n g_i(\widehat{f}(e_i))$ , where  $\widehat{f} = K \otimes_R f$ .*

*Proof.* Since  $M$  is torsion-free, it embeds into a free  $R$ -module and so the homomorphism  $M \rightarrow K \otimes_R M$  is injective. Considering  $x$  as an element of  $K \otimes_R M$ , write  $x = \sum_{j=1}^n a_j e_j$ , where the  $a_j$  are elements of  $K$ . Then a short computation using the definition of the  $g_i$  shows that  $\sum_{i=1}^n g_i(x)e_i = x$ . For the other assertion, pass to the total quotient ring  $K$ . Since  $K \otimes_R M$  is free, Lemma A.3 implies that the trace of  $\widehat{f}$  is the sum of the diagonal elements of a matrix  $(a_{ij})$  representing  $\widehat{f}$ . Since  $g_i(\widehat{f}(e_i)) = a_{ii}$ , the statement follows.  $\square$

**Corollary A.5.** *Assume that  $M$  is a torsion-free module of constant rank and has a trace. If  $\text{rank}(M)$  is invertible in  $R$ , then  $\text{tr}$  is surjective from  $\text{Hom}_R(M, M)$  to  $R$ .*

**Lemma A.6.** *Assume that  $M$  is a torsion-free of constant rank and that  $M$  has a trace. Then we have  $\text{tr} \alpha = \text{ev}$  as homomorphisms from  $M^* \otimes_R M$  to  $R$ .*

*Proof.* For any  $f \in M^*$  and  $x \in M$ , a straightforward computation using Lemma A.4 shows that  $f(x) = \text{tr}(\alpha(f \otimes x))$ .  $\square$

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