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Abstract

This paper extends an existing logic, $\mathcal{L}_N$, to some of the generalized quantifiers of natural language. In contrast to the usual approach, this extension does not require the identity relation. Sommers has suggested that the identity is unnecessary in a logic that properly treats singular terms. This paper lends support to Sommers position.

$\mathcal{L}_N$ is a logic designed for natural language reasoning (see [3]). This paper defines an extension, $\mathcal{L}_{NQ}$, of that logic to include the cardinal quantifiers, **at least n**, and the second-order quantifier, **most**. Because of the limited expressiveness of first-order languages, a complete axiomatization for **most** is not possible. However incompleteness does not negate the usefulness of the axiomatization for natural language reasoning. Theorems, generalizing those of [3], are given. These theorems establish the properties of monotonicity, conservativity, and conversion for $\mathcal{L}_{NQ}$.

These results are of interest in connection with Sommers position that by endowing singular terms with "wild quantity," identity as a logical operator is not needed. This in turn results in a logic that is simpler and more closely conforms to natural language.

1 **Introduction** $\mathcal{L}_N$, a logic designed for natural language reasoning, was presented in [3]. This paper defines an extension, $\mathcal{L}_{NQ}$, of that logic to include the cardinal quantifiers, **at least n**, and the second-order quantifier, **most**. Because of the limited expressiveness of first-order languages, a complete axiomatization for **most** is not possible. Incompleteness does not negate the usefulness of the axiomatization for natural language reasoning however. Theorems, generalizing those of [3], are given. These theorems establish the properties of monotonicity, conservativity, and conversion for $\mathcal{L}_{NQ}$.

The extension is also of interest in another connection. Sommers has taken the position (see "Do We Need Identity?" in [4]) that by endowing singular terms with "wild quantity," e.g., recognizing that **some Socrates is human** is logically equivalent to **all Socrates is human**, identity as a logical operator is not needed. Elimination of the identity relation results in a simpler logic, and one that more closely conforms to natural language.

$\mathcal{L}_N$ incorporates a version of Sommers' position. It has no identity relation. It defines certain predicates as singular. Semantically, singular predicates denote singleton sets of individuals. Syntactically, they are endowed with wild quantity (by axiom S2) and existential import (by axiom S1). It is shown in [3] that the expressiveness of the logical identity relation can be attained in $\mathcal{L}_N$ through the use of schemas.

In a first-order language with identity, the cardinal quantifiers are usually introduced by definition (e.g., [1]). The quantifier **most** cannot be introduced in this way. In

$\mathcal{L}_{NQ}$, both are established by axiom schemas.

Of course, either approach is available in a first-order language with identity. But the demonstration in this paper that $\mathcal{L}_{NQ}$, a first-order language without identity, has sufficient expressiveness to axiomatize these quantifiers lends support to Sommers' position on the identity relation.

2 Definition of the Language

The alphabet of $\mathcal{L}_{NQ}$ consists of the following.

1. Predicate symbols $\mathcal{P} = \mathcal{S} \cup (\bigcup_{j\in\omega} \mathcal{R}_j)$ where $\mathcal{R}_j = \{R_i^j : i \in \omega\}$, $\mathcal{S} = \{S_i : i \in \omega\}$, and $\mathcal{S}$ and the $\mathcal{R}_j$ are mutually disjoint.
2. Selection operators $\{\langle k_1, \ldots, k_n\rangle : n \in (\omega - \{0\}), k_i \in (\omega - \{0\}), 1 \leq i \leq n\}$.
3. Quantifiers **some**, $\{\mathbf{k} : \mathbf{k} \in (\omega - \{0\})\}$, and **most**.
4. Boolean operators $\cap$ and $\bar{\ }$.
5. Parentheses (and).

$\mathcal{L}_{NQ}$ is partitioned into sets of n-ary expressions for $n \in \omega$. These sets are defined to be the smallest satisfying the following conditions.

1. Each $S_i \in \mathcal{S}$ is a unary expression.
2. For all $n \in \omega$, each $R_i^n \in \mathcal{R}_n$ is a n-ary expression.
3. For each predicate symbol $P \in \mathcal{P}$ of arity m, $\langle k_1, \ldots, k_m\rangle P$ is a n-ary expression where $n = max(k_i)_{1\leq i\leq m}$.
4. If X is a n-ary expression then $\overline{(X)}$ is a n-ary expression.
5. If X is a m-ary expression and Y is a l-ary expression then $(X \cap Y)$ is a n-ary expression where $n = max(l, m)$.
6. If X is a unary expression and Y is a $(n + 1)$-ary expression then $(\mathbf{some}XY)$ is a n-ary expression.

7. If X is a unary expression and Y is a $(n+1)$-ary expression then $(\mathbf{k}XY)$ is a n-ary expression for each $\mathbf{k} \in (\omega - \{0\})$.

8. If X is a unary expression and Y is a $(n+1)$-ary expression then $(\mathbf{most}XY)$ is a n-ary expression.

In the sequel, superscripts and parentheses are dropped whenever no confusion can result. Metavariables are used as follows: S ranges over $\mathcal{S}$; R^n ranges over $\mathcal{R}_n$; P ranges over $\mathcal{P}$; X, Y, Z, W, V range over $\mathcal{L}_{NQ}$; and X^n, Y^n, Z^n, W^n, V^n range over n-ary expressions of $\mathcal{L}_{NQ}$. Applying subscripts to these symbols does not change their ranges.

An *interpretation* of $\mathcal{L}_{NQ}$ is a pair $\mathcal{I} = \langle \mathcal{D}, \mathcal{F} \rangle$ where $\mathcal{D}$ is a finite nonempty set and $\mathcal{F}$ is a mapping defined on $\mathcal{P}$ satisfying:

1. for each $S_i \in \mathcal{S}$, $\mathcal{F}(S_i) = \{\langle d \rangle\}$ for some (not necessarily unique) $d \in \mathcal{D}$, and

2. for each $R^n \in \mathcal{R}_n$, $\mathcal{F}(R^n) \subseteq \mathcal{D}^n$.

Let $\alpha = \langle d_1, d_2, \ldots \rangle \in \mathcal{D}^\omega$ (a sequence of individuals). Then $X \in \mathcal{L}_{NQ}$ *is satisfied by* α *in* $\mathcal{I}$ (written $\mathcal{I} \models_\alpha X$) iff one of the following holds:

1. $X \in \mathcal{P}$ with arity n and $\langle d_1, \ldots, d_n \rangle \in \mathcal{F}(X)$

2. $X = \langle k_1, \ldots, k_m \rangle P$ where $P \in \mathcal{P}$ with arity m and $\langle d_{k_1}, \ldots, d_{k_m} \rangle \models P$

3. $X = \overline{Y}$ and $\mathcal{I} \not\models_\alpha Y$

4. $X = Y \cap Z$ and $\mathcal{I} \models_\alpha Y$ and $\mathcal{I} \models_\alpha Z$

5. $X = \mathbf{some}Y^1 Z^{n+1}$ and for some $d \in \mathcal{D}$, $\langle d \rangle \models Y^1$ and $\langle d \rangle \models Z^{n+1}$

6. $X = \mathbf{k}Y^1 Z^{n+1}$ and $card(\{d \in \mathcal{D} : \langle d \rangle \models Y^1$ and $\langle d \rangle \models Z^{n+1}\}) \geq k$

7. $X = \mathbf{most}Y^1 Z^{n+1}$ and $card(\{d \in \mathcal{D} : \langle d \rangle \models Y^1$ and $\langle d \rangle \models Z^{n+1}\}) > card(\{d \in \mathcal{D} : \langle d \rangle \models Y^1$ and $\langle d \rangle \not\models Z^{n+1}\})$

where $\mathcal{I} \not\models_\alpha X$ is an abbreviation for not($\mathcal{I} \models_\alpha X$) and $\langle d_{i_1}, \ldots, d_{i_n} \rangle \models X$ is an abbreviation for $\mathcal{I} \models_{\langle d_{i_1}, \ldots, d_{i_n}, d_1, d_2, \ldots \rangle} X$.

X *is true in* $\mathcal{I}$ (written $\mathcal{I} \models X$) iff $\mathcal{I} \models_\alpha X$ for every $\alpha \in \mathcal{D}^\omega$. X *is valid* (written $\models X$) iff X is true in every interpretation of $\mathcal{L}_{NQ}$. A 0-ary expression of $\mathcal{L}_{NQ}$ is called a *sentence.* A set Γ of sentences *is satisfied in* $\mathcal{I}$ iff each $X \in \Gamma$ is true in $\mathcal{I}$.

The following abbreviations are introduced to improve readability.

1. $\breve{R}^n := \langle n, \ldots, 1 \rangle R^n$

2. $X \cup Y := \overline{(\overline{X} \cap \overline{Y})}$

3. $X \subseteq Y := \overline{(X \cap \overline{Y})}$

4. $X \equiv Y := (X \subseteq Y) \cap (Y \subseteq X)$

5. $T := (S_0 \subseteq S_0)$

6. $\mathbf{some}X_n\mathbf{some}X_{n-1} \cdots \mathbf{some}X_1 Y := (\mathbf{some}X_n(\mathbf{some}X_{n-1} \cdots (\mathbf{some}X_1 Y) \cdots)$

7. $\mathbf{some}X^1Y_n^2 \circ Y_{n-1}^2 \circ \cdots \circ Y_1^2 := (\mathbf{some} \cdots (\mathbf{some}(\mathbf{some}X^1Y_n^2)Y_{n-1}^2) \cdots Y_1^2)$

8. $\mathbf{all}X^1Y := \overline{\mathbf{some}X^1\overline{Y}}$

9. $\mathbf{no}X^1Y := \overline{\mathbf{some}X^1Y}$

10. $\mathbf{!k}X^1Y := \mathbf{k}X^1Y \cap \overline{(\mathbf{k{+}1})X^1Y}$

11. $\bar{\mathbf{k}}X^1Y := \overline{\mathbf{k}X^1Y}$

It is easy to see that:

1. $\mathcal{I} \models_\alpha X \cup Y$ iff $(\mathcal{I} \models_\alpha X$ or $\mathcal{I} \models_\alpha Y)$

2. $\mathcal{I} \models_\alpha X \subseteq Y$ iff $(\mathcal{I} \models_\alpha X$ implies $\mathcal{I} \models_\alpha Y)$

3. $\mathcal{I} \models_\alpha X \equiv Y$ iff $(\mathcal{I} \models_\alpha X$ iff $\mathcal{I} \models_\alpha Y)$

4. $\mathcal{I} \models_\alpha T$ for every $\mathcal{I}$ and α

5. $\mathcal{I} \models_\alpha \mathbf{some}X^1Y_n^2 \circ \cdots \circ Y_1^2$ iff for some $d \in \mathcal{D}$, $\langle d \rangle \models X^1$ and $\langle d \rangle \models Y_n^2 \circ \cdots \circ Y_1^2$ where $\circ$ denotes composition of relations in $\mathcal{I}$

6. $\mathcal{I} \models_\alpha \mathbf{all}X^1Y$ iff for all $d \in \mathcal{D}$, $\langle d \rangle \models X^1$ implies $\langle d \rangle \models Y$

7. $\mathcal{I} \models_\alpha \mathbf{no}X^1Y$ iff for all $d \in \mathcal{D}$, $\langle d \rangle \models X^1$ implies $\langle d \rangle \not\models Y$

8. $\mathcal{I} \models_\alpha \mathbf{!k}X^1Y$ iff $card(\{d \in \mathcal{D} : \langle d \rangle \models X^1$ and $\langle d \rangle \models Y\}) = k$

9. $\mathcal{I} \models_\alpha \bar{\mathbf{k}}X^1Y$ iff $card(\{d \in \mathcal{D} : \langle d \rangle \models X^1$ and $\langle d \rangle \models Y\}) < k$

3 Axiomatization of $\mathcal{L}_{NQ}$

The axiom schemas of $\mathcal{L}_{NQ}$ are the following.

BT. Every schema that can be obtained from a tautologous Boolean wff by uniform substitution of nullary metavariables of $\mathcal{L}_{NQ}$ for sentential variables, $\cap$ for $\wedge$, and $^-$ for $\neg$

C1. $\mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\langle k_1,\ldots,k_m\rangle P \subseteq \mathbf{some}S_{i_{k_m}}\cdots\mathbf{some}S_{i_{k_1}}P$ where P is of arity m and $n = max(k_j)_{1\leq j\leq m}$

C2. $\mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\overline{\langle k_1,\ldots,k_m\rangle P} \subseteq \mathbf{some}S_{i_{k_m}}\cdots\mathbf{some}S_{i_{k_1}}\overline{P}$ where P is of arity m and $n = max(k_j)_{1\leq j\leq m}$

S1. $\mathbf{some}SS$

S2. $\mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\overline{(\mathbf{some}SX^{n+1})} \equiv \mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{some}S\overline{X^{n+1}}$

D. $\mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}(X^m\cap Y^l) \equiv (\mathbf{some}S_{i_m}\cdots\mathbf{some}S_{i_1}X^m\cap\mathbf{some}S_{i_l}\cdots\mathbf{some}S_{i_1}Y^l)$ where $n = max(l,m)$

EG. $(\mathbf{some}SX^1\cap\mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{some}SY^{n+1}) \subseteq \mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{some}X^1Y^{n+1}$

KG1. $\mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{some}X^1Y^{n+1} \equiv \mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{1}X^1Y^{n+1}$

KG2. $(\mathbf{some}SX^1 \cap \mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{some}SY^{n+1} \cap \mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{k}(X^1 \cap \overline{S})Y^{n+1}) \subseteq \mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}(\mathbf{k{+}1})X^1Y^{n+1}$ for each $\mathbf{k} \in (\omega - \{0\})$

MG1. $(\mathbf{some}SX^1 \cap \mathbf{all}S_{i_n}\cdots\mathbf{all}S_{i_1}\mathbf{all}X^1Y^{n+1}) \subseteq \mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{most}X^1Y^{n+1}$

MG2. $(\mathbf{some}S_iX^1\cap\mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{some}S_iY^{n+1}\cap\mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{most}(X^1\cap \overline{S_i} \cap \overline{S_j})Y^{n+1}) \subseteq \mathbf{some}S_{i_n}\cdots\mathbf{some}S_{i_1}\mathbf{most}X^1Y^{n+1}$

The inference rules of $\mathcal{L}_{NQ}$ are the following.

MP. From X^0 and $X^0 \subseteq Y^0$ infer Y^0

EI. From $\overline{(Z^0 \cap \mathbf{some}SX^1 \cap \mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}\mathbf{some}SY^{n+1})}$, where S does not occur in X^1, Y^{n+1}, or Z^0, and is distinct from $S_{i_1}, \ldots, S_{i_n}$, infer $\overline{(Z^0 \cap \mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}\mathbf{some}X^1Y^{n+1})}$

KI. From $\overline{(Z^0 \cap \mathbf{some}SX^1 \cap \mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}\mathbf{some}SY^{n+1} \cap \mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1} \mathbf{k}(X^1 \cap \overline{S})Y^{n+1})}$, where S does not occur in X^1, Y^{n+1}, or Z^0, and is distinct from $S_{i_1}, \ldots, S_{i_n}$, infer $\overline{(Z^0 \cap \mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}(\mathbf{k{+}1})X^1Y^{n+1})}$ for each $\mathbf{k} \in (\omega - \{0\})$

MI. From $\overline{(Z^0 \cap \mathbf{some}S_iX^1 \cap \mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}\mathbf{some}S_iY^{n+1} \cap (\mathbf{all}S_{i_n} \cdots \mathbf{all}S_{i_1}\mathbf{all}\, X^1Y^{n+1} \cup \mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}\mathbf{most}(X^1 \cap \overline{S_i} \cap \overline{S_j})Y^{n+1}))}$, where S_i and S_j do not occur in X^1, Y^{n+1}, or Z^0, and are distinct from $S_{i_1}, \ldots, S_{i_n}$, infer $\overline{(Z^0 \cap \mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}\mathbf{most}X^1Y^{n+1})}$

The set $\mathcal{T}$ of theorems of $\mathcal{L}_{NQ}$ is the smallest set containing the axioms and closed under MP, EI, KI, and MI.

Axiom S2 can also be written

S2. $\mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}\mathbf{all}SX^{n+1} \equiv \mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}\mathbf{some}SX^{n+1}$

In view of this "wild quantity" of singular predicates, $\mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}\mathbf{all}SX^{n+1}$ and $\mathbf{some}S_{i_n} \cdots \mathbf{some}S_{i_1}\mathbf{some}SX^{n+1}$ will usually be written simply $S_{i_n} \cdots S_{i_1}SX^{n+1}$.

The following theorem establishes the soundness of this axiomatization.

THEOREM 1 *$X \in \mathcal{T}$ only if $\models X$.*

proof: Observe that by the definition of satisfaction, $\langle \mathcal{F}(S_{i_1}), \ldots, \mathcal{F}(S_{i_n}) \rangle \models X^n$ iff $\langle \mathcal{F}(S_{i_2}), \ldots, \mathcal{F}(S_{i_n}) \rangle \models S_{i_1} X^n$ iff $\cdots$ iff $\mathcal{I} \models S_{i_n} \cdots S_{i_1} X^n$. From this observation and the definition of validity, it is not difficult to show that the axioms are valid and that validity is preserved by the inference rules. Details will be given only for MG1, MG2 and MI. In this proof, $\mathcal{C}_1 := \{d : \langle d \rangle \models X\}$, $\mathcal{C}_1' := \{d : \langle d \rangle \models X \cap \overline{S_j}\}$, $\mathcal{C}_1'' := \{d : \langle d \rangle \models X \cap \overline{S_j} \cap \overline{S_i}\}$, and $\mathcal{C}_2 := \{d : \langle d, \mathcal{F}(S_{i_1}), \ldots, \mathcal{F}(S_{i_1}) \rangle \models Y\}$.

(i) **Claim**: MG1 is valid.

proof: $\mathcal{I} \models SX \cap S_{i_n} \cdots S_{i_1} \mathbf{all} XY$ iff $\mathcal{I} \models SX$ and $\mathcal{I} \models S_{i_n} \cdots S_{i_1} \mathbf{all} XY$ iff $\langle \mathcal{F}(S) \rangle \models X$ and $\forall d \in \mathcal{D} : \langle d \rangle \models X$ implies $\langle d, \mathcal{F}(S_{i_1}), \ldots, \mathcal{F}(S_{i_1}) \rangle \models Y$. Therefore $card(\mathcal{C}_1 \cap \mathcal{C}_2) \geq 1$ and $card(\mathcal{C}_1 \cap \overline{\mathcal{C}_2}) = 0$. Hence $\mathcal{I} \models S_{i_n} \cdots S_{i_1} \mathbf{most} XY$.

(ii) **Claim**: MG2 is valid.

proof: As above, $\mathcal{I} \models S_i X \cap S_{i_n} \cdots S_{i_1} S_i Y \cap S_{i_n} \cdots S_{i_1} \mathbf{most}(X \cap \overline{S_i} \cap \overline{S_j}) Y$ iff $\langle \mathcal{F}(S_i) \rangle \models X$, $\langle \mathcal{F}(S_i), \mathcal{F}(S_{i_1}), \ldots, \mathcal{F}(S_{i_n}) \rangle \models Y$, and $card(\mathcal{C}_1'' \cap \mathcal{C}_2) > card(\mathcal{C}_1'' \cap \overline{\mathcal{C}_2})$. Since $\mathcal{F}(S_i) \notin \mathcal{C}_1''$ but $\mathcal{F}(S_i) \in \mathcal{C}_1'$, $card(\mathcal{C}_1' \cap \mathcal{C}_2) = card(\mathcal{C}_1'' \cap \mathcal{C}_2) + 1 > card(\mathcal{C}_1'' \cap \overline{\mathcal{C}_2}) + 1 = card(\mathcal{C}_1' \cap \overline{\mathcal{C}_2}) + 1$. Therefore, for any value of $\mathcal{F}(S_j)$, $card(\mathcal{C}_1 \cap \mathcal{C}_2) \geq card(\mathcal{C}_1' \cap \mathcal{C}_2) > card(\mathcal{C}_1' \cap \overline{\mathcal{C}_2}) + 1 \geq card(\mathcal{C}_1 \cap \overline{\mathcal{C}_2})$. Hence, $\mathcal{I} \models S_{i_n} \cdots S_{i_1} \mathbf{most} XY$.

(iii) **Claim**: MI preserves validity.

proof: Suppose $\models \overline{(Z^0 \cap S_i X \cap S_{i_n} \cdots S_{i_1} S_i Y \cap (S_{i_n} \cdots S_{i_1} \mathbf{all} XY \cup S_{i_n} \cdots S_{i_1}} \overline{\mathbf{most}(X \cap \overline{S_i} \cap \overline{S_j})Y))}$, where S_i and S_j do not occur in X, Y, or Z^0, and are distinct from $S_{i_1}, \ldots, S_{i_n}$, but there exist interpretations $\mathcal{I}$ such that $\mathcal{I} \models Z^0 \cap S_{i_n} \cdots S_{i_1} \mathbf{most} XY$. Thus $card(\mathcal{C}_1 \cap \mathcal{C}_2) > card(\mathcal{C}_1 \cap \overline{\mathcal{C}_2}) \geq 0$. Since S_i is *fresh* (i.e., has no other occurrences), there is an interpretation such that $\mathcal{F}(S_i) \in \mathcal{C}_1 \cap \mathcal{C}_2$. Therefore, $\mathcal{I} \models S_i X$ and $\mathcal{I} \models S_{i_n} \cdots S_{i_1} S_i Y$. Now there are two cases to consider.

(a) $card(\mathcal{C}_1 \cap \overline{\mathcal{C}_2}) = 0$.

Then $\mathcal{I} \not\models S_{i_n} \cdots S_{i_1} X\overline{Y}$, i.e., $\mathcal{I} \models \overline{S_{i_n} \cdots S_{i_1} X\overline{Y}}$, which contradicts the assumption.

(b) $card(\mathcal{C}_1 \cap \overline{\mathcal{C}_2}) > 0$.

Then $card(\mathcal{C}_1 \cap \mathcal{C}_2) > 1$. Since S_j is fresh, there is an interpretation such that $\mathcal{F}(S_j) \in \mathcal{C}_1 \cap \overline{\mathcal{C}_2}$. Therefore, $card(\mathcal{C}_1'' \cap \mathcal{C}_2) = card(\mathcal{C}_1 \cap \mathcal{C}_2) - 1$ and $card(\mathcal{C}_1'' \cap \overline{\mathcal{C}_2}) = card(\mathcal{C}_1 \cap \overline{\mathcal{C}_2}) - 1$. Hence $\mathcal{I} \models S_{i_n} \cdots S_{i_1} \mathbf{most}(X \cap \overline{S_i} \cap \overline{S_j})Y$, which again contradicts the assumption.

□

The axiomatization is not complete however. Indeed the quantifier **most** cannot be axiomatized in a first-order language. This is easily shown as follows. (See also [1].)

Suppose **most** is axiomatizable in $\mathcal{L}_{NQ}$. Let $X = \mathbf{most} TB$ and let Γ be a set of sentences such that for any interpretation $\mathcal{I}$ of $\mathcal{L}_{NQ}$, $\mathcal{I} \models X$ iff $\mathcal{I} \models \Gamma$. Let $n = \{0, 1, 2, \ldots, n-1\}$ and $\omega_{odd} = \{1, 3, 5, \ldots\}$. For each $n \in \omega$, define interpretation

$\mathcal{I}_n = \langle n, \mathcal{F}_n \rangle$, where $\mathcal{F}_n(B) = \{0\} \cup n \cap \omega_{odd}$. Obviously, for each $n \in \omega$, $\mathcal{I}_n \models \Gamma$.

Now define $\mathcal{I} = \prod_{n\in\omega} \mathcal{I}_n/F$, where F is a nonprincipal ultrafilter (e.g., an extension of the Frechét filter to an ultrafilter). By Łoš's Theorem (e.g., see [2]), $\mathcal{I} \models \Gamma$. Since F contains no singletons, $!\mathbf{k}TT$ cannot be satisfied in $\mathcal{I}$ for any k. Therefore $\mathcal{I}$ is infinite. Moreover, $\langle (2k+1)/F \rangle \models B$ for every $k \in \omega$. Since both T and B denote infinite sets in $\mathcal{I}$, it follows that $\mathcal{I} \not\models \mathbf{most}TB$, a contradiction.

If the quantifier **most** were eliminated, the axiomatization of the remainder of $\mathcal{L}_{NQ}$ would be complete. The proof closely follows that given in [3]. Alternatively, if interpretations are restricted to some fixed finite upper bound (e.g., by adding the axiom $\bar{N}TT$), the axiomatization is complete. Of course, this is tantamount to accepting incompleteness. In any event, incompleteness does not negate the usefulness of the axiomatization for reasoning about natural language discourse.

4 Theorems

The theorems presented in [3] can be generalized to apply to $\mathcal{L}_{NQ}$. Since the proofs closely follow those given in [3], the theorems will be stated without proof.

The main results are two monotonicity theorems. These theorems establish the monotonicity properties of quantifiers. They subsume the resolution principle. In addition, other properties of natural language quantifiers, including conservativity, are proved.

Before stating the first monotonicity theorem, some definitions are needed.

An occurrence of a subexpression Y in an expression W has *positive (negative) polarity* if that occurrence of Y lies in the scope of an even (odd) number of $^{-}$ operations in W, unless that occurrence of Y is a subexpression of V in $\mathbf{most}VZ$, in which case Y has *both* positive *and* negative polarity.

An occurrence of a subexpression Y^m, where $m \geq 1$, is *governed by X in W* if W is $\mathbf{some}XY^m$, $\mathbf{some}X\overline{Y^m}$, $\mathbf{some}X(Y^m \cap Z^l)$, $\mathbf{k}XY^m$, $\mathbf{k}X\overline{Y^m}$, $\mathbf{k}X(Y^m \cap Z^l)$, $\mathbf{most}XY^m$, $\mathbf{most}X\overline{Y^m}$, $\mathbf{most}X(Y^m \cap Z^l)$, or the complement of one of these expressions. An occurrence of Y^m is *governed by $X_n \cdots X_1$ in W*, where $1 \leq n \leq m$, if V is governed by X_n in W and that occurrence of Y^m is governed by $X_{n-1} \cdots X_1$ in V. An occurrence of Y^m in $\langle k_1, \ldots, k_m \rangle Y^m$ is *governed by $X_{k_m} \cdots X_{k_1}$ in W* if $\langle k_1, \ldots, k_m \rangle Y^m$ is governed by $X_n \cdots X_1$ in W, where $n = max(k_i)_{1 \leq i \leq m}$.

THEOREM 2 (First Monotonicity Theorem) *Let Y^m occur in W with positive (respectively, negative) polarity. Let $(\mathbf{all}T)^m(Y^m \subseteq Z^l)$ (respectively, $(\mathbf{all}T)^m(Z^l \subseteq Y^m)$),* *where* $l \leq m$. *Let* W' *be obtained from* W *by (i) substituting* Z^l *for that occurrence of* Y^m, *(ii) substituting* $\langle k_1, \ldots, k_l \rangle$ *for selection operator* $\langle k_1, \ldots, k_m \rangle$ *on* Y^m, *if any, and (iii) eliminating all occurrences of governing subexpressions that no longer govern after the substitutions in (i) and (ii). Finally, let* **some**TX *for every governing subexpression* X *with an occurrence of negative polarity that was eliminated in (iii). Then* $(\mathbf{all}T)^h(W \subseteq W')$, *where* h *is the arity of* W.

From previous definitions, it follows that if the expression **all**YX occurs with positive (negative) polarity, then the occurrence of Y has negative (positive) polarity while the occurrence of X has positive (negative) polarity; if the expression **no**YX occurs with positive (negative) polarity, then the occurrence of Y and the occurrence of X both have negative (positive) polarity; if the expression **!k**YX occurs with either positive or negative polarity, then the occurrence of Y and the occurrence of X have *both* positive *and* negative polarity; if the expression $\bar{\mathbf{k}}YX$ occurs with positive (negative) polarity, then the occurrence of Y and the occurrence of X both have negative (positive) polarity; if the expression $Y \subseteq X$ occurs with positive (negative) polarity, then the occurrence of Y has negative (positive) polarity while the occurrence of X has positive (negative) polarity; if the expression $Y \cup X$ occurs with positive (negative) polarity, then the occurrence of Y and the occurrence of X both have positive (negative) polarity; and if the expression $Y \equiv X$ occurs with either positive or negative polarity, then the occurrence of Y and the occurrence of X have *both* positive *and* negative polarity. With these provisions, Theorem 2 applies to expressions containing occurrences of defined operators. In this connection, singular predicates require special mention. Since $\mathbf{all}SX := \overline{\mathbf{some}S\overline{X}} \equiv \mathbf{some}S\overline{\overline{X}} \equiv \mathbf{some}SX$, any occurrence of a singular predicate can be taken to have *either* positive *or* negative polarity.

Before the second monotonicity theorem can be presented, a definition is needed.

A subexpression Y^m will be said to *occur disjunctively* in expression W iff (i) $W = \mathbf{all}X_n \cdots \mathbf{all}X_1 Y^m \cup Z$ where $n \leq m$; or (ii) $W = \mathbf{all}X_n \cdots \mathbf{all}X_{k+1}(Z_1 \cup Z_2)$ where $0 \leq k \leq n$ and Y^m occurs disjunctively in Z_1.

THEOREM 3 (Second Monotonicity Theorem) *Let Y^m occur disjunctively in W, governed by $X_k \cdots X_1$. Let W' be obtained from W by replacing that occurrence of Y^m with Z^l ($l \leq m$) and deleting all occurrences of* $\mathbf{all}X_i$ *that no longer govern a subexpression. Let* $\mathbf{some}TX_i$ *for every* $\mathbf{all}X_i$ *that was deleted. Then* $(\mathbf{all}T)^h((W \cap \mathbf{all}X_k \cdots \mathbf{all}X_1(Y^m \subseteq Z^l)) \subseteq W')$, *where h is the arity of W.*

It is easy to see (from the equivalence $(Y^m \subseteq Z^l) \equiv (\overline{Y^m} \cup Z^l))$ that this theorem corresponds to the resolution principle in conventional logic. A corollary provides a rule corresponding to unit resolution.

COROLLARY 4 *(Cancellation Rule) Let Y^m occur disjunctively in W, governed by $X_k \cdots X_1$. Let W' be obtained from W by deleting that occurrence of Y^m and all occurrences of* $\mathbf{all}X_i$ *that no longer govern a subexpression. Let* $\mathbf{some}TX_i$ *for every* $\mathbf{all}X_i$ *that was deleted. Then* $(\mathbf{all}T)^h((W \cap \mathbf{all}X_k \cdots \mathbf{all}X_1\overline{Y^m}) \subseteq W')$, *where h is the arity of W.* □

The final theorems establish the property of conservativity and the the rules for conversion in the case of unary predicates.

THEOREM 5 (Conservativity) *(schema) (i)* $(\mathbf{all}T)^{m-1}\mathbf{some}XY^m \equiv (\mathbf{all}T)^{m-1}\mathbf{some}X(Y^m \cap X)$ *(ii)* $(\mathbf{all}T)^{m-1}\mathbf{all}XY^m \equiv (\mathbf{all}T)^{m-1}\mathbf{all}X(Y^m \cap X)$ *(iii)* $(\mathbf{all}T)^{m-1}\mathbf{k}XY^m \equiv (\mathbf{all}T)^{m-1}\mathbf{k}X(Y^m \cap X)$ *(iv)* $(\mathbf{all}T)^{m-1}\mathbf{most}XY^m \equiv (\mathbf{all}T)^{m-1}\mathbf{most}X(Y^m \cap X)$. $\square$

THEOREM 6 (Conversion) *For unary expressions X and Y, (i)* $\mathbf{some}XY \equiv \mathbf{some}YX$ *(ii)* $\mathbf{all}XY \equiv \mathbf{all}(\overline{Y})\,\overline{X}$ *(iii)* $\mathbf{k}XY \equiv \mathbf{k}YX$ $\square$

5 **Conclusion** This paper generalizes the language $\mathcal{L}_N$ to include the cardinal quantifiers and the second-order quantifier **most**. The axiomatization of $\mathcal{L}_N$ is appropriately extended and the theorems establishing quantifier properties also extended.

The paper does not go on to prove other results involving these quantifiers, since they are for the most part quite straightforward. For example, the common-sense expectations such as

$$(\mathbf{k}XY \cap \overline{2\mathbf{k}}XT) \subseteq \mathbf{most}XY$$

are easily obtained.

The main interest lies in the demonstration that a first-order language without identity has sufficient expressiveness to define these natural language quantifiers.

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