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Methods of Nonparametric Multivariate Ranking and Selection

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ABSTRACT

In a Ranking and Selection problem, a collection of k populations $\{\pi_i\}_{i=1}^k$ is given which follow some (partially) unknown probability distribution P_{X_i} given by a random vector X_i. The problem is to select the "best" of the k populations where "best" is well defined in terms of some unknown population parameter. In many univariate parametric and nonparamentric settings, solutions to these ranking and selection problems exist. In the multivariate case, only parametric solutions have been developed. We have developed several methods for solving nonparametric multivariate ranking and selection problems. The problems considered allow an experimenter to select the "best" populations based on nonparametric notions of dispersion, location, and distribution. For the first two problems, we use Tukey's Halfspace Depth to define these notions. In the last problem, we make use of a multivariate version of the Kolmogorov-Smirnov Statistic for making selections.

Methods of Nonparametric Multivariate Ranking and Selection

by

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Contents

Chapter 1

Ranking and Selection

In this chapter, several Ranking and Selection procedures are illustrated. This provides the reader with an introduction to the pertinent Ranking and Selection concepts that will be used in Chapters 3, 4, and 5. Additionally, this chapter will provide motivation for the procedures that are developed. In Section 1.1, we give a short introduction to the basic setting of a Ranking and Selection problem. Sections 1.2, and 1.3, provide the necessary background for the two approaches that are taken with any Ranking and Selection problem. Section 1.4 introduces two univariate nonparametric procedures.

1.1 Introduction

In a Ranking and Selection problem, a collection of k populations $\{\pi_i\}_{i=1}^{k}$ is given. These populations follow some (partially) unknown probability distribution P_{X_i} given by a random variable X_i. When the context is clear, we let $P_{X_i} = P_i$. Our goal is to select a subset of populations. Of course, any subset will not satisfy us. We want to select the "best" populations. The "best" populations are determined by some unknown parameter $\theta_i \in \mathbb{R}$ for P_i. If we let

$$\theta_{[1]} \geq \theta_{[2]} \geq \cdots \geq \theta_{[k]} \tag{1.1}$$

represent the ordering of the parameters θ_i, we can define the "best" populations in terms of the ordered unknown values of θ_i. It may be that our desire is to select the population with the largest (smallest) value, $\theta_{[1]}$ ($\theta_{[k]}$). Taking this idea further, we may desire to select the populations with the $t < k$ largest (smallest) values, or rank the populations, i.e. select the first "best", second "best", $\ldots, k^{th}$ "best" populations.

Since the values of θ_i are unknown, we can not simply select the populations that correspond to the "best" without some hint as to the correct ordering of the θ_i. As would be expected, a sample of size n is collected from each population, and θ_i is estimated by some function of the sample $\widehat{\theta}_{i,n}$. Ordering these sample values, we have

$$\widehat{\theta}_{[1],n} \geq \widehat{\theta}_{[2],n} \geq \cdots \geq \widehat{\theta}_{[k],n}. \tag{1.2}$$

It is hoped that there is some type of useful relationship between (1.1) and (1.2). By useful, we mean that it will allow us to correctly select (CS_n) the "best" population(s).

Given that we will make this selection based upon less than complete information, being based on a sample, there is a chance that we will make a mistake. Thus, our true goal is to control the chance of making a mistake. That is, we would like to determine a sample size n so that the probability of making a correct selection is greater than some predetermined value i.e.

$$P(CS_n) \geq P^* \in (0, 1). \tag{1.3}$$

Two main approaches to solving this problem exist. The first is known as the Indifference Zone approach; the Subset Selection Approach is the second. Both approaches will be used to determine nonparamentric selection procedures for selecting from among k multivariate populations. In the next few sections we outline some procedures, relating to univariate populations, that will illustrate both types of approaches.

1.2 Indifference Zone Approach

The Indifference Zone Approach can be best described by considering a procedure for selecting the Normally Distributed population with the largest mean. This is commonly referred to as the Normal Means procedure.

1.2.1 Normal Means Procedure

Bechhofer and Sobel first described the Normal Means procedure in [2] and [3]. We are given k normally distributed populations, $\{\pi_i\}_{i=1}^{k}$, with known common standard deviation, σ, and unknown mean μ_i. Since, we are looking for the "best" population, we need some way to define "best".

Definition 1.1. For $i \neq j$, if $\mu_i > \mu_j$, then π_i is said to be **better** than π_j.

Denoting the ordered population means as

$$\mu_{[1]} \geq \mu_{[2]} \geq \cdots \geq \mu_{[k]}, \tag{1.4}$$

we can see that our task is to select the population, π_i, with the mean, $\mu_{[1]}$. Now we ask a question: when do we not care about making a correct selection? If all of the populations have the same mean value, it would make no difference which population is selected. So, we do not need to determine a procedure for this situation. Taking this a step further, suppose there is no practical difference between the populations' means. That is, for some small $\epsilon > 0$, and for all $i = 1, 2, \ldots, k$,

$$\mu_{[1]} \geq \mu_i \geq \mu_{[1]} - \epsilon. \tag{1.5}$$

In a situation like this, there is no reason to care which population is selected. Each population can be considered just as good as another. Consequently, we would be indifferent to which population is selected. This leads to a second question: In what situation would we care about making a correct selection? The obvious answer: we would care when there is a significant difference between the populations' means. Thus, it would be of practical interest for us to make a correct selection whenever the largest population mean is significantly larger than the second largest mean, that is whenever $\mu_{[1]} > \mu_{[2]} + \delta^*$ where $\delta^* > 0$. With this, we can refine the goal given in (1.3). Our goal is to determine a sample size n so that the probability of correctly selecting the population with the largest mean is greater than some predetermined value P^* whenever the largest mean is significantly larger than the second largest i.e.

$$P(CS_n) \geq P^* \in (0,1) \text{ whenever } \mu_{[1]} > \mu_{[2]} + \delta^*. \tag{1.6}$$

This last statement is what is known as the *probability requirement*. We will define the *preference zone* as

$$PZ = \{\vec{\mu}_k = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathbb{R}^k | \mu_{[1]} > \mu_{[2]} + \delta^*\}. \tag{1.7}$$

Its complement will be referred to as the *indifference zone*. Thus, we want a procedure that satisfies the probability requirement whenever the populations are in the preference zone, and are indifferent whenever the populations fall in the indifference zone.

The actual procedure for making our selection is relatively straightforward.

Normal Means Procedure(Bechhofer & Sobel [2, 3]):

The Normal Means Procedure is as follows:

1. Take a simple random sample of size n from each population where

$$n = \left(\frac{h\sigma}{\delta^*}\right)^2, \tag{1.8}$$

   h is the solution to

$$\int_{\infty}^{\infty} \Phi(z+h)^{k-1}\phi(z)\ dz = P^*, \tag{1.9}$$

   and $\Phi(z)$ is the standard Normal cumulative distribution function and $\phi(z)$ is standard Normal density.

2. Estimate μ_i by the sample mean, $\overline{X}_{i,n}$, from each population.

3. Claim that the population π_i corresponding to $\overline{X}_{[1],n}$ is the population with the largest mean $\mu_{[1]}$ where the ordered sample means are denoted by

$$\overline{X}_{[1],n} \geq \overline{X}_{[2],n} \geq \cdots \geq \overline{X}_{[k],n}. \tag{1.10}$$

Now, we outline the derivation of the integral equation in (1.9). Let $\overline{X}_{(i),n}$ represent the sample mean that corresponds to $\mu_{[i]}$. A correct selection is the event where the largest sample mean equals the sample mean produced by the population with the largest population mean,

$$CS_n = \left\{ \overline{X}_{[1],n} = \overline{X}_{(1),n} \right\}. \tag{1.11}$$

Equivalently, we could say that a correct selection occurs when the sample mean from the population with the largest mean is equal to the maximum of all the sample means calculated.

$$CS_n = \left\{ \overline{X}_{(1),n} = \max_{i=1,2,\ldots,k} \overline{X}_{i,n} \right\}. \tag{1.12}$$

By standardizing $\overline{X}_{(i),n}$, so that

$$Z_i = \frac{\sqrt{n}\left(\overline{X}_{(i),n} - \mu_{[i]}\right)}{\sigma}, \tag{1.13}$$

we see that

$$\begin{aligned} P(CS_n) &= P\left(\overline{X}_{[1],n} = \overline{X}_{(1),n}\right) \\ &= P\left(\overline{X}_{(1),n} = \max_{i=1,2,\ldots,k} \overline{X}_{i,n}\right) \\ &= \int_{\infty}^{\infty} \prod_{i=2,\ldots,k} P\left(z + \frac{\sqrt{n}(\mu_{[1]} - \mu_{[i]})}{\sigma} > Z_i\right)\, dF_{Z_1(z)}. \end{aligned} \tag{1.14}$$

Now, we introduce the *least favorable configuration.* It will be the configuration of $(\mu_1, \mu_2, \ldots, \mu_k)$ in the preference zone where (1.14) is minimized. It can be shown that this occurs when

$$\mu_{[1]} - \delta^* = \mu_{[2]} = \cdots = \mu_{[k]}. \tag{1.15}$$

Conceptually, this should be the configuration that should make our decision making process the most difficult. Except for the "best" population, all populations have the same mean, and are as close as possible to being the "best" population. Consequently, we can see that

$$
\begin{aligned}
(1.14) \geq \inf_{\vec{\mu}_k \in PZ} \int_{\infty}^{\infty} \prod_{i=2,\ldots,k} P\left(z + \frac{\sqrt{n}(\mu_{[1]} - \mu_{[i]})}{\sigma} > Z_i\right) \, dF_{Z_1(z)} \qquad (1.16) \\
= \int_{\infty}^{\infty} \prod_{i=2,\ldots,k} P\left(z + \frac{\sqrt{n}\delta^*}{\sigma} > Z_i\right) \, dF_{Z_1(z)} \\
= \int_{\infty}^{\infty} \Phi(z+h)^{k-1} \phi(z) \, dz.
\end{aligned}
$$

It should be noted that defining the indifference zone serves both a mathematical as well as a practical use. Without it, the infimum in (1.16) would be over all of $\mathbb{R}^k$. In which case, the infimum would be attained when all means have the same value. In that case, we can only state that $P(CS_n) \geq k^{-1}$. Thus, in our current situation, we will take $P^* \in (\frac{1}{k}, 1)$, to insure that the probability of making correct selection is better than a random guess.

With this procedure in hand, we have reviewed the basic concepts that cover the indifference zone approach. However, before moving to the Subset Selection Approach, we will introduce some more procedures that use the Indifference Zone approach to illustrate other necessary concepts, and to motivate some of the procedures described in later chapters.

1.2.2 Two-Stage Normal Means Procedure

Our goal in this section is the same as in the previous one; we want to select the Normally distributed population with the largest mean. However, we assume that the common standard deviation is no longer known. The procedure for this situation was first given, in more generality, by Bechhofer, Sobel, & Dunnett in [30]. This procedure will be completed by sampling in two stages. This is necessary because an estimate of the common unknown standard deviation is needed. In fact, Dudewicz has shown in [5], that no single stage procedure, independent of the variance, exists for making this type of selection. The procedure is as follows:

Two-Stage Normal Means Procedure (Bechhofer, Sobel, & Dunnett [30]):

Stage 1:

(a) Take a sample of size n_0 from each population.

(b) Compute the pooled sample standard deviation s_p.

(c) Determine N the total sample size that must be taken from each population where $n = k(n_0 - 1)$,

$$N = \max\left\{ n_0, 2s_p^2 \left[\frac{h}{\delta*} \right]^2 \right\}, \tag{1.17}$$

$$b_{ij} = \begin{cases} 2(k-1)/k & \text{if } i = j \\ \\ -2/k & \text{if } i \neq j \end{cases},$$

$$C = \frac{\Gamma[\frac{1}{2}(n+k-1)]}{\sqrt{k}\left(\frac{1}{2}n\pi\right)^{\frac{1}{2}(k-1)}\Gamma\left(\frac{1}{2}n\right)} \quad (1.18)$$

and h is the solution to

$$\int_{-\infty}^{h} \cdots \int_{-\infty}^{h} C\left\{1 + \frac{1}{n}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} b_{ij}t_i t_j\right\}^{-\frac{1}{2}(n+k-1)} dt_1\, dt_2 \ldots dt_{k-1} = P^*. \quad (1.19)$$

Stage 2:

(a) Take a sample of size $N - n_0$ from each population.

(b) Calculate $\overline{X}_{i,N}$ for each population.

(c) Claim that the population, π_i corresponding to $\overline{X}_{[i],N}$ is the population with mean $\mu_{[k]}$.

The important concept to take away from this is that of using two stages. The first stage estimates an unknown parameter and determines a necessary sample size. The second stage determines the population for which we are looking.

Other variations of the Two-stage Normal Means Procedure exist. In the next procedure, the equality of the populations' standard deviations is not assumed. The procedure that follows was first given by Dudewicz and Dalal in [5].

Two-Stage Normal Means Procedure (Dudewicz & Dalal [5]):

Stage 1:

(a) Take a sample of size n_0 from each population.

(b) Compute the sample standard deviation s_i for each population.

(c) Determine n_i the total sample size that must be taken from population π_i where $F_{n_0}(z)$ is the cumulative students-t distribution with $n_0 - 1$ degrees of freedom, $f_{n_0}(z)$ is its density, h is the solution to

$$\int_{-\infty}^{\infty} F_{n_0}^{k-1}(z+h) f_{n_0}(z)\ dz = P^*, \tag{1.20}$$

and

$$n_i = \max\left\{ n_0 + 1, \left[\frac{s_i h}{\delta^*}\right]^2 \right\}. \tag{1.21}$$

Stage 2:

(a) Take a sample of size $n_i - n_0$ from each population.

(b) Calculate a weighted sample mean $\widetilde{X}_{i,n_i}$ for each population based upon the combined sample of size n_i from π_i where

$$\widetilde{X}_{i,n_i} = \sum_{j=1}^{n_i} a_{i,j} X_{i,j}, \tag{1.22}$$

and $a_{i,j}$ in (1.22) are chosen so that

(i) $\sum_{j=1}^{n_i} a_{i,j} = 1$

(ii) $a_{i,1} = \cdots = a_{i,n_0}$

(iii) $s_i^2 \sum_{j=1}^{n_i} a_{i,1}^2 = (\frac{\delta^*}{h})^2$.

(c) Claim that the population π_i corresponding to the largest weighted mean, $\widetilde{X}_{[i],n_i}$, is the population with the largest mean, $\mu_{[k]}$.

This procedure is different in two ways from the previous one. First, our sample from each population could be of a different size. Secondly, we do not use the intuitive estimate for the mean; instead we use $\widetilde{X}_{i,n_i}$.

1.2.3 Normal Variances Procedure

The preceding procedures selected populations based on the location of the population's distribution. However, we are not limited to comparing only locations. In this section, we compare distributions based upon their dispersion. Since we will continue to look at Normally distributed populations, dispersion would be measured using the population variance. This is the goal of the Normal Variances Procedures described by Bechhofer and Sobel in [3].

As before, we have k Normally distributed populations, π_i with mean μ_i and variance σ_i^2. The goal of this type of procedure is to select the Normally distributed population with the smallest variance with probability at least P^* whenever $\delta^*\sigma_{[2]}^2 \geq \sigma_{[1]}^2$. This procedure can be completed in a single stage, in a manner similar to the Normal Means Procedure.

Normal Variances Procedure(Bechhofer & Sobel [3]):

1. Take a simple random sample of size n from each population where n is the smallest positive integer to satisfy

$$\int_0^{\infty} \left[1 - G_{n-1}\left(\frac{z}{\delta^*}\right)\right]^{k-1} g_{n-1}(z)\, dz \geq P^*, \tag{1.23}$$

where $G_{n-1}(z)$ is the cumulative distribution function of a χ^2 random variable with $n-1$ degrees of freedom, and $g_{n-1}(z)$ is the corresponding density.

2. Estimate σ_i^2 by the sample variance, $s_{i,n}^2$, from each population.

3. Claim that the population π_i corresponding to $s_{[i],n}^2$ is the population with the largest mean $\sigma_{[1]}$ where the ordered sample variances are denoted by

$$s_{[1],n}^2 \leq s_{[2],n}^2 \leq \cdots \leq s_{[k],n}^2. \tag{1.24}$$

1.2.4 Remarks

Remark 1.1. The procedures presented illustrate the manner in which only the "best" population should be selected; the procedures can be made more general. They can be reformulated to select the first two "best", the first three "best", and so on. Additionally, we have considered the case of the largest mean, or the smallest variance. But, these procedures can be reformulated to consider the smallest mean, or the largest variance.

Remark 1.2. These are not the only procedures available for selecting populations based on their means. Procedures based upon other parameters exist; as well as procedures based on populations with nonnormal distributions. These have been outlined in [8] and [9].

1.3 Subset Selection Approach

With the subset selection approach to Ranking and Selection problems, our goal is to select a subset of the populations that contains the "best" population(s). In many applications, subset selection procedures are meant to eliminate populations from study. Thus, the "best" populations could be considered the "good" populations worthy of possible further study. More accurately, our goal is to select a subset of the populations that contains all of the "good" population(s). Therefore, given a collection of k populations, $\{\pi_i\}_{i=1}^k$, we assume $\{\pi_i\}_{i=1}^k$ can be partitioned into two subsets, G and B, where

$$G = \{\text{the ``good'' populations}\} \tag{1.25}$$

and

$$B = G^c. \tag{1.26}$$

Specifically, the goal of a subset selection procedure is to select a set $\widehat{G}_n \subset \{\pi_i\}_{i=1}^k$ such that

$$G \subset \widehat{G}_n \tag{1.27}$$

and

$$P(G \subset \widehat{G}_n) \geq P^*. \tag{1.28}$$

How do we define the "good" populations? In many cases, these are defined as those that are better than some standard, or control, population. So, G may be defined as

$$G = \{\pi_i | \pi_i \text{ is better than a standard(control).}\}. \tag{1.29}$$

We will illustrate these ideas in the next sections. Section 1.3.1 considers selecting populations that are better than a standard population. Section 1.3.2 considers selecting those populations better than a control population. Subset selection procedures were first studied by Gupta in [10].

1.3.1 Normal Means Selection with Respect to a Standard

Again, we consider the case of k Normally distributed populations. We assume that their means, μ_i are unknown, but that their common standard deviation, σ, is known. Our goal is to select a subset of all populations that includes those populations whose mean is better than a given standard mean, μ_0. If $\mu_i \geq \mu_0$, a population will be considered better than the standard . Thus, the "good" populations are

$$G = \{\pi_i | \mu_i \geq \mu_0\}. \tag{1.30}$$

The subset Normal Means procedure is as follows:

Subset Normal Means Selection with Respect to a Standard(Gupta [10]):

1. Take a simple random sample of size n from each population.

2. Estimate μ_i with the sample mean, $\overline{X}_{i,n}$, from each population.

3. Select all populations that are members of

$$\widehat{G}_n = \left\{ \pi_i \mid \overline{X}_{i,n} \geq \mu_0 - \delta^* \frac{\sigma}{\sqrt{n}} \right\} \tag{1.31}$$

   where δ^* is the solution to

$$\Phi(\delta^*) = (P^*)^{\frac{1}{k}} . \tag{1.32}$$

4. Claim that the populations in G are contained in $\widehat{G}_n$.

This procedure will satisfy the *probability requirement*,

$$P\left(G \subset \widehat{G}_n\right) \geq P^* \in \left(.5^k, 1\right). \tag{1.33}$$

It should be noticed that the procedure allows the sample size to be chosen arbitrarily. But, once the sample size is chosen, the selection rule is fixed based upon the value chosen for P^*. This is seen as follows. First, a correct selection (CS) of population occurs whenever $G \subset \widehat{G}_n$. Thus, we can calculate a lower bound for the probability of correct selection as follows:

$$
\begin{aligned}
P(CS) &= P\left(G \subset \widehat{G}_n\right) && (1.34)\\
&= P\left(\overline{X}_{i,n} \geq \mu_0 - \delta^* \frac{\sigma}{\sqrt{n}}, i \in G\right) && (1.35)\\
&= \prod_{i \in G} P\left(\overline{X}_{i,n} \geq \mu_0 - \delta^* \frac{\sigma}{\sqrt{n}}\right) && (1.36)\\
&= \prod_{i \in G} P\left(Z_i \geq \frac{\sqrt{n}}{\sigma}(\mu_0 - \mu_i) - \delta^*\right) && (1.37)\\
&\geq \prod_{i \in G} P\left(Z \leq \delta^*\right) && (1.38)\\
&\geq \Phi(\delta^*)^k. && (1.39)
\end{aligned}
$$

(1.36) is due to the independent sampling from each population. By letting

$$
Z_i = \frac{\sqrt{n}(\overline{X}_{i,n} - \mu_i)}{\sigma}, \qquad (1.40)
$$

we have (1.37). Since $\mu_i \geq \mu_0$ for all $i \in G$, we minimize (1.37) when $\mu_i = \mu_0$ for all $\delta^* i \in G$. This gives us (1.38), and removes the sample size n from consideration. Finally, we minimize further by setting $G = \{\pi_i\}_{i=1}^k$. Setting (1.39) equal to P^* gives

(1.32).

1.3.2 Normal Means Selection with respect to a Control

When selecting with respect to a control, we are given a collection of $k+1$ Normally distributed populations, $\{\pi_i\}_{i=0}^{k}$. The means, μ_i, are unknown, and standard deviation, σ, is known and the same for each population. π_0 will be designated as the control population. The goal is to correctly select all populations that are better than π_0. The populations that are better than π_0 are those such that $\mu_i \geq \mu_0$. The procedure for making our selections is similar to the procedure given in Section 1.3.1.

Subset Normal Means Selection with Respect to a Control (Gupta [10]):

1. Take a simple random sample of size n from each population, including π_0.

2. Estimate μ_i with the sample mean, $\overline{X}_{i,n}$, from each population.

3. Select all populations that are members of

$$\widehat{G}_n = \left\{ \pi_i \mid \overline{X}_{i,n} \geq \overline{X}_{0,n} - \delta^* \frac{\sigma}{\sqrt{n}} \right\} \tag{1.41}$$

   where δ^* is the solution to

$$\int_{-\infty}^{\infty} \Phi^k(z+\delta^*)\phi(z)\ dz = P^*. \tag{1.42}$$

4. Claim that the populations in G are contained in $\widehat{G}_n$.

The full derivation of this procedure can be found in [10].

1.3.3 Remarks

Remark 1.3. As with the procedures considered in section 1.2, we have considered only two of many possible procedures. We have only considered the cases where the common standard deviation was known and sample sizes were equal. These procedures can be modified to consider populations with different standard deviations, both known and unknown, as well as unequal sample sizes. We have only considered the means of Normally distributed populations. But, we need not restrict ourselves to simply normal populations. In fact, we need not restrict ourselves to defining "good" populations based on the mean. Many other parameters can be used.

Remark 1.4. As stated above, the sample sizes in the two subset procedures could be determined arbitrarily. These procedures would be most useful, when the sample of size n has already been collected from each population, and the experimenter needs to determine a selection rule that will meet their probability requirement, i.e. determine δ^*.

Remark 1.5. A procedure could be devised in which we always determine that $\widehat{G}_n = \Omega$. This would meet any probability requirement. It would also make this procedure useless. Thus, it is also important to consider, in some way, the expected size of $\widehat{G}_n$, $E(\#\widehat{G}_n)$ where $\#A$ denotes the cardinality of a set A. It would be preferable to use a procedure in which $|G|\leq E(|\widehat{G}_n|) \geq |G|+\epsilon$ for some $\epsilon > 0$.

Remark 1.6. The subset selection procedures given here are meant to familiarize us with making selections against a known standard, or a known population. However, our goal could have been to select a subset of populations that contains the population with the largest mean. Many methods of this sort exist.

1.4 Nonparametric Procedures

In this section, we present two nonparametric univariate Ranking and Selection procedures. We will outline procedures that make selections based on either the location of, or the dispersion of, the given populations. We will describe the location of a distribution using the α-quantile, and the dispersion using the Inter-(α, β) Range. When considering a univariate distribution, P_i, we will denote its cumulative distribution function as $F_i(x) = P(X_i \leq x)$, $x \in \mathbb{R}$.

Definition 1.2. For $\alpha \in (0, 1)$, the α-quantile of a distribution F is defined to be

$$x_\alpha(F) = \inf \{x \in \mathbb{R} \mid F(x) = \alpha\} . \tag{1.43}$$

Definition 1.3. For $0 < \alpha < \frac{1}{2} < \beta < 1$, the inter-$(\alpha, \beta)$ Range of F is defined to be

$$Q_{\alpha,\beta}(F) = x_\beta(F) - x_\alpha(F). \tag{1.44}$$

These are generalizations of the usual nonparametric measures of location and dispersion. The median of F is $x_{.5}(F)$, and the interquartile range is equal to $Q_{.25,.75}(F)$.

1.4.1 Largest α-quantile Procedure

Given k populations, with absolutely continuous distributions F_i, the goal of this section is to select the population with the largest α-quantile. This procedure was originally developed by Sobel in [28]. We define an ordering on the populations as follows:

Definition 1.4. For fixed $\alpha \in (0,1)$, given two populations π_1, π_2 with cumulative distribution functions F_1, F_2, then π_1 is said to be **better** than $\pi_2 (\pi_1 \preceq \pi_2)$ if and only if

$$x_\alpha(F_1) \leq x_\alpha(F_2). \tag{1.45}$$

This definition also induces an ordering on distributions. This will be denoted by $F_1 \preceq F_2$. Let $F_{[1]} \preceq F_{[2]} \preceq \cdots \preceq F_{[k]}$ be the correct ordering of the distributions. As with the procedures outlined in section 1.2, we want to choose a population whenever the "best" population is sufficiently different than the rest. In this case, we will prefer to make a correct selection whenever $x_\beta(F_{[k]})$ is the largest not only for $\beta = \alpha$, but also for all $\beta \in (\alpha - \epsilon, \alpha + \epsilon)$. Specifically, the preference zone is defined as

$$PZ = \{(F_1, F_2, \ldots, F_k) | d \geq \delta^*\} \tag{1.46}$$

where $\delta^*, \epsilon > 0$ are constants decided upon by the experimenter,

$$I = \left[x_{\alpha-\epsilon}(F_{[k]}), x_{\alpha+\epsilon}(F_{[k]})\right], \tag{1.47}$$

$$\underline{F}(x) = \min_{i=1,\ldots,k-1} F_{[i]}(x), \tag{1.48}$$

and

$$d = \inf_{x \in I}(\underline{F}(x) - \overline{F}(x)). \tag{1.49}$$

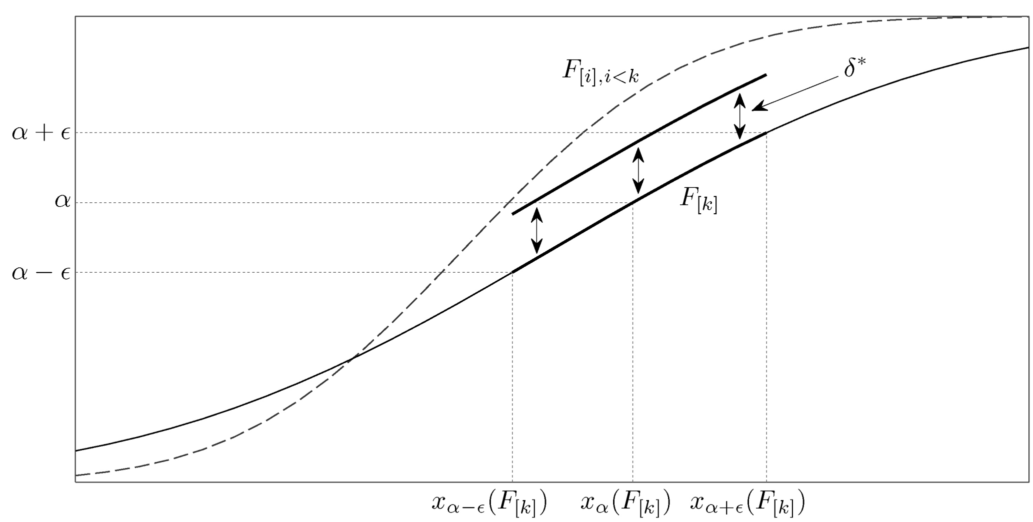


Figure 1.1: Preference Zone: Largest α-quantile

With our preference zone defined, the actual procedure for making a selection is relatively straightforward.

Largest α-quantile Procedure (Sobel [28]) :

1. Take a sample of size n from each population such that n satisfies

$$\int_{\beta-\epsilon^*}^{\beta+\epsilon^*} \int_{\alpha-\epsilon^*}^{\alpha+\epsilon^*} \left[C \int_{v_0-d^*}^{1} \int_{0}^{u_0+d^*} u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s}\, du\, dv \right]^{k-1} \times u_0^{r-1}(v_0-u_0)^{s-r-1}(1-v_0)^{n-s}\, du_0\, dv_0 = P^* \quad (1.50)$$

with $r = (n+1)\alpha, s = (n+1)\beta$, and

$$C = \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(s-r)\Gamma(n-s+1)}.$$

2. For each population, estimate $x_\alpha(F_i)$ with the sample αn order statistic, $\hat{x}_\alpha(F_{i,n})$, where

$$F_{i,n}(x) = \frac{\sum_{j=1}^{n} I_{\{X_{i,j} \le x\}}}{n} \quad (1.51)$$

and $I_A(x)$ is an indicator function,

$$I_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases} \quad (1.52)$$

3. Claim that the population π_i corresponding to $\hat{x}_\alpha(F_{[k],n})$ is the population with

the largest α-quantile $x_\alpha(F_{[k]})$ where the ordered sample means are denoted by

$$\overline{X}_{[1],n} \geq \overline{X}_{[2],n} \geq \cdots \geq \overline{X}_{[k],n}. \tag{1.53}$$

1.4.2 Smallest Inter-(α, β) Range Procedure

This section reviews a procedure given by Sobel in [29] that selects the least dispersed of k populations with absolutely continuous distributions, F_i.

Definition 1.5. If F_1 and F_2 are the c.d.f. for two populations, then F_1 *is less dispersed than* F_2 ($F_1 \preceq F_2$) if and only if for a fixed $\alpha \in (0, \frac{1}{2}), \beta \in (\frac{1}{2}, 1)$,

$$Q_{\alpha,\beta}(F_1) \leq Q_{\alpha,\beta}(F_2). \tag{1.54}$$

Thus, if $F_{[1]} \preceq F_{[2]} \preceq \cdots \preceq F_{[k]}$ is the correct ordering of the distributions being considered, the goal of this procedure is select the population π_i corresponding to $F_{[1]}$ subject to the probability requirement $P(CS) \geq P^*$ whenever the populations fall in the preference zone. The preference zone for this procedure has similarities to the one used in Section 1.4.1. It is defined as

$$PZ = \{(F_1, F_2, \ldots, F_k) | \delta \geq \delta^*\} \tag{1.55}$$

where $\delta^*, \epsilon > 0$ are constants decided upon by the experimenter so that $0 < \epsilon <$

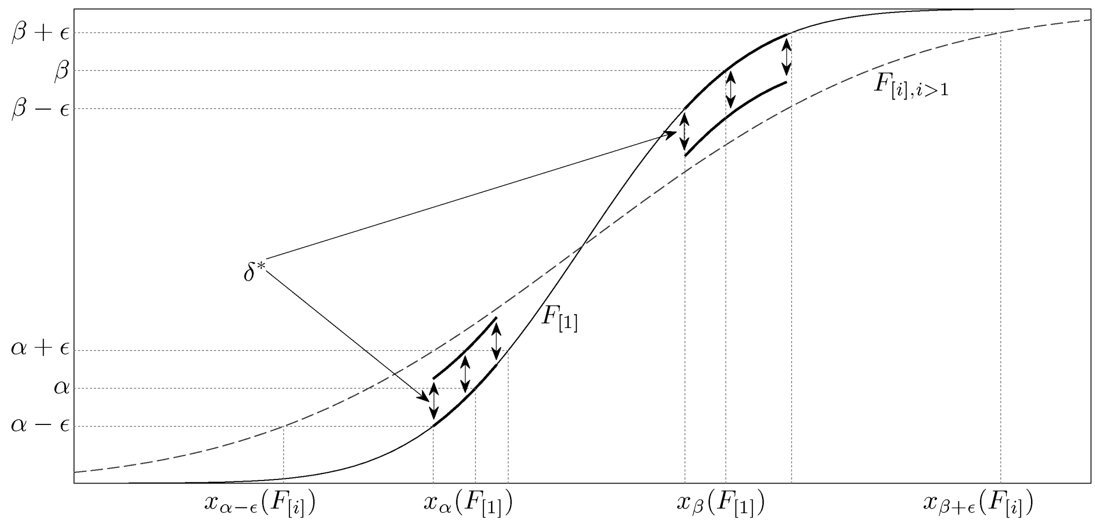


Figure 1.2: Preference Zone: Smallest Inter-(α, β)-range

$\min\{\alpha, \frac{1}{2}-\alpha, 1-\beta, \beta-\frac{1}{2}\}$,

$$I_1 = [x_{\alpha-\epsilon}(F_{[1]}), x_{\alpha+\epsilon}(F_{[1]})], \tag{1.56}$$

$$I_2 = [x_{\beta-\epsilon}(F_{[1]}), x_{\beta+\epsilon}(F_{[1]})], \tag{1.57}$$

and

$$\delta = \min_{i=2,\cdots,k} \left\{ \inf_{x\in I_1} F_{[i]} - F_{[1]}, \inf_{x\in I_2} F_{[1]} - F_{[i]} \right\}. \tag{1.58}$$

The procedure is as follows.

Smallest Inter-(α, β) Range (Sobel [29]) :

1. Take a simple random sample of size n from each population where $G(p) = \frac{n!}{(r-1)!(n-r)!}\int_0^p x^{r-1}(1-x)^{n-r}$ and n satisfies

$$(k-1)\int_{\alpha+\delta^*-\epsilon}^{\alpha+\delta^*+\epsilon} G^k(v)[1-G(v-\delta^*)]\,dG(v) + G^{k-1}(\alpha-\epsilon+\delta^*)[1-G(\alpha-\epsilon)] = P^*. \quad (1.59)$$

   (To simplify matters, α could be taken to be a rational, and n possibly increased so that αn is an integer.)

2. For each population, estimate $Q_{\alpha,\beta}(F_i)$ with the sample Inter-(α, β) Range of F_i,

$$\widehat{Q}_{\alpha,\beta}(F_{i,n}) = \hat{x}_\beta(F_{i,n}) - \hat{x}_\alpha(F_{i,n}). \quad (1.60)$$

3. Claim that the population π_i corresponding to $\widehat{Q}_{\alpha,\beta}(F_{[1]i,n})$ is the population with the smallest α-quantile $Q_{\alpha,\beta}(F_{[1],n})$ where the ordered sample Inter-(α, β) Ranges are denoted by

$$\widehat{Q}_{\alpha,\beta}(F_{[1],n}) \le \widehat{Q}_{\alpha,\beta}(F_{[2],n}) \le \cdots \le \widehat{Q}_{\alpha,\beta}(F_{[k],n}). \quad (1.61)$$

As with the previous procedure, α, β are taken to be rational numbers, and n may be increased so that $(n+1)\alpha$ and $(n+1)\beta$ are integers.

1.4.3 Remarks

Remark 1.7. Both methods outlined above use the Indifference Zone approach. Subset selection approaches can be found in [19] and [29].

Remark 1.8. The procedure described in Section 1.4.2 deals with the dispersion of the different populations. Dispersion can generally be considered separately from location. It should be noted that this procedure deals with distributions that have the same location in the sense that the intervals, $[x_\alpha(F_i), x_\beta(F_i)]$, must be nested. In the procedure described in Chapter 3, a multivariate procedure is developed, that does not require the populations to be nested in any manner.

1.5 Multivariate Procedures

Until this point, we have restricted our review to populations with univariate distributions. This section will outline some of the possible setups for selecting among multivariate procedures. Procedures have been developed that make a selection based upon the location, the dispersion, and other characteristics of a distribution. It will be assumed that the populations follow a multivariate Normal Distribution in $\mathbb{R}^d, d \geq 2$. We denote the mean vector, and the dispersion matrix for the distribution P_{X_i} by $\vec{\mu}_i$ and Σ_i.

1.5.1 Largest Mahalanobis Distance

In this section, we review the setup for procedures presented by Alam and Rizvi in [1].

Definition 1.6. The Mahalanobis Distance of a point $x \in \mathbb{R}^d$ from the origin is $d(x) = x'\Sigma^{-1}x$.

This goal is to select the population among k that are given whose mean vector is the furthest from the origin with respect to the Mahalanobis Distance. Therefore, we may define an ordering on the populations as follows:

Definition 1.7. Given two populations π_1, π_2 with mean vectors $\vec{\mu}_1$ and $\vec{\mu}_2$, then π_1 is said to be **better** than π_2 if and only if $d(\vec{\mu}_1) > d(\vec{\mu}_2)$.

With this definition, we may order the populations from nearest to farthest. Thinking of the origin as a target, the population with the largest Mahalanobis Distance from the origin can be thought of as the most off-target. The actual procedures follow the same patterns as those given before, and so we omit them. However, it should be noted that different procedures exist based upon whether the dispersion matrix is known, or unknown. These procedures will use the non-central χ^2 distribution, or non-central F distribution, for determining sample sizes.

1.5.2 Smallest Generalized Variance

In [6], Eaton describes one possible setup for selecting from several multivariate Normal distributions. Populations are selected based upon the size of the generalized variance of their distributions.

Definition 1.8. The Generalized Variance of a distribution P_X with dispersion matrix Σ is $|\det(\Sigma)|$.

The generalized variance is one possible method for measuring the dispersion of a given distribution. In Chapter 3, we use a similar idea to measure the dispersion of a distribution.

1.6 Concluding Remarks

In this chapter, several Ranking and Selection procedures were illustrated. In Section 1.1, a short introduction to the basic setting of a Ranking and Selection problem was given. Section 1.2 introduced the Indifference Zone approach using the Normal Means Procedure. As a natural follow-up, the Two-Stage Normal Means Procedure was given. This was given in an attempt to illustrate how an initial sample can be used to estimate some unknown parameter, which can then be used to determine the necessary sample size needed in order to make a selection. In Section 1.3, subset selection procedures with respect to a standard were reviewed. This was meant to provide a motivation for the formulation of the procedure given in Chapter 5. Section 1.4

illustrates two nonparametric univariate techniques for making a selection based upon location (α-quantile) and dispersion (Inter-(α, β) range). The procedures outlined in Chapter 3 can not be considered a generalization of those outlined in Section 1.4.2, but they can be considered to have the same intention, selection based upon dispersion. The same can be said for the procedures given in Chapter 4 when compared to Section 1.4.1. They both considered the location of the given distributions, but the idea of location will be different. These differences will come from the manner in which we define concepts that will be used to describe the dispersion, location, and type of distributions being considered. In most cases, these concepts will be defined with the use of Depth Functions: the topic of our next chapter.

Chapter 2

Data Depth

2.1 Introduction to Data Depth

Data Depth is the study of depth functions. Ideally, a depth function is a function that measures how “deep” a point is with respect to a given probability distribution. “Deep” should be understood in the everyday sense of the word. Thus, a depth function can convey a sense of how centered (buried), or how outlying (near the surface), a point is.

Definition 2.1 (Serfling [34])**.** If $\mathcal{P}$ is the collection of all probability distributions, a *depth function* is any bounded, nonnegative mapping

$$D(\,\cdot\,;\,\cdot\,) : \mathbb{R}^d \times \mathcal{P} \to \mathbb{R} \tag{2.1}$$

that provides a probability based center-outward ordering of points on $\mathbb{R}^d$, $d \geq 1$.

Let $P_X \in \mathcal{P}$ be a probability distribution given by a random vector $X \in \mathbb{R}^d$. To be consistent with our everyday understanding of the words "depth" and "deep", it has been argued that a depth function should have certain properties[34].

Properties 2.1. *Useful Properties for Depth Functions:*

(i) $D(Ax + b; P_{AX+b}) = D(x; P_X)$ *where* A *is a* $d \times d$ *nonsingular matrix , and* b *is any* $d \times 1$ *vector . (The depth of a point should not depend upon the coordinate system being used.)*

(ii) If $P_X \in \mathcal{P}$ *has center* θ*, then* $D(\theta; P_X) = \sup_{x \in \mathbb{R}^d} D(x; P_X)$*. (If a distribution has a well-defined center, then it should have maximal depth.)*

(iii) If θ *is the deepest point for any* $P_x \in \mathcal{P}$*, then* $D(x; P_X) \leq D(\theta + t(x - \theta); P_X)$ *for* $t \in [0, 1]$*. (Depth is a decreasing function along any ray away from the deepest point.)*

(iv) $D(x; P_X) \to 0$ *as* $\|x\|_2 \to \infty$ *for any* $P_X \in \mathcal{P}$*. The farther a point is away from the center, the shallower the points will become.*

Many depth functions have been proposed. We shall define a few of them. A more comprehensive review by Liu, Parelius, and Singh can be found in [15].

Example 2.1. The Mahalanobis Depth of a point $x \in \mathbb{R}^d$ is defined as

$$MD(x; P_X) = \frac{1}{1 + (x - \mu_{P_X})' \Sigma_{P_X}^{-1} (x - \mu_{P_X})}. \tag{2.2}$$

The Mahalanobis Depth has all of the properties listed above. However, it does not exist for all probability distributions. If either μ_{P_X}, or Σ_{P_X}, do not exist for $P_X, MD(x; P_X)$ will not exist.

Example 2.2. The Simplicial Depth (Liu, [14]) of a point $x \in \mathbb{R}^d$ is defined as

$$SD(x; P_X) = P(x \in S[X_1, X_2, \ldots, X_{d+1}]) \tag{2.3}$$

where $S[X_1, X_2, \ldots, X_{d+1}]$ is the closed simplex of $d+1$ random observations from P_X. When given a sample of n data points, $\{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^d$, the simplicial depth of a point x^* can be computed as follows:

1. Construct all triangles using data points as vertices.

2. Determine the proportion of triangles that contain x^*.

2.2 Tukey's Halfspace Depth

The final depth function that will be introduced will be the first that was proposed in the literature, Tukey's Halfspace Depth. In [35], Zuo and Serfling show that it has the four properties listed above. It is named for Tukey [32]. However, Hodges used a version of the bivariate halfspace depth, without calling it a depth function, to conduct a bivariate sign test in [11]. It is arguably the most studied depth function. It will be the depth function that will be used from this point forward.

Definition 2.2. The halfspace depth[11, 17, 32] of a point $x \in \mathbb{R}^d$ with respect to $P_X \in \mathcal{P}$ is

$$D(x; P_X) = \inf\{P_X(H) \mid x \in H, H \textit{ is a closed half-space}\},$$

$$= \inf_{u \in U} P_X(H_{x,u}),$$

where $H_{x,u} = \{y \in \mathbb{R}^d \mid u'y \geq u'x, u \in U\}$ and $U = \{u \in \mathbb{R}^d \| \|u\|_2 = 1\}$.

2.2.1 Properties

This section will introduce many properties of the halfspace depth.

Theorem 2.2. *(Massé, [17]) Suppose P_X is absolutely continuous, then $D(x; P_X)$ is a continuous function of x.*

By definition, the halfspace depth of a point x is an infimum over all closed half spaces that contain x. This can be changed to the infimum over all closed half spaces that contain x on their boundary. Also, in general, the infimum can not be replaced by a minimum since there does not necessarily exist a halfspace that attains the infimum value.

Example 2.3. Let $P := (P_1 + P_2)/2$ where P_1 is the standard normal distribution on the x-axis, and P_2 is a point mass at $(0, 2)$. The depth of $(0, 1)$ is $\frac{1}{4}$, but no halfspace attains this probability.

However, under some conditions, the infimum is attained.

Theorem 2.3 (Massé, [17])**.** *Suppose P_X is absolutely continuous, then there exists $u \in U$,*

$$D(x; P_X) = P_X(H_{x,u}). \tag{2.4}$$

When this infimum is attained, it may or may not be attained uniquely.

Definition 2.3. The set of minimal directions for x with respect to P_X is

$$T(x) = \{u \in U | P_X(H_{x,u}) = D(x; P_X)\}. \tag{2.5}$$

Example 2.4. Consider the bivariate standard normal distribution Z. $P_Z(H_{(0,0),u}) = \frac{1}{2}$ for all $u \in U$. Thus, $T((0,0)) = U$. However, $P_Z(H_{(1,0),u}) \geq P_Z(H_{(1,0),(1,0)})$ for all $u \in U$. Therefore, $T((1,0)) = \{(1,0)\}$.

Of course, the extreme cases are not the only possibilities.

Example 2.5. Suppose $P((0,-1)) = P((0,1)) = \frac{1}{2}$. Then $D((0,0); P) = \frac{1}{2}$ and $T((0,0)) = U \backslash \{(1,0), (-1,0)\}$.

For any probability distribution, the halfspace depth is naturally bounded above by 1. This upper bound is obtained when the distribution consists of a single point mass.

Definition 2.4. The maximal depth of a probability distribution P_X is

$$\alpha^* = \sup_{x \in \mathbb{R}^d} D(x; P_X). \tag{2.6}$$

Our next theorem tells us that this depth is attained.

Theorem 2.4 (Rousseeuw & Ruts, [23])**.** *There exists at least one* $x^* \in \mathbb{R}^d$ *such that* $D(x^*; P_X) = \alpha^*$.

In fact, it may be attained by more than one point. Any point along the line segment between $(0, -1)$ and $(0, 1)$ in Example 2.5 attains the maximal depth of $\frac{1}{2}$ for this distribution. If we are willing to assume that a distribution is absolutely continuous, we can place tighter bounds upon the maximal depth of a distribution.

Theorem 2.5 (Rousseeuw & Ruts, [23])**.** *If* P_X *is absolutely continuous, then*

$$\frac{1}{d+1} \leq \alpha^* \leq \frac{1}{2}. \tag{2.7}$$

Besides bounding the halfspace depth, the maximal depth can also be used to classify some absolutely continuous probability distributions.

Definition 2.5. A distribution is said to be *angularly symmetric* about a point θ if and only if $P_X(\theta + A) = P_X(\theta - A)$ for any Borel set $A \subset \mathbb{R}^d$.

Theorem 2.6 (Rousseeuw & Struyf, [24])**.** *If* P_X *is an absolutely continuous distribution, then* P_X *is angularly symmetric if and only if* $\sup_{x \in \mathbb{R}^d} D(x; P_X) = \frac{1}{2}$.

Besides using a single point to classify a type of distribution, under certain circumstances it has been shown that the halfspace depth characterizes a distribution.

Theorem 2.7. *Suppose that at least one of the following conditions is satisfied:*

(i) P_X *has compact support. [13]*

(ii) P_X *is an empirical distribution. [31]*

(iii) $D(x; P_X)$ *has smooth depth contours. [12]*

Then, the halfspace depth characterizes the distribution P_X*.*

Hence, if we know $D(x; P_X)$ for all $x \in \mathbb{R}^d$, then we can theoretically reconstruct the distribution of P_X. Next, we have one last set of properties.

Theorem 2.8 (Zou & Serfling, [35])**.** *The halfspace depth has all the Useful Properties for Depth Functions:*

(i) $D(Ax + b; P_{AX+b}) = D(x; P_X)$ *where* A *is a* $d \times d$ *nonsingular matrix , and* b *is any* $d \times 1$ *vector .*

(ii) For any $P_X \in \mathcal{P}$ *with center* θ*, then* $D(\theta; P_X) = \alpha^*$.

(iii) If θ *is the deepest point for any* $P_X \in \mathcal{P}$*, then* $D(x; P_X) \leq D(\theta + t(x - \theta); P_X)$ *for* $t \in [0, 1]$.

(iv) $D(x; P_X) \to 0$ *as* $\|x\|_2 \to \infty$ *for any* $P_X \in \mathcal{P}$*.*

2.2.2 Convergence

In this section, we give some convergence results.

Definition 2.6. Let $X_1, X_2, \ldots, X_n$ be a simple random sample from P_X, and $B \subset \mathbb{R}^d$ be any Borel set, then the *empirical distribution* of P_X is ,

$$\widehat{P}_{X,n}(B) = \frac{1}{n}\sum_{j=1}^{n} I_B(X_j) \tag{2.8}$$

where $I_B(x)$ is the indicator function for B. When it will not lead to confusion, the sample size n will be suppressed in the notation.

Definition 2.7. The *empirical depth* of a point $x \in \mathbb{R}^d$ with respect to P_X is defined to be

$$\begin{aligned} D_n(x; P_X) &:= D(x; \widehat{P}_{X,n}) \\ &= \inf\{\widehat{P}_{X,n}(H) \mid x \in H, H \textit{ is a closed half-space}\}. \end{aligned}$$

The first result tells us that the empirical depth function of P_X converges uniformly to its population version, almost surely.

Theorem 2.9 (Donoho & Gasko, [4])**.** *For any $P \in \mathcal{P}$,*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} |D_n(x; P_X) - D(x; P_X)| \overset{a.s.}{\to} 0. \tag{2.9}$$

This is done by showing that

$$\sup_{x\in\mathbb{R}^d} |D_n(x; P_X) - D(x; P_X)| \leq \sup_{H\in\mathcal{H}} |P_n(H) - P(H)| \tag{2.10}$$

where $\mathcal{H}$ is the collection of all halfspaces in $\mathbb{R}^d$. The next result is about the distributional convergence of a class of points with respect to a distribution. But, first we need some definitions.

Definition 2.8 (Massé, [18])**.** A point $x \in \mathbb{R}^d$ is called *P-Smooth*, if $D(x; P_X) = 0$ or the cardinality of $T(x)$ is equal to 1.

Definition 2.9 (Massé, [18])**.** P_X is called *locally regular*, if the following conditions hold:

(i) $P_X(\partial H) = 0$ for all closed half-spaces H,

(ii) For every x of positive depth, either $T(x)$ is finite or $T(x) = U$.

Any absolutely continuous distribution will meet the first condition.

Theorem 2.10 (Massé, [18])**.** *Suppose P_X is locally regular, and for fixed x*

(i) x is P_X-smooth,

(ii) $\alpha = D(x; P) > 0$

then

$$\sqrt{n}[D_n(x; P_X) - D(x; P_X)] \xrightarrow{d} N\left(0, \alpha(1-\alpha)\right). \tag{2.11}$$

2.2.3 Halfspace Depth Contours

The previous section looked at the behavior of the halfspace depth at a single point. This section looks at collections of points.

Definition 2.10. For $\alpha \in [0, \alpha^*]$, the *α-trimmed depth-region* of P is defined to be

$$D^\alpha(P_X) = \{x \in \mathbb{R}^d | D(x; P_X) \geq \alpha\}, \tag{2.12}$$

$$= \bigcap \{H \mid H \text{ is a closed halfspace}, P_X(H) > 1 - \alpha\}. \tag{2.13}$$

Example 2.6 (Rousseeuw & Ruts [23])**.** The *α-trimmed depth-region* for some bivariate distributions.

(i) If $U \sim \text{Uniform}([0,1] \times [0,1])$ then

$$D^\alpha(P_U) = \left\{(x, y) \in [0,1] \times [0,1] \mid \min(x, 1-x) \min(y, 1-y) \geq \frac{\alpha}{2}\right\} \tag{2.14}$$

(ii) If $Z \sim \text{Normal}(0,0,1,1,0)$, then

$$D^\alpha(P_Z) = \left\{(x, y) \mid x^2 + y^2 \leq (\Phi^{-1}(1-\alpha))^2\right\} \tag{2.15}$$

(iii) If C is bivariate Cauchy, then

$$D^\alpha(P_C) = \left\{(x, y) \mid \max\{|x|, |y|\} \leq \tan\left[\sqrt{\pi(\frac{1}{2} - \alpha)}\right]\right\}. \tag{2.16}$$

We will denote the empirical α-trimmed depth-regions of P_X, $D_n^\alpha(P_X)$, by $D_n^\alpha(P_X)$. The α-trimmed depth-regions have many useful properties. In cases, where we have several distributions given by random vectors $X_1, X_2, \ldots, X_k$, we denote $D_n^\alpha(P_{X_i})$ by

$D^{\alpha}_{i,n}$.

Theorem 2.11 (Zou & Serfling [35]). *Properties of α-trimmed depth-regions.*

(i) If A is a $d \times d$ nonsingular matrix, and b is any $d \times 1$ vector, then $D^{\alpha}(P_{AX+b}) = AD^{\alpha}(P_X) + b$ and $D^{\alpha}_n(P_{AX+b}) = AD^{\alpha}_n(P_X) + b$.

(ii) If $\alpha_1 \geq \alpha_2$, then $D^{\alpha_1}(P_X) \subseteq D^{\alpha}_2(P_X)$ and $D^{\alpha_1}_n(P_X) \subseteq D^{\alpha_2}_n(P_X)$.

(iii) $D^{\alpha}(P_X)$ and $D^{\alpha}_n(P_X)$ are connected sets in $\mathbb{R}^d$.

(iv) For $\alpha > 0$, $D^{\alpha}(P_X)$ are compact. If P_X is absolutely continuous, then $D^{\alpha}_n(P_X)$ is compact also.

Property (i) says that the α-trimmed depth-regions are affine equivariant. Let $\alpha_p = \sup\{\alpha \mid P(D^{\alpha}(P_X)) \geq p\}$.

Definition 2.11. The p^{th}*-central region* of P_X is $D^{\alpha_p}(P_X)$.

Figures 2.1, 2.2, and 2.3, represent p^{th}-central regions for 5000 data points randomly drawn from three different distributions, Normal, Uniform, and Cauchy. Property (ii) says that the depth regions are nested, which gives p^{th}*-central region* a rather intuitive meaning. The p^{th}*-central region* is the smallest, by set containment, α-trimmed depth-regions that contains at least probability p. Property (iv), is especially important to ensure that this next term takes on a finite value.

Definition 2.12. For $p \in (0, 1)$, the p^{th}*-volume of* P_X, $V^p(P_X)$, is defined to be

$$V^p(P_X) = \inf\{Volume(D^{\alpha}(P_X)) \mid P(D^{\alpha}(P_X)) \geq p,\ 0 < \alpha < \alpha^*\} \tag{2.17}$$

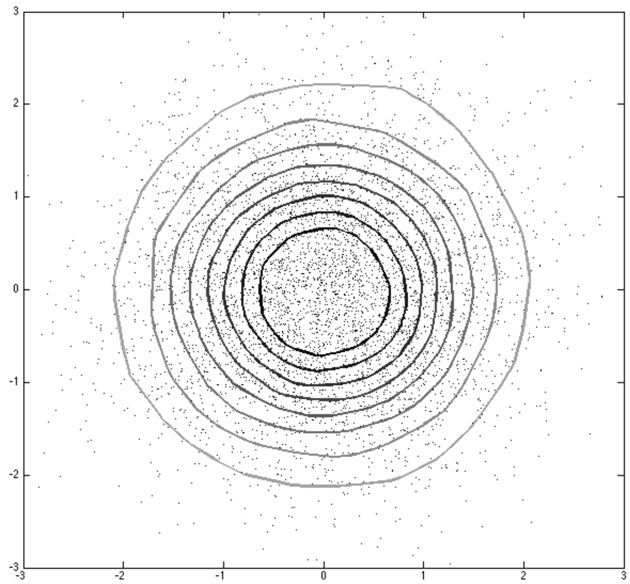


Figure 2.1: Normal(0,0,1,1,0)
$p = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$

where α^* is the maximal depth described in 2.4.

Because the α-trimmed depth-regions are nested, it is not hard to see that the p^{th}-*volume* of P_X is equal to the volume of the p^{th}-*central region*. If we let $p = 0.5$, $V^p(P_X)$ is the volume of the central 50% of P_X: much like the Interquartile Range describes the length of the central 50% of a univariate distribution. Thus, $V^p(P_X)$ can be seen to be a measure of the dispersion of a distribution. Letting p vary between 0 and 1, we can get a sense of how the probability is dispersed.

Example 2.7. For $d = 2$, the p^{th}-*volume* of several distributions:

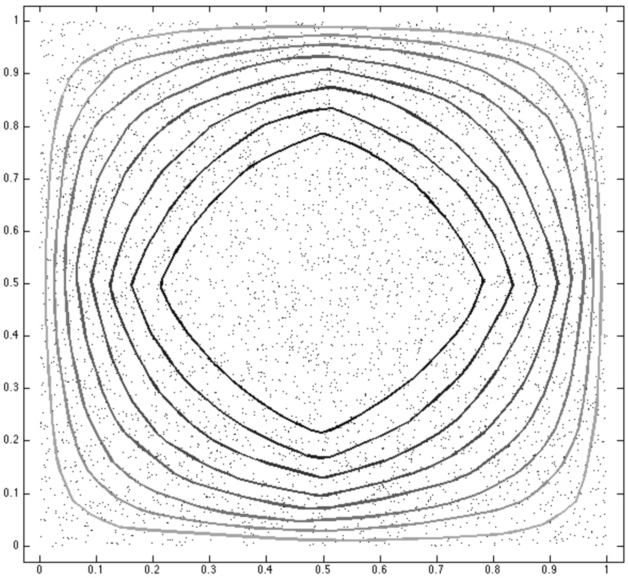


Figure 2.2: Uniform $[0,1]\times[0,1]$
$p = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$

(i) If $U \sim \text{Uniform}([0,1]\times[0,1])$ then

$$V^p(P_U) = p; \tag{2.18}$$

(ii) If $Z \sim \text{Normal}(0,0,1,1,0)$, then

$$V^p(P_Z) = -2\pi\ln(1-p); \tag{2.19}$$

(iii) If C is bivariate Cauchy, then

$$V^p(P_C) = 4\tan^2(\sqrt{p}\pi/2). \tag{2.20}$$

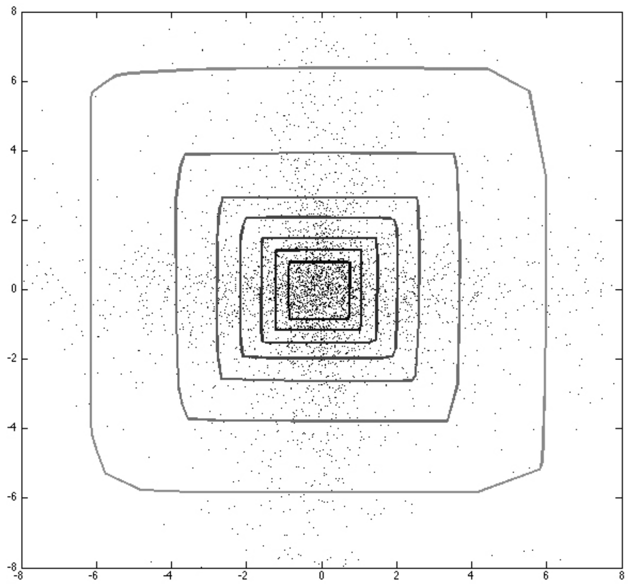


Figure 2.3: Cauchy $(0, 1)$
$p = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$

2.2.4 Convergence of Depth Regions

This section reviews some needed convergence results for depth regions. The first result tells us that as n increases, the empirical α-trimmed depth-regions of P_X become close to their population counterpart.

Theorem 2.12 (Zuo & Serfling [35]). *For any $P_X \in \mathcal{P}, \epsilon > 0, \delta < \epsilon, \alpha \geq 0$ and $\alpha_n \to \alpha$,*

(i) there exists N_ϵ such that for all $n \geq N_\epsilon$

$$D^{\alpha+\epsilon} \subset D_n^{\alpha_n+\delta} \subset D_n^{\alpha_n} \subset D_n^{\alpha_n-\delta} \subset D^{\alpha-\epsilon} \ a.s. \tag{2.21}$$

(ii) if $P(\{x \in \mathbb{R}^d \mid D(x; P_X) = \alpha\}) = 0$, *then as* $n \to \infty, D_n^{\alpha_n} \overset{a.s.}{\to} D^\alpha$.

The second result we review tells us the asymptotic distribution of p^{th}*-semi-empirical volume of* P_X.

Definition 2.13. For $p \in (0, 1)$, the p^{th}*-semi-empirical volume of* $P\dot{}X$, $\widetilde{V}_n^p(P_X)$, is defined to be

$$\widetilde{V}_n^p = \inf\{Volume(D^\alpha(P_X)) \mid P_n(D^\alpha(P_X)) \geq p,\ 0 < \alpha < \alpha^*\}. \tag{2.22}$$

This function is called semi-empirical for a reason. With a close examination of the definition, it becomes apparent that this value, $\widetilde{V}_n^p$, can not be computed directly from a sample alone. Information about the population distribution is needed. However, it can be shown that under certain conditions, we have an asymptotically Normal Distribution.

Theorem 2.13 (Serfling [25])**.** *Assume that*

(i) P_X *is absolutely continuous;*

(ii) $D(x; P_X)$ *is continuous in* x, *vanishes outside the support of* P_X, $D(x; P_X) \to 0$ *as* $\|x\| \to \infty$, *and* $\{x \mid D(x; P_X) = \alpha\} \neq \phi$ *for* $0 < \alpha < \alpha^*$;

(iii) V^p *is finite, strictly increasing, and with derivative* $v^p > 0$.

Then for fixed $p \in (0,1)$, *as* $n \to \infty$

$$\frac{\sqrt{n}\left(\widetilde{V}_n^p - V^p\right)}{v^p} \xrightarrow{d} N(0, p(1-p)).$$

2.3 Concluding Remarks

In this chapter, we introduced the concept of Data Depth, and provided a few examples of existing depth functions. Then, we focused on the properties of Tukey's Halfspace Depth that will be useful in developing the procedures of Chapters 3 and 4. We chose the Tukey's Halfspace Depth because it is the oldest, and most studied, depth function. Therefore, many results exist that facilitate its use. Additionally, the definition of the halfspace depth is intuitively easy to understand, and relates nicely to the the univariate quantile. However, the disadvantage it has is that its computation can be quite complex. However, this is not to say, that the Halfspace Depth cannot be computed. Several algorithms exist for computing, or approximating, the Halfspace Depth in any dimension. In the following chapters, we limited ourselves to using existing MatLab code in our simulations. This limited our simulations to $\mathbb{R}^2$. However, using other algorithms, the forthcoming simulations could be repeated in higher dimension.

Chapter 3

Dispersion

In this chapter, we develop four procedures for selecting the population with the least dispersed distribution. Let $\Omega = \{\pi_i\}_{i=1}^k$ be a collection of k populations that follow an absolutely continuous distributions P_{X_i} given by a random vectors $X_i \in \mathbb{R}^d, d \geq 1$. Recall, the *$\alpha$-trimmed depth-regions* of P,

$$D^\alpha(P_{X_i}) = \{x \in \mathbb{R}^d \mid D(x; P_{X_i}) \geq \alpha\},$$

and for $p \in (0,1)$ the p^{th}-volume of P_{X_i},

$$V^p(P_{X_i}) = \inf\{Volume(D^\alpha(P_{X_i})) \mid P_{X_i}(D^\alpha(P_{X_i})) \geq p,\ 0 < \alpha < \alpha^*\}$$

where $\alpha^* = \sup_{x \in \mathbb{R}^d} D(x; P)$. To simplify notation, when it will not cause confusion, we let $P_i = P_{X_i}$ and $V_i^p = V^p(P_{X_i})$.

Definition 3.1. Given populations π_1 and π_2 with distributions P_1 and P_2, π_1 is said to be *less dispersed* than π_2 at level p , ($\pi_1 \preceq \pi_2$), if and only if for fixed $p \in (0, 1)$,

$$V_1^p \leq V_2^p. \tag{3.1}$$

Letting $V_{[1]}^p \leq V_{[2]}^p \leq \cdots \leq V_{[k]}^p$ represent the ordered population volumes, this induces an ordering on Ω, $\pi_{[1]} \preceq \pi_{[2]} \preceq \cdots \preceq \pi_{[k]}$), from least dispersed to most dispersed where $\pi_{[i]}$ is the population with the p^{th}-volume $V_{[i]}^p$. Figure 3.1 illustrates the difference in dispersion for populations. The regions in black represent the .5-central region.

3.1 Goal: Selecting the Least Dispersed

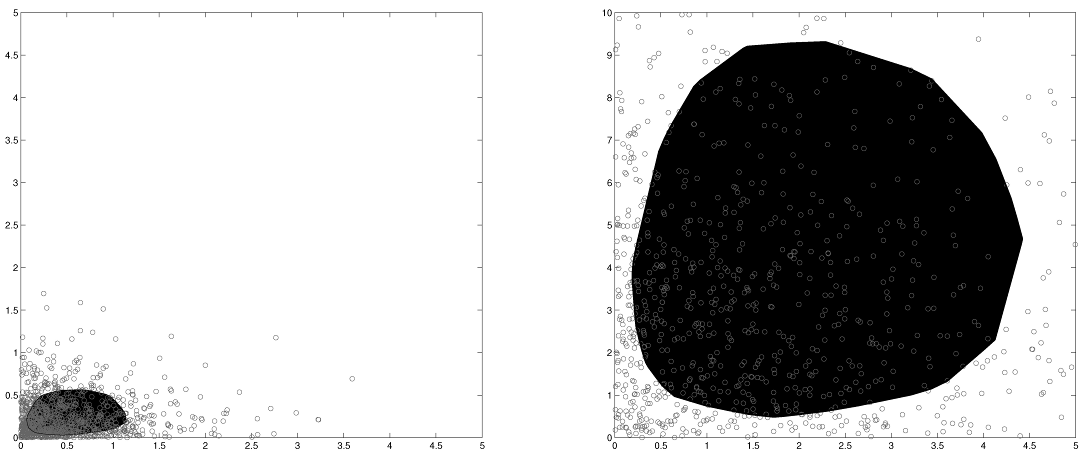
Our goal in this chapter is to develop procedures for selecting $\pi_{[1]}$, the least dispersed population. Two different types of indifference zone will be defined. Let $D_{i,n}^{\alpha}$ denote $D^{\alpha}(P_{i,n})$.

Definition 3.2. For $p \in (0, 1)$, the p^{th}-*empirical volume* of P_i, $\widehat{V}_{i,n}^p$ is defined to be

$$\widehat{V}_{i,n}^p = \inf\{Volume(D_{i,n}^{\alpha}) \mid P_{i,n}(D_{i,n}^{\alpha}) \geq p,\ 0 < \alpha < \alpha_i^*\} \tag{3.2}$$

where α^X is the maximal depth for the empirical distribution of P_i.

Specifically, it is the volume of the smallest *empirically* determined p^{th} central



(a) $V^{.5} = .46$ (b) $V^{.5} = 29.50$

Figure 3.1: Volumes of Exponential Depth Regions

region with *empirical* probability at least p. Since we hope to select the least dispersed population, we hope that the smallest empirical volume is produced by the population with the smallest actual volume. Let $\widehat{V}^p_{[1],n} \leq \widehat{V}^p_{[2],n} \leq \cdots \leq \widehat{V}^p_{[k],n}$ denote the ordered p^{th}-empirical volumes, and $\widehat{V}^p_{(i),n}$ represent the sample volume corresponding to the i^{th} smallest population volume. A correct selection (CS_n) based on a sample of size n is the event

$$CS_n = \Big\{\pi_{[1]} \text{ is selected.}\Big\} \tag{3.3}$$

$$= \Big\{\text{The least dispersed population is selected.}\Big\} \tag{3.4}$$

$$= \Big\{\widehat{V}^p_{[1],n} = \widehat{V}^p_{(1),n}\Big\}. \tag{3.5}$$

While our decision making process is now clear, the correctness of our decision is uncertain. We need to control this uncertainty in some manner. Thus, our goal is to determine a procedure that will make a correct selection with a probability at least $P^* \in (k^{-1}, 1)$.

3.2 Assumptions:

All four types of procedures assume the conditions for Theorem 2.13 hold. To review, these assumptions are

1. P_i is absolutely continuous, $i = 1, \ldots, k$;

2. V_i^p is finite, strictly increasing, and with derivative $v_i^p > 0, p \in (0, 1)$.

3.3 Procedures:

As has been said, four procedures will be defined. Two procedures will be defined in a manner that allows for a single sample of size n to be taken from each population. However, to complete these procedures, some additional population parameters are assumed to be known. In practice, knowledge of these parameters would be unrealistic. Therefore, we also define two procedures that will be conducted in two stages. The first stage will be used to make a consistent estimate of the necessary parameters. The second stage will be used to make our decision. The other distinction between the procedures will come in the definition of the preference zone. In one set of procedures, the preference zone will be defined using a difference; in the other set, with a ratio. The difference-based procedures will use the results of Theorem 2.13 in a more natural way. While the ratio-based procedures will compare populations in a more natural manner. The ratio-based results will compare the ratio of the dispersions of populations. Justifications for these procedures will be given in Section 3.4.

3.3.1 Single-Stage Difference-based Selection of the Least Dispersed Population:

Two similar pairs of procedures will be defined. The difference between the procedures will be based on the information that is available regarding the derivatives, v_i^p, of the functions V_i^p. The assumptions regarding knowledge of v_i^p will be listed with their respective procedure. Regardless of the assumptions, the preference zone will be the same. We define the preference zone, for fixed $p \in (0,1)$ and $\delta^* > 0$, as

$$PZ = \left\{ (V_1^p, V_2^p, \ldots, V_k^p) \mid V_{[2]}^p - V_{[1]}^p > \delta^* \right\}. \tag{3.6}$$

Procedure R_{V1a}**:**

For procedure R_{V1a}, it will be assumed that the values of v_i^p are known for all populations. R_{V1a} will be conducted as follows:

1. Take a sample of size n from each population, where

$$n = \left\lceil \left(\frac{h v_{[k]}^p}{\delta^*} \right)^2 p(1-p) \right\rceil \tag{3.7}$$

and h is the solution to

$$\int_{-\infty}^{h} \prod_{i=2}^{k} P\left(Z > \frac{v_{(1)}^{p}}{v_{(i)}^{p}}(z-h)\right) \phi(z)dz = P^* \tag{3.8}$$

and Z is a standard normal random variable.

2. Calculate the p^{th} empirical volume, $\widehat{V}_{i,n}^{p}$ for each population.

3. Declare that the population π_i with sample volume $\widehat{V}_{[1],n}^{p}$ is $\pi_{[1]}$.

This procedure will asymptotically satisfy the probability requirement

$$P(CS_n \mid R_{V1a}) \geq P^* \text{ whenever } V_{[2]}^{p} - V_{[1]}^{p} > \delta^*.$$

Procedure R_{V1b}:

For procedure R_{V1b}, it will be assumed that there exist v_* and v^* such that $0 < v_* \leq v_i^p \leq v^*$ for $i = 1, \ldots, k. R_{V1b}$ will be conducted as follows:

1. Take a sample of size n from each population, where

$$n = \left\lceil \left(\frac{hv^*}{\delta^*}\right)^2 (p(1-p)) \right\rceil, \tag{3.9}$$

and h is the solution to

$$\int_{-\infty}^{h} P\left(Z > \frac{v_*}{v^*}(z-h)\right)^{k-1} \phi(z)dz = P^* \tag{3.10}$$

and Z is a standard normal random variable.

2. Calculate the p^{th} empirical volume, $\widehat{V}^p_{i,n}$ for each population.

3. Declare that the population π_i with sample volume $\widehat{V}^p_{[1],n}$ is $\pi_{[1]}$.

This procedure will asymptotically satisfy the probability requirement

$$P(CS_n \mid R_{V1b}) \geq P^* \text{ whenever } V^p_{[2]} - V^p_{[1]} > \delta^*. \tag{3.11}$$

3.3.2 Two-Stage Difference-based Selection of the Least Dispersed Population:

Two procedures will be given. The second is similar to the first, simply more conservative in its sample size calculation. Two stages will be required due to the nature of our approach to the problem. We make no assumptions regarding prior knowledge of the derivatives of V^p_i. The first stage will be used to make some estimates regarding these unknown derivatives. The second stage will be used to make our decision. We let $v_{i,n}$ denote a consistent estimator of v_i^p based on a sample of size n. The procedures are as follows.

Procedure R_{V2a}:

Stage 1:

(a) Take a sample of size n_1 from each population.

(b) Calculate v_{i,n_1}^p for each population.

(c) Determine a total sample size

$$n = \max\left\{ n_1, \left\lceil \left(\frac{h v_{[k],n_1}^p}{\delta^*} \right)^2 (p(1-p)) \right\rceil \right\} \tag{3.12}$$

where h is the solution to

$$\int_{-\infty}^{h} \prod_{i=2}^{k} P\left(Z > \frac{v_{[1],n_1}^p}{v_{[i],n_1}^p}(z-h) \right) \phi(z)dz = P^* \tag{3.13}$$

and Z is a standard normal random variable.

Stage 2:

(a) Take a sample of size $n_2 = n - n_1$ from each population, if $n_2 > 0$.

(b) Calculate the p^{th} empirical volume, $\widehat{V}_{i,n}^p$ for each population.

(c) Declare that the population π_i with sample volume $\widehat{V}_{[1],n}^p$ is $\pi_{[1]}$.

This procedure will satisfy the probability requirement

$$P(CS_n \mid R_{V2a}) \geq P^* \text{ whenever } V^p_{[2]} - V^p_{[1]} > \delta^*. \tag{3.14}$$

Procedure R_{V2b}**:**

Stage 1:

(a) Take a sample of size n_1 from each population.

(b) Calculate v^p_{i,n_1} for each population.

(c) Determine a total sample size

$$n = \max\left\{ n_1, \left\lceil \left(\frac{hv^p_{(k),n_1}}{\delta^*} \right)^2 (p(1-p)) \right\rceil \right\} \tag{3.15}$$

where h is the solution to

$$\int_{-\infty}^{h} P\left(Z > \frac{v^p_{[1],n_1}}{v^p_{[k],n_1}} (z-h) \right)^{k-1} \phi(z)dz = P^* \tag{3.16}$$

and Z is a standard normal random variable.

Stage 2:

(a) Take a sample of size $n_2 = n - n_1$ from each population, if $n_2 > 0$.

(b) Calculate the p^{th} empirical volume, $\widehat{V}_{i,n}^p$ for each population.

(c) Declare that the population π_i with sample volume $\widehat{V}_{[1],n}^p$ is $\pi_{[1]}$.

This procedure will satisfy the probability requirement

$$P(CS_n \mid R_{V2b}) \geq P^* \text{ whenever } V_{[2]}^p - V_{[1]}^p > \delta^*. \tag{3.17}$$

3.3.3 Single-Stage Ratio-based Selection of the Least Dispersed Population:

The procedures outlined in this section will be similar to those given in Section 3.3.1. They will be single stage procedures that assume some knowledge of the derivative of the volume functionals. The difference comes in the definition of the preference zone. We define the preference zone, for fixed $p \in (0,1), \delta^* > 1$, and $\beta > 0$ as

$$PZ = \left\{(V_1^p, V_2^p, \ldots, V_k^p) \mid V_{[2]}^p / V_{[1]}^p > \delta^*, V_{[1]}^p > \beta\right\}. \tag{3.18}$$

Procedure R_{V3a}:

For procedure R_{V3a}, it will be assumed that the values of v_i^p are known for all populations. R_{V3a} will be conducted as follows:

1. Take a sample of size n from each population, where

$$n = \left\lceil \left(\frac{h v^p_{[k]}}{(\delta^* - 1)\beta} \right)^2 p(1-p) \right\rceil, \tag{3.19}$$

and h is the solution to

$$\int_{-\infty}^{h} \prod_{i=2}^{k} P\left(Z > \frac{v^p_{(1)}}{v^p_{(i)}}(z-h) \right) \phi(z)dz = P^* \tag{3.20}$$

and Z is a standard normal random variable.

2. Calculate the p^{th} empirical volume, $\widehat{V}^p_{i,n}$ for each population.

3. Declare that the population π_i with sample volume $\widehat{V}^p_{[1],n}$ is $\pi_{[1]}$.

This procedure will satisfy the probability requirement

$$P(CS_n \mid R_{V3b}) \geq P^* \text{ whenever } V^p_{[2]}/V^p_{[1]} > \delta^* \text{ and } V^p_{[1]} > \beta. \tag{3.21}$$

Procedure R_{V3b}:

For procedure R_{V3b}, it will be assumed that there exist known v_* and v^* such that $0 < v_* \leq v^p_i \leq v^*$ for $i = 1, \ldots, k$. R_{V3a} will be conducted as follows:

1. Take a sample of size n from each population, where

$$n = \left\lceil \left(\frac{hv^*}{(\delta^* - 1)\beta} \right)^2 p(1-p) \right\rceil, \tag{3.22}$$

   and h is the solution to

$$\int_{-\infty}^{h} P\left(Z > \frac{v_*}{v^*}(z-h) \right)^{k-1} \phi(z)dz = P^* \tag{3.23}$$

   and Z is a standard normal random variable.

2. Calculate the p^{th} empirical volume, $\widehat{V}_{i,n}^p$ for each population.

3. Declare that the population π_i with sample volume $\widehat{V}_{[1],n}^p$ is $\pi_{[1]}$.

This procedure will satisfy the probability requirement

$$P(CS_n \mid R_{V3b}) \geq P^* \text{ whenever } V_{[2]}^p / V_{[1]}^p > \delta^* \text{ and } V_{[1]}^p > \beta. \tag{3.24}$$

3.3.4 Two-Stage Ratio-based Selection of the Least Dispersed Population:

The procedures put forth in this section mirror those given in Section 3.3.2. Thus, we will not assume any preexisting information is known about the derivatives of the volume functionals. However, we will be considering a variation of the preference zone given in Section 3.3.3. We define the preference zone, for fixed $p \in (0, 1)$, and $\delta^* > 1$, as

$$PZ = \left\{ (V_1^p, V_2^p, \ldots, V_k^p) \mid V_{[2]}^p / V_{[1]}^p > \delta^* \right\}. \tag{3.25}$$

Procedure R_{V4a}**:**

Stage 1:

(a) Take a sample of size n_1 from each population.

(b) Calculate v_{i,n_1}^p for each population.

(c) Calculate $\widehat{V}_{i,n_1}^p$ for each population.

(d) Determine a total sample size

$$n = \max \left\{ n_1, \left\lceil \left(\frac{h v_{[k],n_1}^p}{(\delta^* - 1)\widehat{V}_{[1],n_1}^p} \right)^2 (p(1-p)) \right\rceil \right\} \tag{3.26}$$

where h is the solution to

$$\int_{-\infty}^{h} \prod_{i=2}^{k} P\left(Z > \frac{v_{[1],n_1}^p}{v_{[i],n_1}^p}(z-h) \right) \phi(z) dz = P^* \tag{3.27}$$

and Z is a standard normal random variable.

Stage 2:

(a) Take a sample of size $n_2 = n - n_1$ from each population, if $n_2 > 0$.

(b) Calculate the p^{th} empirical volume, $\widehat{V}^p_{i,n}$ for each population.

(c) Declare that the population π_i with sample volume $\widehat{V}^p_{[1],n}$ is $\pi_{[1]}$.

This procedure will satisfy the probability requirement

$$P(CS_n \mid R_{V4a}) \geq P^* \text{ whenever } V^p_{[2]}/V^p_{[1]} > \delta^*. \tag{3.28}$$

Procedure R_{V4b}:

Stage 1:

(a) Take a sample of size n_1 from each population.

(b) Calculate v^p_{i,n_1} for each population.

(c) Calculate $\widehat{V}^p_{i,n_1}$ for each population.

(d) Determine a total sample size

$$n = \max\left\{ n_1, \left\lceil \left(\frac{hv^p_{[k],n_1}}{(\delta^* - 1)\widehat{V}^p_{[1],n_1}} \right)^2 (p(1-p)) \right\rceil \right\} \tag{3.29}$$

where h is the solution to

$$\int_{-\infty}^{h} P\left(Z > \frac{v^p_{[1],n_1}}{v^p_{[k],n_1}}(z-h) \right)^{k-1} \phi(z)dz = P^* \tag{3.30}$$

and Z is a standard normal random variable.

Stage 2:

(a) Take a sample of size $n_2 = n - n_1$ from each population, if $n_2 > 0$.

(b) Calculate the p^{th} empirical volume, $\widehat{V}^p_{i,n}$ for each population.

(c) Declare that the population π_i with sample volume $\widehat{V}^p_{[1],n}$ is $\pi_{[1]}$.

This procedure will satisfy the probability requirement

$$P(CS_n \mid R_{V4b}) \geq P^* \text{ whenever } V^p_{[2]}/V^p_{[1]} > \delta^*. \tag{3.31}$$

Of all the procedures, the two stage procedures would seem to be the most reasonable. Prior knowledge of the derivatives v^p_i seems unlikely. When considering both two stage procedures, the ratio-based procedures would seem the most reasonable since we are looking at the dispersion of the populations under consideration.

3.4 Proofs:

3.4.1 General Results

Each procedure outlined in Section 3.3 is based on the asymptotic normality result given in Theorem 2.13. Under the assumptions of Theorem 2.13, we have that

$$\frac{\sqrt{n}(\widetilde{V}^p_n - V^p)}{v^p} \xrightarrow{d} N(0, p(1-p))$$

for $0 < p < 1$ where $\widetilde{V}_n^p$ is the p^{th}*-semi-empirical volume,*

$$\widetilde{V}_n^p = \inf\{Volume(D^\alpha(P)) | P_n(D^\alpha(P)) \geq p,\ 0 < \alpha < \alpha^*\}.$$

A close examination of the definition of $\widetilde{V}_n^p$ and $\widehat{V}_{i,n}^p$ will show that they are not the same. If the underlying distribution is unknown, $\widetilde{V}_n^p$ is not computable. Knowledge of the underlying distribution is needed in order to determine the sets $D^\alpha(P)$. On the other hand, $\widehat{V}_{i,n}^p$ was defined to be computable, and to be an almost sure approximation of $\widetilde{V}_n^p$. This will be shown using the next few results. With these forthcoming results, we will be able to define an alternative correct selection as

$$\widetilde{CS}_n = \left\{\widetilde{V}_{[1],n}^p = \widetilde{V}_{(1),n}^p\right\}. \tag{3.32}$$

Then we will show $\lim_{n\to\infty} \left| P(\widetilde{CS}_n) - P(CS_n) \right| \to 0$.

Lemma 3.1. *If* $\alpha_n \to \alpha$ *and* $P(\{x \in \mathbb{R}^d | D(x;P) = \alpha\}) = 0$ *then* $P_n(D_n^{\alpha_n}) \to P(D^\alpha)$ *almost surely.*

Proof. Let $\epsilon > 0$. According to [35], for $\epsilon > 0$. there exists N_ϵ such that $D^{\alpha+\epsilon} \subset D_n^{\alpha_n} \subset D^{\alpha-\epsilon}$ for all $n \geq N_\epsilon$. Therefore, $P_n(D^{\alpha+\epsilon}) \leq P_n(D_n^{\alpha_n}) \leq P_n(D^{\alpha-\epsilon})$. By the strong law of large numbers, we have

$$P(D^{\alpha+\epsilon}) \leq \liminf P_n(D_n^{\alpha_n}) \leq \limsup P_n(D_n^{\alpha_n}) \leq P(D^{\alpha-\epsilon}).$$

Let $\epsilon \to 0$. Using continuity from above and below,

$$P(D^{\alpha+\epsilon}) \to P(\{x \in \mathbb{R}^d | D(x;P) > \alpha\}), \text{and}$$
$$P(D^{\alpha-\epsilon}) \to P(D^{\alpha}).$$

Since $P(\{x \in \mathbb{R}^d | D(x;P) = \alpha\}) = 0$, we see that $P_n(D_n^{\alpha_n}) \overset{a.s.}{\to} P(D^{\alpha})$. □

Lemma 3.2. *Let $\alpha_p = \sup\{\alpha | P(D^{\alpha}) \geq p\}$, and $\widehat{\alpha}_{p,n} = \sup\{\alpha | P_n(D_n^{\alpha}) \geq p\}$. If P is absolutely continuous, $\alpha \in (0, \alpha^*)$, and $P(D^{\alpha})$ is strictly decreasing in α, then $\widehat{\alpha}_{p,n} \to \alpha_p$.*

Proof. Since $\widehat{\alpha}_{p,n} \in [0,1]$, then $\beta = \liminf \widehat{\alpha}_{p,n} \in [0,1]$, and there exists a subsequence $\widehat{\alpha}_{p,n_j} \to \beta$. Suppose $\beta < \alpha_p$. Let $0 < \delta^* < |\alpha_p - \beta|/10$. Then there exists $J_\delta^* > 0$ such that for all $j \geq J_\delta^*$, $|\widehat{\alpha}_{p,n_j} - \beta| < \delta^*$. Now, we have $\widehat{\alpha}_{p,n_j} < \beta + \delta^* < \alpha_p$ for all $j \geq J_\delta^*$ and $D_n^{\alpha_{p,n_j}} \supset D_n^{\beta+\delta^*} \supset D_n^{\alpha_p}$ [35]. Thus,

$$P_{n_j}(D_{n_j}^{\alpha_{p,n_j}}) \geq p > P_{n_j}(D_{n_j}^{\beta+\delta^*}) \geq P_{n_j}(D_{n_j}^{\alpha_p}).$$

Letting $j \to \infty$,

$$p \geq P(D^{\beta+\delta^*}) \geq P(D^{\alpha_p}) \geq p.$$

This contradicts our assumption that $P(D^{\alpha})$ is strictly decreasing in α. Therefore, $\beta \geq \alpha_p$.

Next, assume that $\beta = \limsup \widehat{\alpha}_{p,n} > \alpha_p$. Again, we have $\widehat{\alpha}_{p,n_j} \to \beta$. Therefore,

$P_{n_j}(D_{n_j}^{\alpha_{p,n_j}}) \to P(D^\beta)$. Since $P_{n_j}(D_{n_j}^{\alpha_{p,n_j}}) \geq p$, then $P(D^\beta) \geq p$. By definition, if $\alpha > \alpha_p$, then $P(D^\alpha) < p$, and so we have a contradiction. Therefore, $\beta \leq \alpha_p$ and $\widehat{\alpha}_{p,n} \to \alpha_p$. □

Lemma 3.3. *Let* $p \in (0,1), \alpha_p = \sup\{\alpha | P(D^\alpha) \geq p\}$, *and* $\widetilde{\alpha}_{p,n} = \sup\{\alpha | P_n(D^\alpha) \geq p\}$. *If* P *is absolutely continuous,* $\alpha \in (0, \alpha^*)$, *and* $P(D^\alpha)$ *is strictly decreasing in* α, *then* $\widetilde{\alpha}_{p,n} \to \alpha_p$.

Proof. The proof is similar to the one provided in the Lemma 3.2. □

Lemma 3.4. *If* P *is absolutely continuous, and* $Volume(\{D(x; P) = \alpha_p\}) = 0$, *then* $\widehat{V}_n^p \to V^p$ *and* $\widetilde{V}_n^p \to V^p$ *almost surely for* $p \in (0,1)$.

Proof. For the first case, notice that $\widehat{V}_n^p = Vol(D_n^{\widehat{\alpha}_{p,n}})$, and $V^p = Vol(D^{\alpha_p})$. Let $\epsilon > 0$. Since $\widehat{\alpha}_{p,n} \to \alpha_p$ and by [35], there exists $N_\epsilon > 0$ such that $D^{\alpha_p+\epsilon} \subset D_n^{\widehat{\alpha}_{p,n}} \subset D^{\alpha_p-\epsilon}$ for all $n \geq N_\epsilon$. Thus, $Vol(D^{\alpha_p+\epsilon}) \leq Vol(D_n^{\widehat{\alpha}_{p,n}}) \leq Vol(D^{\alpha_p-\epsilon})$ for all $n \geq N_\epsilon$. Letting $n \to \infty$, $Vol(D^{\alpha_p+\epsilon}) \leq \liminf Vol(D_n^{\widehat{\alpha}_{p,n}}) \leq \limsup Vol(D_n^{\widehat{\alpha}_{p,n}}) \leq Vol(D^{\alpha_p-\epsilon})$. Letting $\epsilon \to 0$, with continuity from above and below, we have $Vol(\{D(x; P) > \alpha_p\}) \leq \liminf Vol(D_n^{\widehat{\alpha}_{p,n}}) \leq \limsup Vol(D_n^{\widehat{\alpha}_{p,n}}) \leq Vol(D^{\alpha_p})$. Under our assumption that $Vol(\{D(x; P) = \alpha_p\}) = 0$, we have the desired result. The proof for $\widetilde{V}_n^p$ is similar. □

Proposition 3.5. *If* P *is absolutely continuous,* $p \in (0,1)$, *and* $Volume(\{D(x; P) = \alpha_p\}) = 0$, *then* $\left|P(\widetilde{CS}_n) - P(CS_n)\right| \to 0$.

Proof. We will show that $P(\liminf\{\widetilde{CS}_n \cap CS_n\}) = 1$. Let A_i be the event that $\{V^p_{i,n} \to V^p_i\} \cap \{\widetilde{V}^p_{i,n} \to V^p_i\}$. Using the previous lemma, $P(A_i) = 1,\ i = \ldots, k$. By assumption, our populations are independent, thus $P(\cap_{i=1}^k A_i) = 1$. Let $\epsilon \in (0, \min\{V_{(i+1)} - V_{(i)}\}/2)$. Take $\omega \in \cap_{i=1}^k A_i$. By the previous lemma, there exists N_ϵ such that $|V^p_{i,n}(\omega) \to V^p_i(\omega)| < \epsilon$ and $|\widetilde{V}^p_{i,n}(\omega) \to V^p_i(\omega)| < \epsilon$ for $i = 1, \ldots, k$ and all $n > N_\epsilon$. This implies $V^p_{[1],n}(\omega) = V^p_{(1),n}(\omega)$ and $\widetilde{V}^p_{[1],n}(\omega) = \widetilde{V}^p_{(1),n}(\omega)$ for all $n > N_\epsilon$. Therefore, $\omega \in \{\widetilde{CS}_n \cap CS_n\}$ for all but finitely many n, and so $\omega \in \liminf\{\widetilde{CS}_n \cap CS_n\}$. Using Fatou's lemma, we have

$$\begin{aligned}
1 =& P(\cap_{i=1}^k A_i) \\
&\leq P(\liminf\{\widetilde{CS}_n \cap CS_n\}) \\
&\leq \liminf P(\{\widetilde{CS}_n \cap CS_n\}) \\
&\leq 1.
\end{aligned}$$

□

This provides us with a justification for using the semi-empirical volumes as a replacement for determining the necessary sample size. Additionally, it can be seen that $P(CS_n)$ can be bounded below by using an event that is very nearly $\widetilde{CS}_n$.

Proposition 3.6. *If for all i, P_{X_i} is absolutely continuous, V^p_i is finite, strictly increasing, with derivative $v^p_i > 0, p \in (0, 1)$, and $V^p_{[2]} - V^p_{[1]} > \delta^* > 0$ then for a fixed*

p

$$P(CS_n) \geq P\left(\widetilde{V}^{p+\tau^*}_{(1),n} < \widetilde{V}^{p-\tau^*}_{(i),n},\ i = 2, \ldots, k\right) \tag{3.33}$$

for some $\tau^* > 0$.

Proof. Since V_i^p is finite, strictly increasing, and differentiable, therefore V_i^p is continuous in p for all i. Let $\tau^* > 0$ be such that $V^{p-\tau^*}_{[i]} - V^{p+\tau^*}_{[1]} > \frac{\delta^*}{2}$ for all $i = 2, \ldots, k$. Such a τ exists for each i, and so we take the smallest. Notice, that once one such a τ^* is found, this holds for all values less than τ^*. Using the continuity of V_i^p again, we know $Volume(\{D(x; P_{X_i}) = \alpha_p\}) = 0$ for all i. Therefore, by Lemma 3.4, $\widehat{V}^p_{i,n} \to V^p$ and $\widetilde{V}^p_{i,n} \to V^p$ almost surely for all p. Thus, almost surely for all n large enough $\widetilde{V}^{p-\tau}_{(i),n} < \widehat{V}^p_{(i),n} < \widetilde{V}^{p+\tau}_{(i),n}$ for all i. In particular, for all n large enough, each of these values is within $\frac{\delta^*}{4}$ of their respective limits. Thus, for each $i = 2, \ldots, k$ if $\widetilde{V}^{p+\tau^*}_{(1),n} < \widetilde{V}^{p-\tau^*}_{(i),n}$, then

$$\widehat{V}^p_{(1),n} < \widetilde{V}^{p+\tau}_{(1),n} < \widetilde{V}^{p-\tau^*}_{(i),n} < \widehat{V}^p_{(i),n}. \tag{3.34}$$

Intersecting these events, we get the desired result. □

If we let $n \to \infty$ in the previous result, we see that we can approximate a lower bound on $P(CS_n)$ with $P\left(\widetilde{V}^{p+\tau^*}_{(1),n} < \widetilde{V}^{p-\tau^*}_{(i),n},\ i = 2, \ldots, k\right)$ for any τ^* small enough. Therefore, for τ^* sufficiently small, we essentially have , $P(CS_n) \geq P(\widetilde{CS}_n)$.

3.4.2 Difference-based Results:

Procedure R_{V1a}**:**

Theorem 3.7. *Given k populations π_i with distributions P_i, if the following hold*

1. *P_i is absolutely continuous, $i = 1, \ldots, k$;*
2. *V_i^p is finite, strictly increasing, and with derivative $v_i^p > 0$.*
3. *v_i^p are known for all populations.*
4. *$V_{[2]}^p - V_{[1]}^p > \delta^*$*

then

$$P(\widetilde{CS}_n | R_{V1a}) \geq \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_{i,n}\right) dP_{Z_{1,n}} \tag{3.35}$$

where

$$Z_{i,n} = \frac{n^{-\frac{1}{2}}\left(\widetilde{V}_{(i),n}^p - V_{[i]}^p\right)}{v_{(i)}^p \sqrt{p(1-p)}} \text{ and } h = \frac{\delta^*}{v_{[k]}^p}\sqrt{\frac{n}{p(1-p)}}. \tag{3.36}$$

Proof. We shall see this as follows.

$$\begin{aligned}
P(\widetilde{CS}_n | R_{V1a}) &= P\left(\widetilde{V}_{(1),n}^p = \widetilde{V}_{[1]n}^p\right) \\
&= P\left(\widetilde{V}_{(1),n}^p < \widetilde{V}_{(i),n}^p,\ i = 2, \ldots, k\right) \\
&= P\left(Z_{1,n} < \frac{v_{(i)}^p}{v_{(1)}^p} Z_{i,n} + \frac{V_{[i]}^p - V_{[1]}^p}{v_{(1)}^p}\sqrt{\frac{n}{p(1-p)}},\ i = 2, \ldots, k\right) \\
&= \int_{-\infty}^{\infty} P\left(z - \frac{V_{[i]}^p - V_{[1]}^p}{v_{(1)}^p}\sqrt{\frac{n}{p(1-p)}} < \frac{v_{(i)}^p}{v_{(1)}^p} Z_{i,n},\ i = 2, \ldots, k\right) dP_{Z_{1,n}}
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - \frac{V_{[i]}^p - V_{[1]}^p}{v_{(1)}^p} \sqrt{\frac{n}{p(1-p)}} < \frac{v_{(i)}^p}{v_{(1)}^p} Z_{i,n} \right) dP_{Z_{1,n}} \tag{3.37}$$

$$\geq \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - \frac{\delta^*}{v_{[k]}^p} \sqrt{\frac{n}{p(1-p)}} < \frac{v_{(i)}^p}{v_{(1)}^p} Z_{i,n} \right) dP_{Z_{1,n}} \tag{3.38}$$

$$\geq \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_{i,n} \right) dP_{Z_{1,n}} \tag{3.39}$$

It is easy to see that (3.37) takes its minimum over the entire preference zone when

$$V_{[k]}^p = \cdots = V_{[2]}^p = V_{[1]}^p + \delta^*. \tag{3.40}$$

This partially explains (3.38). However, looking at (3.37), we do not know the actual correspondence between $v_{(i)}^p$ and the given values v_i. Therefore, we further reduce matters by replacing $v_{(1)}^p$ with $v_{[k]}^p$ on the left hand side of the inner event described in (3.38). Substitute h and we have the desired result. □

Ideally, at this point we would set (3.39) equal to P^* and determine the value of h that will be used to determine our sample size n. Unfortunately, we have two problems. The first is that the distributions involved depend upon n. This leaves these distributions unknown to us exactly. Which makes solving an integral equation using (3.39) impossible for us. Fortunately, our procedure does not have us solve an equation where (3.39) is set equal to P^*. This can be corrected using Theorem 2.13, which will allow us to approximate (3.39) using normal distributions. We take care of the first problem with the next proposition. However, since we will use the facts of this proof several times, we will provide a lemma of the necessary result.

Lemma 3.8. *Let X_n be a sequence of random variables that converges in distribution to a random variable X. Further, for $i = 1, \dots, k$, let $f_{i,n}, f_i$ be bounded functions such that $f_{i,n}$ converges uniformly to continuous f_i, then $Ef_n(X_n) \to Ef(X)$ where $f_n = \prod_{i=1}^k f_{i,n}$ and $f = \prod_{i=1}^k f_i$.*

Proof. Since the collections of $f_{i,n}$ and f_i are bounded, and uniformly convergent, this implies that f_n converges uniformly to f. Thus, we consider

$$|Ef_n(X_n) - Ef(X)| \leq E|f_n(X_n) - f(X_n)| + |Ef(X_n) - Ef(X)|. \tag{3.41}$$

For large enough n, the first term on the right can be made less than $\epsilon/2$ by the uniform convergence of f_n to f. As for the second term, we apply an alternative definition for convergence in distribution, which says that $X_n \xrightarrow{d} X$ if and only if $|Ef(X_n) - Ef(X)| \to 0$ for all continuous bounded functions f. As defined, f is the product of bounded continuous functions, thus it is bounded and continuous. □

Proposition 3.9. *For fixed $h > 0$, as n goes to infinity,*

$$\int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_{i,n}\right) dP_{Z_{1,n}} \to \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_i\right) dP_{Z_1} \tag{3.42}$$

where $\{Z_i\}$ are a collection of independent standard normal random variables.

Proof. Referring to Lemma 3.8, let $f_{i,n}(z) = P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_{i,n}\right)$, $f_i = P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_i\right)$

for $i = 2, \ldots, k$, $X_n = Z_{1,n}$, and $X = Z_1$. By Glivenko-Cantelli, the $f_{i,n}$ converge uniformly to f_i almost surely, and each is bounded. As stated in Theorem 2.13, X_n converges in distribution to X. □

With this result, we can now approximate $\int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_{i,n}\right) dP_{Z_{1,n}}$ by $\int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_i\right) dP_{Z_1}$ provided we assume that n is large enough. This leaves us with our second problem. We would like to solve

$$\int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_i\right) dP_{Z_1} = P^* \tag{3.43}$$

for h. However, the $\frac{v_{(i)}^p}{v_{(1)}^p}$ terms remain as a scaling factor. Since the actual correspondences between the derivatives and the populations are unknown, we cannot know the correct configuration to use. We will remove this problem by splitting the integral into parts A^* and B^* where

$$A^*(h) + B^*(h) = \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_i\right) dP_{Z_1} \tag{3.44}$$

$$A^*(h) = \int_{-\infty}^{h} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_i\right) dP_{Z_1}, \tag{3.45}$$

and

$$B^*(h) = \int_{h}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_i\right) dP_{Z_1}. \tag{3.46}$$

By the dominated convergence theorem, it should be apparent that as $h \to \infty$ both

$A \to 1$ and $B \to 0$. Also, a quick sketch of a normal distribution, will show that

$$A^*(h) \geq A(h) \tag{3.47}$$

and

$$B^*(h) \geq B(h) \tag{3.48}$$

where

$$A(h) = \int_{-\infty}^{h} \prod_{i=2}^{k} P\left(z - h < \frac{v_{[i]}^p}{v_{[1]}^p} Z_i\right) dP_{Z_1} \tag{3.49}$$

and

$$B(h) = \int_{-\infty}^{h} \prod_{i=1}^{k-1} P\left(z - h < \frac{v_{[i]}^p}{v_{[k]}^p} Z_i\right) dP_{Z_1}. \tag{3.50}$$

With this, we have enough information to determine h by solving

$$A(h) + B(h) = P^*. \tag{3.51}$$

This equation will be used to determine the necessary sample size when a computer is being used. If this is to be solved using Table 6.1, we would use $A(h) = P^*$. This will give a slightly higher sample size than necessary.

At this point, we would like to put the previous results into the context of Procedure R_{V1a}. Procedure R_{V1a} starts by solving $A(h) = P^*$, an equation which has a solution by the next lemma.

Lemma 3.10. *If $h_1 < h_2$, and $A(h)$ is defined as in* (3.49)*, then $A(h_1) < A(h_2)$.*

Proof. Let $f_{i,n}(z,h) = P\left(z - h < \frac{v^p_{[i]}}{v^p_{[1]}} Z_i\right)$. For any values of z and h, $f_{i,n}(z,h) > 0$. Also, for any fixed z, $f_{i,n}(z,h_1) < f_{i,n}(z,h_2)$. Taking a product and multiplying by a standard normal density, we have that

$$\prod_{i=1}^{k} f_{i,n}(z,h_1)\phi(z) < \prod_{i=1}^{k} f_{i,n}(z,h_2)\phi(z). \tag{3.52}$$

Since both sides of the inequality are positive, we have

$$\begin{aligned} \int_{-\infty}^{h_1} \prod_{i=1}^{k} f_{i,n}(z,h_1)\phi(z)\,dz &\leq \int_{-\infty}^{h_1} \prod_{i=1}^{k} f_{i,n}(z,h_2)\phi(z)\,dz && (3.53)\\ &< \int_{-\infty}^{h_2} \prod_{i=1}^{k} f_{i,n}(z,h_2)\phi(z)\,dz. && (3.54) \end{aligned}$$

□

This solution to $A(h) = P^*$ produces a constant h, which is fixed by P^* and not dependent on n. This completes the first part of the first step of Procedure R_{V1a}. Now we apply Proposition 3.9 which says for a fixed h, which is what we have, for large n:

$$\int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v^p_{(i)}}{v^p_{(1)}} Z_{i,n}\right) dP_{Z_{1,n}} \approx \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v^p_{(i)}}{v^p_{(1)}} Z_i\right) dP_{Z_1}. \tag{3.55}$$

Now we will make the assumption that any sample size n that is determined will be large enough for (3.55) to hold. Regardless, we have an h for the right hand side of the result in Proposition 3.9. Since our procedure meets all the conditions for Theorem 3.7, the inequality at the end of Theorem 3.7 holds true with that particular h that we found by solving $A(h) = P^*$. This allows us to determine a sample size. Thus, h and n only become related once Theorem 3.7 is applied. But, prior to that point, h is simply a fixed constant solution to $A(h) = P^*$. This completes the justification for Procedure R_{V1a}. Similar justifications will be used in the following procedures.

Procedure R_{V1b}:

In this section, we consider Procedure R_{V1b}, a variant of Procedure R_{V1a}.

Theorem 3.11. *Given k populations π_i with distributions P_i, if the following hold*

1. *P_i is absolutely continuous, $i = 1, \ldots, k$;*

2. *V_i^p is finite, strictly increasing, and with derivative $v_i^p > 0$;*

3. *there exist v_* and v^* such that $0 < v_* \leq v_i^p \leq v^*$ for $i = 1, \ldots, k$;*

4. *$V_{[2]}^p - V_{[1]}^p > \delta^*$.*

then

$$P(\widetilde{CS}_n | R_{V1b}) \geq \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i)}^p}{v_{(1)}^p} Z_{i,n}\right) dP_{Z_{1,n}} \tag{3.56}$$

where

$$h = \frac{\delta^*}{v^*} \sqrt{\frac{n}{p(1-p)}}. \tag{3.57}$$

Proof. The proof is nearly identical to that of Theorem (3.7) with one exception. At (3.38), we will use v^* instead of $v^p_{[k]}$ to further reduce the value of the integral. $\square$

As with Procedure R_{V1_a}, we want to determine h subject to

$$\int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v^p_{(i)}}{v^p_{(1)}} Z_{i,n}\right) dP_{Z_{1,n}} = P^*. \tag{3.58}$$

We have the same two problems to contend with as before. The only difference is that we have bounds only on the derivatives, and h is defined slightly differently. However, the same reasoning allows us, for large enough n, the approximation

$$\int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v^p_{(i)}}{v^p_{(1)}} Z_{i,n}\right) dP_{Z_{1,n}} \approx \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v^p_{(i)}}{v^p_{(1)}} Z_i\right) dP_{Z_1}. \tag{3.59}$$

Splitting the right hand side as we did before, we have

$$A^*(h) \geq A(h) \tag{3.60}$$

and

$$B^*(h) \geq B(h) \tag{3.61}$$

where

$$A(h) = \int_{-\infty}^{h} P\left(z - h < \frac{v^*}{v_*} Z_2\right)^{k-1} dP_{Z_1} \tag{3.62}$$

and

$$B(h) = \int_{-\infty}^{h} P\left(z - h < \frac{v_*}{v^*} Z_2\right)^{k-1} dP_{Z_1}. \tag{3.63}$$

Again, we solve $A(h) + B(h) = P^*$ or $A(h) = P^*$ depending upon whether computer software, or Table is being used.

The justification of the Two-Stage Difference-based Procedures will be fairly similar to the justification given for the Single-Stage Difference-based Procedures. Essentially, we will find an approximation to an approximation by way of Slutsky's Theorem. First, we state a useful fact based upon Theorem 2.13.

Lemma 3.12. *If $v^p_{(i),n_1}$ is a consistent estimator of $v^p_{(i)}$, then*

$$\lim_{n_1\to\infty} \lim_{n\to\infty} Z_{i,n_1,n} \xrightarrow{d} N(0,1) \tag{3.64}$$

where

$$Z_{i,n_1,n} = \frac{\sqrt{n}\left(\widetilde{V}^p_{(i),n} - V^p_{[i]}\right)}{v^p_{(i),n_1}\sqrt{p(1-p)}}. \tag{3.65}$$

Proof. An application of Theorem 2.13 tells us that $\lim_{n\to\infty} Z_{i,n_1,n} \xrightarrow{d} N\left(0, \frac{v^p_{(i),n_1}}{v^p_{(i)}}\right)$. Since $v^p_{(i),n_1}$ is a consistent estimator of $v^p_{(i)}$, we apply Slutsky's Theorem. □

Proposition 3.13. *For fixed $h > 0$, $\theta \neq 0$,*

$$\lim_{n_1\to\infty} \lim_{n\to\infty} \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \theta Z_{i,n_1,n}\right) dP_{Z_{1,n_1,n}} \to \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \theta Z_i\right) dP_{Z_1} \tag{3.66}$$

where $\{Z_i\}$ are a collection of independent standard normal random variables.

Proof. Apply Lemmas 3.8 and 3.12. $\square$

Procedure R_{V2a}:

Theorem 3.14. *Given k populations π_i with distributions P_i, if the following hold*

1. *P_i is absolutely continuous, $i = 1, \ldots, k$;*

2. *V_i^p is finite, strictly increasing, and with derivative $v_i^p > 0$;*

3. *$V_{[2]}^p - V_{[1]}^p > \delta^*$.*

then

$$P(\widetilde{CS}_n | R_{V2a}) \geq \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i),n_1}^p}{v_{(1),n_1}^p} Z_{i,n_1}\right) dP_{Z_{1,n_1}}. \tag{3.67}$$

Proof. By replacing $v_{(i)}^p$ with $v_{(i),n_1}^p$, the proof is identical to that of Theorem 3.7. $\square$

Now, we follow the same type of argument as before. First, we approximate the previous result.

Proposition 3.15. *For fixed $h > 0$, as n goes to infinity,*

$$\int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i),n_1}^p}{v_{(1),n_1}^p} Z_{i,n_1}\right) dP_{Z_{1,n_1}} \to \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v_{(i),n_1}^p}{v_{(1),n_1}^p} Z_i\right) dP_{Z_1} \tag{3.68}$$

almost surely where $\{Z_i\}$ are a collection of independent standard normal random variables.

Proof. This follows by Lemma 3.12 and an argument similar to that given for Proposition 3.9. □

The final justification for Procedure R_{V2a} follows the same argument given after Proposition 3.9, on pages 68 - 73, by simply using $v^p_{(i)}$ with $v^p_{(i),n_1}$ in (3.44),(3.45), and (3.46). Then we substitute $v^p_{(i),n_1}$ with $v^p_{[i],n_1}$ in (3.47), and (3.48).

Procedure R_{V2b}:

Similar to Procedure R_{V1b}, Procedure R_{V2b} is a variant of Procedure R_{V2a}. As such, the necessary results are simple reformulations of those given for Procedure R_{V2a}. Therefore, they have been omitted.

3.4.3 Ratio-based Results:

In this section, the results necessary to justify the Ratio-based procedures will be given. Mainly, this will consist of a variation of Theorem 3.7 because this will show how the ratio is incorporated into the calculations. Once this is done, all the necessary results are slight modifications to the procedures in the previous section.

Theorem 3.16. *Given k populations π_i with distributions P_i, if the following hold*

1. *P_i is absolutely continuous, $i = 1, \ldots, k$;*

2. *V^p_i is finite, strictly increasing, and with derivative $v^p_i > 0$;*

3. *v^p_i are known for all populations;*

4. $V^p_{[2]}/V^p_{[1]} > \delta^*$*;*

5. $V^p_{[1]} > \beta$*.*

then

$$P(\widetilde{CS}_n|R_{V3a}) \geq \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v^p_{(i)}}{v^p_{(1)}} Z_{i,n}\right) dP_{Z_{1,n}} \tag{3.69}$$

where

$$h = \frac{(\delta^* - 1)\beta}{v^p_{[k]}} \sqrt{\frac{n}{p(1-p)}}. \tag{3.70}$$

Proof. We will see this as follows.

$$\begin{aligned}
P(\widetilde{CS}_n|R_{V3a}) &= P\left(\widetilde{V}^p_{(1),n} = \widetilde{V}^p_{[1]n}\right) \\
&= P\left(\widetilde{V}^p_{(1),n} < \widetilde{V}^p_{(i),n},\ i = 2, \ldots, k\right) \\
&= P\left(Z_{1,n} < \frac{v^p_{(i)}}{v^p_{(1)}} Z_{i,n} + \frac{V^p_{[i]} - V^p_{[1]}}{v^p_{(1)}} \sqrt{\frac{n}{p(1-p)}},\ i = 2, \ldots, k\right) \\
&= \int_{-\infty}^{\infty} P\left(z - \frac{V^p_{[i]} - V^p_{[1]}}{v^p_{(1)}} \sqrt{\frac{n}{p(1-p)}} < \frac{v^p_{(i)}}{v^p_{(1)}} Z_{i,n},\ i = 2, \ldots, k\right) dP_{Z_{1,n}} \\
&= \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - \frac{V^p_{[i]} - V^p_{[1]}}{v^p_{(1)}} \sqrt{\frac{n}{p(1-p)}} < \frac{v^p_{(i)}}{v^p_{(1)}} Z_{i,n}\right) dP_{Z_{1,n}} && (3.71) \\
&= \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - \frac{(\frac{V^p_{[i]}}{V^p_{[1]}} - 1)V^p_{[1]}}{v^p_{(1)}} \sqrt{\frac{n}{p(1-p)}} < \frac{v^p_{(i)}}{v^p_{(1)}} Z_{i,n}\right) dP_{Z_{1,n}} && (3.72) \\
&\geq \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - \frac{(\delta^* - 1)\beta}{v^p_{[k]}} \sqrt{\frac{n}{p(1-p)}} < \frac{v^p_{(i)}}{v^p_{(1)}} Z_{i,n}\right) dP_{Z_{1,n}} && (3.73) \\
&\geq \int_{-\infty}^{\infty} \prod_{i=2}^{k} P\left(z - h < \frac{v^p_{(i)}}{v^p_{(1)}} Z_{i,n}\right) dP_{Z_{1,n}} && (3.74)
\end{aligned}$$

It should be easy to see that (3.73) takes its minimum over the entire preference zone when

$$V^p_{[k]} = \cdots = V^p_{[2]} = \delta^* V^p_{[1]} = \delta^* \beta. \tag{3.75}$$

As before, this partially explains (3.72). However, we still do not know the actual correspondence between $v^p_{(i)}$ and the given values v_i. Therefore, we further reduce matters by replacing $v^p_{(1)}$ with $v^p_{[k]}$ on the left hand side of the inner event described in (3.72). Substitute h, and we have the desired result. □

As should be noticed, except for the definition of h, this is the exact same integral as given in Theorem 3.7. Thus, the arguments following Theorem 3.7 still hold as far as determining h are concerned. So, we will omit these arguments. In fact, if the ratio is introduced into the arguments of the previous section in a manner corresponding to that given in Theorem 3.16, we can omit the corresponding Ratio-based results. However, we must take note of two things. First, similar to the previous proofs, we will eventually split this integral into two pieces $A(h)$ and $B(h)$ defined in a similar fashion as in the previous proofs. We then have a choice of solving either $A(h) + B(h) = P^*$ or $A(h) = P^*$. Either will suffice; the second would be more conservative. The shorter was included in the statement of the procedure simply because it was shorter. Secondly, we make one remark regarding the equation used to determine the secondary sample size in Procedures R_{V4a} and R_{V4b}. In these procedures, the secondary sample size is determined using

$$n = \max\left\{ n_1, \left\lceil \left(\frac{hv^p_{[k],n_1}}{(\delta^* - 1)\widehat{V}^p_{[1],n_1}} \right)^2 (p(1-p)) \right\rceil \right\}. \tag{3.76}$$

However, a straight modification of the Difference based procedures would produce

$$n = \max\left\{ n_1, \left\lceil \left(\frac{hv^p_{[k],n_1}}{(\delta^* - 1)V^p_{[1]}} \right)^2 (p(1-p)) \right\rceil \right\}. \tag{3.77}$$

This depends upon the unknown value of $V^p_{[1]}$. This difficulty is bypassed by using $\widehat{V}^p_{[1],n_1}$, an almost sure approximation of $V^p_{[1]}$.

3.5 Simulations:

Since these procedures are based on asymptotic results, it is reasonable to check the "usefulness" of these results using simulations. By "usefulness", we mean to check how close our Simulated Probability of Correct Selection, $\widehat{P}^*$, is to our desired P^*. We had several choices to make in order to accomplish this. First, since the two-stage procedures are the most realistic, we focused our attention on them. Hence, our simulations focus on Procedures $R_{V2a}, R_{V2b}, R_{V4a}$, and R_{V4b}. As mentioned in all the proofs to the procedures, we always have a choice of integral equations to solve. We chose to use the integral equation of the form $A(h) + B(h) = P^*$, over $A(h) = P^*$. The second is easier to use, if computations are being done by hand, but is more conservative. If the results produced using $A(h) + B(h) = P^*$ were favorable, then using $A(h) = P^*$ would only increase our sample size. Later in our simulations, we noticed that the contribution of $B(h)$ to the sum is negligible. Next, we decided to select the least dispersed of $k = 3$ populations. Fourth, we decided to run 10,560 iterations of this selection process. Since we were looking at a two-stage procedure, we decided to use an initial sample size of $n_1 = 50$. No rationalization for using 50 can be given. However, it is necessary that n_1 is greater than the dimension of the space. In our case, we simulate only in $\mathbb{R}^2$. Therefore, $n_1 \geq 3$. But, it would be advisable to use something larger. With a larger initial sample size, a more usable picture of the distribution is formed in which to compute the derivative. It was decided to use $p = .5$ for all simulations, $V_{[1]}^p = 1$. This would allow an easy comparison between the difference-based ($\delta^* = .25$) and ratio-based ($\delta^* = 1.25$) procedures. Also, conceptually, it can be thought of as a multivariate interquartile range; it captures the central 50% of the data. Next, we decided to use only bivariate Normal, Cauchy, and Uniform distributions. This is so we can know when a correct selection is made. Using Example 2.7 we have an explicit method for calculating V^p for these three distribution types. Finally, as a computational necessity, a limit was placed on the total sample size that could be used. In cases where the total sample size was determined to be in excess of 10,000, we used 10,000 as the total sample size.

The actual simulation proceeded in several steps. The first was to decide upon the exact configuration of the distributions being used. There are $3^3 = 27$ permutations of these three distributions. For each iteration, one of these permutations was chosen. The first in the list would become the least dispersed population. Once a configuration of populations was made, a sample of size $n_1 = 50$ was randomly generated from each of the distributions in the given permutation. Using the information in Example 2.7, the data points were scaled so that the least dispersed first population came from a population with $V^p_{[1]} = 1$, and the others came from a population with $V^p_{[2]} = V^p_{[3]} = 1.25$.

At this point in the simulation, it is necessary to consistently estimate the derivatives for the distributions. This was done with the help of two results. In [26], attention is shifted from estimating v^p_i to the estimation of the density function, f_i, of P_i. That is, if $f_{i,n}$ is a consistent estimator of f_i, then

$$v^p_{i,n_1} = \frac{1}{\mathit{average\ of\ } f_{i,n_1}(X_i) \mathit{\ over\ } X_i \mathit{\ on\ boundary\ of\ } D_n^{\widehat{\alpha}_{p,n}}}. \tag{3.78}$$

A consistent estimator of f_i is given in [16],

$$f_{i,n} = \left(\frac{k_n - 1}{n}\right)\left(\frac{d\ \Gamma(d/2)}{2r^d_{k_n}\,\pi^{d/2}}\right) \tag{3.79}$$

where the sequence of integers $k_n \to \infty$ as $n \to \infty$, and $\lim_{n\to\infty} \frac{k_n}{n} = 0$ and r_{k_n} is the Euclidean distance from X_i to the k_n^{th} nearest data point. Based on a recommendation in [16], we took $k_n = \lceil \sqrt{n}\, \rceil$. In the case of Procedures R_{V4a} and R_{V4b}, $V^p_{[1]}$ is estimated with $\widehat{V}^p_{[1]}$. Computation of $\widehat{V}^p_{[1]}$ was done with the assistance of two existing MatLab programs: halfspacedepth.m [21, 33]and ISODepth.m [22, 33]. The first computes the halfspace depth of a point with respect to a bivariate dataset. The second determines the contour of any predetermined p^{th}-central region of a data set.

At this point in the process, the secondary sample size was determined using the relevant procedure. With this the secondary sample was taken from each population and rescaled as necessary. The empirical volumes would then be computed based upon both samples. At this point, it was then determined if the smallest empirical volume came from the first population in the list. If yes, then we would have a correct selection. If no, then we had an incorrect selection. Tables 3.1 and 3.2 give the results for the two-stage simulations.

Recall, that at this point the difference between the results in Table 3.1 and Table 3.2 is the result of the difference in assumptions. In Table 3.1 , we are looking at the difference between population volumes, while in Table 3.2 , we are considering the ratio of population volumes. This required two different methods for determining the final total sample size. We should notice that our simulated probabilities of correct selection are significantly higher than than our desired P^* value. This is most likely accounted for by the fact that we determined our probability of a correct selection using the semi-empirical volumes, but estimated these volumes using the empirical volumes. Additionally, it might indicate the need to determine if there exist sharper inequalities that can be used in determining a lower bounding integral.

Table 3.1: Two-Stage Difference-based Procedures:
$k = 3, n_1 = 50, V_{[2]}^{.5} - V_{[1]}^{.5} = 0.25$

R_{V2a}

P^*	$\widehat{P^*}$	$\bar{n}$
0.60	0.813	604
0.65	0.858	746
0.70	0.890	1000
0.75	0.912	1237
0.80	0.938	1538
0.85	0.957	1812
0.90	0.969	2162
0.95	0.968	2501

R_{V2b}

P^*	$\widehat{P^*}$	$\bar{n}$
0.60	0.800	569
0.65	0.848	784
0.70	0.882	997
0.75	0.917	1203
0.80	0.936	1563
0.85	0.956	1845
0.90	0.967	2155
0.95	0.967	2475

$n \geq 10,000$

P^*	R_{V2a}	R_{V2b}
0.60	71/71	61/61
0.65	97/97	119/119
0.70	195/195	195/195
0.75	277/278	254/255
0.80	380/380	416/420
0.85	486/489	538/538
0.90	648/652	631/632
0.95	685/688	689/691

Least Dispersed Population by Type:

Normal

P^*	R_{V2a}	R_{V2b}
0.60	0.79	0.78
0.65	0.83	0.83
0.70	0.88	0.87
0.75	0.91	0.91
0.80	0.93	0.93
0.85	0.96	0.96
0.90	0.97	0.97
0.95	0.98	0.97

Uniform

P^*	R_{V2a}	R_{V2b}
0.60	0.82	0.81
0.65	0.88	0.86
0.70	0.89	0.89
0.75	0.92	0.93
0.80	0.94	0.94
0.85	0.97	0.96
0.90	0.98	0.98
0.95	0.98	0.98

Cauchy

P^*	R_{V2a}	R_{V2b}
0.60	0.83	0.81
0.65	0.87	0.85
0.70	0.89	0.89
0.75	0.90	0.91
0.80	0.94	0.93
0.85	0.95	0.95
0.90	0.96	0.96
0.95	0.95	0.95

3.6 Concluding Remarks:

In this chapter, we presented multiple procedures for selecting the least dispersed of several populations. This was accomplished using the volume of regions defined by the halfspace depth. The key to determining the probability of a correct selection cames from the distributional convergence result presented in Theorem 2.13. This allowed us to use a normal approximation to determine the sample size necessary

Table 3.2: Two-Stage Ratio-based Procedures:
$k = 3, n_1 = 50, V_{[2]}^{.5}/V_{[1]}^{.5} > 1.25$

R_{V4a}

P^*	$\widehat{P^*}$	$\bar{n}$
0.60	0.850	889
0.65	0.898	1110
0.70	0.924	1363
0.75	0.947	1699
0.80	0.963	2020
0.85	0.975	2413
0.90	0.983	2782
0.95	0.976	3248

R_{V4b}

P^*	$\widehat{P^*}$	$\bar{n}$
0.60	0.847	862
0.65	0.891	1112
0.70	0.918	1366
0.75	0.943	1692
0.80	0.961	2038
0.85	0.975	2379
0.90	0.979	2790
0.95	0.975	3212

$n \geq 10,000$

P^*	R_{V4a}	R_{V4b}
0.60	170/170	148/149
0.65	250/252	248/248
0.70	340/340	333/333
0.75	494/495	487/489
0.80	647/648	647/650
0.85	854/858	853/857
0.90	1027/1029	1011/1014
0.95	1260/1260	1166/1170

Least Dispersed Population by Type:

Normal

P^*	R_{V4a}	R_{V4b}
0.60	0.83	0.82
0.65	0.89	0.88
0.70	0.92	0.90
0.75	0.93	0.94
0.80	0.96	0.95
0.85	0.97	0.97
0.90	0.99	0.98
0.95	0.99	0.98

Uniform

P^*	R_{V4a}	R_{V4b}
0.60	0.86	0.86
0.65	0.92	0.91
0.70	0.94	0.94
0.75	0.96	0.96
0.80	0.98	0.97
0.85	0.98	0.98
0.90	0.99	0.99
0.95	0.99	0.98

Cauchy

P^*	R_{V4a}	R_{V4b}
0.60	0.85	0.86
0.65	0.89	0.89
0.70	0.91	0.91
0.75	0.95	0.94
0.80	0.96	0.96
0.85	0.97	0.97
0.90	0.97	0.97
0.95	0.96	0.96

to meet our probability requirement. This led to what apparently could be called a conservative procedure, since our simulated probability of a correct selection was much higher than we would expect. If the exact distribution of $\widehat{V}_n^p$ could be found, instead of using the estimate of $\widetilde{V}_n^p$, a less conservative procedure could probably be given. Lastly, an obvious extension would be to develop subset selection procedures. This could lead to possible procedures:

- Select all populations π_i with a smaller dispersion than a standard or control, population π_0, or
- Select a subset that contains the least dispersed population.

These would be of some value also.

Chapter 4

Location

In this chapter, we develop four procedures for selecting the population with the "most centered" distribution. Thus, we will outline procedures that select a population based upon the location of the distribution. Our populations will be defined in the same manner as in Chapter 3. We let $\{\pi_i\}_{i=1}^k$ be a collection of $k \in \mathbb{Z}^+$ populations that follow absolutely continuous distributions P_{X_i} given by a random vectors $X_i \in \mathbb{R}^d$, $d \geq 1$. When it will not cause confusion, P_i will denote P_{X_i}. Now, we need to recall the following facts from Chapter 2. First, the halfspace depth of a point $x \in \mathbb{R}^d$ with respect to a distribution P_i is

$$D(x; P_i) = \inf\{P_{X_i}(H) | x \in H, H \text{ is a closed halfspace}\}. \tag{4.1}$$

This will also be denoted by $D_i(x)$. Secondly, the maximal depth of a distribution P_i is

$$\alpha_i^* = \sup_{x \in \mathbb{R}^d} D(x; P_i). \tag{4.2}$$

Finally, we should remember that the maximal depth α_i^* of an absolutely continuous distribution P_i is contained in the interval $[\frac{1}{d+1}, \frac{1}{2}]$.

Based on the properties outlined in Chapter 2, it should be clear that the depth of a point $x \in \mathbb{R}^d$ gives a measure of how central, or how outlying, the point x is. In other words, points with a higher depth can be thought of as more central than points with a lower depth. Points with lower depth can be thought of as being more outlying than points with higher depth. This will be the basis for our method of comparing, and ordering different populations. In this chapter, we will assume that a specific point $y \in \mathbb{R}^d$ has been selected. This point will be our target point. Figure 4.1 illustrates this idea. Among our k populations, we will want to select the population where the depth of y is greatest. Since the depth of y will be important, we denote it by

$$\alpha_i = D(y; P_i). \tag{4.3}$$

The results of these procedures can then be interpreted in two ways. First, we select the population where y has the greatest depth. Secondly, we select the population which is centered the most at y. Using either of these interpretations, we would hope to define an ordering on the the populations in Ω, by simply stating that a population

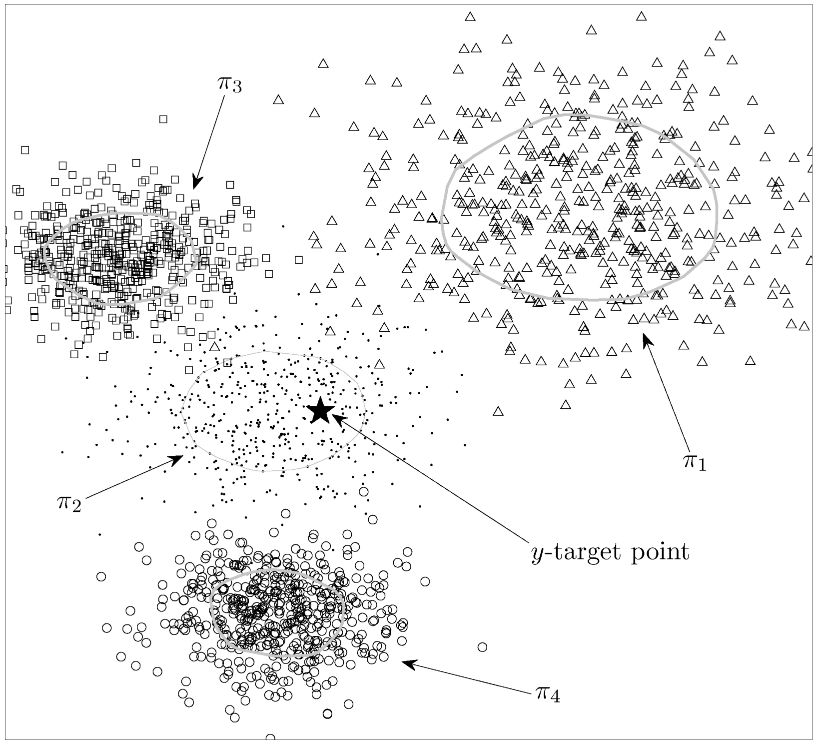


Figure 4.1: Five hundred random observations from four populations $\{\pi_i\}_{i=1}^4$ were taken. Clearly, the target point y is most centrally located among the observations from π_2.

with higher α_i is better. This fails to have any real meaning because the maximal depth for two distinct populations may not be the same. Therefore, if we are to order two populations based on the depth of a single point, we would need to standardize the depth of the given point to allow for a fair comparison.

Definition 4.1. The *relative (halfspace) depth* of $y \in \mathbb{R}^d$ with respect to P_i is defined to be

$$\beta_i = \frac{\alpha_i}{\alpha_i^*}. \tag{4.4}$$

This will also be referred to as the relative population depth of P_i.

This allows us to compare two populations on an equal footing.

Definition 4.2. Given populations π_1, π_2, the point $y \in \mathbb{R}^d$, and depth function $D(\cdot;\cdot)$, π_1 is said to be *relatively more central* than π_2 with respect to y ($\pi_1 \preceq \pi_2$), if and only if

$$\beta_1 > \beta_2. \tag{4.5}$$

As an alternative to this definition, if we are willing to assume something more, we can define the following.

Definition 4.3. Given populations π_1, π_2, the point $y \in \mathbb{R}^d$, and depth function $D(\cdot;\cdot)$, such that $\alpha_1^* = \alpha_2^*$, π_1 is said to be *more central* than π_2 with respect to y,($\pi_1 \preceq \pi_2$), if and only if

$$\alpha_1 > \alpha_2. \tag{4.6}$$

Notice that these definitions produce an identical ranking of populations with the maximal depths are all the same. This is not necessarily the case when the maximal depths are not the same.

When the maximal depth for a collection of populations is the same for all populations, we will refer to α_i as the *population depth* of P_i. If we want to define what it means for one population to be (relatively) better than another, we replace the (relatively) more central with (relatively) better in the above definitions. Letting $\alpha_{[1]} \leq \alpha_{[2]} \leq \ldots \leq \alpha_{[k]}$ $\left(\beta_{[1]} \leq \beta_{[2]} \leq \ldots \leq \beta_{[k]}\right)$ represent the ordered (relative) population depths induces an ordering on Ω, $\pi_{[1]} \preceq \pi_{[2]} \preceq \cdots \preceq \pi_{[k]}$), from least central to most central where $\pi_{[i]}$ is the population with (relative) population depth $\alpha_{[i]}(\beta_{[i]})$. Of course, the ordering in the first case is only meaningful when the maximal depth for all populations under consideration is the same.

4.1 Goal:

Our goal in this chapter is to develop procedures for selecting $\pi_{[k]}$, the (relatively) most central population. As in Chapter 3, we will need to define some empirical versions of the parameters of interest. Recall, from Chapter 2, that the empirical depth of a point $x \in \mathbb{R}^d$ with respect to P_i is defined to be

$$\begin{aligned} D_n(x; P_i) &:= D(x; \widehat{P}_{i,n}) \\ &= \inf\{\widehat{P}_{i,n}(H) \mid x \in H, H \textit{ is a closed half-space}\} \end{aligned} \tag{4.7}$$

where $\widehat{P}_{i,n}$ is the empirical distribution of P_i based on an i.i.d. sample of size n.

Definition 4.4. The *empirical (halfspace) depth* of $x \in \mathbb{R}^d$ with respect to P_i based on a sample of size n is defined to be

$$\widehat{\alpha}_{i,n} = D(x; \widehat{P}_{i,n}). \tag{4.8}$$

When considering a collection of populations whose maximal depth is equal and a fixed point y, $\widehat{\alpha}_{i,n} = D(y; \widehat{P}_{i,n})$ will be referred to as the *empirical population depth.*

Definition 4.5. The *empirical maximal (halfspace) depth* of P_i is defined to be

$$\widehat{\alpha}^*_{i,n} = \max_{x \in \mathbb{R}^d} D(x; \widehat{P}_{i,n}). \tag{4.9}$$

Definition 4.6. The *empirical relative (halfspace) depth* of $y \in \mathbb{R}^d$ with respect to P_i based on a sample of size n is defined to be

$$\widehat{\beta}_{i,n} = \frac{\widehat{\alpha}_{i,n}}{\widehat{\alpha}^*_{i,n}}. \tag{4.10}$$

This will also be referred to as the empirical relative population depth of P_i.

When it is clear from the context, the sample size will be omitted from the notation in the previous definitions.

Using these definitions, we propose four possible procedures for selecting the population such that $y \in \mathbb{R}^d$ is (relatively) most central. Alternatively, we say that we are looking for the population with the greatest (relative) population depth. Since we hope to select the (relatively) most central population, we will define procedures in the hope that the largest empirical (relative) population depth is produced by the population with the actual largest (relative) population depth. Let $\widehat{\alpha}_{[1],} \leq \widehat{\alpha}_{[2],n} \leq \cdots \leq \widehat{\alpha}_{[k],n}$ $\left(\widehat{\beta}_{[1],n} \leq \widehat{\beta}_{[2],n} \leq \cdots \leq \widehat{\beta}_{[k],n}\right)$ denote the ordered empirical (relative) depths, and $\widehat{\alpha}_{(i),n}(\widehat{\beta}_{(i),n})$ represent the empirical (relative) depth that corresponds to $\pi_{[i]}$, the population with depth $\alpha_{[i]}(\beta_{[i]})$. A correct selection ($CS_n$) based on a sample of size n will be the event that the population π_i with (relative) depth

$\alpha_{[k]}$ $\left(\beta_{[k]}\right)$ is selected, i.e.,

$$CS_n = \left\{\pi_{[k]} \text{ is selected.}\right\} \tag{4.11}$$

$$= \left\{\text{The most central population is selected.}\right\} \tag{4.12}$$

$$= \left\{\widehat{\alpha}_{[k],n} = \widehat{\alpha}_{(k),n}\right\}, \tag{4.13}$$

or

$$CS_n = \left\{\text{The relatively most central population is selected.}\right\} \tag{4.14}$$

$$= \left\{\widehat{\beta}_{[k],n} = \widehat{\beta}_{(k),n}\right\} \text{ respectively.} \tag{4.15}$$

Now that we have defined our decision making process, we need to control the uncertainty in the process. Therefore, our goal is to define a procedure such that $P(CS_n) \geq P^* \in (k^{-1}, 1)$.

4.2 Assumptions:

Certain assumptions will be needed for all the procedures that will follow. These assumption are made so that Theorem 2.10 can be applied. To review, these assumptions are

1. P_i is locally regular, $i = 1, \ldots, k$,

2. y is P_i-smooth for each π_i.

4.3 Procedures

Similar to Chapter 3, four types of procedures will be presented. The first two procedures can be completed in a single stage. One will make use of the relative depth, and the other will not. The final two procedures will be two stage procedures. Again, one will use the relative depth, and the other will assume that the maximal depth is the same for all populations being considered. The motivation that differentiates the two-stage and the single stage procedures mimics that from Chapter 3. The single stage procedures make use of some previous information about some unknown population parameters. The two-stage procedures use the first stage to estimate the unknown parameters. Justifications for all the proposed procedures are given in Section 4.4.

4.3.1 Single-Stage selection of the Most Relatively Central Population:

The goal for our first procedure, Procedure R_{RC1}, will be to select the $\pi_{[k]}$, the population associated with $\beta_{[k]}$. The preference zone for R_{RC1} is defined as

$$PZ_{RC1} = \left\{(\beta_1, \beta_2, \ldots, \beta_k) \mid \delta\beta_{[k]} > \beta_{[k-1]}, \beta_{[1]} > \epsilon\right\} \tag{4.16}$$

where $1 > \delta > \epsilon > 0$ are preselected.

Procedure R_{RC1}:

1. Take a sample of size n from each population where

$$n = \max\left\{\left\lceil\left(\frac{2-\epsilon}{\epsilon}\right)\left(\frac{h}{(1-\delta)}\right)^2\right\rceil, d+1\right\}, \tag{4.17}$$

   and h is the solution to

$$\int_{-\infty}^{h} P\left(z-h < Z\sqrt{\frac{d+1-\epsilon}{\epsilon}}\right)^{k-1} \phi(z)\,dz = P^*. \tag{4.18}$$

2. Calculate the empirical relative depth $\widehat{\beta}_{i,n}$ for each population.

3. Declare that the population π_i with empirical depth $\widehat{\beta}_{[k],n}$ is $\pi_{[k]}$.

This procedure will asymptotically satisfy the probability requirement

$$P(CS_n \mid R_{RC1}) \geq P^* \text{ whenever } \beta_{[k]} > \delta\beta_{[k-1]} \text{ and } \beta_{[1]} > \epsilon.$$

4.3.2 Single-Stage selection of the Most Central Population:

The goal for our next procedure, Procedure R_{C1}, will be to select the $\pi_{[k]}$, the population associated with $\alpha_{[k]}$. This procedure makes the assumption that α_i^* is the same for all populations under consideration. This is not an unrealistic assumption.

Recall, a distribution is said to be *angularly symmetric* about a point θ if and only if $P(\theta+A) = P(\theta-A)$ for any Borel set in $\mathbb{R}^d$. If, for example, all the populations being considered are known to be angularly symmetric, then the maximal depth for all the populations is equal to $\frac{1}{2}$ [24]. In this case, we no longer need to estimate α^*; we can select the population such that y is most central, without having to standardize.

The preference zone for R_{C1} is defined as

$$PZ_{C1} = \left\{(\alpha_1, \alpha_2, \ldots, \alpha_k) \mid \delta\alpha_{[k]} > \alpha_{[k-1]}, \alpha_{[1]} > \epsilon\right\} \tag{4.19}$$

where $\frac{1}{2} > \frac{\delta}{2} > \epsilon > 0$ are preselected.

Procedure R_{C1}:

1. Take a sample of size n from each population where

$$n = \max\left\{\left\lceil \left(\frac{1-\epsilon}{\epsilon}\right)\left(\frac{h}{1-\delta}\right)^2 \right\rceil, d+1\right\}, \tag{4.20}$$

   and h is the solution to

$$\int_{-\infty}^{h} P\left(z - h < Z\sqrt{\delta(2-\delta)}\right)^{k-1} \phi(z)dz = P^*. \tag{4.21}$$

2. Calculate the empirical relative depth $\widehat{\alpha}_{i,n}$ for each population.

3. Declare that the population π_i with empirical depth $\widehat{\alpha}_{[k],n}$ is $\pi_{[k]}$.

This procedure will asymptotically satisfy the probability requirement

$$P(CS_n \mid R_{C1}) \geq P^* \text{ whenever } \delta\alpha_{[k]} > \alpha_{[k-1]} \text{ and } \alpha_{[1]} > \epsilon.$$

4.3.3 Two-Stage selection of the Most Relatively Central Population:

For the third procedure, Procedure R_{RC2}, our goal is the same. We want to select $\pi_{[k]}$, the population associated with $\beta_{[k]}$. The preference zone for R_{RC2} is defined as

$$PZ_{RC1} = \left\{ (\beta_1, \beta_2, \ldots, \beta_k) \mid \delta\beta_{[k]} > \beta_{[k-1]} \right\} \tag{4.22}$$

where $\delta \in (0, 1)$ is preselected.

Procedure R_{RC2}:

Stage 1:

(a) Take a sample of size $n_1 > d + 1$ from each population.

(b) Calculate an estimate of $\beta_{[1]}$,

$$\widehat{\epsilon}_{\beta,n_1} = \min_{i=1,\cdots,k} \left\{ \widehat{\beta}_i \right\}. \tag{4.23}$$

(c) Determine a total sample size

$$n = \max\left\{\left[\frac{(d+1-\widehat{\epsilon}_{\beta,n_1})h}{(1-\delta)\widehat{\epsilon}_{\beta,n_1}}\right]^2, n_1\right\} \tag{4.24}$$

where h is the solution to

$$\int_{-\infty}^{h} P\left(z-h < Z\sqrt{\frac{d+1-\widehat{\epsilon}_{\beta,n_1}}{\widehat{\epsilon}_{\beta,n_1}}}\right)^{k-1} \phi(z)dz = P^*. \tag{4.25}$$

Stage 2:

(a) Take a sample of size $n_2 = n - n_1$ from each population, if $n_2 > 0$.

(b) Calculate $\widehat{\beta}_{i,n}$ for each population.

(c) Declare that the population π_i with sample volume $\widehat{\beta}_{[k],n}$ is $\pi_{[k]}$.

This procedure will asymptotically satisfy the probability requirement

$$P(CS_n \mid R_{RC1}) \geq P^* \text{ whenever } \delta\beta_{[k]} > \beta_{[k-1]}.$$

4.3.4 Two-Stage selection of the Most Central Population:

For our final procedure, R_{C2}, our goal is to select the population associated with $\alpha_{[k]}$, assuming that the maximal depth is the same for all populations. The preference zone for R_{C2} is defined as

$$PZ_{C2} = \left\{(\alpha_1, \alpha_2, \ldots, \alpha_k) \mid \delta\alpha_{[k]} > \alpha_{[k-1]}\right\} \tag{4.26}$$

where $\delta \in (0, 1)$ is preselected.

Procedure R_{C2}:

Stage 1:

(a) Take a sample of size $n_1 > d + 1$ from each population.

(b) Calculate an estimate of $\alpha_{[1]}$,

$$\widehat{\epsilon}_{\alpha,n_1} = \min_{i=1,\cdots,k} \{\widehat{\alpha}_i\}. \tag{4.27}$$

(c) Determine a total sample size

$$n = \max\left\{\left(\frac{1-\widehat{\epsilon}_{\alpha,n_1}}{\widehat{\epsilon}_{\alpha,n_1}}\right)\left(\frac{h}{1-\delta}\right)^2, n_1\right\} \tag{4.28}$$

where h is the solution to

$$\int_{-\infty}^{h} P\left(z - h < Z\sqrt{\delta(2-\delta)}\right)^{k-1} \phi(z)dz = P^*. \tag{4.29}$$

Stage 2:

(a) Take a sample of size $n_2 = n - n_1$ from each population, if $n_2 > 0$.

(b) Calculate $\widehat{\alpha}_{i,n}$ for each population.

(c) Declare that the population π_i with sample volume $\widehat{\alpha}_{[k],n}$ is $\pi_{[k]}$.

This procedure will asymptotically satisfy the probability requirement

$$P(CS_n \mid R_{C2}) \geq P^* \text{ whenever } \delta\alpha_{[k]} > \alpha_{[k-1]}.$$

4.4 Proofs

Our approach to justifying these procedures will follow the same pattern as those given in Chapter 3. In Section 4.4.1, we begin by defining an alternative version of a correct selection, $\widetilde{CS}_n$, and show that for large n, $P(\widetilde{CS}_n) \approx P(CS_n)$. Also, we show some inequalities that will be used in later sections. In Sections 4.4.2, 4.4.2, 4.4.3, and 4.4.3, we show how the integral equation included with each procedure can be viewed as an asymptotic lower bound for $P(\widetilde{CS}_n)$ and how that helps to determine the necessary sample size for making a decision.

4.4.1 General Results

Definition 4.7. The *semi-empirical relative (halfspace) depth* of $y \in \mathbb{R}^d$ with respect to P_i based on a sample of size n is defined to be

$$\widetilde{\beta}_{i,n} = \frac{\widehat{\alpha}_{i,n}}{\alpha_i^*}. \tag{4.30}$$

This will also be referred to as the semi-empirical relative depth of P_i.

Notice that in order to compute the semi-empirical relative depth of P_i, the maximal depth α_i^* must be known. However, with α_i^* we may define an alternative correct selection as

$$\widetilde{CS}_n = \left\{ \widetilde{\beta}_{[k],n} = \widetilde{\beta}_{(k),n} \right\}. \tag{4.31}$$

To see that $P(\widetilde{CS}_n) \approx P(CS_n)$ for large n will take a few steps.

Lemma 4.1. $\lim_{n\to\infty} \widehat{\alpha}^*_{i,n} = \alpha_i^*$ *almost surely.*

Proof. By 2.4, there exists at least one $x^* \in \mathbb{R}^d$ such that $D(x^*; P) = \alpha^*$. By 2.9, $\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} |D_n(x; P) - D(x; P)| = 0$ almost surely. Thus, $\lim_{n\to\infty} D_n(x^*; P) = \alpha^*$. Let x_n be any sequence of points such that $D_n(x_n; P) = \widehat{\alpha}^*_{i,n}$, then $\widehat{\alpha}^*_{i,n} \geq D_n(x^*; P)$ for all n. Therefore, $\lim_{n\to\infty} \widehat{\alpha}^*_{i,n} \geq \alpha^*$ almost surely. But, $\lim_{n\to\infty} D_n(x; P) \leq \alpha^*$ for all $x \in \mathbb{R}^d$. So, $\lim_{n\to\infty} \widehat{\alpha}^*_{i,n} \leq \alpha^*$ almost surely. □

Lemma 4.2. $\lim_{n\to\infty} \widehat{\alpha}_{i,n} = \alpha_i$ *almost surely.*

Proof. Refer to Theorem 2.9. □

Corollary 4.3. *Almost surely, as* $n \to \infty$,

1. $\widehat{\beta}_{i,n} \to \beta_i$.

2. $\widetilde{\beta}_{i,n} \to \beta_i$.

3. $\widehat{\epsilon}_n = \min_{i=1,\cdots,k}\left\{\widehat{\beta}_{i,n}\right\} \to \beta_{[1]}$.

Proposition 4.4. *As* $n \to \infty, |P(\widetilde{CS}_n) - P(\widehat{CS}_n)| \to 0$.

Proof. We consider the relatively most central case first. Let $\gamma = \min_{i\neq j}|\beta_i - \beta_j|$ and $\gamma/5 > \epsilon > 0$. By Theorem 2.9, it is shown that almost surely

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^d}|D_n(x) - D(x)| = 0. \tag{4.32}$$

Therefore, for all $n \geq N$, and for all i, $|\widetilde{\beta}_i - \beta_i| < \epsilon$ and $|\widehat{\beta}_i - \beta_i| < \epsilon$ almost surely. Thus for large enough n, $|\widetilde{\beta}_{(i),n} - \beta_{[i]}| < \epsilon$ and $|\widehat{\beta}_{(i)} - \beta_{[i]}| < \epsilon$ almost surely for all i. In which case, $P(\widetilde{CS}_n), P(\widehat{CS}_n) \to 1$. For the most central case, the same proof works by replacing β with α. □

Now, we shift our focus to some inequalities that will be used.

Lemma 4.5. *If there exists* $\epsilon > 0$ *such that*

$$1 > \beta_{[k]} > \beta_{[i]} \geq \epsilon > 0 \tag{4.33}$$

for $i = 1, \ldots, k-1$, *then*

$$\frac{1}{\nu} \leq \left(\frac{\alpha^*_{(k)}}{\alpha^*_{(i)}}\right) \sqrt{\frac{\alpha_{(i)}(1-\alpha_{(i)})}{\alpha_{(k)}(1-\alpha_{(k)})}} \leq \nu \tag{4.34}$$

where $\nu = \sqrt{\frac{d+1-\epsilon}{\epsilon}}$.

Proof. By definition, $\beta_i = \frac{\alpha_i}{\alpha^*_i}$. Letting $\alpha_{(i)}$ ($\alpha^*_{(i)}$) denote the value of α_i (α^*_i) corresponding to the population with relative depth $\beta_{[i]}$, we see that $\beta_{[i]} = \frac{\alpha_{(i)}}{\alpha^*_{(i)}}$. By applying the fact that $\alpha^*_i \in \left(\frac{1}{d+1}, \frac{1}{2}\right)$ to the definition of $\beta_{[i]}$, we have

$$\frac{1}{2} \geq \alpha^*_{(i)} \geq \alpha_{(i)} \geq \alpha^*_{(i)}\epsilon \geq \frac{\epsilon}{d+1}, \tag{4.35}$$

or equivalently

$$\frac{1}{2} \leq 1-\alpha^*_{(i)} \leq 1-\alpha_{(i)} \leq 1-\alpha^*_{(i)}\epsilon \leq \frac{d+1-\epsilon}{d+1}. \tag{4.36}$$

Inverting this, we have

$$2 \geq \frac{1}{1-\alpha_{(i)}} \geq \frac{d+1}{d+1-\epsilon}. \tag{4.37}$$

Multilplying through by $\alpha^*_{(i)}$, and using $\alpha^*_{(i)} \in \left(\frac{1}{d+1}, \frac{1}{2}\right)$ again, we have

$$1 \geq \frac{\alpha^*_{(i)}}{1-\alpha_{(i)}} \geq \frac{1}{d+1-\epsilon}. \tag{4.38}$$

Next, we multiply through by $\beta_{[i]}^{-1}$, and use the assumption that $\beta_{[i]} > \epsilon$ to get

$$\frac{1}{\epsilon} \geq \frac{(\alpha^*_{(i)})^2}{\alpha_{(i)}(1-\alpha_{(i)})} \geq \frac{1}{d+1-\epsilon} \tag{4.39}$$

for all i. For $i = k$, we have

$$d+1-\epsilon \geq \frac{\alpha_{(k)}(1-\alpha_{(k)})}{\left(\alpha^*_{(k)}\right)^2} \geq \epsilon. \tag{4.40}$$

Finally, we multiply the last two inequalities, (4.39)(4.40), and take square roots. □

The next two lemmas will be used to give a lower bound for the probability of a correct selection for Procedure R_{C1}. That proof will involve considering a slight alteration of the setup of Procedure R_{C1}.

Lemma 4.6. *Given k absolutely continuous distributions, if $\alpha_{[1]} \geq \frac{\epsilon}{2}$, then $\beta_{[1]} \geq \epsilon$.*

Proof. Recall Theorem 2.5; given an absolutely continuous distribution, $\alpha^* \leq \frac{1}{2}$. Therefore,

$$\beta_{[i]} = \frac{\alpha_{(i)}}{\alpha^*_{(i)}} \geq \frac{\alpha_{[1]}}{\alpha^*_{[k]}} \geq \frac{\frac{\epsilon}{2}}{\frac{1}{2}} = \epsilon. \tag{4.41}$$

□

Lemma 4.7. *Given k absolutely continuous distributions, if $\alpha_{[1]} \geq \frac{\epsilon}{2}$, then*

$$\frac{\alpha^*_{(k)}\beta_{[k]}}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}} \geq \sqrt{\frac{\epsilon}{2-\epsilon}}. \tag{4.42}$$

Proof. Given that $\beta_{[k]} = \frac{\alpha_{(k)}}{\alpha^*_{(k)}}$, and that $g(x) = \sqrt{\frac{x}{1-x}}$ is an increasing function in x on $(0, \frac{1}{2})$, we have

$$\frac{\alpha^*_{(k)}\beta_{[k]}}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}} = \frac{\alpha_{(k)}}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}} = f(\alpha_{(k)}) \geq f(\alpha_{[1]}) \geq f\left(\frac{\epsilon}{2}\right). \tag{4.43}$$

□

Lemma 4.8. *If there exists* $1 > \delta, \epsilon > 0$ *such that*

$$1/2 > \alpha_{[k]} > \delta\alpha_{[k]} > \alpha_{[i]} \geq \epsilon > 0 \tag{4.44}$$

for $i = 1, \ldots, k-1$*, then*

$$4\epsilon(1-\epsilon) \leq \frac{\alpha_{[i]}(1-\alpha_{[i]})}{\alpha_{[k]}(1-\alpha_{[k]})} \leq \delta(2-\delta). \tag{4.45}$$

Proof. Since $f(y) = y(1-y)$ is increasing on $(0, 1/2)$, it follows that for $i = 1, \ldots, k-1$,

$$\epsilon(1-\epsilon) = f(\epsilon) \leq f(\alpha_{[i]}) \leq f(\delta\alpha_{[k]}) = \delta\alpha_{[k]}(1-\delta\alpha_{[k]}), \tag{4.46}$$

and

$$\epsilon(1-\epsilon) \leq f(\alpha_{[k]}) \leq f(1/2) = 1/4. \tag{4.47}$$

Combining these inequalities, we have

$$4\epsilon(1-\epsilon) \leq \frac{f(\alpha_{[i]})}{f(\alpha_{[k]})} = \frac{\delta(1-\delta\alpha_{[k]})}{(1-\alpha_{[k]})}. \tag{4.48}$$

Since the right hand side is an increasing function in $\alpha_{[k]} \leq 1/2$, we get the desired result. □

4.4.2 Single-Stage Results:

Procedure R_{C1}:

We begin by looking at what could be considered another procedure. We will call it R'_{C1}. The essential difference between Procedures R_{C1} and R'_{C1} is in the definition of the preference zone. In Procedure R'_{C1} the preference zone will be defined with $\alpha_{[1]} \geq \frac{\epsilon}{2}$ instead of $\beta_{[1]} \geq \epsilon$.

Theorem 4.9. *Given k populations π_i with distributions P_i, if the following hold*

1. *P_i is locally regular, $i = 1, \ldots, k$,*

2. *y is P_i-smooth for each π_i,*

3. *Let $\delta \in (0,1)$ such that $\delta\beta_{[k]} > \beta_{[k-1]}$,*

4. *Let $\epsilon > 0$ such that $\alpha_{[1]} \geq \frac{\epsilon}{2}$,*

then

$$P(\widetilde{CS}_n | R'_{C1}) \geq \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{[i]} Z_{i,n}\right) dP_{Z_{k,n}} \tag{4.49}$$

where for $i = 2, 3, \ldots, k$, $\widehat{\alpha}_{(i)}$ *is the empirical depth corresponding to population* $\pi_{[i]}$,

$$Z_{i,n} = \frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{(i)})}{\sqrt{\alpha_{(i)}(1 - \alpha_{(i)})}}, \tag{4.50}$$

$$\Theta_{(i)} = \left(\frac{\alpha^*_{(k)}}{\alpha^*_{(i)}}\right)\sqrt{\frac{\alpha_{(i)}(1 - \alpha_{(i)})}{\alpha_{(k)}(1 - \alpha_{(k)})}}, \tag{4.51}$$

and

$$h = (1 - \delta)\sqrt{\frac{n\epsilon}{2 - \epsilon}}. \tag{4.52}$$

Proof. To begin, we notice that

$$\begin{aligned} Z_{i,n} &= \frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{(i)})}{\sqrt{\alpha_{(i)}(1 - \alpha_{(i)})}} && (4.53)\\ &= \frac{\alpha^*_{(i)}}{\alpha^*_{(i)}}\frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{(i)})}{\sqrt{\alpha_{(i)}(1 - \alpha_{(i)})}} && (4.54)\\ &= \frac{\sqrt{n}\alpha^*_{(i)}}{\sqrt{\alpha_{(i)}(1 - \alpha_{(i)})}}\left(\frac{\widehat{\alpha}_{(i)}}{\alpha^*_{(i)}} - \frac{\alpha_{(i)}}{\alpha^*_{(i)}}\right). && (4.55) \end{aligned}$$

Also, for $i = 1, \ldots, k-1$, if

$$\widetilde{\beta}_{(k),n} = \frac{\widehat{\alpha}_{(k)}}{\alpha^*_{(k)}} > \frac{\widehat{\alpha}_{(i)}}{\alpha^*_{(i)}} = \widetilde{\beta}_{(i),n} \tag{4.56}$$

then

$$Z_{k,n} > \Theta_{(i)}Z_{i,n} + \left(\frac{\sqrt{n}\alpha^*_{(k)}\left(\beta_{[i]} - \beta_{[k]}\right)}{\sqrt{\alpha_{(k)}(1 - \alpha_{(k)})}}\right). \tag{4.57}$$

This can be seen as follows:

$$
\begin{aligned}
Z_{k,n} &= \frac{\sqrt{n}(\widehat{\alpha}_{(k)} - \alpha_{(k)})}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}} && (4.58)\\
&> \sqrt{n}\left(\frac{\widehat{\alpha}_{(i)}}{\alpha^*_{(i)}} - \frac{\alpha_{(k)}}{\alpha^*_{(k)}}\right)\left(\frac{\alpha^*_{(k)}}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}}\right) && (4.59)\\
&= \sqrt{n}\left(\frac{\widehat{\alpha}_{(i)}}{\alpha^*_{(i)}} - \frac{\alpha_{(i)}}{\alpha^*_{(i)}} + \frac{\alpha_{(i)}}{\alpha^*_{(i)}} - \frac{\alpha_{(k)}}{\alpha^*_{(k)}}\right)\left(\frac{\alpha^*_{(k)}}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}}\right) && (4.60)\\
&= \left(\frac{\widehat{\alpha}_{(i)}}{\alpha^*_{(i)}} - \frac{\alpha_{(i)}}{\alpha^*_{(i)}}\right)\left(\frac{\sqrt{n}\alpha^*_{(k)}}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}}\right) + \left(\frac{\sqrt{n}\alpha^*_{(k)}\left(\beta_{[i]} - \beta_{[k]}\right)}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}}\right) && (4.61)\\
&= \left(\frac{\widehat{\alpha}_{(i)}}{\alpha^*_{(i)}} - \frac{\alpha_{(i)}}{\alpha^*_{(i)}}\right)\left(\frac{\sqrt{n}\alpha^*_{(i)}}{\sqrt{\alpha_{(i)}(1-\alpha_{(i)})}}\right)\left(\frac{\alpha^*_{(k)}\sqrt{\alpha_{(i)}(1-\alpha_{(i)})}}{\alpha^*_{(i)}\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}}\right)\\
&\quad + \left(\frac{\sqrt{n}\alpha^*_{(k)}\left(\beta_{[i]} - \beta_{[k]}\right)}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}}\right) && (4.62)\\
&= \left(\frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{(i)})}{\sqrt{\alpha_{(i)}(1-\alpha_{(i)})}}\right)\left(\frac{\alpha^*_{(k)}\sqrt{\alpha_{(i)}(1-\alpha_{(i)})}}{\alpha^*_{(i)}\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}}\right) + \left(\frac{\sqrt{n}\alpha^*_{(k)}\left(\beta_{[i]} - \beta_{[k]}\right)}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}}\right) && (4.63)\\
&= \Theta_{(i)} Z_{i,n} + \left(\frac{\sqrt{n}\alpha^*_{(k)}\left(\beta_{[i]} - \beta_{[k]}\right)}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}}\right). && (4.64)
\end{aligned}
$$

Now, we consider $P(\widetilde{CS}_n | R'_{C1})$,

$$
\begin{aligned}
P(\widetilde{CS}_n | R'_{C1}) &= P\left(\widetilde{\beta}_{[k],n} = \widetilde{\beta}_{(k),n}\right)\\
&= P\left(\widetilde{\beta}_{(k),n} > \widetilde{\beta}_{(i),n},\ i = 1, \ldots, k-1\right)\\
&= P\left(\frac{\widehat{\alpha}_{(k)}}{\alpha^*_{(k)}} > \frac{\widehat{\alpha}_{(i)}}{\alpha^*_{(i)}},\ i = 1, \ldots, k-1\right) && (4.65)\\
&= P\left(Z_{k,n} > \Theta_{(i)} Z_{i,n} + \left(\frac{\sqrt{n}\alpha^*_{(k)}\left(\beta_{[i]} - \beta_{[k]}\right)}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}}\right),\ i = 1, \ldots, k-1\right)\\
& && (4.66)
\end{aligned}
$$

$$= \int \prod_{i=1}^{k-1} P\left(z + \frac{\sqrt{n}\alpha^*_{(k)}\left(\beta_{[k]} - \beta_{[i]}\right)}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}} > \Theta_{[j]} Z_{i,n}\right) dP_{Z_{1,n}}. \tag{4.67}$$

By the assumption $\delta\beta_{[k]} > \beta_{[i]}$ for all i, we see that

$$\beta_{[k]} - \beta_{[i]} \geq \beta_{[k]}(1-\delta). \tag{4.68}$$

Consequently,

$$P(\widetilde{CS}_n | R'_{C1}) = \int \prod_{i=1}^{k-1} P\left(z + \frac{\sqrt{n}\alpha^*_{(k)}\left(\beta_{[k]} - \beta_{[i]}\right)}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}} > \Theta_{[j]} Z_{i,n}\right) dP_{Z_{1,n}} \tag{4.69}$$

$$\geq \int \prod_{i=1}^{k-1} P\left(z + \frac{\sqrt{n}\alpha^*_{(k)}\beta_{[k]}(1-\delta)}{\sqrt{\alpha_{(k)}(1-\alpha_{(k)})}} > \Theta_{[j]} Z_{i,n}\right) dP_{Z_{1,n}}. \tag{4.70}$$

Applying Lemma 4.7, this becomes

$$\geq \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + (1-\delta)\sqrt{\frac{n\epsilon}{2-\epsilon}} > \Theta_{(i)} Z_{i,n}\right) dP_{Z_{1,n}} \tag{4.71}$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{(i)} Z_{i,n}\right) dP_{Z_{1,n}}. \tag{4.72}$$

□

At this point, we would like to use (4.72) to ascertain our necessary sample size by determining h so that

$$\int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{(i)} Z_{i,n}\right) dP_{Z_{1,n}} = P^*. \tag{4.73}$$

We have the same two types of problems we encountered in Chapter 3. (4.73) depends on n, and $\Theta_{(i)}$ is unknown. We tackle these problems, following the example laid out in Chapter 3 on pages 68 - 73. First, we show that the lefthand side of (4.73) has a large sample approximation.

Proposition 4.10. *For fixed $h > 0$, as n goes to infinity,*

$$\int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{(i)} Z_{i,n}\right) dP_{Z_{1,n}} \to \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{(i)} Z_i\right) dP_{Z_1} \tag{4.74}$$

where $\{Z_i\}$ are a collection of independent standard normal random variables.

Proof. The proof follows from the proof for Theorem 3.9. □

With this result, we can now approximate $\int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{(i)} Z_{i,n}\right) dP_{Z_{1,n}}$ with $\int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{(i)} Z_i\right) dP_{Z_1}$ provided we assume that n is sufficiently large. Now, we consider our second problem. The values of $\Theta_{(i)}$ are unknown and remain as a scaling factor. However, bounds for $\Theta_{(i)}$ exist, and are given in Lemma 4.5. We will remove this problem by splitting the integral into parts A^* and B^* where

$$A^*(h) + B^*(h) = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{(i)} Z_i\right) \, dP_{Z_1} \tag{4.75}$$

$$A^*(h) = \int_{-h}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{(i)} Z_i\right) \, dP_{Z_1}, \tag{4.76}$$

and

$$B^*(h) = \int_{-\infty}^{-h} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{(i)} Z_i\right)\, dP_{Z_1}. \tag{4.77}$$

A quick sketch of a normal distribution, will show that

$$A^*(h) \geq A(h) \tag{4.78}$$

and

$$B^*(h) \geq B(h). \tag{4.79}$$

where

$$A(h) = \int_{-h}^{\infty} P\left(z + h > \sqrt{\frac{d+1-\epsilon}{\epsilon}} Z_2\right)^{k-1} dP_{Z_1} \tag{4.80}$$

and

$$B(h) = \int_{-\infty}^{-h} P\left(z + h > \sqrt{\frac{\epsilon}{d+1-\epsilon}} Z_2\right)^{k-1} dP_{Z_1}. \tag{4.81}$$

Also, by the dominated convergence theorem, it should be apparent that $A \to 1$ and $B \to 0$ as $h \to \infty$. With this, we have enough information to determine h by solving

$$A(h) + B(h) = P^*. \tag{4.82}$$

As in Chapter 3, we may solve either $A(h) + B(h) = P^*$ or $A(h) = P^*$. The former will be used in computer implementations, the latter if we are using tabulated values.

However, if we have been following carefully, we will see that we have only shown that $P(\widetilde{CS}_n|R'_{C1}) \geq P^*$, and not what we intended, $P(\widetilde{CS}_n|R_{C1}) \geq P^*$.

Theorem 4.11. *Given k populations π_i with distributions P_i, if the following hold*

1. *P_i is locally regular, $i = 1, \ldots, k$,*

2. *y is P_i-smooth for each π_i,*

3. *Let $\delta \in (0,1)$ such that $\delta\beta_{[k]} > \beta_{[k-1]}$,*

4. *Let $\epsilon > 0$ such that $\beta_{[1]} \geq \epsilon$,*

then

$$P(\widetilde{CS}_n|R_{C1}) \geq \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{[i]} Z_{i,n}\ \right) dP_{Z_{k,n}} \tag{4.83}$$

where for $i = 2, 3, \ldots, k$, $\widehat{\alpha}_{(i)}$ is the empirical depth corresponding to population $\pi_{[i]}$,

$$Z_{i,n} = \frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{(i)})}{\sqrt{\alpha_{(i)}(1 - \alpha_{(i)})}}, \tag{4.84}$$

$$\Theta_{(i)} = \left(\frac{\alpha^*_{(k)}}{\alpha^*_{(i)}}\right) \sqrt{\frac{\alpha_{(i)}(1 - \alpha_{(i)})}{\alpha_{(k)}(1 - \alpha_{(k)})}}, \tag{4.85}$$

and

$$h = (1 - \delta)\sqrt{\frac{n\epsilon}{2 - \epsilon}}. \tag{4.86}$$

Proof. By Lemma 4.6, if $\alpha_{[1]} \geq \frac{\epsilon}{2}$ then $\beta_{[1]} \geq \epsilon$. Therefore, $P(\widetilde{CS}_n|R'_{C1}) \leq P(\widetilde{CS}_n|R_{C1})$.

□

Procedure R_{C2}:

Theorem 4.12. *Given k populations π_i with distributions P_i, if the following hold*

1. *P_i is locally regular, $i = 1, \ldots, k$,*

2. *y is P_i-smooth for each π_i,*

3. *Let $\delta \in (0, 1)$ be such that $\delta\alpha_{[k]} > \alpha_{[k-1]}$,*

4. *Let $\epsilon > 0$ be such that $\alpha_{[1]} \geq \epsilon$,*

5. *$\alpha_1^* = \alpha_2^* = \cdots = \alpha_k^*$,*

then

$$P(\widetilde{CS}_n | R_{C2}) \geq \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{[i]} Z_{i,n}\ \right) dP_{Z_{k,n}} \tag{4.87}$$

where for $i = 2, 3, \ldots, k$, $\widehat{\alpha}_{(i)}$ is the empirical depth corresponding to population $\pi_{[i]}$,

$$Z_{i,n} = \frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{[i]})}{\sqrt{\alpha_{[i]}(1 - \alpha_{[i]})}}, \tag{4.88}$$

$$\Theta_{[i]} = \sqrt{\frac{\alpha_{[i]}(1 - \alpha_{[i]})}{\alpha_{[k]}(1 - \alpha_{[k]})}} \text{ for i=1,\ldots,k-1} \tag{4.89}$$

and

$$h = (1 - \delta)\sqrt{\frac{n\epsilon}{1 - \epsilon}}. \tag{4.90}$$

Proof. The proof follows the same pattern as that given for Theorem 4.11. First, we show that for $i = 1, \ldots, k-1$

$$\widehat{\alpha}_{(k)} > \widehat{\alpha}_{(i)} \tag{4.91}$$

then

$$Z_{k,n} + \frac{\sqrt{n}(\alpha_{[k]} - \alpha_{[i]})}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}} > \Theta_{[i]} Z_{i,n}. \tag{4.92}$$

This is because

$$\begin{aligned}
Z_{,n} &= \frac{\sqrt{n}(\widehat{\alpha}_{(k)} - \alpha_{[k]})}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}} && (4.93)\\
&> \frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{[k]})}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}} && (4.94)\\
&= \frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{[i]} + \alpha_{[i]} - \alpha_{[k]})}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}} && (4.95)\\
&= \frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{[i]})}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}} + \frac{\sqrt{n}(\alpha_{[i]} - \alpha_{[k]})}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}} && (4.96)\\
&= \frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{[i]})}{\sqrt{\alpha_{[i]}(1-\alpha_{[i]})}} \sqrt{\frac{\alpha_{[i]}(1-\alpha_{[i]})}{\alpha_{[k]}(1-\alpha_{[k]})}} + \frac{\sqrt{n}(\alpha_{[i]} - \alpha_{[k]})}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}} && (4.97)\\
&= \Theta_{[i]} Z_{i,n} - \frac{\sqrt{n}(\alpha_{[k]} - \alpha_{[i]})}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}}. && (4.98)
\end{aligned}$$

This shows that

$$\begin{aligned}
P(CS_n | R_{C2}) &= P\Big(\widehat{\alpha}_{[k]} = \widehat{\alpha}_{(k)}\Big) && (4.99)\\
&= P\Big(\widehat{\alpha}_{(k)} > \widehat{\alpha}_{(i)}, i = 1, \ldots, k-1\Big). && (4.100)\\
&= \int \prod_{i=1}^{k-1} P\left(z + \frac{\sqrt{n}(\alpha_{[k]} - \alpha_{[i]})}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}} > \Theta_{[i]} Z_{i,n}\right) dP_{Z_{1,n}}. && (4.101)
\end{aligned}$$

Using (4.47), and the fact that

$$\alpha_{[k]} - \alpha_{[i]} > (1-\delta)\alpha_{[k]}, \tag{4.102}$$

we have that

$$\begin{aligned}
\frac{\sqrt{n}(\alpha_{[k]} - \alpha_{[i]})}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}} &\geq \sqrt{n}\frac{(1-\delta)\alpha_{[k]}}{\sqrt{\alpha_{[k]}(1-\alpha_{[k]})}} && (4.103)\\
&= \sqrt{n}(1-\delta)\sqrt{\frac{\alpha_{[k]}}{1-\alpha_{[k]}}} && (4.104)\\
&\geq \sqrt{n}(1-\delta)\sqrt{\frac{\epsilon}{1-\epsilon}} && (4.105)\\
&= h. && (4.106)
\end{aligned}$$

The inequality in (4.105) is because $\frac{\alpha_{[k]}}{1-\alpha_{[k]}}$ is an increasing function in $\alpha_{[k]}$. Therefore,

$$P(\widetilde{CS}_n|R_{C2}) \geq \int \prod_{i=1}^{k-1} P\left(z + h > \Theta_{[i]} Z_{i,n}\right) dP_{Z_{1,n}}. \tag{4.107}$$

□

Now, we would like to approximate a solution, h, to

$$\int \prod_{i=1}^{k-1} P\left(z + h > \Theta_{[i]} Z_{i,n}\right) dP_{Z_{1,n}} = P^*. \tag{4.108}$$

As before, this will be done in two steps. First, applying Proposition 4.10, we see that for large h

$$\int \prod_{i=1}^{k-1} P\left(z+h > \Theta_{[i]} Z_{i,n}\right) dP_{Z_{1,n}} \approx \int \prod_{i=1}^{k-1} P\left(z+h > \Theta_{[i]} Z_i\right) dP_{Z_1}. \tag{4.109}$$

We then split the integral into parts A^* and B^* where

$$A^*(h) + B^*(h) = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z+h > \Theta_{[i]} Z_i\right)\, dP_{Z_1} \tag{4.110}$$

$$A^*(h) = \int_{-h}^{\infty} \prod_{i=1}^{k-1} P\left(z+h > \Theta_{(i)} Z_i\right)\, dP_{Z_1}, \tag{4.111}$$

and

$$B^*(h) = \int_{-\infty}^{-h} \prod_{i=1}^{k-1} P\left(z+h > \Theta_{(i)} Z_i\right)\, dP_{Z_1}. \tag{4.112}$$

A quick sketch of a normal distribution, and an application of Lemma 4.8 will show that

$$A^*(h) \geq A(h) \tag{4.113}$$

and

$$B^*(h) \geq B(h) \tag{4.114}$$

where

$$A(h) = \int_{-h}^{\infty} P\left(z+h > \delta(2-\delta) Z_2\right)^{k-1}\, dP_{Z_1} \tag{4.115}$$

and

$$B(h) = \int_{-\infty}^{-h} P\left(z + h > 4\epsilon(1-\epsilon)Z_2\right)^{k-1}\, dP_{Z_1}. \tag{4.116}$$

Again, we see that $A \to 1$ and $B \to 0$ as $h \to \infty$. With this, we have enough information to determine h by solving either $A(h) + B(h) = P^*$ or $A(h) = P^*$

4.4.3 Two-Stage Results:

The justifications of the Two-Stage procedures follows from those provided for their single stage counterparts, with some slight modifications. We detail them here.

Procedure R_{RC2}:

Theorem 4.13. *Given k populations π_i with distributions P_i, if the following hold*

1. *P_i is locally regular, $i = 1, \ldots, k$,*

2. *y is P_i-smooth for each π_i,*

3. *Let $\delta \in (0,1)$ such that $\delta\beta_{[k]} > \beta_{[k-1]}$,*

4. *$\beta_{[1]} > 0$,*

then

$$P(\widetilde{CS}_n | R_{C2}) \geq \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} P\left(z + h > \Theta_{[i]} Z_{i,n}\right) dP_{Z_{k,n}} \tag{4.117}$$

where for $i = 2, 3, \ldots, k$, $\widehat{\alpha}_{(i)}$ *is the empirical depth corresponding to population* $\pi_{[i]}$,

$$Z_{i,n} = \frac{\sqrt{n}(\widehat{\alpha}_{(i)} - \alpha_{(i)})}{\sqrt{\alpha_{(i)}(1 - \alpha_{(i)})}}, \tag{4.118}$$

$$\Theta_{(i)} = \left(\frac{\alpha^*_{(k)}}{\alpha^*_{(i)}}\right) \sqrt{\frac{\alpha_{(i)}(1 - \alpha_{(i)})}{\alpha_{(k)}(1 - \alpha_{(k)})}}, \tag{4.119}$$

and

$$h = (1 - \delta)\sqrt{\frac{n\beta_{[1]}}{2 - \beta_{[1]}}}. \tag{4.120}$$

Proof. Let $\epsilon = \beta_{[1]}$. Use Theorem 4.11. □

Fixing h, if we follow the reasoning from Section 4.4.2, we eventually will reach the conclusion that we should solve

$$\int_{-h}^{\infty} P\left(z + h > \sqrt{\frac{d + 1 - \epsilon}{\epsilon}} Z_2\right)^{k-1} dP_{Z_1} = P^* \tag{4.121}$$

where $\epsilon = \beta_{[1]}$. By corollary 4.3, and Slutsky's Theorem, we know that

$$\widehat{\epsilon}_{\beta,n_1} Z \xrightarrow{d} \epsilon Z. \tag{4.122}$$

Therefore, for $n_1 >> 0$, we can approximate

$$\int_{-h}^{\infty} P\left(z + h > \sqrt{\frac{d + 1 - \epsilon}{\epsilon}} Z_2\right)^{k-1} dP_{Z_1} \tag{4.123}$$

with

$$\int_{-h}^{\infty} P\left(z+h>\sqrt{\frac{d+1-\widehat{\epsilon}_{\beta,n_1}}{\widehat{\epsilon}_{\beta,n_1}}}Z_2\right)^{k-1} dP_{Z_1}. \tag{4.124}$$

And so, we determine h using

$$\int_{-h}^{\infty} P\left(z+h>\sqrt{\frac{d+1-\widehat{\epsilon}_{\beta,n_1}}{\widehat{\epsilon}_{\beta,n_1}}}Z_2\right)^{k-1} dP_{Z_1} = P^*. \tag{4.125}$$

With h determined, we would like to determine our sample size n by solving

$$h = (1-\delta)\sqrt{\frac{n\beta_{[1]}}{2-\beta_{[1]}}}. \tag{4.126}$$

But, we still have a dependence on the unknown value of $\beta_{[1]}$. To work around this, we use the approximate

$$h \approx (1-\delta)\sqrt{\frac{n\widehat{\epsilon}_{\beta,n_1}}{2-\widehat{\epsilon}_{\beta,n_1}}}. \tag{4.127}$$

Procedure R_{C2}:

Letting $\epsilon = \alpha_{[1]}$ in Section 4.4.2, nothing changes in our justification of the procedure, with one exception. In that Section 4.4.2, our sample size is determined using the relation

$$h = (1-\delta)\sqrt{\frac{n\epsilon}{1-\epsilon}} \tag{4.128}$$

with h defined as the solution to 4.21. This relation is dependent on the unknown value of $\alpha_{[1]}$. However, we remedy this in the same manner we used with Procedure R_{RC2}. We determine our sample size using the approximation

$$h \approx (1-\delta)\sqrt{\frac{n\widehat{\epsilon}_{\alpha,n_1}}{1-\widehat{\epsilon}_{\alpha,n_1}}}. \tag{4.129}$$

4.5 Simulations

Our simulations, in this section, fall into two categories based on the category of procedure used. We begin with the procedures that assume that the maximal depth is the same for all populations.

Table 4.1: Single Stage Procedures R_{C_1}:
$k = 3$, $\delta = .95$
$\delta\alpha_{[3]} = \alpha_{[2]} = \epsilon$

$\alpha_{[3]} = .45$			$\alpha_{[3]} = .25$			$\alpha_{[3]} = .05$		
P^*	$\widehat{P^*}$	n	P^*	$\widehat{P^*}$	n	P^*	$\widehat{P^*}$	n
0.60	0.602	420	0.60	0.615	1021	0.60	0.626	6744
0.65	0.665	598	0.65	0.667	1448	0.65	0.668	9405
0.70	0.720	820	0.70	0.717	1980	0.70	0.712	12709
0.75	0.771	1100	0.75	0.763	2649	0.75	0.770	16861
0.80	0.824	1461	0.80	0.821	3513	0.80	0.823	22211
0.85	0.877	1948	0.85	0.866	4677	0.85	0.867	29430
0.90	0.920	2661	0.90	0.914	6385	0.90	0.917	40038
0.95	0.960	3930	0.95	0.963	9423	0.95	0.959	58941

The results can be seen in Table 4.1. Three bivariate normal populations were used. With the exception of choosing the origin, any point in the plane could be used as the target point. Therefore, from iteration to iteration, we randomly selected a different target point. The results in Table 4.1 are based on 10,000 iterations. As an optimal worst case configuration, we chose to use a configuration where $\delta\alpha_{[3]} = \alpha_{[2]} = \alpha_{[1]} = \epsilon$. From this we can see, the smallest possible sample size that would accomplish our goal. The results in Table 4.1 make use of our knowledge of $\alpha_{[1]}$ by allowing us to set $\epsilon = \alpha_{[1]}$. The results show that the empirical probability of a correct selection, $\widehat{P}^*$, is very close to the desired P^*. In reality, it is most likely that the value of $\alpha_{[1]}$ is unknown to us. Consequently, a wise experimenter would try to place a reasonable lower bound on $\alpha_{[1]}$ by making ϵ reasonably small. This necessarily increases the size of the sample that is collected from each population, but the probability requirement would still be met. This can be seen most visibly, as the value of $\alpha_{[3]}$ decreased in Table 4.1. Since δ was held constant, we see that the increase in sample size is driven by the value of the smallest population depth. Table 4.2 considers 2500 iterations of Procedures R_{C_1} using $k = 5, 10$ bivariate Normal populations. As we can see, they are reasonably close to the desired value of P^*. For an experimenter that would like to control the sample size, and would like to guess a lower bound on $\alpha_{[1]}$, they can use Procedures R_{C_2}. Provided that $\alpha_{[1]}$ is larger than a value of ϵ that would be used as a lower bound by the experimenter, there can be a dramatic savings in sample size collected. This can be seen in Table 4.3. Table 4.3 is based on 10,000 iterations of initial samples of size 50. $\bar{n}$ is the estimated mean sample size when $\alpha_{[1]} = .35, .25$

Table 4.2: Single Stage Procedures R_{C_1}:
$\alpha_{[3]} = .45$, $\delta = .95$
$\delta\alpha_{[3]} = \alpha_{[2]} = \epsilon$

	$k = 5$			$k = 10$	
P^*	$\widehat{P^*}$	n	P^*	$\widehat{P^*}$	n
0.60	0.626	944	0.60	0.641	1675
0.65	0.690	1188	0.65	0.705	1980
0.70	0.728	1476	0.70	0.754	2330
0.75	0.796	1823	0.75	0.812	2740
0.80	0.845	2254	0.80	0.842	3238
0.85	0.890	2816	0.85	0.882	3874
0.90	0.924	3615	0.90	0.937	4759
0.95	0.966	4993	0.95	0.970	6250

and $\alpha_{[3]} = .45$. n is the single stage sample size if $\epsilon = .05$ is used. On average, $\bar{n}$ is much lower than n. This indicates an advantage to the two stage procedure.

Table 4.3: Two Stage Procedures R_{C_2}:
$k = 3$, $\delta = .95$
$\alpha_{[3]} = .45, \alpha_{[2]} = \delta\alpha_{[3]}$, $\epsilon = .05$

	$\alpha_{[1]} = .35$				$\alpha_{[1]} = .25$		
P^*	$\bar{n}$	n	$se_{\bar{n}}$	P^*	$\bar{n}$	n	$se_{\bar{n}}$
0.60	842	6380	7	0.60	1324	6380	16
0.65	1213	8901	11	0.65	1869	8901	22
0.70	1644	12032	15	0.70	2559	12032	32
0.75	2222	15967	19	0.75	3389	15967	40
0.80	2944	21037	25	0.80	4539	21037	54
0.85	3914	27878	33	0.85	5993	27878	74
0.90	5288	37931	44	0.90	8225	37931	99
0.95	7810	55844	72	0.95	11911	55844	134

In Tables 4.4 and 4.5, we apply Procedure R_{RC_1} to a situation where the maximal depth is not the same for all populations. To do this, we compared a bivariate exponential population to some normally distributed populations. In Table 4.4, three populations were compared at varying values of δ. While in Table 4.5, $k = 5, 10$

populations were compared for $\delta = .9$. Each simulation was run 10,000 times.

Table 4.4: Single Stage Procedures R_{RC_1}:
Exponential vs Normal
$k = 3, \alpha_{[3]} = .9, \alpha_{[2]} = \delta\alpha_{[3]}$,

$\delta = .9$			$\delta = .8$			$\delta = .7$		
P^*	$\widehat{P}^*$	n	P^*	$\widehat{P}^*$	n	P^*	$\widehat{P}^*$	n
0.60	0.584	273	0.60	0.689	95	0.60	0.775	60
0.65	0.623	366	0.65	0.749	127	0.65	0.839	80
0.70	0.682	480	0.70	0.805	166	0.70	0.892	104
0.75	0.743	623	0.75	0.855	214	0.75	0.931	134
0.80	0.795	806	0.80	0.906	277	0.80	0.961	173
0.85	0.852	1052	0.85	0.942	360	0.85	0.983	225
0.90	0.909	1412	0.90	0.972	483	0.90	0.995	301
0.95	0.963	2050	0.95	0.993	700	0.95	1.000	436

Table 4.5: Single Stage Procedures R_{RC_1}:
Exponential vs Normal
$\delta = .9, \alpha_{[3]} = .9, \alpha_{[2]} = \delta\alpha_{[3]}$,

k=5			k=10		
P^*	$\widehat{P}^*$	n	P^*	$\widehat{P}^*$	n
0.60	0.617	619	0.60	0.659	1124
0.65	0.670	747	0.65	0.713	1281
0.70	0.716	896	0.70	0.767	1458
0.75	0.774	1073	0.75	0.802	1664
0.80	0.827	1291	0.80	0.851	1911
0.85	0.876	1573	0.85	0.893	2224
0.90	0.922	1971	0.90	0.927	2657
0.95	0.966	2655	0.95	0.968	3384

4.6 Concluding Remarks

In this chapter, we outlined several procedures for selecting a population whose distribution is "most centered" at a target point $y \in \mathbb{R}^d$. Procedures R_{RC1} and R_{C1}

allowed this to be completed by taking a single sample. Procedures R_{RC2} and R_{C2} required that our sample be taken in two stages. In practice, the second approach seems the more reasonable, if no previous knowledge of the populations under consideration exists. We should note what may or may not be considered a restriction on the use of this procedure. All of these procedures assume that the populations under consideration have distributions that are *locally regular.* This restriction would mean that we may only consider a subset of all absolutely continuous populations when applying this procedure.

To conclude our remarks, it should be possible to produce subset procedures of this type. Namely,

- Select all populations π_i that are "more central" than a standard or control, population π_0, or

- Select a subset that contains the "most central" population.

In our next chapter, we develop procedures for selection of all populations whose distribution is similar to some given collection of distributions.

Chapter 5

Distribution

In this chapter, we are given k populations, $\{\pi_i\}_{i=1}^k$, with unknown distributions, $F_{X_i} \in \mathcal{F}$ where $\mathcal{F}$ is a subset of all probability distributions. We develop a general procedure for selecting all populations with a distribution that is considered desirable by an experimenter. There will be two differences between this chapter and the previous. The first one will be our focus on univariate examples. In the previous two chapters, some form of a univariate procedure already existed, but a multivariate procedure did not. For the procedures of this chapter, there does not exist a similarly motivated univariate procedure. Therefore, the procedure of this chapter will be outlined keeping this in mind. Most of the examples will be univariate in nature. This will mainly be due to the fact that exact expressions can be attained for some pertinent values in the univariate setup. It is not to say, that an "exact" multivariate example cannot be given. The second difference from the previous chapters will be our avoidance of any type of depth function in our selection procedures. This was done to keep the procedures as simple as possible. The use of a depth function is possible with these procedures, but would only add complexity to the procedures. As a replacement for the depth function, and to maintain generality, we will make use of the multivariate cumulative distribution function. For $d = 1$, this reduces to a standard univariate cumulative distribution function.

Remark 5.1. A depth-based selection procedure for selecting populations based on their distribution would make use of Theorems 2.7 and 2.9. Theorems 2.7 states that, under certain assumptions, the halfspace depth characterizes the underlying distributions. Theorem 2.9 says that the empirical depth function converges uniformly to its population counterpart almost surely. Taken together, we could conclude that if an empirical depth function is "close" to some given depth function, then it would be reasonable to conclude that the underlying distribution for the empirical depth function is "close" to the underlying distribution for the given depth function. Of course, what we mean by "close" would need to be defined. However, the proof of Theorem 2.9 indicates a more direct path to this conclusion, that does not make use of depth functions. This is the path we have taken.

Definition 5.1. The *cumulative multivariate distribution* of a random vector $X_i = (X_{i,1}, \ldots, X_{i,d})^T$, $d \geq 1$, is

$$F_{X_i}(x) = P(X_{i,1} \leq x_1, \ldots, X_{i,d} \leq x_d) \tag{5.1}$$

where $x = (x_1, \dots, x_d)^T \in \mathbb{R}^d$. When it will not cause confusion, F_i will denote F_{X_i}.

Returning to our problem, we need to decide what we mean by desirable. Consider the following setup. We have an experimenter who has k populations and the experimenter knows that the distributions, F_i, of each populations falls within $\mathcal{F} \subset \mathcal{P}_d$, where $\mathcal{P}_d$ the collection of all distributions on $\mathbb{R}^d$. This means the experimenter has an idea of the form of all the population's distributions. However, the experimenter would like to narrow down the study to a subset of populations which has a more specific set of properties i.e. the experimenter has defined a set $D \subset \mathcal{F}$, so that

$$D = \{F_X \in \mathcal{F} \mid F_X \text{ is a desirable distribution.}\}. \tag{5.2}$$

The experimenter would like to select all populations that fall into a set G, where

$$G = \{\pi_i | F_i \in D\}.$$

From this basic set up, it should be apparent that we will use the subset selection approach described in Chapter 1. As a complement to G, we define

$$B = \{\pi_i | F_i \notin D\}.$$

As a means of determining when a population is desirable or not, we need a manner of measuring how far a given population's distribution is from the distributions in D.

Definition 5.2. The *distance* of F_X from D is defined to be

$$\epsilon(F_X, D) = \inf_{F_Y \in D} \sup_{y \in \mathbb{R}^d} |F_X(y) - F_Y(y)|. \tag{5.3}$$

Using this, we form the closure of D,

$$\bar{D} = \{F_X \in \mathcal{F} \mid \epsilon(F_X, D) = 0\},$$

and redefine G as $G = \{\pi_i | \epsilon(F_X, \bar{D}) = 0\}$. In all examples, it will be the case that $\bar{D} = D$. Thus, to ease notation, ϵ_i will denote $\epsilon(F_i, \bar{D})$

Remark 5.2. This redefinition of G may reclassify a population as desirable. This can happen for a population whose distribution $F_i \notin D$, but $F_i \in \mathcal{F}$ and $\epsilon_i = 0$. We will not consider this a problem. Based on our distance measure, such an F_i is almost indistinguishable from some sequence of distributions in D. As such, we should consider it desirable also.

Similarly, an experimenter can define,

$$U = \{F_X \in \mathcal{F} \mid F_X \text{ is a undesirable distribution.}\}, \tag{5.4}$$

a set disjoint from, but not necessarily complementary to, $\bar{D}$. There may be distributions in $\mathcal{F}$ to whose classification the experimenter is indifferent. This allows us to redefine B, as $B = \{\pi_i | \epsilon_i \geq \epsilon^*\}$ where

$$\epsilon^* = \inf_{F_X \in U} \inf_{F_Y \in D} \sup_{y \in \mathbb{R}^d} |F_X(y) - F_Y(y)|. \tag{5.5}$$

We may also define the closure of U,

$$\bar{U} = \{F_X \in \mathcal{F} \mid \epsilon(F_X, \bar{D}) \geq \epsilon^*$$

5.1 Goal:

The goal of our procedure will be to select a subset of populations, $\widehat{G}_n$, that contains all populations in G. Therefore, a correct selection (CS_n) will be defined as

$$CS_n = \left\{ G \subseteq \widehat{G}_n \right\}. \tag{5.6}$$

Thus, it will become necessary to estimate the distance of a population's distribution from $\bar{D}$. To do this, we need to define the empirical cumulative multivariate distribution.

Definition 5.3. Let $X_{i,1}, \ldots, X_{i,n}$ be independent and identically distributed random vectors in $\mathbb{R}^d$ with component random variables $X_{i,j_1}, \ldots, X_{i,j_d}, j = 1, \ldots, n$. Then the *empirical cumulative multivariate distribution* of a sample from population π_i is defined to be

$$F_{X_{i},n}(y) = \frac{\sum_{j=1}^{n} \prod_{k=1}^{d} 1_{\{X_{i,j_k} \leq y_k\}}(y)}{n}$$

where $y = (y_1, \ldots, y_d)^T \in \mathbb{R}^d$. As before, when appropriate, to ease notation, $F_{i,n}(y)$ will be used to denote $F_{X_{i},n}(y)$. Additionally, abusing notation, $F_{i,n}(y)$ will denote both a random variable, and observed value. Its usage will be clear from the context.

Definition 5.4. The *empirical distance* of F_i from $\bar{D}$ is defined to be

$$\widehat{\epsilon}(X_{i,1}, \ldots, X_{i,n}, \bar{D}) = \inf_{F_Y \in \bar{D}} \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_Y(y)|. \tag{5.7}$$

To ease notation, $\widehat{\epsilon}_{i,n}$ will denote $\widehat{\epsilon}(X_{i,1}, \ldots, X_{i,n}, \bar{D})$.

With this distance, we would like to make a selection in such a way that allows us to control the probability of a correct selection whenever G is non-empty. Thus, we need our procedures to satisfy the requirement that the probability of a correct selection is at least P^*:

$$P(CS_n) = P\left(G \subseteq \widehat{G}_n\right) \geq P^* \in (0, 1). \tag{5.8}$$

5.2 Assumptions:

We make the following assumptions:

1. $\mathcal{F} \subset$ all absolutely continuous distributions on $\mathbb{R}^d$.

2. If $F_X, F_Y \in \bar{D}$, then

$$\sup_{y \in \mathbb{R}^d} |F_{Y,n}(y) - F_Y(y)| \stackrel{d}{=} \sup_{y \in \mathbb{R}^d} |F_{X,n}(y) - F_X(y)|. \tag{5.9}$$

3. If $F_X, F_Y \in \bar{U}$, then

$$\sup_{y \in \mathbb{R}^d} |F_{Y,n}(y) - F_Y(y)| \stackrel{d}{=} \sup_{y \in \mathbb{R}^d} |F_{X,n}(y) - F_X(y)|. \tag{5.10}$$

4. $\epsilon^* > 0$.

5. $\bar{D}$ is defined so that either $\widehat{\epsilon}_{i,n}$ is measurable, or there exists $\bar{D}^* \subset \bar{D}$, so that $\widehat{\epsilon}(X_{i,1}, \ldots, X_{i,n}, \bar{D}^*)$ is measurable and $\widehat{\epsilon}_{i,n} = \widehat{\epsilon}(X_{i,1}, \ldots, X_{i,n}, \bar{D}^*)$ almost surely.

The first two assumptions are meant to assist in the computation of a lower bound on the probability of a correct selection. This bound will be computed using the distribution of

$$KS_{i,n} = \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_i(y)|.$$

In the univariate case, when $d = 1$, the second assumption is unnecessary, since $KS_{i,n}$ has the same distribution for all continuous univariate distributions. However, in the multivariate case, the distribution of $KS_{i,n}$ is not known, and is dependent upon F_i. Thus, it will be necessary to simulate the distribution of $KS_{i,n}$. Since exact form of F_i is unknown, we will need the second assumption to allow us to simulate the distribution of $KS_{i,n}$. Under the second assumption, we may choose any member of D to simulate the distribution of $KS_{i,n}$, and will not need to use the exact form of F_i.

Example 5.1. Let X be a random vector in $\mathbb{R}^d$, where $X = (X_1, \ldots, X_d)$. Let $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$, and define

$$\mathcal{F} = \{F_{X+r} \mid r \in \mathbb{R}^d\}. \tag{5.11}$$

Then,

$$\sup_{y\in\mathbb{R}^d} |F_{X_n+r}(y) - F_{X+r}(y)| = \sup_{y\in\mathbb{R}^d} |F_{X_n}(y-r) - F_X(y-r)| \tag{5.12}$$

$$\stackrel{d}{=} \sup_{x\in\mathbb{R}^d} |F_{X_n}(x) - F_X(x)|. \tag{5.13}$$

The third and fourth assumptions are only necessary if a statement about the efficiency of the procedure is needed. If $\epsilon^* = 0$, on average, we would expect to select all the populations, which would defeat the purpose of our trying to screen out undesirable populations. By assuming that $\epsilon^* > 0$, we are essentially saying that there are three classes of distributions, the desirables, the undesirables, and those to whose classification we are indifferent. These would be distributions, F_X, such that $\epsilon(F_X, \bar{D}) \in (0, \epsilon^*)$.

5.3 Procedures:

This section outlines procedures for selecting a random subset $\widehat{G}_n$ that contains all distributions that are in G. Since there is a slight difference between the univariate and the multivariate versions, we list them separately. Justifications for all the proposed procedures are given in Section 5.4.

5.3.1 Selection among Univariate distributions

When considering the univariate case, we will approximate the distribution of

$$KS_n = \sqrt{n} \sup_{y \in \mathbb{R}} |F_n(x) - F(x)|,$$

using

$$KS = \lim_{n \to \infty} \sqrt{n} \sup_{y \in \mathbb{R}} |F_n(x) - F(x)|$$

which has a known distribution. Tabled values of KS can be found in [27]. An excellent approximation can be found using $n \geq 100$.

Procedure R_{SU}:

1. Determine a value δ so that $P(KS < \delta) \geq (P^*)^{\frac{1}{k}}$.

2. Select a pair n, and $\tau > 0$, so that $\sqrt{n}\tau \geq \delta$.

3. Take a random sample of size n from each population.

4. Compute $\widehat{\epsilon}_{i,n}$ for each population.

5. Find all populations with empirical desirability less than τ, i.e. determine the set

$$\widehat{G}_n = \{\pi \mid \widehat{\epsilon}_{i,n} \leq \tau\}. \tag{5.14}$$

6. Claim with probability at least P^* that all the populations in G have been selected.

Efficiency of Procedure R_{SU}

If $\#A$ denotes the cardinality of a set A, and S denotes $\#\widehat{G}_n$, then the efficiency of Procedure R_{SU} can be described by considering the expected sample subset size, $E(S)$, of $\widehat{G}_n$. Given n, τ, ϵ^*, and P^*, we can state that

$$\#G \cdot (P^*)^{\frac{1}{k}} \leq E(S) \leq \#G + \#B \cdot P\left(KS \geq \sqrt{n}(\epsilon^* - \tau)\right). \tag{5.15}$$

5.3.2 Selecting among Multivariate Distributions

This procedure is basically a two stage procedure. Although we sample each population only once, the first stage requires us to simulate a distribution from one of the distributions in $\bar{D}$.

Procedure R_{SM}:

Stage 1:

(a) Select any distribution $F_D \in D$.

(b) Simulate the distribution of $KS_{D,n} = \sup_{y \in \mathbb{R}^d} |F_{D,n}(y) - F_D(y)|$.

(c) Select $\tau > 0$ so that $\widehat{P}(KS_{D,n} < \tau) \geq (P^*)^{\frac{1}{k}}$.

Stage 2:

(a) Take a sample of size n from each population.

(b) Calculate $\widehat{\epsilon}_{i,n}$ for each sample.

(c) Find all populations with empirical desirability less than τ, i.e. determine the set

$$\widehat{G}_n = \{\pi \mid \widehat{\epsilon}_{i,n} < \tau\}. \tag{5.16}$$

(d) Claim with probability at least P^* (approximately) that all the populations in G have been selected.

Efficiency of Procedure R_{SM}

As with the univariate procedure, we wish to measure efficiency with the expected subset size of $\widehat{G}_n$. Given n, τ, ϵ^*, and P^*, we can state, approximately, that

$$\#G \cdot (P^*)^{\frac{1}{k}} \leq E(S) \leq \#G + \#B \cdot \widehat{P}\left(KS_{U,n} \geq \epsilon^* - \tau\right), \tag{5.17}$$

where $KS_{U,n} = \sup_{y \in \mathbb{R}^d} |F_{U,n}(y) - F_U(y)|$ and F_U is any distribution in U.

5.4 Proofs

5.4.1 General Results:

This section will cover the necessary justifications for the procedures given above. We start with Procedure R_{SU}.

Theorem 5.1. *Given k independent populations with distributions $F_i \in \mathcal{F}$, if the following hold:*

1. $\tau > 0$,
2. *G is non-empty,*

then

$$P(CS_n) \geq P\left(\sup_{y \in \mathbb{R}^d} |F_n(y) - F(y)| < \tau\right)^k. \tag{5.18}$$

Proof. By definition, if $F_i \in \bar{D}$ then

$$\inf_{F_Y \in \bar{D}} \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_Y(y)| \leq \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_Y(y)| \tag{5.19}$$

for all $F_Y \in \bar{D}$. In particular,

$$\inf_{F_Y \in \bar{D}} \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_Y(y)| \leq \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_i(y)| \tag{5.20}$$

for all F_i, such that $\pi_i \in G$. Thus, if

$$KS_{i,n} = \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_i(y)| < \tau, \tag{5.21}$$

then

$$\widehat{\epsilon}_{i,n} = \inf_{F_Y \in \bar{D}} \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_Y(y)| < \tau. \tag{5.22}$$

Therefore,

$$P(CS_n) = P(G \subset \widehat{G}_n) \tag{5.23}$$

$$= P\left(\widehat{\epsilon}_{i,n} < \tau,\, \pi_i \in G\right). \tag{5.24}$$

Since $\widehat{\epsilon}_{i,n}$ are independent random variables, we have our next equality:

$$= \prod_{\pi_i \in G} P\left(\widehat{\epsilon}_{i,n} < \tau\right) \tag{5.25}$$

$$\geq \prod_{\pi_i \in G} P\left(KS_{i,n} < \tau\right) \tag{5.26}$$

$$\geq P\left(KS_n < \tau\right)^k. \tag{5.27}$$

When $d = 1$, $\sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_i(y)|$ has the same distribution regardless of the underlying distribution. Thus, we can drop the subscript i. Then F represents any distribution in $\mathcal{F}$. If $d \geq 2$, we have assumed that $\sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_i(y)|$ has the same distribution. This allows us to drop the subscript again. □

In the proof of Theorem 5.1, (5.27) will be used to complete the justification of the probability statements in Procedures R_{SU}, and R_{SM}. For Procedure R_{SU}, we see that

$$P\left(\sup_{y\in\mathbb{R}^d}|F_n(y)-F(y)|<\tau\right)=P\left(\sqrt{n}\sup_{y\in\mathbb{R}^d}|F_n(y)-F(y)|<\sqrt{n}\tau\right) \tag{5.28}$$

$$\approx P\left(KS<\delta\right) \tag{5.29}$$

where $\delta=\sqrt{n}\tau$ and n is large. Hence, if δ is selected as in Procedure R_{SU}, we will achieve our probability requirement. For the higher dimension case, we use our simulated distribution for $\sup_{y\in\mathbb{R}^d}|F_n(y)-F(y)|$ to determine τ as in Procedure R_{SM}. Thus, in (5.27)

$$P\left(\sup_{y\in\mathbb{R}^d}|F_n(y)-F(y)|<\tau\right)\approx P_m\left(\sup_{y\in\mathbb{R}^d}|F_n(y)-F(y)|<\tau\right). \tag{5.30}$$

In either case, we will use an approximation to determine the value of τ.

To conclude this section, we derive the bounds on the expected subset size, $E(S)$.

Theorem 5.2. *Under the assumptions of Theorem 5.1, if* $\epsilon^*>0$ *then*

$$\#G\cdot(P^*)^{\frac{1}{k}}\leq E(S)\leq\#G+\#B\cdot P\left(\sup_{y\in\mathbb{R}}|F_n(y)-F(y)|\geq(\epsilon^*-\tau)\right).$$

Proof. Since $S = \#\widehat{G}_n$, S may also be defined as

$$S = \sum_{i=1}^{k} 1_{\{\pi_i \text{ is selected.}\}}. \tag{5.31}$$

Therefore,

$$E(S) = \sum_{i=1}^{k} P\left(\pi_i \text{ is selected.}\right) \tag{5.32}$$
$$= \sum_{\pi_i \in G}^{k} P\left(\pi_i \text{ is selected.}\right) + \sum_{\pi_i \in B}^{k} P\left(\pi_i \text{ is selected.}\right). \tag{5.33}$$

Using the proof of Theorem 5.1, we see that $(P^*)^{\frac{1}{k}} \leq P\left(\pi_i \text{ is selected.}\right)$ for $\pi_i \in G$. For $\pi_i \in B$, it is obvious that $P\left(\pi_i \text{ is selected.}\right) \geq 0$. This provides the lefthand inequality. As for the right hand inequality, $1 \geq P\left(\pi_i \text{ is selected.}\right)$ for $\pi_i \in G$. As for $\pi_i \in B$, consider the following:

If $\pi_i \in B$, then $\inf_{F_Y \in \bar{D}} \sup_{y \in \mathbb{R}^d} |F_i(y) - F_Y(y)| = \beta \geq \epsilon^*$. Thus, for all $F_Y \in \bar{D}$,

$$\beta \leq \sup_{y \in \mathbb{R}^d} |F_i(y) - F_Y(y)| \tag{5.34}$$

$$\leq \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_i(y)| + \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_Y(y)|. \tag{5.35}$$

Since the first summand in (5.35) does not depend on F_Y, it is considered constant. Hence,

$$\beta \leq \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_i(y)| + \inf_{F_Y \in \bar{D}} \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_Y(y)|. \tag{5.36}$$

Now, if $\pi_i \in B$ is selected, then

$$\inf_{F_Y \in \bar{D}} \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_Y(y)| < \tau. \tag{5.37}$$

Consequently,

$$\beta \leq \sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_i(y)| + \tau. \tag{5.38}$$

Therefore,

$$P(\pi_i \text{ is selected}, \pi_i \in B) \leq P\left(\sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_i(y)| \geq \beta - \tau\right) \tag{5.39}$$

$$\leq P\left(\sup_{y \in \mathbb{R}^d} |F_{i,n}(y) - F_i(y)| \geq \epsilon^* - \tau\right). \tag{5.40}$$

We complete the proof by dropping the subscripts related to the populations, as we have assumed that the distribution of $\sup_{y \in \mathbb{R}} |F_n(y) - F(y)|$ is invariant for $F \in \bar{U}$. □

5.5 Examples & Simulations

In this section, we look at some examples that implement these procedures. We will first consider the univariate situation. Suppose we would like to apply Procedure R_{SU} so that

$$P(CS_n | R_{SU}) \geq P^* \text{ and } E(S) \leq \#G + 1 \tag{5.41}$$

whenever G is nonempty. In the worst case scenario, $\#G = 1$. So, we need to determine n and τ so that

$$P(KS < \sqrt{n}\tau) \geq (P^*)^{\frac{1}{k}} \tag{5.42}$$

and

$$P\left(KS > \sqrt{n}(\epsilon^* - \tau)\right) \leq \frac{1}{k-1}. \tag{5.43}$$

This amounts to solving the following system of equations:

$$\begin{cases} \sqrt{n}\tau = \delta_1 \\ \sqrt{n}(\epsilon^* - \tau) = \delta_2 \end{cases}$$

where δ_1 is a solution to (5.42), and δ_2 is a solution to (5.43). Explicitly, the solution is

$$\begin{cases} \tau = \frac{\epsilon^* \delta_1}{\delta_1 + \delta_2} \\ n = \left(\frac{\delta_1 + \delta_2}{\epsilon^*}\right)^2. \end{cases}$$

By using [27], Table 5.1 gives us different pairings of τ and n for $P^* = .8, .9$, $\epsilon^* = .05, .1$, and $k = 3, 4, 5, 10, 15, 20$. Table 5.1 will be used in all of the following univariate examples. We begin with a selection from some uniform distributions.

Example 5.2. In this situation, we will assume that all populations under consider-

Table 5.1: Selecting among Distributions R_{SU}:

	$P^* = .8$				$P^* = .9$			
	$\epsilon^* = .1$		$\epsilon^* = .05$		$\epsilon^* = .1$		$\epsilon^* = .05$	
k	n	τ	n	τ	n	τ	n	τ
3	450	0.0608	1798	0.0304	511	0.0633	2044	0.0316
4	529	0.0587	2116	0.0293	591	0.0609	2362	0.0305
5	581	0.0577	2324	0.0288	641	0.0597	2561	0.0298
10	735	0.0554	2938	0.0277	801	0.0572	3204	0.0286
15	824	0.0547	3275	0.0274	889	0.0564	3553	0.0282
20	883	0.0545	3529	0.0273	943	0.0560	3770	0.0280

ation will have a Uniform distribution on $[0, r], r > 0$; the desirable populations have $r \leq \alpha$; and the undesirables have $r \geq \beta$. Now, we let $0 < \alpha < \beta$, and

$$\mathcal{F} = \{F_{rY} | Y \sim \text{Uniform}[0, 1], r > 0\} \tag{5.44}$$

$$D = \{F_{rY} \in \mathcal{F} | r \in (0, \alpha]\} \tag{5.45}$$

$$U = \{F_{rY} \in \mathcal{F} | r \in [\beta, \infty)\}. \tag{5.46}$$

We would like to find $\bar{D}$ and $\bar{U}$. In this case, they are equal to D and U, respectively. Since the argument is similar, we will only show that $\bar{D} = D$. To do this, take $F_X \in \bar{D}$. Since, $\bar{D} \subset \mathcal{F}$, we know that $X = r'Y$ for some $r' > 0$. We need to determine r' so that $\inf_{r \in (0,\alpha]} \sup_{y \in \mathbb{R}^d} |F_{r'Y}(y) - F_{rY}(y)| = 0$. Now, if $r' \in (0, \alpha]$, it is easy to see that this equation is satisfied. So, we need to check that there are no values of $r' > \alpha$ that satisfy this equation. Suppose that $r' > \alpha$, then $r' > r$ and

$$|F_{r'Y}(y) - F_{rY}(y)| = \begin{cases} 0 & \text{if } y \leq 0 \\ y\left(\frac{1}{r} - \frac{1}{r'}\right) & \text{if } y \in (0, r] \\ 1 - \frac{y}{r'} & \text{if } y \in (r, r'] \\ 0 & \text{if } y > r'. \end{cases} \tag{5.47}$$

Therefore, $\sup_{y \in \mathbb{R}^d} |F_{r'Y}(y) - F_{rY}(y)| = 1 - \frac{r}{r'}$, and

$$\inf_{r \in (0,\alpha]} \sup_{y \in \mathbb{R}^d} |F_{r'Y}(y) - F_{rY}(y)| = 1 - \frac{\alpha}{r'} > 0. \tag{5.48}$$

Consequently, $\bar{D} = D$. This also tells us that for any $F_X \in \bar{U}$,

$$\inf_{r \in (0,\alpha]} \sup_{y \in \mathbb{R}^d} |F_X(y) - F_{rY}(y)| \geq 1 - \frac{\alpha}{\beta} = \epsilon^*. \tag{5.49}$$

In Table 5.2, we let $\epsilon^* = .1$ where $\alpha = 1$, and $\beta = \frac{10}{9}$, and used Table 5.1 to determine n and τ. Simulations were iterated 1000 times each. For each k, we used varying numbers of desirable populations. For all the undesirable populations we used Uniform $[0, \beta]$ distributions. Also, for each desirable population, we randomly selected a value, r_i, giving each desirable population, π_i, a Uniform $[0, r_i]$ distribution. In each case, $\widehat{E}(S)$ was closer to the lower bound for $E(S)$ than it was to the upper bound. Additionally, our simulated probability of correct selection, $\hat{P}$, was much higher than

Table 5.2: Uniform Distributions, 1000 iterations, $\epsilon^* = .1$, $\alpha = 1$

$P^* = .8$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.992	0.992	0.988	0.988	0.986	0.986	0.989	0.989
2	0.975	1.975	0.985	1.985	0.989	1.989	0.979	1.979
3	0.967	2.966	0.958	2.958	0.976	2.976	0.977	2.977
4			0.962	3.961	0.960	3.960	0.976	3.976
5					0.958	4.957	0.964	4.964
10							0.927	9.923

$P^* = .9$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.995	0.995	0.996	0.996	0.997	0.997	0.996	0.996
2	0.987	1.987	0.986	1.986	0.984	1.983	0.990	1.990
3	0.977	2.977	0.976	2.976	0.978	2.978	0.990	2.990
4			0.971	3.971	0.969	3.969	0.977	3.977
5					0.969	4.969	0.970	4.969
10							0.941	9.939

Table 5.3: $\bar{D} = \{\text{U}[0,1]\}$ vs $\bar{U} = \{\text{U}[0,\beta]\}$, $\epsilon^* = .1$, $\beta \leq .9$ or $\beta \geq \frac{10}{9}$

$P^* = .8$

	3		4		5		10	
$\#G$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.932	0.932	0.949	0.949	0.961	0.961	0.979	0.979
2	0.865	1.862	0.905	1.903	0.915	1.914	0.959	1.958
3	0.810	2.797	0.857	2.851	0.881	2.876	0.938	2.937
4			0.810	3.795	0.847	3.838	0.919	3.916
5					0.812	4.797	0.895	4.891
10							0.805	9.785

$P^* = .9$

	3		4		5		10	
$\#G$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.967	0.967	0.974	0.974	0.979	0.979	0.990	0.990
2	0.934	1.932	0.954	1.954	0.963	1.963	0.981	1.981
3	0.905	2.902	0.927	2.925	0.942	2.941	0.968	2.968
4			0.902	3.899	0.922	3.919	0.963	3.962
5					0.906	4.903	0.953	4.952
10							0.904	9.899

P^*. This large "discrepancy" can be easily explained. First, the procedure is designed to only give a lower bound on the probability of a correct selection, which it does provide. Secondly, we should only reach this lower bound when three conditions are met:

1. $\#G = k$,
2. $\bar{D} = \{F_0\}$, and
3. every population in G has a distribution equal to F_0.

In one case, if $\#G = m < k$, and the final two hold, then $P(CS_n) = (P^*)^{\frac{m}{k}} \geq P^*$. This is demonstrated in Table 5.3, where $\mathcal{F} = \{F_{rY} | Y \sim \text{Uniform}[0,1], r > 0\}$ and $\bar{D} = \{F_Y | Y \sim \text{Uniform}[0,1]\}$. In another case, say that $\{F_0\} \subsetneq \bar{D}$, but the other conditions hold. Then, the distribution of $\widehat{\epsilon}_{0,n}$ is not the same as $\inf_{F_Y \in \{F_0\}} \sup_{y \in \mathbb{R}^d} |F_{0,n}(y) - F_Y(y)| = \sup_{y \in \mathbb{R}^d} |F_{0,n}(y) - F_0(y)|$. In fact, the distribution of $\widehat{\epsilon}_{0,n}$ will depend upon the makeup of $\bar{D}$. To see this, consider the fact that if $\bar{D}_0 = \{F_0\}$, and $\bar{D}_j$ is such that $\bar{D}_0 \subset \bar{D}_j \subset \bar{D}_{j+1} \subset \bar{D}$ for all $j = 1, 2, \ldots$, then

$$P\left(\inf_{F_Y \in \bar{D}_j} \sup_{y \in \mathbb{R}^d} |F_{0,n}(y) - F_Y(y)| \leq \tau\right) < P\left(\inf_{F_Y \in \bar{D}_{j+1}} \sup_{y \in \mathbb{R}^d} |F_{0,n}(y) - F_Y(y)| \leq \tau\right).$$

For our next example, we consider a class of Univariate Normal distributions.

Example 5.3. Let $\mathcal{F}$ be the set of all Univariate Normal Distributions with mean $\mu \geq 0$ and standard deviation $\sigma = 1$. Let $D = \{F_Y \mid Y = Z + \mu,\ Z \sim N(0,1), \mu \in [0, \alpha]\}$.

Table 5.4: Normal Distributions, 1000 iterations, $\epsilon^* = .1$, $\alpha = 1$

$P^* = .8$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.999	1.004	1.000	1.005	0.999	1.002	1.000	1.002
2	0.998	2.003	0.999	1.999	1.000	2.004	1.000	2.001
3	0.995	2.995	0.997	2.998	1.000	3.001	0.998	2.998
4			0.996	3.996	0.998	3.998	1.000	4.001
5					0.997	4.997	0.998	4.999
10							0.999	9.999

$P^* = .9$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.998	0.998	1.000	1.003	1.000	1.002	1.000	1.002
2	0.966	1.966	1.000	2.000	0.999	2.002	1.000	2.000
3	0.952	2.951	0.995	2.995	0.999	3.000	1.000	3.002
4			0.997	3.997	0.998	3.999	0.999	3.999
5					0.999	4.999	1.000	5.001
10							0.999	9.999

In this case, $\sup_{y\in\mathbb{R}}|F_X(y) - F_Y(y)|$ attains its maximum at $y = \frac{\mu_X+\mu_Y}{2}$ where $\mu_X \geq 0$ and $\mu_Y \in [0,\alpha]$. Therefore,

$$\sup_{y\in\mathbb{R}}|F_X(y) - F_Y(y)|= \left|P\left(Z \leq \frac{\mu_X - \mu_Y}{2}\right) - P\left(Z \leq \frac{\mu_Y - \mu_X}{2}\right)\right|. \tag{5.50}$$

Thus, $\inf_{F_Y\in D}\sup_{y\in\mathbb{R}}|F_X(y) - F_Y(y)|= 0$ if and only if $\mu_X \in [0,\alpha]$. Further, if $\mu_Y > \alpha$, then $\inf_{F_Y\in D}\sup_{y\in\mathbb{R}}|F_X(y) - F_Y(y)|= \left|P\left(Z \leq \frac{\mu_X-\alpha}{2}\right) - P\left(Z \leq \frac{\alpha-\mu_X}{2}\right)\right|$. Consequently, if $U = \{F_X \mid X = Z + \mu, Z \sim N(0,1), \mu \in [\beta,\infty)\}, \beta > \alpha$, then $\epsilon^* = \left|P\left(Z \leq \frac{\beta-\alpha}{2}\right) - P\left(Z \leq \frac{\alpha-\beta}{2}\right)\right|$. Table 5.4 shows the results for a simulation based on different values of P^*, k and $\#G$. Our results are similar to those given in Example 5.2. Table 5.5 gives a slight variation of this setup. It considers $\bar{D}$ to consist of only a standard Normal Distribution. In this case, we get simulated probabilities that approximate the desired lower bound, a result similar to that found in Table 5.3.

For our last univariate example, we expand $\mathcal{F}$ beyond a single type of distribution. This will illustrate the power of these techniques, since our desirable populations will be those with a distribution that is "close" to a certain class, without being a member of that class.

Example 5.4. Let $\mathcal{F}$ be the set of all absolutely continuous univariate distributions. Let $\beta \in (0,1)$, and define

$$D = \left\{F_X \in \mathcal{F}\,\middle|\, \inf_{(\mu,\sigma)\in\mathbb{R}\times\mathbb{R}^+}\sup_{y\in\mathbb{R}}|F_X(y) - F_{N(\mu,\sigma)}(y)|\leq \beta\right\}. \tag{5.51}$$

Table 5.5: $\bar{D} = \{\mathrm{N}[0,1]\}$ vs $\bar{U} = \{\mathrm{N}[\mu,1]\}$, $\epsilon^* = .1$

$P^* = .8$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.929	0.934	0.938	0.950	0.964	0.968	0.972	0.976
2	0.883	1.883	0.903	1.903	0.925	1.929	0.950	1.952
3	0.809	2.796	0.838	2.831	0.905	2.905	0.928	2.928
4			0.807	3.788	0.834	3.825	0.924	3.925
5					0.812	4.795	0.904	4.903
10							0.814	9.802

$P^* = .9$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.965	0.973	0.971	0.982	0.984	0.989	0.991	0.995
2	0.949	1.954	0.972	1.977	0.962	1.966	0.985	1.986
3	0.916	2.912	0.929	2.930	0.952	2.954	0.984	2.988
4			0.912	3.911	0.934	3.935	0.965	3.964
5					0.915	4.912	0.947	4.946
10							0.913	9.909

If $\{G_j(y)\}_{j=1}^{\infty}$ represents a sequence of absolutely continuous distribution functions that converge to a non-degenerate normal distribution, then $\{G_j(y)\}_{j=N}^{\infty}$, the tail of the sequence, is included in D. So, D contains those absolutely continuous distributions that are "close" to a Normal distribution. Additionally, we should notice that $\bar{D} = D$. This can be seen as follows. First, if $F_X \in \bar{D}$, then there exists a sequence $F_{Y_j} \in D$ so that $\lim_{j\to\infty} \sup_{y\in\mathbb{R}} |F_{Y_j} - F_X| = 0$. Since $F_{Y_j} \in D$, there exists a sequence of normal distributions $F_{Z_{j,k}}(y)$ so that $Z_{j,k} \sim N(\mu_{j,k}, \sigma_{j,k})$ and

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} |F_{Y_j} - F_{Z_{j,k}}| = \alpha_j \in [0, \beta]. \tag{5.52}$$

Because $\{\alpha_j\} \subset [0, \beta]$, there exists a subsequence, α_{j_r}, such that $\alpha_{j_r} \to \alpha \in [0, \beta]$. Therefore, we may create a sequence of distributions, $F_{Z_{j_r,k_r}}(y)$, where k_r is taken so that $\sup_{y\in\mathbb{R}} |F_{Y_j} - F_{Z_{j_r,k_r}}| \in (\alpha_{j_r} - {j_r}^{-1}, \alpha_{j_r} + {j_r}^{-1})$. Using the triangle inequality, we see that

$$\lim_{j_r\to\infty} \sup \sup_{y\in\mathbb{R}} |F_X - F_{Z_{j_r,k_r}}| \leq \sup_{y\in\mathbb{R}} |F_X - Y_{j_r}| + \sup_{y\in\mathbb{R}} |F_{Y_{j_r}} - F_{Z_{j_r,k_r}}| \tag{5.53}$$

$$\to \alpha \in [0, \beta]. \tag{5.54}$$

Therefore, there is a further subsequence of $\sup_{y\in\mathbb{R}} |F_X - F_{Z_{j_r,k_r}}|$, that converges to some value in $[0, \beta]$. Hence, $F_X \in D$.

In this situation, it will be quite useful that $\bar{D} = D$. It means that we can make our selections based up comparison with Normal distributions only, instead of all distributions in $\bar{D}$. While, still a large selection, at least it is limited in form. What is meant by this? Our selections will be based on

$$\tilde{\epsilon}_{i,n} = \inf_{(\mu,\sigma)\in\mathbb{R}\times\mathbb{R}^+} \sup_{y\in\mathbb{R}} |F_{i,n}(y) - F_{Z(\mu,\sigma)}(y)|, \tag{5.55}$$

instead of $\hat{\epsilon}_{i,n}$. We replace $\widehat{G}_n$ with $\widetilde{G}_n = \{\pi_i \mid \tilde{\epsilon}_{i,n} \leq \tau + \beta\}$. However, our lower bound for our probability of a correct selection will not change. This follows a similar argument given in the proof of Theorem 5.1. Namely, that for all $(\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^+$

$$\begin{aligned} \tilde{\epsilon}_{i,n} &\leq \sup_{y\in\mathbb{R}} |F_{i,n}(y) - F_{Z(\mu,\sigma)}(y)| && (5.56) \\ &\leq \sup_{y\in\mathbb{R}} |F_{i,n}(y) - F_i(y)| + \sup_{y\in\mathbb{R}} |F_i(y) - F_{Z(\mu,\sigma)}(y)|. && (5.57) \end{aligned}$$

Which implies that $\tilde{\epsilon}_{i,n} \leq \sup_{y\in\mathbb{R}} |F_{i,n}(y) - F_i(y)| + \beta$ for $F_i \in D$. Now, if $\sup_{y\in\mathbb{R}} |F_{i,n}(y) - F_i(y)| < \tau$, then $\tilde{\epsilon}_{i,n} \leq \tau + \beta$. Thus, if we alter our selection statistic from $\widehat{\epsilon}_{i,n}$ to $\tilde{\epsilon}_{i,n}$, we need only modify our selection rule to $\widehat{G}_n = \{\pi_i \mid \tilde{\epsilon}_{i,n} \leq \beta + \tau\}$. This allows us to maintain our desired probability of a correct selection, while making fewer comparisons. At this point, we should select the "undesirables"; these are determined in the same manner as before, with the added restriction that, once defined, ϵ^* must be greater than β. This must be done in order to allow us to control the expected subset size. If ϵ^* is decided to be less than β, the upper bound on the expected subset size will equal to k. While this is only an upper bound, it is not ideal. Defining U so that $\epsilon^* > \beta$ allows for τ to be selected so that $\tau \in (0, \epsilon^* - \beta)$. τ chosen in this manner can be used to control the expected subset size. To demonstrate this example in a more straightforward manner, we altered the definition of $\mathcal{F}$. Let $\mathcal{F} = \{F_X \mid X \sim N(0,1) \text{ or } X \sim t_r\,, r > 0\}$. In this case, $\beta = .01$. Thus, in this case, $\bar{D} = \{F_X \mid X \sim N(0,1) \text{ or } X \sim t_r\,, r \geq 15\}$. If $\epsilon^* = .1$, then $\bar{U} = \{F_X \mid X \sim t_r\,, 1.19 \geq r > 0\}$. In the definition, of both $\bar{D}$ and $\bar{U}$, the value of r is an approximation based on comparing a standard normal cumulative distribution with t-distributions with varying degrees of freedom. For the simulations found in Table 5.6, all populations in G were given t_{15} distributions, while all those in B were given $t_{1.19}$ distributions. Regardless, they were only compared with the standard normal cumulative distribution. In Table 5.6, we only consider $P^* = .8$. This was to allow us to make selections based upon two rules, $\tilde{\epsilon}_{i,n} \leq \tau$ and $\tilde{\epsilon}_{i,n} < \tau + \beta$. From a previous argument, we know that the second rule will maintain our desired probability of correct selection. (When using $\tilde{\epsilon}_{i,n} \leq \tau$ as a rule, it does appear that its usage is not detrimental to our cause. However, the distribution of $\tilde{\epsilon}_{i,n}$ is not known and so we don't know how $P(\tilde{\epsilon}_{i,n} \leq \tau)$ compares to $P(\widehat{\epsilon}_{i,n} \leq \tau)$.) This is not a problem when using $\tilde{\epsilon}_{i,n} < \tau + \beta$, since we know that $P(\tilde{\epsilon}_{i,n} \leq \tau) \geq P(\widehat{\epsilon}_{i,n} \leq \tau)$. Again, each configuration was iterated 1000 times.

Now, we look to a multivariate implementation. In the multivariate case, we consider two setups using multivariate Uniformly distributed populations. One setup is similar to that used for the simulations that produced 5.3. The other will be similar to the simulation of Table 5.3. Both will fall into our final example.

Table 5.6: $\bar{D} = \{N(0,1) \text{ or } r \geq 15\}$ vs $\bar{U} = \{t_r\,, 1.19 \geq r > 0\}$, $\epsilon^* = .1$

τ

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.924	0.924	0.946	0.946	0.946	0.946	0.974	0.974
2	0.875	1.869	0.900	1.897	0.908	1.906	0.965	1.965
3	0.798	2.779	0.819	2.807	0.850	2.841	0.931	2.928
4			0.796	3.774	0.836	3.829	0.912	3.908
5					0.784	4.769	0.883	4.876
10							0.772	9.735

$\tau + \beta$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.975	0.975	0.990	0.990	0.984	0.984	0.998	0.998
2	0.949	1.949	0.962	1.961	0.983	1.983	0.991	1.991
3	0.913	2.911	0.956	2.954	0.968	2.968	0.986	2.986
4			0.929	3.927	0.954	3.954	0.992	3.992
5					0.946	4.944	0.975	4.975
10							0.960	9.959

Example 5.5. For our final example, our desirable set will be of the form

$$\bar{D} = \{F_Y \mid Y \sim \text{Uniform}[0, r_1] \times [0, r_2],\ 0 < r_1 \le \alpha_1, 0 < r_2 \le \alpha_2\}$$

and the undesirable set will be

$$\bar{U} = \{F_Y \mid Y \sim \text{Uniform}[0, r_1] \times [0, r_2],\ r_1 \ge \beta_1 > \alpha_1 \text{ or } r_2 \ge \beta_2 > \alpha_2\}.$$

With a little patience, it can be shown that

$$\epsilon^* = \min\left\{1 - \frac{\alpha_1}{\beta_1}, 1 - \frac{\alpha_1\alpha_2}{\beta_1\beta_2}, 1 - \frac{\alpha_2}{\beta_2}\right\} \tag{5.58}$$

$$= \min\left\{1 - \frac{\alpha_1}{\beta_1}, 1 - \frac{\alpha_2}{\beta_2}\right\}. \tag{5.59}$$

Thus, letting $\alpha_1 = \alpha_2 = 1$, and $\beta_1 = \beta_2 = \frac{10}{9}$, then $\epsilon^* = .1$. This will be the case in our set of simulations. This required us to use the two-stage Procedure R_{SM}. In the first stage, we needed to determine the value of τ that would be used to discriminate in our selections. Additionally, we wanted to restrict our expected subset size so that $E(S) \le \#G + 1$. With both of these restrictions, we would be able to determine a common sample size, n, as well as τ. This was accomplished by simulating $KS_{D,n}$ for F_D, a Uniform $[0, 1] \times [0, 1]$ distribution, and for $n = 500, 600, \ldots, 1900$ where each sample distribution was based upon 5000 values of $KS_{D,n}$. Since an exact computation of $KS_{D,n}$ is not possible, as it is the supremum over all points $[0, 1]^2$, a grid of 6400 evenly spaced points in this range was used to search for the value of $KS_{D,n}$ for each iteration. With these simulated distributions, n and τ were determined for $P^* = .8, .9$. Using these values, the second stage was completed. In this stage, we took samples from $k = 3, 4, 5, 10$ populations where, as appropriate, $\#G = 1, 2, 3, 4, 5, 10$. To simplify the computations, all populations in G had distributions F_Y where F_Y has a Uniform$[0, 1]^2$. And, we restricted our "search" to distributions F_X where F_X has a Uniform$[0, r_1] \times [0, 1]$ where $r_1 = .90, .91, .92, \ldots, 1.00$. This was done to simulate the fact that we technically do not know the exact form of the populations in G, but significantly reduced the region that we would check. As for the populations in B, these all had Uniform $[0, \frac{10}{9}] \times [0, 1]$ distributions. In Table 5.7, we see the results of these simulations. As in the univariate case, we achieve results that are much higher than desired in all cases.

Our last set of simulations, whose results are shown in Table 5.8, modified our desirable set so that

$$\bar{D} = \{F_Y \mid Y \sim \text{Uniform}[0, 1] \times [0, 1]\},$$

while the undesirable set remained the same. In this case, our estimated probability of correct selection, $\hat{P}$, is much closer to the desired value of $P^* = .8, .9$ when $\#G = k$. It may seem that our estimates are a little low in some cases, but we should remember that our selection procedure is based on the approximate distribution of $KS_{D,n}$, and not the exact distribution. Thus, we should not expect the results to be as consistent

Table 5.7: $\bar{D} = \{\text{Uniform}[0, r_1] \times [0, r_2],\ 0 < r_1 \leq 1, 0 < r_2 \leq 1\}$ vs $\{\text{Uniform}[0, r_1] \times [0, r_2],\ r_1 \geq \frac{10}{9} \text{ or } r_2 \geq \frac{10}{9}\}$

$P^* = .8$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.957	0.958	0.965	0.965	0.981	0.981	0.980	0.980
2	0.920	1.919	0.931	1.929	0.954	1.953	0.966	1.964
3	0.871	2.863	0.925	2.923	0.933	2.933	0.970	2.970
4			0.875	3.871	0.899	3.894	0.946	3.944
5					0.898	4.887	0.946	4.944
10							0.872	9.864

$P^* = .9$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.981	0.982	0.988	0.988	0.987	0.987	0.996	0.996
2	0.973	1.972	0.973	1.973	0.978	1.978	0.990	1.990
3	0.933	2.932	0.955	2.955	0.966	2.966	0.978	2.978
4			0.940	3.940	0.958	3.957	0.974	3.974
5					0.935	4.933	0.970	4.970
10							0.960	9.956

as those given in the univariate case.

Table 5.8: $\bar{D} = \{\text{Uniform}[0,1]^2$, vs Uniform$[0,r_1] \times [0,r_2]$, $r_1 \geq \frac{10}{9}$ or $r_2 \geq \frac{10}{9}\}$

$P^* = .8$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.918	0.918	0.948	0.948	0.957	0.957	0.975	0.975
2	0.855	1.847	0.887	1.885	0.929	1.929	0.949	1.948
3	0.797	2.781	0.848	2.836	0.889	2.887	0.915	2.913
4			0.785	3.768	0.854	3.846	0.909	3.908
5					0.809	4.794	0.874	4.870
10							0.774	9.750

$P^* = .9$

	3		4		5		10	
#G	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$	$\hat{P}$	$\widehat{E}(S)$
1	0.955	0.955	0.963	0.963	0.967	0.967	0.991	0.991
2	0.927	1.926	0.944	1.943	0.952	1.952	0.982	1.982
3	0.882	2.871	0.931	2.929	0.944	2.945	0.981	2.981
4			0.887	3.883	0.926	3.923	0.958	3.958
5					0.897	4.893	0.964	4.962
10							0.916	9.913

5.6 Concluding Remarks

In this chapter, we presented procedures for selecting populations with certain distributional properties that can be applied to both univariate and multivariate populations. In the univariate case, an exact procedure was developed making use of the Kolmogorov-Smirnov Statistic. While in the multivariate case, an approximate procedure was developed. This approximation was necessary since the multivariate version of the Kolmogorov-Smirnov Statistic has a distribution that varies depending upon the populations being considered. This requires us to simulate its distribution.

Chapter 6

Tables & Code

This chapter includes tables that can be used for determining the necessary sample size for some of the procedures in Chapters 3 and 4. Some of the procedures presented in these chapters required determining an h value based on an equation of the form

$$\int_{-\infty}^{h} P\left(Z > \nu^{-1}(z-h)\right)^{k-1} \phi(z)dz = P^* \tag{6.1}$$

where $\nu = \frac{v^*}{v_*}$ or $\nu = \frac{v_{[k],n_1}}{v_{[1],n_1}}$. The following tables give an approximate value of h that satisfies integral equations of this form. With h determined, the sample size for each population could be found. The tables that follow can be used in several cases, namely when $k = 2, \ldots, 10$; for various values of P^*, as indicated indicated in the lefthand column. The ratio of the (estimated) derivatives, ν, is listed along the top row. Subsequent to the tables, MatLab code that will allow for the determination of h for other values of k, P^*, and ν is included. It is called **SolveForH1**. It works for all $k = 2, \ldots, 100$, $P^* \in (.6, .999)$ and $\nu \in [1, 100]$.

A separate code, **SolveForH2**, is included for determining h based upon an equation of the form

$$\int_{-\infty}^{h} \prod_{i=2}^{k} P\left(Z > \frac{a_1}{a_i}(z-h)\right) \phi(z) dz = P^* \tag{6.2}$$

where $a_i = v_{[i],n_1}$ or $a_i = v_{[i]}$.

Now, we present a few examples that will illustrate the determination of a sample size. The examples are for making selections based on dispersion, but could just as easily be for the situations described in Chapter 4.

Example 6.1. Suppose we are given 5 populations from which to select from that take values in $\mathbb{R}^3$. We wish to be 97.5% confident that we will select the population with the smallest .7-central region, by volume, whenever $\frac{V_{[2]}^{.7}}{V_{[1]}^{.7}} > 1.1$.

Without further information, we will use the two-stage ratio-based procedure R_{V4b}. Thus, first we take a sample of size $n_1 = 75$ from each population. With this, we compute $v_{i,75}$ and $\widehat{V}_{i,75}^{.7}$ for each of the five populations. Suppose $v_{i,75} = .7, .98, .8, .88, .91$ and $\widehat{V}_{i,75}^{.7} = 9, 3, 2, 54, .5$. Then $P^* = .95$, $\delta = 1.1$, $\widehat{V}_{[1],75}^{.7} = .5$, $p = .7$, $v_{[1],75} = .7$, and $v_{[5],75} = .98$. Now, we need to determine h as in (6.2). h will be determined by $\nu = .98/.7 = 3$. Referring to the third row on Page 172, we find $\nu = 1.4$ and $P^* = .975$. This corresponds to $h = 5.2504$. Using **SolveForH1**, we would use

$$h = \text{SolveForH1}(MinDeriv, MaxDeriv, k, PStar) \tag{6.3}$$

$$= \text{SolveForH1}(.7, .98, 5, .975) \tag{6.4}$$

$$= 4.2504. \tag{6.5}$$

Using **SolveForH2**, we would use

$$h = \text{SolveForH2}(Derivatives, PStar) \tag{6.6}$$

$$= \text{SolveForH2}([.7, .98, .8, .88, .91], .975) \tag{6.7}$$

$$= 4.0149. \tag{6.8}$$

Thus, we would need to take an additional sample of size $n_2 = n - 75$ from each population, where n is the larger of $n_1 = 75$ and

$$\left\lceil \left(\frac{hv_{[k],n_1}}{(\delta - 1)\widehat{V}^{p}_{[1],n_1}} \right)^2 (p(1-p)) \right\rceil = \left\lceil \left(\frac{4.2504(.98)}{(1.1-1).5} \right)^2 (.7(1-.7)) \right\rceil \tag{6.9}$$

$$= 1,457. \tag{6.10}$$

Using the value of h from SolveForH2, we would only need a sample of size 1,300 from each population. This is still quite large, but it is a dramatic savings.

6.0.1 Tables

Table 6.1: Values of h that satisfy (6.1)

$k = 2$

$P^*\backslash\nu$	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59
0.600	0.4571	0.4611	0.4652	0.4692	0.4732	0.4772	0.4812	0.4852	0.4892	0.4932
0.650	0.5949	0.5991	0.6034	0.6077	0.6119	0.6162	0.6204	0.6247	0.6289	0.6332
0.700	0.7403	0.7449	0.7494	0.7539	0.7585	0.7630	0.7675	0.7721	0.7766	0.7812
0.750	0.8976	0.9025	0.9073	0.9121	0.9170	0.9218	0.9267	0.9316	0.9365	0.9414
0.800	1.0732	1.0784	1.0836	1.0888	1.0940	1.0993	1.1045	1.1098	1.1151	1.1204
0.850	1.2784	1.2840	1.2897	1.2954	1.3011	1.3068	1.3126	1.3184	1.3242	1.3300
0.900	1.5374	1.5437	1.5500	1.5563	1.5627	1.5692	1.5756	1.5821	1.5887	1.5953
0.950	1.9234	1.9307	1.9380	1.9449	1.9530	1.9606	1.9682	1.9760	1.9838	1.9916
0.975	2.2600	2.2684	2.2768	2.2853	2.2940	2.3027	2.3115	2.3205	2.3295	2.3386
0.990	2.6539	2.6636	2.6734	2.6833	2.6935	2.7036	2.7140	2.7245	2.7354	2.7459
0.995	2.9235	2.9342	2.9450	2.9560	2.9678	2.9784	2.9899	3.0016	3.0135	3.0255

$P^*\backslash\nu$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69
0.600	0.4972	0.5011	0.5051	0.5090	0.5130	0.5169	0.5209	0.5248	0.5287	0.5326
0.650	0.6374	0.6417	0.6459	0.6501	0.6544	0.6586	0.6628	0.6670	0.6712	0.6754
0.700	0.7857	0.7903	0.7948	0.7993	0.8039	0.8085	0.8130	0.8176	0.8221	0.8266
0.750	0.9463	0.9512	0.9561	0.9610	0.9659	0.9709	0.9758	0.9807	0.9857	0.9906
0.800	1.1257	1.1311	1.1364	1.1418	1.1472	1.1526	1.1580	1.1634	1.1689	1.1743
0.850	1.3359	1.3418	1.3477	1.3537	1.3596	1.3656	1.3717	1.3777	1.3838	1.3898
0.900	1.6019	1.6086	1.6153	1.6221	1.6289	1.6357	1.6426	1.6495	1.6565	1.6635
0.950	1.9996	2.0076	2.0156	2.0238	2.0320	2.0403	2.0486	2.0571	2.0655	2.0741
0.975	2.3478	2.3572	2.3666	2.3761	2.3857	2.3955	2.4053	2.4152	2.4252	2.4364
0.990	2.7568	2.7679	2.7790	2.7903	2.8018	2.8134	2.8252	2.8370	2.8490	2.8612
0.995	3.0376	3.0500	3.0625	3.0751	3.0879	3.1019	3.1140	3.1274	3.1407	3.1544

$P^*\backslash\nu$	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79
0.600	0.5365	0.5404	0.5443	0.5481	0.5520	0.5558	0.5597	0.5635	0.5673	0.5712
0.650	0.6796	0.6838	0.6880	0.6922	0.6964	0.7006	0.7047	0.7089	0.7130	0.7172
0.700	0.8312	0.8357	0.8403	0.8448	0.8494	0.8539	0.8585	0.8630	0.8675	0.8721
0.750	0.9956	1.0006	1.0055	1.0105	1.0155	1.0205	1.0255	1.0304	1.0354	1.0404
0.800	1.1798	1.1852	1.1907	1.1962	1.2017	1.2073	1.2128	1.2183	1.2239	1.2295
0.850	1.3960	1.4021	1.4083	1.4144	1.4207	1.4269	1.4331	1.4394	1.4457	1.4520
0.900	1.6705	1.6776	1.6847	1.6918	1.6990	1.7063	1.7135	1.7209	1.7282	1.7356
0.950	2.0828	2.0914	2.1003	2.1090	2.1179	2.1269	2.1360	2.1451	2.1541	2.1635
0.975	2.4456	2.4560	2.4663	2.4768	2.4874	2.4981	2.5090	2.5199	2.5309	2.5420
0.990	2.8734	2.8859	2.8984	2.9110	2.9239	2.9368	2.9499	2.9631	2.9765	2.9899
0.995	3.1681	3.1821	3.1962	3.2104	3.2253	3.2394	3.2540	3.2689	3.2839	3.2991

$P^*\backslash\nu$	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
0.600	0.5750	0.5788	0.5825	0.5863	0.5901	0.5939	0.5976	0.6013	0.6051	0.6088
0.650	0.7213	0.7255	0.7296	0.7337	0.7379	0.7420	0.7461	0.7502	0.7543	0.7583
0.700	0.8766	0.8812	0.8857	0.8902	0.8947	0.8993	0.9038	0.9083	0.9128	0.9173
0.750	1.0454	1.0504	1.0554	1.0605	1.0655	1.0705	1.0755	1.0805	1.0856	1.0906
0.800	1.2350	1.2406	1.2462	1.2519	1.2575	1.2631	1.2688	1.2744	1.2801	1.2857
0.850	1.4584	1.4647	1.4711	1.4775	1.4839	1.4904	1.4968	1.5033	1.5098	1.5163
0.900	1.7430	1.7505	1.7580	1.7655	1.7731	1.7807	1.7883	1.7960	1.8038	1.8115
0.950	2.1728	2.1822	2.1917	2.2012	2.2107	2.2204	2.2301	2.2399	2.2497	2.2596
0.975	2.5532	2.5644	2.5758	2.5873	2.5989	2.6106	2.6223	2.6341	2.6461	2.6582
0.990	3.0035	3.0172	3.0310	3.0450	3.0591	3.0733	3.0877	3.1021	3.1167	3.1314
0.995	3.3144	3.3297	3.3453	3.3611	3.3769	3.3929	3.4091	3.4253	3.4417	3.4581

$P^*\backslash\nu$	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.600	0.6125	0.6162	0.6199	0.6236	0.6272	0.6309	0.6346	0.6382	0.6418	0.6455
0.650	0.7624	0.7665	0.7706	0.7746	0.7787	0.7827	0.7867	0.7908	0.7948	0.7988
0.700	0.9218	0.9263	0.9308	0.9353	0.9398	0.9443	0.9488	0.9533	0.9578	0.9623
0.750	1.0956	1.1007	1.1057	1.1107	1.1158	1.1208	1.1258	1.1309	1.1359	1.1410
0.800	1.2914	1.2971	1.3028	1.3085	1.3142	1.3200	1.3257	1.3315	1.3372	1.3430
0.850	1.5229	1.5294	1.5357	1.5426	1.5493	1.5559	1.5625	1.5693	1.5760	1.5827
0.900	1.8193	1.8272	1.8350	1.8429	1.8509	1.8589	1.8669	1.8749	1.8830	1.8923
0.950	2.2696	2.2797	2.2898	2.3000	2.3102	2.3205	2.3309	2.3413	2.3518	2.3624
0.975	2.6703	2.6825	2.6948	2.7072	2.7197	2.7323	2.7450	2.7577	2.7706	2.7835
0.990	3.1462	3.1611	3.1761	3.1913	3.2065	3.2219	3.2374	3.2530	3.2686	3.2844
0.995	3.4749	3.4917	3.5086	3.5256	3.5427	3.5600	3.5774	3.5949	3.6125	3.6302

$P^*\backslash\nu$	0.991	0.992	0.993	0.994	0.995	0.996	0.997	0.998	0.999	1.000
0.600	0.6458	0.6462	0.6465	0.6469	0.6473	0.6476	0.6480	0.6483	0.6487	0.6491
0.650	0.7992	0.7996	0.8000	0.8004	0.8008	0.8012	0.8016	0.8020	0.8024	0.8028
0.700	0.9627	0.9632	0.9636	0.9641	0.9645	0.9650	0.9654	0.9659	0.9663	0.9667
0.750	1.1415	1.1420	1.1425	1.1430	1.1435	1.1440	1.1445	1.1450	1.1455	1.1460
0.800	1.3436	1.3441	1.3447	1.3453	1.3459	1.3464	1.3470	1.3476	1.3482	1.3488
0.850	1.5834	1.5840	1.5847	1.5854	1.5861	1.5867	1.5874	1.5881	1.5888	1.5895
0.900	1.8919	1.8928	1.8936	1.8944	1.8952	1.8960	1.8968	1.8977	1.8985	1.8993
0.950	2.3634	2.3645	2.3656	2.3666	2.3677	2.3687	2.3698	2.3708	2.3719	2.3730
0.975	2.7848	2.7861	2.7874	2.7887	2.7900	2.7913	2.7926	2.7939	2.7952	2.7965
0.990	3.2860	3.3070	3.2892	3.2907	3.2923	3.2939	3.2952	3.2971	3.2987	3.3003
0.995	3.6320	3.6335	3.6355	3.6373	3.6392	3.6409	3.6427	3.6445	3.6463	3.6487

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Table 6.1: $k = 2$

$P^*\backslash\nu$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
0.600	0.6491	0.6846	0.7192	0.7528	0.7854	0.8172	0.8480	0.8781	0.9073	0.9359
0.650	0.8028	0.8425	0.8814	0.9195	0.9569	0.9936	1.0297	1.0650	1.0998	1.1340
0.700	0.9667	1.0113	1.0554	1.0991	1.1425	1.1853	1.2279	1.2700	1.3119	1.3535
0.750	1.1460	1.1966	1.2472	1.2979	1.3487	1.3995	1.4504	1.5015	1.5527	1.6041
0.800	1.3488	1.4069	1.4659	1.5257	1.5863	1.6476	1.7097	1.7727	1.8366	1.9013
0.850	1.5895	1.6579	1.7282	1.8004	1.8744	1.9502	2.0278	2.1073	2.1885	2.2715
0.900	1.8993	1.9828	2.0698	2.1602	2.2539	2.3507	2.4507	2.5534	2.6589	2.7669
0.950	2.3730	2.4826	2.5982	2.7193	2.8455	2.9762	3.1109	3.2490	3.3901	3.5335
0.975	2.7965	2.9311	3.0733	3.2221	3.3767	3.5361	3.6997	3.8665	4.0363	4.2084
0.990	3.3003	3.4645	3.6373	3.8173	4.0033	4.1943	4.3896	4.5884	4.7903	4.9948
0.995	3.6487	3.8320	4.0249	4.2252	4.4318	4.6438	4.8601	5.0804	5.3040	5.5306

$P^*\backslash\nu$	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
0.600	0.9637	0.9909	1.0174	1.0434	1.0689	1.0938	1.1183	1.1423	1.1659	1.1891
0.650	1.1677	1.2009	1.2336	1.2659	1.2979	1.3295	1.3608	1.3918	1.4225	1.4531
0.700	1.3949	1.4360	1.4771	1.5180	1.5588	1.5997	1.6405	1.6815	1.7225	1.7636
0.750	1.6558	1.7078	1.7602	1.8129	1.8661	1.9198	1.9740	2.0288	2.0841	2.1401
0.800	1.9671	2.0338	2.1016	2.1704	2.2403	2.3112	2.3831	2.4561	2.5300	2.6048
0.850	2.3563	2.4428	2.5309	2.6205	2.7116	2.8039	2.8974	2.9920	3.0875	3.1838
0.900	2.8770	2.9891	3.1030	3.2183	3.3349	3.4526	3.5713	3.6908	3.8109	3.9316
0.950	3.6791	3.8265	3.9753	4.1255	4.2767	4.4290	4.5821	4.7359	4.8905	5.0457
0.975	4.3827	4.5588	4.7365	4.9156	5.0959	5.2774	5.4599	5.6432	5.8274	6.0123
0.990	5.2019	5.4109	5.6219	5.8344	6.0485	6.2639	6.4805	6.6981	6.9168	7.1363
0.995	5.7598	5.9912	6.2248	6.4601	6.6977	6.9356	7.1757	7.4165	7.6585	7.9016

$P^*\backslash\nu$	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
0.600	1.2119	1.2344	1.2566	1.2785	1.3000	1.3214	1.3424	1.3633	1.3839	1.4043
0.650	1.4834	1.5136	1.5437	1.5737	1.6035	1.6333	1.6631	1.6928	1.7225	1.7523
0.700	1.8049	1.8464	1.8881	1.9301	1.9724	2.0149	2.0578	2.1011	2.1447	2.1886
0.750	2.1967	2.2539	2.3118	2.3703	2.4295	2.4893	2.5497	2.6106	2.6722	2.7342
0.800	2.6806	2.7571	2.8344	2.9124	2.9909	3.0701	3.1497	3.2297	3.3101	3.3909
0.850	3.2808	3.3783	3.4765	3.5750	3.6740	3.7733	3.8729	3.9727	4.0728	4.1730
0.900	4.0528	4.1745	4.2966	4.4191	4.5419	4.6649	4.7883	4.9119	5.0357	5.1597
0.950	5.2015	5.3578	5.5145	5.6718	5.8294	5.9873	6.1457	6.3043	6.4632	6.6225
0.975	6.1979	6.3842	6.5712	6.7584	6.9461	7.1343	7.3230	7.5120	7.7015	7.8912
0.990	7.3572	7.5776	7.7993	8.0217	8.2445	8.4680	8.6919	8.9161	9.1412	9.3662
0.995	8.1455	8.3903	8.6358	8.8818	9.1287	9.3755	9.6240	9.8726	10.1215	10.3711

$P^*\backslash\nu$	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
0.600	1.4245	1.4446	1.4645	1.4842	1.5038	1.5233	1.5427	1.5620	1.5812	1.6002
0.650	1.7821	1.8119	1.8418	1.8718	1.9019	1.9321	1.9625	1.9930	2.0236	2.0544
0.700	2.2330	2.2778	2.3229	2.3685	2.4144	2.4608	2.5075	2.5547	2.6022	2.6500
0.750	2.7968	2.8598	2.9232	2.9870	3.0511	3.1156	3.1803	3.2453	3.3105	3.3759
0.800	3.4719	3.5532	3.6346	3.7163	3.7981	3.8801	3.9622	4.0444	4.1267	4.2091
0.850	4.2734	4.3740	4.4747	4.5755	4.6766	4.7777	4.8790	4.9803	5.0817	5.1832
0.900	5.2840	5.4084	5.5330	5.6577	5.7826	5.9077	6.0328	6.1581	6.2835	6.4091
0.950	6.7819	6.9416	7.1015	7.2616	7.4219	7.5824	7.7431	7.9034	8.0648	8.2259
0.975	8.0811	8.2714	8.4619	8.6527	8.8437	9.0350	9.2264	9.4180	9.6098	9.8019
0.990	9.5919	9.8176	10.0440	10.2702	10.4969	10.7241	10.9511	11.1787	11.4064	11.6339
0.995	10.6203	10.8706	11.1207	11.3716	11.6228	11.8741	12.1255	12.3773	12.6293	12.8815

$P^*\backslash\nu$	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
0.600	1.6192	1.6381	1.6570	1.6758	1.6946	1.7133	1.7320	1.7507	1.7694	1.7880
0.650	2.0854	2.1166	2.1480	2.1796	2.2114	2.2434	2.2757	2.3082	2.3409	2.3739
0.700	2.6983	2.7468	2.7957	2.8448	2.8943	2.9440	2.9939	3.0441	3.0944	3.1450
0.750	3.4415	3.5072	3.5730	3.6390	3.7051	3.7712	3.8374	3.9037	3.9701	4.0365
0.800	4.2915	4.3741	4.4567	4.5393	4.6220	4.7048	4.7876	4.8705	4.9534	5.0364
0.850	5.2848	5.3865	5.4882	5.5900	5.6919	5.7938	5.8959	5.9979	6.1000	6.2022
0.900	6.5346	6.6604	6.7862	6.9121	7.0381	7.1641	7.2902	7.4164	7.5427	7.6690
0.950	8.3871	8.5485	8.7100	8.8716	9.0332	9.1950	9.3569	9.5188	9.6810	9.8431
0.975	9.9976	10.1861	10.3785	10.5712	10.7638	10.9564	11.1494	11.3424	11.5356	11.7287
0.990	11.8621	12.0903	12.3186	12.5474	12.7764	13.0046	13.2336	13.4627	13.6916	13.9213
0.995	13.1343	13.3868	13.6398	13.8966	14.1455	14.3994	14.6529	14.9062	15.1601	15.4142

$P^*\backslash\nu$	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9
0.600	1.8067	1.8253	1.8440	1.8627	1.8814	1.9001	1.9189	1.9376	1.9565	1.9753
0.650	2.4071	2.4405	2.4742	2.5081	2.5423	2.5767	2.6113	2.6461	2.6812	2.7165
0.700	3.1957	3.2466	3.2976	3.3487	3.3999	3.4513	3.5027	3.5542	3.6058	3.6574
0.750	4.1030	4.1695	4.2360	4.3026	4.3692	4.4358	4.5025	4.5692	4.6359	4.7026
0.800	5.1194	5.2024	5.2855	5.3686	5.4517	5.5349	5.6181	5.7013	5.7846	5.8678
0.850	6.3044	6.4066	6.5090	6.6113	6.7137	6.8161	6.9185	7.0210	7.1235	7.2261
0.900	7.7956	7.9218	8.0480	8.1737	8.3015	8.4281	8.5548	8.6815	8.8083	8.9351
0.950	10.0053	10.1676	10.3304	10.4923	10.6548	10.8173	10.9799	11.1426	11.3053	11.4681
0.975	11.9220	12.1143	12.3088	12.5024	12.6960	12.8897	13.0834	13.2772	13.4711	13.6650
0.990	14.1506	14.3796	14.6099	14.8394	15.0692	15.2993	15.5280	15.7592	15.9894	16.2194
0.995	15.6693	15.9222	16.1768	16.4307	16.6852	16.9401	17.1947	17.4493	17.7028	17.9590

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Table 6.1: $k = 2$

$P^*\backslash\nu$	7.0	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
0.600	1.9943	2.0133	2.0323	2.0514	2.0706	2.0899	2.1092	2.1286	2.1481	2.1677
0.650	2.7519	2.7876	2.8235	2.8595	2.8958	2.9322	2.9687	3.0054	3.0422	3.0790
0.700	3.7091	3.7608	3.8128	3.8645	3.9163	3.9682	4.0201	4.0720	4.1240	4.1760
0.750	4.7694	4.8362	4.9030	4.9698	5.0366	5.1034	5.1703	5.2372	5.3040	5.3710
0.800	5.9512	6.0345	6.1178	6.2012	6.2846	6.3680	6.4514	6.5349	6.6184	6.7019
0.850	7.3287	7.4313	7.5340	7.6366	7.7393	7.8420	7.9448	8.0477	8.1504	8.2532
0.900	9.0619	9.1888	9.3157	9.4427	9.5697	9.6967	9.8237	9.9508	10.0779	10.2050
0.950	11.6309	11.7937	11.9566	12.1189	12.2825	12.4455	12.6086	12.7718	12.9349	13.0980
0.975	13.8591	14.0531	14.2472	14.4410	14.6356	14.8298	15.0241	15.2183	15.4127	15.6074
0.990	16.4499	16.6802	16.9104	17.1410	17.3715	17.6019	17.8327	18.0632	18.2941	18.5248
0.995	18.2140	18.4688	18.7240	18.9788	19.2350	19.4893	19.7452	20.0005	20.2561	20.5110

$P^*\backslash\nu$	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9
0.600	2.1874	2.2072	2.2271	2.2471	2.2672	2.2874	2.3077	2.3281	2.3488	2.3693
0.650	3.1162	3.1534	3.1907	3.2281	3.2656	3.3032	3.3408	3.3786	3.4164	3.4542
0.700	4.2280	4.2800	4.3320	4.3841	4.4361	4.4881	4.5403	4.5923	4.6444	4.6966
0.750	5.4379	5.5049	5.5718	5.6388	5.7057	5.7728	5.8397	5.9067	5.9737	6.0407
0.800	6.7854	6.8689	6.9524	7.0360	7.1195	7.2031	7.2867	7.3703	7.4540	7.5376
0.850	8.3560	8.4588	8.5617	8.6646	8.7675	8.8704	8.9734	9.0763	9.1793	9.2823
0.900	10.3322	10.4594	10.5866	10.7138	10.8411	10.9683	11.0956	11.2229	11.3502	11.4776
0.950	13.2613	13.4245	13.5876	13.7510	13.9144	14.0777	14.2411	14.4045	14.5679	14.7313
0.975	15.8017	15.9962	16.1907	16.3854	16.5800	16.7804	16.9692	17.1639	17.3587	17.5535
0.990	18.7556	18.9864	19.2173	19.4497	19.6797	19.9102	20.1412	20.3729	20.6035	20.8347
0.995	20.7671	21.0282	21.2775	21.5336	21.7899	22.0455	22.3011	22.5571	22.8013	23.0692

$P^*\backslash\nu$	9.0	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
0.600	2.3901	2.4110	2.4320	2.4531	2.4744	2.4958	2.5173	2.5389	2.5607	2.5825
0.650	3.4921	3.5300	3.5680	3.6061	3.6441	3.6822	3.7204	3.7585	3.7967	3.8349
0.700	4.7487	4.8008	4.8529	4.9051	4.9572	5.0093	5.0615	5.1136	5.1658	5.2180
0.750	6.1078	6.1748	6.2419	6.3089	6.3760	6.4431	6.5101	6.5772	6.6441	6.7114
0.800	7.6212	7.7048	7.7885	7.8722	7.9559	8.0396	8.1233	8.2070	8.2907	8.3744
0.850	9.3849	9.4883	9.5914	9.6944	9.7974	9.9005	10.0036	10.1067	10.2098	10.3129
0.900	11.6050	11.7323	11.8597	11.9871	12.1146	12.2418	12.3694	12.4969	12.6243	12.7520
0.950	14.8948	15.0583	15.2218	15.3857	15.5489	15.7125	15.8761	16.0396	16.2033	16.3669
0.975	17.7482	17.9430	18.1379	18.3327	18.5280	18.7225	18.9174	19.1124	19.3140	19.5024
0.990	21.0661	21.2974	21.5289	21.7600	21.9907	22.2223	22.4536	22.6851	22.9168	23.1479
0.995	23.3250	23.5813	23.8374	24.0931	24.3492	24.6058	24.8619	25.1177	25.3741	25.6309

$P^*\backslash\nu$	10	15	20	25	30	35	40	45	50	55
0.600	2.6045	3.8100	5.0741	6.3388	7.6046	8.8701	10.1371	11.4034	12.6699	13.9364
0.650	3.8731	5.7927	7.7160	9.6407	11.5661	13.4916	15.4177	17.3437	19.2699	21.1961
0.700	5.2701	7.8835	10.5011	13.1204	15.7408	18.3614	20.9822	23.6039	26.2408	28.8467
0.750	6.7786	10.1398	13.5067	16.8757	20.2459	23.6168	26.9880	30.3595	33.7315	37.1030
0.800	8.4582	12.6523	16.8537	21.0564	25.2624	29.4695	33.6754	37.8823	42.0896	46.2969
0.850	10.4160	15.5810	20.7546	25.9316	31.1103	36.2899	41.4702	46.6511	51.8321	57.0132
0.900	12.8794	19.2660	25.6631	32.0645	38.4678	44.8730	51.2781	57.6841	64.0903	70.4971
0.950	16.5306	24.7276	32.9381	41.1542	49.3730	57.5933	65.8148	74.0364	82.2592	90.4816
0.975	19.6973	29.4641	39.2480	49.0384	58.8310	68.6275	78.4226	88.2197	98.0175	107.8161
0.990	23.3795	34.9728	46.5850	58.2055	69.8291	81.4581	93.0814	104.7110	116.3251	127.9698
0.995	25.8866	38.7256	51.5808	64.4476	77.3155	90.1895	103.0646	115.9414	128.8190	141.6962

$P^*\backslash\nu$	60	65	70	75	80	85	90	95	100
0.600	15.2026	16.4695	17.7360	19.0027	20.2693	21.5361	22.8027	24.0694	25.3359
0.650	23.1224	25.0488	26.9750	28.9016	30.8281	32.7543	34.6812	36.6076	38.5337
0.700	31.4684	34.0901	36.7117	39.3335	41.9551	44.5769	47.1994	49.8209	52.4426
0.750	40.4751	43.8457	47.2190	50.5920	53.9634	57.3356	60.7083	64.0805	67.4525
0.800	50.5042	54.7119	58.9196	63.1272	67.3349	71.5430	75.7489	79.9584	84.1673
0.850	62.1947	67.3761	72.5577	77.7391	82.9213	88.1030	93.2849	98.4667	103.6487
0.900	76.9037	83.3157	89.7179	96.1365	102.5319	108.9386	115.3466	121.7505	128.1617
0.950	98.7046	106.9283	115.1511	123.3751	131.5985	139.8221	148.0463	156.2700	164.4938
0.975	117.5769	127.4120	137.2117	147.0097	156.8092	166.6081	176.4079	186.2064	196.0067
0.990	139.6016	151.2282	162.8582	174.4909	186.1227	197.7529	209.3838	220.7813	232.6424
0.995	154.5757	167.4497	180.3249	193.2050	206.0812	218.9614	231.8462	244.7461	257.5973

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Table 6.1: $k = 3$

$P^*\backslash\nu$	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59
0.600	0.6025	0.6096	0.6166	0.6237	0.6307	0.6378	0.6448	0.6519	0.6589	0.6660
0.650	0.7420	0.7494	0.7568	0.7641	0.7715	0.7789	0.7863	0.7937	0.8010	0.8084
0.700	0.8893	0.8970	0.9047	0.9124	0.9201	0.9279	0.9356	0.9434	0.9511	0.9589
0.750	1.0484	1.0565	1.0646	1.0727	1.0808	1.0890	1.0971	1.1053	1.1135	1.1217
0.800	1.2260	1.2345	1.2430	1.2515	1.2601	1.2687	1.2774	1.2861	1.2947	1.3035
0.850	1.4333	1.4423	1.4514	1.4605	1.4696	1.4788	1.4880	1.4973	1.5066	1.5160
0.900	1.6947	1.7045	1.7143	1.7235	1.7340	1.7439	1.7540	1.7639	1.7741	1.7843
0.950	2.0835	2.0944	2.1053	2.1163	2.1273	2.1385	2.1498	2.1611	2.1725	2.1839
0.975	2.4219	2.4338	2.4458	2.4578	2.4700	2.4822	2.4946	2.5071	2.5196	2.5323
0.990	2.8166	2.8298	2.8428	2.8564	2.8699	2.8835	2.8973	2.9112	2.9252	2.9394
0.995	3.0862	3.1000	3.1143	3.1287	3.1431	3.1577	3.1725	3.1875	3.2023	3.2176

$P^*\backslash\nu$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69
0.600	0.6730	0.6801	0.6871	0.6941	0.7012	0.7082	0.7152	0.7223	0.7293	0.7363
0.650	0.8159	0.8232	0.8307	0.8381	0.8455	0.8529	0.8603	0.8677	0.8752	0.8826
0.700	0.9667	0.9745	0.9823	0.9902	0.9980	1.0058	1.0137	1.0215	1.0294	1.0373
0.750	1.1299	1.1382	1.1465	1.1547	1.1630	1.1714	1.1797	1.1881	1.1964	1.2048
0.800	1.3122	1.3210	1.3298	1.3386	1.3475	1.3563	1.3652	1.3742	1.3831	1.3921
0.850	1.5253	1.5347	1.5442	1.5537	1.5632	1.5728	1.5824	1.5920	1.6017	1.6114
0.900	1.7945	1.8048	1.8151	1.8255	1.8359	1.8464	1.8569	1.8675	1.8782	1.8888
0.950	2.1955	2.2071	2.2188	2.2306	2.2425	2.2544	2.2664	2.2785	2.2906	2.3029
0.975	2.5451	2.5580	2.5710	2.5841	2.5972	2.6100	2.6239	2.6374	2.6509	2.6646
0.990	2.9537	2.9681	2.9826	2.9973	3.0121	3.0270	3.0421	3.0572	3.0725	3.0879
0.995	3.2330	3.2485	3.2642	3.2800	3.2959	3.3119	3.3282	3.3444	3.3609	3.3776

$P^*\backslash\nu$	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79
0.600	0.7433	0.7504	0.7574	0.7644	0.7714	0.7784	0.7854	0.7924	0.7994	0.8064
0.650	0.8900	0.8975	0.9049	0.9124	0.9198	0.9272	0.9347	0.9421	0.9496	0.9570
0.700	1.0452	1.0531	1.0611	1.0689	1.0768	1.0847	1.0927	1.1006	1.1084	1.1165
0.750	1.2132	1.2216	1.2301	1.2385	1.2470	1.2555	1.2640	1.2725	1.2810	1.2895
0.800	1.4011	1.4101	1.4192	1.4283	1.4374	1.4465	1.4557	1.4649	1.4741	1.4833
0.850	1.6212	1.6310	1.6407	1.6506	1.6605	1.6705	1.6804	1.6904	1.7004	1.7105
0.900	1.8996	1.9104	1.9212	1.9321	1.9430	1.9540	1.9651	1.9762	1.9873	1.9985
0.950	2.3152	2.3276	2.3401	2.3526	2.3652	2.3779	2.3906	2.4034	2.4163	2.4293
0.975	2.6784	2.6922	2.7062	2.7202	2.7343	2.7486	2.7629	2.7773	2.7918	2.8064
0.990	3.1034	3.1190	3.1347	3.1507	3.1666	3.1827	3.1989	3.2152	3.2317	3.2482
0.995	3.3943	3.4112	3.4282	3.4453	3.4626	3.4801	3.4975	3.5152	3.5330	3.5508

$P^*\backslash\nu$	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
0.600	0.8134	0.8204	0.8274	0.8343	0.8413	0.8483	0.8553	0.8622	0.8692	0.8762
0.650	0.9645	0.9719	0.9794	0.9869	0.9943	1.0018	1.0092	1.0167	1.0242	1.0317
0.700	1.1245	1.1325	1.1405	1.1485	1.1562	1.1645	1.1725	1.1805	1.1885	1.1966
0.750	1.2981	1.3067	1.3153	1.3239	1.3325	1.3411	1.3498	1.3584	1.3671	1.3758
0.800	1.4925	1.5018	1.5111	1.5204	1.5298	1.5391	1.5485	1.5579	1.5674	1.5768
0.850	1.7206	1.7307	1.7409	1.7511	1.7614	1.7716	1.7819	1.7923	1.8026	1.8130
0.900	2.0097	2.0210	2.0324	2.0437	2.0552	2.0666	2.0782	2.0897	2.1013	2.1130
0.950	2.4424	2.4554	2.4686	2.4818	2.4952	2.5085	2.5220	2.5355	2.5490	2.5627
0.975	2.8210	2.8358	2.8506	2.8656	2.8806	2.8957	2.9109	2.9262	2.9415	2.9569
0.990	3.2648	3.2816	3.2984	3.3154	3.3325	3.3496	3.3670	3.3843	3.4017	3.4195
0.995	3.5688	3.5870	3.6047	3.6236	3.6421	3.6607	3.6794	3.6981	3.7172	3.7362

$P^*\backslash\nu$	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.600	0.8831	0.8901	0.8970	0.9040	0.9110	0.9179	0.9248	0.9317	0.9387	0.9456
0.650	1.0391	1.0466	1.0541	1.0616	1.0690	1.0765	1.0840	1.0915	1.0990	1.1065
0.700	1.2046	1.2127	1.2207	1.2288	1.2369	1.2450	1.2531	1.2612	1.2693	1.2774
0.750	1.3845	1.3932	1.4019	1.4107	1.4195	1.4282	1.4370	1.4458	1.4546	1.4635
0.800	1.5863	1.5958	1.6053	1.6149	1.6244	1.6340	1.6436	1.6533	1.6629	1.6726
0.850	1.8235	1.8340	1.8444	1.8550	1.8655	1.8761	1.8868	1.8974	1.9081	1.9188
0.900	2.1247	2.1365	2.1483	2.1601	2.1720	2.1839	2.1959	2.2079	2.2200	2.2321
0.950	2.5764	2.5901	2.6040	2.6179	2.6318	2.6458	2.6599	2.6741	2.6883	2.7026
0.975	2.9724	2.9882	3.0037	3.0194	3.0351	3.0511	3.0671	3.0831	3.0993	3.1155
0.990	3.4370	3.4547	3.4726	3.4905	3.5085	3.5268	3.5449	3.5632	3.5815	3.6000
0.995	3.7554	3.7745	3.7934	3.8134	3.8330	3.8526	3.8724	3.8923	3.9121	3.9322

$P^*\backslash\nu$	0.991	0.992	0.993	0.994	0.995	0.996	0.997	0.998	0.999	1.000
0.600	0.9463	0.9470	0.9477	0.9484	0.9491	0.9498	0.9504	0.9511	0.9518	0.9525
0.650	1.1072	1.1080	1.1087	1.1094	1.1102	1.1109	1.1117	1.1124	1.1132	1.1139
0.700	1.2782	1.2790	1.2798	1.2806	1.2814	1.2823	1.2831	1.2839	1.2847	1.2855
0.750	1.4644	1.4652	1.4661	1.4670	1.4679	1.4688	1.4697	1.4706	1.4714	1.4723
0.800	1.6736	1.6746	1.6756	1.6765	1.6775	1.6784	1.6794	1.6804	1.6813	1.6823
0.850	1.9199	1.9210	1.9220	1.9231	1.9242	1.9253	1.9263	1.9274	1.9285	1.9296
0.900	2.2333	2.2345	2.2357	2.2369	2.2381	2.2394	2.2406	2.2419	2.2430	2.2442
0.950	2.7039	2.7054	2.7068	2.7083	2.7097	2.7111	2.7125	2.7140	2.7154	2.7168
0.975	3.1171	3.1187	3.1203	3.1220	3.1236	3.1252	3.1268	3.1285	3.1301	3.1317
0.990	3.6067	3.6038	3.6056	3.6074	3.6093	3.6111	3.6130	3.6149	3.6167	3.6185
0.995	3.9342	3.9356	3.9383	3.9403	3.9422	3.9444	3.9463	3.9482	3.9503	3.9520

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Table 6.1: $k = 3$

$P^*\backslash\nu$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
0.600	0.9525	1.0216	1.0904	1.1588	1.2271	1.2953	1.3633	1.4314	1.4994	1.5676
0.650	1.1139	1.1889	1.2640	1.3394	1.4149	1.4908	1.5669	1.6434	1.7203	1.7977
0.700	1.2855	1.3671	1.4494	1.5324	1.6162	1.7007	1.7859	1.8719	1.9588	2.0464
0.750	1.4723	1.5616	1.6520	1.7439	1.8370	1.9314	2.0270	2.1237	2.2217	2.3207
0.800	1.6823	1.7805	1.8807	1.9829	2.0869	2.1928	2.3003	2.4095	2.5200	2.6319
0.850	1.9296	2.0388	2.1508	2.2656	2.3830	2.5026	2.6244	2.7480	2.8734	3.0003
0.900	2.2442	2.3680	2.4956	2.6268	2.7612	2.8985	3.0383	3.1803	3.3243	3.4700
0.950	2.7168	2.8631	3.0145	3.1706	3.3306	3.4942	3.6608	3.8300	4.0007	4.1752
0.975	3.1317	3.2980	3.4705	3.6483	3.8307	4.0172	4.2071	4.4003	4.5960	4.7942
0.990	3.6185	3.8084	4.0056	4.2093	4.4177	4.6313	4.8489	5.0701	5.2946	5.5220
0.995	3.9520	4.1585	4.3724	4.5936	4.8206	5.0527	5.2894	5.5303	5.7745	6.0219

$P^*\backslash\nu$	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
0.600	1.6359	1.7043	1.7731	1.8420	1.9113	1.9809	2.0509	2.1212	2.1918	2.2628
0.650	1.8755	1.9538	2.0327	2.1120	2.1919	2.2722	2.3531	2.4344	2.5161	2.5983
0.700	2.1347	2.2239	2.3137	2.4043	2.4955	2.5872	2.6796	2.7724	2.8657	2.9595
0.750	2.4207	2.5217	2.6235	2.7262	2.8296	2.9335	3.0381	3.1432	3.2487	3.3546
0.800	2.7450	2.8592	2.9743	3.0903	3.2070	3.3244	3.4423	3.5608	3.6797	3.7990
0.850	3.1286	3.2580	3.3884	3.5198	3.6519	3.7847	3.9182	4.0522	4.1867	4.3217
0.900	3.6172	3.7656	3.9152	4.0659	4.2174	4.3699	4.5229	4.6766	4.8310	4.9859
0.950	4.3505	4.5275	4.7058	4.8854	5.0662	5.2480	5.4306	5.6142	5.7985	5.9834
0.975	4.9944	5.1965	5.4003	5.6055	5.8122	6.0199	6.2289	6.4388	6.6496	6.8612
0.990	5.7515	5.9834	6.2174	6.4531	6.6904	6.9291	7.1690	7.4101	7.6521	7.8955
0.995	6.2720	6.5246	6.7794	7.0362	7.2948	7.5546	7.8160	8.0785	8.3450	8.6076

$P^*\backslash\nu$	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
0.600	2.3341	2.4058	2.4778	2.5501	2.6227	2.6955	2.7686	2.8419	2.9153	2.9890
0.650	2.6808	2.7637	2.8469	2.9304	3.0141	3.0980	3.1822	3.2665	3.3510	3.4356
0.700	3.0535	3.1479	3.2426	3.3375	3.4326	3.5279	3.6234	3.7191	3.8149	3.9108
0.750	3.4608	3.5674	3.6742	3.7813	3.8886	3.9961	4.1037	4.2115	4.3195	4.4276
0.800	3.9187	4.0387	4.1590	4.2796	4.4004	4.5214	4.6427	4.7634	4.8857	5.0075
0.850	4.4571	4.5929	4.7290	4.8654	5.0021	5.1391	5.2763	5.4138	5.5515	5.6894
0.900	5.1412	5.2971	5.4534	5.6100	5.7670	5.9244	6.0820	6.2399	6.3981	6.5566
0.950	6.1690	6.3552	6.5419	6.7291	6.9170	7.1049	7.2933	7.4821	7.6712	7.8607
0.975	7.0735	7.2866	7.5002	7.7144	7.9292	8.1446	8.3602	8.5764	8.7928	9.0097
0.990	8.1396	8.3844	8.6299	8.8761	9.1231	9.3704	9.6185	9.8669	10.1158	10.3649
0.995	8.8734	9.1402	9.4078	9.6762	9.9452	10.2148	10.4865	10.7557	11.0271	11.2987

$P^*\backslash\nu$	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
0.600	3.0628	3.1368	3.2108	3.2850	3.3593	3.4337	3.5081	3.5826	3.6571	3.7318
0.650	3.5203	3.6051	3.6900	3.7750	3.8601	3.9452	4.0302	4.1157	4.2010	4.2863
0.700	4.0068	4.1030	4.1992	4.2955	4.3918	4.4883	4.5848	4.6814	4.7781	4.8748
0.750	4.5358	4.6441	4.7526	4.8612	4.9698	5.0786	5.1874	5.2962	5.4053	5.5143
0.800	5.1294	5.2517	5.3737	5.4960	5.6184	5.7410	5.8637	5.9864	6.1092	6.2321
0.850	5.8274	5.9657	6.1040	6.2426	6.3813	6.5201	6.6590	6.7981	6.9373	7.0765
0.900	6.7151	6.8740	7.0331	7.1923	7.3517	7.5113	7.6710	7.8308	7.9908	8.1509
0.950	8.0504	8.2404	8.4306	8.6210	8.8117	9.0026	9.1939	9.3848	9.5765	9.7679
0.975	9.2269	9.4443	9.6622	9.8802	10.0984	10.3171	10.5358	10.7548	10.9741	11.1934
0.990	10.6146	10.8648	11.1155	11.3658	11.6169	11.8682	12.1193	12.3715	12.6235	12.8757
0.995	11.5709	11.8433	12.1162	12.3894	12.6633	12.9370	13.2110	13.4864	13.7566	14.0350

$P^*\backslash\nu$	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
0.600	3.8064	3.8811	3.9558	4.0306	4.1052	4.1803	4.2550	4.3298	4.4047	4.4796
0.650	4.3717	4.4572	4.5426	4.6281	4.7137	4.7992	4.8848	4.9704	5.0561	5.1418
0.700	4.9715	5.0683	5.1652	5.2621	5.3590	5.4560	5.5530	5.6501	5.7471	5.8443
0.750	5.6234	5.7326	5.8419	5.9511	6.0605	6.1699	6.2793	6.3888	6.4983	6.6079
0.800	6.3551	6.4782	6.6014	6.7246	6.8479	6.9712	7.0946	7.2181	7.3416	7.4652
0.850	7.2159	7.3554	7.4949	7.6346	7.7743	7.9141	8.0540	8.1939	8.3338	8.4739
0.900	8.3112	8.4716	8.6320	8.7926	8.9532	9.1140	9.2748	9.4357	9.5967	9.7578
0.950	9.9597	10.1516	10.3435	10.5357	10.7279	10.9202	11.1127	11.3053	11.4980	11.6907
0.975	11.4148	11.6326	11.8526	12.0726	12.2927	12.5130	12.7334	12.9538	13.1745	13.3953
0.990	13.1281	13.3808	13.6336	13.8866	14.1398	14.3930	14.6460	14.9000	15.1538	15.4076
0.995	14.3106	14.5857	14.8612	15.1369	15.4127	15.6889	15.9647	16.2419	16.5178	16.7945

$P^*\backslash\nu$	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9
0.600	4.5545	4.6294	4.7044	4.7793	4.8543	4.9293	5.0043	5.0793	5.1543	5.2293
0.650	5.2275	5.3132	5.3989	5.4847	5.5705	5.6563	5.7421	5.8279	5.9138	5.9996
0.700	5.9414	6.0385	6.1357	6.2330	6.3303	6.4281	6.5249	6.6222	6.7196	6.8169
0.750	6.7175	6.8271	6.9368	7.0465	7.1563	7.2661	7.3759	7.4857	7.5955	7.7055
0.800	7.5888	7.7125	7.8361	7.9599	8.0837	8.2075	8.3316	8.4552	8.5792	8.7031
0.850	8.6140	8.7541	8.8944	9.0346	9.1749	9.3153	9.4557	9.5962	9.7367	9.8772
0.900	9.9189	10.0801	10.2414	10.4027	10.5641	10.7255	10.8870	11.0486	11.2102	11.3717
0.950	11.8835	12.0765	12.2695	12.4626	12.6557	12.8490	13.0423	13.2356	13.4290	13.6225
0.975	13.6162	13.8372	14.0582	14.2794	14.5005	14.7220	14.9432	15.1647	15.3862	15.6077
0.990	15.6616	15.9157	16.1700	16.4242	16.6785	16.9333	17.1875	17.4422	17.6970	17.9517
0.995	17.0713	17.3482	17.6252	17.9114	18.1796	18.4572	18.7347	19.0115	19.2898	19.5675

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Table 6.1: $k = 3$

$P^*\backslash\nu$	7.0	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
0.600	5.3044	5.3794	5.4545	5.5296	5.6046	5.6797	5.7548	5.8299	5.9051	5.9801
0.650	6.0855	6.1714	6.2573	6.3432	6.4292	6.5151	6.6011	6.6871	6.7731	6.8590
0.700	6.9143	7.0117	7.1092	7.2066	7.3041	7.4015	7.4990	7.5965	7.6941	7.7916
0.750	7.8154	7.9253	8.0353	8.1453	8.2553	8.3653	8.4753	8.5854	8.6955	8.8056
0.800	8.8271	8.9511	9.0752	9.1992	9.3233	9.4474	9.5716	9.6957	9.8199	9.9441
0.850	10.0177	10.1583	10.2991	10.4396	10.5803	10.7210	10.8618	11.0026	11.1434	11.2841
0.900	11.5335	11.6952	11.8569	12.0186	12.1806	12.3425	12.5044	12.6663	12.8283	12.9903
0.950	13.8160	14.0096	14.2032	14.3969	14.5906	14.7843	14.9782	15.1720	15.3660	15.5599
0.975	15.8294	16.0508	16.2731	16.4947	16.7167	16.9386	17.1606	17.3826	17.6046	17.8267
0.990	18.2065	18.4628	18.7165	18.9715	19.2270	19.4820	19.7375	19.9930	20.2482	20.5063
0.995	19.8456	20.1230	20.4024	20.6792	20.9572	21.2356	21.5136	21.7921	22.0702	22.3485

$P^*\backslash\nu$	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9
0.600	6.0553	6.1304	6.2056	6.2807	6.3558	6.4310	6.5062	6.5813	6.6565	6.7317
0.650	6.9450	7.0310	7.1171	7.2031	7.2891	7.3752	7.4612	7.5473	7.6334	7.7195
0.700	7.8892	7.9867	8.0843	8.1819	8.2795	8.3771	8.4747	8.5724	8.6700	8.7677
0.750	8.9157	9.0258	9.1360	9.2461	9.3563	9.4665	9.5767	9.6865	9.7971	9.9074
0.800	10.0685	10.1926	10.3169	10.4412	10.5655	10.6898	10.8141	10.9385	11.0628	11.1872
0.850	11.4250	11.5659	11.7068	11.8477	11.9887	12.1296	12.2706	12.4117	12.5527	12.6937
0.900	13.1523	13.3144	13.4765	13.6386	13.8008	13.9629	14.1248	14.2873	14.4495	14.6115
0.950	15.7541	15.9479	16.1419	16.3360	16.5302	16.7242	16.9184	17.1126	17.3069	17.5011
0.975	18.0488	18.2711	18.4935	18.7158	18.9380	19.1605	19.3829	19.6053	19.8278	20.0501
0.990	20.7591	21.0145	21.2702	21.5273	21.7813	22.0371	22.2929	22.5490	22.8050	23.0605
0.995	22.6272	22.9057	23.1842	23.4632	23.7424	24.0205	24.2996	24.5782	24.8570	25.1356

$P^*\backslash\nu$	9.0	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
0.600	6.8069	6.8821	6.9572	7.0324	7.1077	7.1828	7.2581	7.3333	7.4085	7.4838
0.650	7.8056	7.8916	7.9777	8.0639	8.1500	8.2361	8.3222	8.4083	8.4945	8.5806
0.700	8.8653	8.9630	9.0607	9.1583	9.2560	9.3538	9.4507	9.5492	9.6469	9.7446
0.750	10.0176	10.1279	10.2382	10.3485	10.4587	10.5691	10.6794	10.7897	10.9000	11.0104
0.800	11.3116	11.4360	11.5605	11.6849	11.8094	11.9339	12.0583	12.1828	12.3077	12.4318
0.850	12.8348	12.9758	13.1169	13.2580	13.3992	13.5403	13.6810	13.8227	13.9637	14.1050
0.900	14.7739	14.9364	15.0988	15.2611	15.4235	15.5858	15.7482	15.9106	16.0731	16.2356
0.950	17.6954	17.8897	18.0840	18.2784	18.4727	18.6672	18.8616	19.0560	19.2505	19.4454
0.975	20.2728	20.4953	20.7172	20.9406	21.1632	21.3860	21.6086	21.8311	22.0540	22.2769
0.990	23.3165	23.5723	23.8282	24.0843	24.3404	24.5965	24.8525	25.1083	25.3650	25.6213
0.995	25.4143	25.6935	25.9726	26.2516	26.5316	26.8098	27.0889	27.3684	27.6472	27.9267

$P^*\backslash\nu$	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0	55.0
0.600	7.5589	11.3233	15.0907	18.8591	22.6282	26.3977	30.1675	33.9372	37.7072	41.4772
0.650	8.6668	12.9785	17.2946	21.6122	25.9309	30.2500	34.5694	38.8891	43.2088	47.5287
0.700	9.8424	14.7352	19.6336	24.5343	29.4361	34.3386	39.2416	44.1447	49.0481	53.9514
0.750	11.1207	16.6455	22.1774	27.7121	33.2483	38.7859	44.3227	49.8602	55.3990	60.9367
0.800	12.5563	18.7912	25.0346	31.2815	37.5301	43.7797	50.0299	56.2806	62.5316	68.7827
0.850	14.2462	21.3170	28.3983	35.4837	42.5712	49.6600	56.7493	63.8394	70.9297	78.0203
0.900	16.3982	24.5338	32.6824	40.8360	48.9920	57.1494	65.3077	73.4667	81.6261	89.7858
0.950	19.6395	29.3807	39.1371	48.9006	58.6668	68.4345	78.2036	87.9738	97.7441	107.5149
0.975	22.4994	33.6580	44.8336	56.0183	67.2056	78.3952	89.5860	100.7781	111.9706	123.1628
0.990	25.8772	38.7085	51.5635	64.4257	77.2913	90.1600	103.0310	115.9015	128.7733	141.6470
0.995	28.2060	42.1930	56.2010	70.2214	84.2459	98.2713	112.3033	126.3305	140.3636	154.3907

$P^*\backslash\nu$	60.0	65.0	70.0	75.0	80.0	85.0	90.0	95.0	100.0
0.600	45.2473	49.0177	52.7877	56.5578	60.3279	64.0984	67.8680	71.6387	75.4089
0.650	51.8486	56.1687	60.4887	64.8090	69.1290	73.4491	77.7693	82.0895	86.4099
0.700	58.8551	63.7587	68.6625	73.5663	78.4703	83.3741	88.2779	93.1817	98.0858
0.750	66.4756	72.0131	77.5506	83.0905	88.6291	94.1678	99.7083	105.2452	110.7840
0.800	75.0339	81.2854	87.5373	93.7887	100.0402	106.2921	112.5438	118.7955	125.0473
0.850	85.1110	92.2018	99.2933	106.3840	113.4768	120.5665	127.6578	134.7490	141.8403
0.900	97.9459	106.1059	114.2689	122.4266	130.5869	138.7475	146.9077	155.0693	163.2294
0.950	117.2866	127.0570	136.8295	146.6005	156.3727	166.1439	175.9171	185.6884	195.4601
0.975	134.3564	145.5498	156.7439	167.9366	179.1341	190.3255	201.5194	212.7127	223.9071
0.990	154.5186	167.3924	180.2660	193.1384	206.0124	218.8859	231.7602	244.6353	257.5241
0.995	168.4258	182.4628	196.4848	210.5174	224.5465	238.5780	252.6189	266.6437	280.6778

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Table 6.1: $k = 4$

$P^*\backslash\nu$	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59
0.600	0.7081	0.7174	0.7266	0.7358	0.7451	0.7543	0.7636	0.7728	0.7821	0.7914
0.650	0.8478	0.8574	0.8670	0.8765	0.8861	0.8957	0.9053	0.9149	0.9245	0.9341
0.700	0.9952	1.0051	1.0150	1.0249	1.0349	1.0448	1.0548	1.0648	1.0747	1.0847
0.750	1.1545	1.1647	1.1750	1.1853	1.1956	1.2060	1.2163	1.2267	1.2371	1.2476
0.800	1.3320	1.3427	1.3534	1.3641	1.3749	1.3857	1.3965	1.4074	1.4183	1.4292
0.850	1.5391	1.5503	1.5616	1.5728	1.5842	1.5955	1.6069	1.6184	1.6298	1.6414
0.900	1.8002	1.8121	1.8240	1.8360	1.8480	1.8601	1.8722	1.8844	1.8966	1.9089
0.950	2.1880	2.2008	2.2138	2.2269	2.2400	2.2532	2.2664	2.2798	2.2932	2.3067
0.975	2.5250	2.5388	2.5527	2.5668	2.5809	2.5951	2.6094	2.6238	2.6383	2.6528
0.990	2.9178	2.9327	2.9479	2.9630	2.9783	2.9937	3.0092	3.0249	3.0407	3.0566
0.995	3.1857	3.2014	3.2173	3.2334	3.2495	3.2657	3.2821	3.2987	3.3154	3.3322

$P^*\backslash\nu$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69
0.600	0.8006	0.8099	0.8192	0.8284	0.8377	0.8470	0.8563	0.8655	0.8748	0.8841
0.650	0.9437	0.9534	0.9630	0.9726	0.9823	0.9920	1.0016	1.0113	1.0210	1.0307
0.700	1.0948	1.1048	1.1148	1.1249	1.1350	1.1451	1.1552	1.1653	1.1754	1.1856
0.750	1.2580	1.2685	1.2790	1.2895	1.3001	1.3106	1.3212	1.3318	1.3424	1.3531
0.800	1.4402	1.4512	1.4622	1.4732	1.4843	1.4954	1.5065	1.5177	1.5289	1.5401
0.850	1.6529	1.6645	1.6762	1.6878	1.6996	1.7113	1.7231	1.7349	1.7468	1.7587
0.900	1.9213	1.9336	1.9461	1.9586	1.9712	1.9838	1.9964	2.0091	2.0219	2.0347
0.950	2.3202	2.3339	2.3476	2.3614	2.3752	2.3891	2.4031	2.4172	2.4314	2.4456
0.975	2.6675	2.6822	2.6971	2.7121	2.7271	2.7422	2.7574	2.7727	2.7881	2.8036
0.990	3.0725	3.0887	3.1049	3.1213	3.1378	3.1543	3.1710	3.1878	3.2047	3.2217
0.995	3.3495	3.3663	3.3835	3.4009	3.4183	3.4358	3.4537	3.4714	3.4894	3.5074

$P^*\backslash\nu$	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79
0.600	0.8934	0.9027	0.9120	0.9212	0.9305	0.9398	0.9491	0.9584	0.9677	0.9770
0.650	1.0404	1.0501	1.0598	1.0696	1.0793	1.0890	1.0988	1.1085	1.1183	1.1280
0.700	1.1957	1.2059	1.2161	1.2263	1.2365	1.2467	1.2569	1.2672	1.2774	1.2877
0.750	1.3637	1.3744	1.3851	1.3959	1.4066	1.4174	1.4281	1.4389	1.4498	1.4606
0.800	1.5513	1.5626	1.5739	1.5852	1.5966	1.6080	1.6194	1.6308	1.6423	1.6538
0.850	1.7707	1.7826	1.7947	1.8067	1.8188	1.8309	1.8431	1.8553	1.8675	1.8798
0.900	2.0476	2.0605	2.0734	2.0864	2.0995	2.1126	2.1258	2.1389	2.1522	2.1655
0.950	2.4598	2.4742	2.4886	2.5031	2.5176	2.5323	2.5469	2.5617	2.5765	2.5914
0.975	2.8202	2.8348	2.8505	2.8664	2.8823	2.8982	2.9143	2.9305	2.9467	2.9630
0.990	3.2387	3.2560	3.2733	3.2908	3.3084	3.3259	3.3437	3.3615	3.3794	3.3975
0.995	3.5257	3.5440	3.5624	3.5809	3.5997	3.6184	3.6374	3.6569	3.6755	3.6948

$P^*\backslash\nu$	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
0.600	0.9863	0.9956	1.0049	1.0142	1.0234	1.0328	1.0421	1.0514	1.0608	1.0701
0.650	1.1378	1.1476	1.1574	1.1672	1.1770	1.1868	1.1966	1.2064	1.2162	1.2260
0.700	1.2980	1.3083	1.3186	1.3289	1.3393	1.3496	1.3600	1.3703	1.3807	1.3911
0.750	1.4714	1.4823	1.4932	1.5041	1.5151	1.5260	1.5370	1.5480	1.5590	1.5700
0.800	1.6653	1.6768	1.6884	1.7000	1.7116	1.7232	1.7349	1.7466	1.7583	1.7701
0.850	1.8921	1.9045	1.9168	1.9292	1.9417	1.9542	1.9667	1.9792	1.9918	2.0044
0.900	2.1788	2.1922	2.2057	2.2191	2.2327	2.2462	2.2598	2.2735	2.2872	2.3009
0.950	2.6063	2.6213	2.6364	2.6515	2.6667	2.6820	2.6973	2.7127	2.7281	2.7436
0.975	2.9794	2.9959	3.0125	3.0290	3.0457	3.0625	3.0793	3.0963	3.1132	3.1303
0.990	3.4156	3.4338	3.4522	3.4706	3.4892	3.5077	3.5264	3.5452	3.5641	3.5830
0.995	3.7141	3.7335	3.7531	3.7728	3.7926	3.8125	3.8324	3.8525	3.8728	3.8930

$P^*\backslash\nu$	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.600	1.0794	1.0887	1.0980	1.1073	1.1167	1.1260	1.1353	1.1446	1.1540	1.1633
0.650	1.2359	1.2457	1.2556	1.2654	1.2753	1.2851	1.2950	1.3049	1.3148	1.3247
0.700	1.4015	1.4119	1.4223	1.4328	1.4432	1.4537	1.4642	1.4746	1.4852	1.4957
0.750	1.5810	1.5921	1.6031	1.6142	1.6253	1.6365	1.6476	1.6588	1.6699	1.6811
0.800	1.7818	1.7936	1.8054	1.8173	1.8291	1.8410	1.8529	1.8649	1.8768	1.8888
0.850	2.0171	2.0297	2.0424	2.0552	2.0680	2.0808	2.0936	2.1065	2.1194	2.1323
0.900	2.3147	2.3286	2.3423	2.3563	2.3703	2.3843	2.3983	2.4124	2.4265	2.4407
0.950	2.7594	2.7748	2.7904	2.8061	2.8219	2.8377	2.8537	2.8696	2.8856	2.9016
0.975	3.1474	3.1647	3.1820	3.1992	3.2167	3.2342	3.2518	3.2694	3.2871	3.3048
0.990	3.6021	3.6212	3.6405	3.6598	3.6791	3.6986	3.7182	3.7379	3.7576	3.7774
0.995	3.9136	3.9340	3.9545	3.9752	3.9957	4.0168	4.0378	4.0589	4.0800	4.1014

$P^*\backslash\nu$	0.991	0.992	0.993	0.994	0.995	0.996	0.997	0.998	0.999	1.000
0.600	1.1642	1.1651	1.1661	1.1670	1.1679	1.1689	1.1698	1.1707	1.1717	1.1726
0.650	1.3257	1.3267	1.3276	1.3286	1.3296	1.3306	1.3316	1.3326	1.3336	1.3346
0.700	1.4967	1.4978	1.4988	1.4999	1.5009	1.5020	1.5030	1.5041	1.5051	1.5062
0.750	1.6823	1.6834	1.6845	1.6856	1.6867	1.6879	1.6890	1.6901	1.6912	1.6923
0.800	1.8900	1.8912	1.8924	1.8936	1.8948	1.8960	1.8972	1.8984	1.8996	1.9008
0.850	2.1336	2.1349	2.1362	2.1375	2.1388	2.1401	2.1414	2.1427	2.1439	2.1452
0.900	2.4421	2.4435	2.4449	2.4464	2.4478	2.4492	2.4506	2.4520	2.4535	2.4549
0.950	2.9032	2.9049	2.9065	2.9081	2.9097	2.9113	2.9129	2.9145	2.9161	2.9177
0.975	3.3066	3.3084	3.3102	3.3120	3.3137	3.3155	3.3173	3.3191	3.3209	3.3227
0.990	3.7793	3.7813	3.7833	3.7853	3.7874	3.7893	3.7912	3.7933	3.7952	3.7971
0.995	4.1033	4.1055	4.1076	4.1097	4.1119	4.1140	4.1162	4.1183	4.1204	4.1226

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Table 6.1: $k = 4$

$P^*\backslash\nu$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
0.600	1.1726	1.2660	1.3596	1.4533	1.5474	1.6417	1.7363	1.8313	1.9266	2.0222
0.650	1.3346	1.4339	1.5338	1.6344	1.7355	1.8373	1.9397	2.0426	2.1462	2.2502
0.700	1.5062	1.6120	1.7188	1.8267	1.9356	2.0454	2.1561	2.2677	2.3800	2.4931
0.750	1.6923	1.8053	1.9199	2.0359	2.1533	2.2721	2.3920	2.5130	2.6350	2.7579
0.800	1.9008	2.0221	2.1454	2.2708	2.3979	2.5267	2.6571	2.7887	2.9216	3.0555
0.850	2.1452	2.2764	2.4104	2.5468	2.6856	2.8263	2.9690	3.1132	3.2589	3.4059
0.900	2.4549	2.5990	2.7466	2.8973	3.0510	3.2071	3.3655	3.5259	3.6880	3.8517
0.950	2.9177	3.0816	3.2501	3.4227	3.5990	3.7786	3.9611	4.1460	4.3333	4.5225
0.975	3.3227	3.5042	3.6915	3.8837	4.0804	4.2809	4.4849	4.6919	4.9017	5.1139
0.990	3.7971	4.0001	4.2098	4.4257	4.6467	4.8724	5.1024	5.3360	5.5728	5.8123
0.995	4.1226	4.3404	4.5659	4.7981	5.0365	5.2798	5.5279	5.7800	6.0358	6.2947

$P^*\backslash\nu$	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
0.600	2.1182	2.2145	2.3112	2.4083	2.5056	2.6032	2.7010	2.7991	2.8974	2.9959
0.650	2.3548	2.4599	2.5655	2.6714	2.7777	2.8844	2.9914	3.0986	3.2061	3.3139
0.700	2.6069	2.7213	2.8362	2.9516	3.0675	3.1838	3.3004	3.4173	3.5346	3.6521
0.750	2.8816	3.0061	3.1312	3.2569	3.3831	3.5098	3.6369	3.7644	3.8923	4.0205
0.800	3.1905	3.3263	3.4628	3.6001	3.7380	3.8764	4.0154	4.1549	4.2946	4.4348
0.850	3.5541	3.7033	3.8534	4.0043	4.1560	4.3083	4.4613	4.6149	4.7689	4.9234
0.900	4.0169	4.1832	4.3507	4.5193	4.6887	4.8590	5.0300	5.2017	5.3740	5.5469
0.950	4.7135	4.9062	5.1003	5.2962	5.4923	5.6899	5.8884	6.0880	6.2882	6.4892
0.975	5.3282	5.5445	5.7622	5.9820	6.2029	6.4250	6.6483	6.8727	7.0979	7.3240
0.990	6.0547	6.2993	6.5459	6.7944	7.0445	7.2959	7.5488	7.8027	8.0581	8.3142
0.995	6.5564	6.8208	7.0873	7.3558	7.6263	7.8983	8.1718	8.4465	8.7225	8.9995

$P^*\backslash\nu$	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
0.600	3.0945	3.1933	3.2923	3.3913	3.4905	3.5898	3.6891	3.7885	3.8880	3.9876
0.650	3.4218	3.5299	3.6382	3.7466	3.8551	3.9638	4.0725	4.1814	4.2904	4.3994
0.700	3.7698	3.8878	4.0060	4.1243	4.2428	4.3615	4.4803	4.5992	4.7183	4.8374
0.750	4.1490	4.2777	4.4066	4.5358	4.6652	4.7948	4.9246	5.0545	5.1846	5.3149
0.800	4.5754	4.7163	4.8575	4.9990	5.1407	5.2827	5.4248	5.5672	5.7098	5.8525
0.850	5.0783	5.2337	5.3894	5.5454	5.7018	5.8584	6.0153	6.1724	6.3300	6.4874
0.900	5.7203	5.8942	6.0686	6.2433	6.4185	6.5940	6.7698	6.9459	7.1222	7.2989
0.950	6.6908	6.8932	7.0960	7.2993	7.5031	7.7074	7.9120	8.1170	8.3224	8.5281
0.975	7.5510	7.7786	8.0069	8.2358	8.4652	8.6951	8.9255	9.1560	9.3877	9.6189
0.990	8.5713	8.8292	9.0879	9.3474	9.6073	9.8681	10.1291	10.3908	10.6529	10.9158
0.995	9.2780	9.5565	9.8365	10.1170	10.3984	10.6803	10.9630	11.2462	11.5297	11.8138

$P^*\backslash\nu$	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
0.600	4.0872	4.1869	4.2867	4.3864	4.4863	4.5861	4.6860	4.7860	4.8860	4.9860
0.650	4.5086	4.6178	4.7271	4.8364	4.9458	5.0553	5.1648	5.2744	5.3841	5.4937
0.700	4.9567	5.0761	5.1956	5.3151	5.4348	5.5545	5.6743	5.7942	5.9141	6.0341
0.750	5.4452	5.5758	5.7064	5.8371	5.9679	6.0988	6.2299	6.3610	6.4922	6.6234
0.800	5.9954	6.1385	6.2817	6.4250	6.5685	6.7121	6.8558	6.9996	7.1435	7.2875
0.850	6.6451	6.8031	6.9612	7.1191	7.2780	7.4366	7.5953	7.7542	7.9131	8.0722
0.900	7.4758	7.6529	7.8302	8.0075	8.1854	8.3633	8.5413	8.7195	8.8978	9.0763
0.950	8.7340	8.9404	9.1469	9.3537	9.5607	9.7679	9.9754	10.1830	10.3907	10.5988
0.975	9.8513	10.0836	10.3162	10.5491	10.7823	11.0158	11.2494	11.4833	11.7174	11.9517
0.990	11.1755	11.4420	11.7057	11.9698	12.2340	12.4989	12.7640	13.0291	13.2945	13.5603
0.995	12.0979	12.3831	12.6685	12.9543	13.2403	13.5251	13.8134	14.1039	14.3885	14.6748

$P^*\backslash\nu$	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
0.600	5.0860	5.1861	5.2862	5.3863	5.4865	5.5867	5.6869	5.7871	5.8873	5.9876
0.650	5.6035	5.7132	5.8230	5.9329	6.0428	6.1527	6.2626	6.3726	6.4826	6.5927
0.700	6.1541	6.2742	6.3944	6.5146	6.6348	6.7551	6.8755	6.9958	7.1163	7.2367
0.750	6.7547	6.8862	7.0176	7.1491	7.2807	7.4124	7.5441	7.6758	7.8076	7.9395
0.800	7.4316	7.5757	7.7200	7.8643	8.0087	8.1532	8.2977	8.4423	8.5869	8.7316
0.850	8.2314	8.3907	8.5501	8.7096	8.8692	9.0288	9.1885	9.3484	9.5082	9.6682
0.900	9.2549	9.4336	9.6124	9.7914	9.9704	10.1495	10.3288	10.5081	10.6875	10.8669
0.950	10.8069	11.0152	11.2236	11.4321	11.6408	11.8496	12.0586	12.2676	12.4767	12.6859
0.975	12.1862	12.4208	12.6556	12.8906	13.1257	13.3610	13.5964	13.8319	14.0675	14.3033
0.990	13.8261	14.0920	14.3582	14.6247	14.8914	15.1581	15.4251	15.6923	15.9595	16.2268
0.995	14.9628	15.2506	15.5397	15.8270	16.1155	16.4037	16.6929	16.9810	17.2712	17.5605

$P^*\backslash\nu$	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9
0.600	6.0879	6.1882	6.2885	6.3889	6.4892	6.5896	6.6899	6.7903	6.8908	6.9912
0.650	6.7027	6.8128	6.9229	7.0331	7.1432	7.2534	7.3636	7.4738	7.5840	7.6943
0.700	7.3572	7.4778	7.5983	7.7189	7.8395	7.9601	8.0808	8.2015	8.3222	8.4429
0.750	8.0713	8.2033	8.3352	8.4672	8.5993	8.7313	8.8634	8.9956	9.1277	9.2599
0.800	8.8764	9.0212	9.1662	9.3109	9.4559	9.6008	9.7459	9.8909	10.0361	10.1811
0.850	9.8282	9.9882	10.1483	10.3085	10.4688	10.6290	10.7893	10.9497	11.1100	11.2705
0.900	11.0465	11.2262	11.4059	11.5856	11.7655	11.9454	12.1253	12.3056	12.4854	12.6655
0.950	12.8952	13.1046	13.3141	13.5238	13.7336	13.9432	14.1530	14.3629	14.5728	14.7828
0.975	14.5391	14.7751	15.0111	15.2473	15.4836	15.7202	15.9563	16.1928	16.4294	16.6659
0.990	16.4943	16.7619	17.0295	17.2975	17.5653	17.8322	18.1014	18.3694	18.6379	18.9065
0.995	17.8501	18.1396	18.4292	18.7191	19.0087	19.2989	19.5845	19.8795	20.1698	20.4612

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Table 6.1: $k = 4$

$P^*\backslash\nu$	7.0	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
0.600	7.0916	7.1921	7.2925	7.3930	7.4934	7.5939	7.6944	7.7949	7.8955	7.9960
0.650	7.8046	7.9148	8.0251	8.1355	8.2458	8.3562	8.4663	8.5769	8.6873	8.7980
0.700	8.5635	8.6845	8.8053	8.9261	9.0469	9.1678	9.2887	9.4096	9.5305	9.6514
0.750	9.3921	9.5244	9.6567	9.7890	9.9213	10.0537	10.1860	10.3184	10.4508	10.5833
0.800	10.3264	10.4715	10.6167	10.7620	10.9073	11.0526	11.1979	11.3429	11.4887	11.6341
0.850	11.4310	11.5915	11.7521	11.9127	12.0733	12.2341	12.3947	12.5554	12.7161	12.8770
0.900	12.8456	13.0246	13.2061	13.3863	13.5666	13.7470	13.9274	14.1079	14.2884	14.4689
0.950	14.9930	15.2029	15.4131	15.6234	15.8336	16.0439	16.2543	16.4647	16.6751	16.8857
0.975	16.9026	17.1395	17.3763	17.6132	17.8502	18.0872	18.3242	18.5614	18.7985	19.0357
0.990	19.1699	19.4434	19.7120	19.9812	20.2493	20.5181	20.7870	21.0559	21.3253	21.5939
0.995	20.7506	21.0412	21.3314	21.6223	21.9139	22.2043	22.4953	22.7860	23.0771	23.3682

$P^*\backslash\nu$	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9
0.600	8.0965	8.1970	8.2976	8.3981	8.4987	8.5993	8.6998	8.8004	8.9010	9.0016
0.650	8.9081	9.0185	9.1289	9.2394	9.3498	9.4603	9.5708	9.6812	9.7917	9.9022
0.700	9.7723	9.8933	10.0143	10.1353	10.2563	10.3773	10.4983	10.6193	10.7404	10.8615
0.750	10.7157	10.8481	10.9806	11.1132	11.2456	11.3782	11.5107	11.6433	11.7758	11.9084
0.800	11.7795	11.9250	12.0704	12.2159	12.3614	12.5070	12.6526	12.7981	12.9438	13.0893
0.850	13.0378	13.1986	13.3595	13.5204	13.6813	13.8422	14.0032	14.1642	14.3252	14.4863
0.900	14.6494	14.8300	15.0106	15.1912	15.3718	15.5525	15.7332	15.9139	16.0947	16.2755
0.950	17.0962	17.3068	17.5174	17.7280	17.9387	18.1494	18.3601	18.5709	18.8057	18.9926
0.975	19.2730	19.5103	19.7476	19.9850	20.2224	20.4599	20.6975	20.9350	21.1726	21.4101
0.990	21.8630	22.1323	22.4015	22.6706	22.9392	23.2092	23.4787	23.7479	24.0176	24.2862
0.995	23.6597	23.9506	24.2417	24.5332	24.8248	25.1161	25.4075	25.6994	25.9907	26.2821

$P^*\backslash\nu$	9.0	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
0.600	9.1022	9.2028	9.3034	9.4040	9.5046	9.6053	9.7059	9.8066	9.9072	10.0078
0.650	10.0127	10.1233	10.2338	10.3443	10.4548	10.5654	10.6759	10.7865	10.8979	11.0076
0.700	10.9825	11.1035	11.2245	11.3457	11.4668	11.5880	11.7091	11.8302	11.9518	12.0725
0.750	12.0410	12.1738	12.3063	12.4389	12.5715	12.7042	12.8369	12.9696	13.1023	13.2350
0.800	13.2349	13.3806	13.5262	13.6719	13.8176	13.9633	14.1090	14.2547	14.4004	14.5462
0.850	14.6472	14.8083	14.9694	15.1305	15.2916	15.4527	15.6138	15.7750	15.9362	16.0973
0.900	16.4563	16.6371	16.8180	16.9989	17.1797	17.3607	17.5416	17.7225	17.9035	18.0845
0.950	19.2034	19.4143	19.6253	19.8362	20.0471	20.2581	20.4692	20.6802	20.8913	21.1023
0.975	21.6478	21.8855	22.1232	22.3611	22.5987	22.8364	23.0775	23.3121	23.5500	23.7879
0.990	24.5564	24.8260	25.0956	25.3657	25.6382	25.9046	26.1744	26.4441	26.7139	26.9837
0.995	26.5741	26.8655	27.1574	27.4492	27.7412	28.0330	28.3247	28.6168	28.9085	29.2007

$P^*\backslash\nu$	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0	55.0
0.600	10.1085	15.1446	20.1843	25.2254	30.2673	35.3096	40.3521	45.3948	50.4377	55.4807
0.650	11.1182	16.6521	22.1911	27.7319	33.2737	38.8163	44.3591	49.9022	55.4455	60.9889
0.700	12.1936	18.2580	24.3288	30.4021	36.4768	42.5522	48.6281	54.7043	60.7802	66.8572
0.750	13.3677	20.0114	26.6630	33.3179	39.9744	46.6322	53.2897	59.9482	66.6068	73.2656
0.800	14.6919	21.9896	29.2967	36.6078	43.9208	51.2349	58.5496	65.8649	73.1804	80.4962
0.850	16.2585	24.3302	32.4132	40.5010	48.5909	56.6822	64.7744	72.8670	80.9602	89.0534
0.900	18.2655	27.3295	36.4072	45.4903	54.5760	63.6634	72.7519	81.8411	90.9308	100.0206
0.950	21.3134	31.8858	42.4749	53.0706	63.6699	74.2712	84.8736	95.4766	106.0803	116.6845
0.975	24.0258	35.9414	47.8763	59.8191	71.7646	83.7146	95.6648	107.6164	119.5677	131.5205
0.990	27.2536	40.7684	54.3064	67.8522	81.4033	94.9569	108.5106	122.0664	135.6234	149.1810
0.995	29.4925	44.1177	58.7665	73.4245	88.0898	102.7565	117.4208	132.0949	146.7624	161.4328

$P^*\backslash\nu$	60.0	65.0	70.0	75.0	80.0	85.0	90.0	95.0	100.0
0.600	60.5235	65.5667	70.6098	75.6529	80.6961	85.7393	90.7823	95.8257	100.8688
0.650	66.5324	72.0759	77.6195	83.1632	88.7067	94.2506	99.7943	105.3383	110.8818
0.700	72.9340	79.0107	85.0876	91.1646	97.2412	103.3185	109.3954	115.4725	121.5497
0.750	79.9249	86.5836	93.2429	99.9020	106.5614	113.2208	119.8802	126.5395	133.1990
0.800	87.8123	95.1284	102.4449	109.7610	117.0774	124.3941	131.7102	139.0268	146.3434
0.850	97.1471	105.2405	113.3348	121.4286	129.5164	137.6169	145.7075	153.8054	161.8984
0.900	109.1105	118.2111	127.2914	136.3822	145.4729	154.5632	163.6823	172.7455	181.8363
0.950	127.2885	137.8937	148.4984	159.1039	169.7092	180.3140	190.9193	201.5247	212.1309
0.975	143.4728	155.4261	167.3799	179.3340	191.2852	203.2393	215.1935	227.1478	239.1017
0.990	162.7392	176.2944	189.8545	203.4130	216.9716	230.5314	244.0862	257.6495	271.2056
0.995	176.1059	190.7775	205.4456	220.1195	234.7895	249.4631	264.1386	278.8092	293.4866

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Table 6.1: $k = 5$

$P^*\backslash\nu$	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59
0.600	0.7875	0.7983	0.8092	0.8201	0.8309	0.8418	0.8527	0.8636	0.8744	0.8853
0.650	0.9269	0.9381	0.9492	0.9604	0.9716	0.9828	0.9940	1.0052	1.0164	1.0277
0.700	1.0740	1.0855	1.0969	1.1084	1.1200	1.1315	1.1431	1.1546	1.1662	1.1778
0.750	1.2328	1.2446	1.2565	1.2683	1.2802	1.2922	1.3041	1.3161	1.3280	1.3400
0.800	1.4098	1.4220	1.4343	1.4466	1.4589	1.4712	1.4836	1.4960	1.5085	1.5209
0.850	1.6163	1.6290	1.6418	1.6545	1.6674	1.6802	1.6931	1.7061	1.7190	1.7321
0.900	1.8764	1.8898	1.9031	1.9166	1.9300	1.9436	1.9571	1.9708	1.9844	1.9982
0.950	2.2626	2.2769	2.2912	2.3056	2.3201	2.3347	2.3493	2.3640	2.3787	2.3936
0.975	2.5981	2.6132	2.6284	2.6438	2.6591	2.6746	2.6902	2.7058	2.7216	2.7374
0.990	2.9889	3.0051	3.0213	3.0377	3.0545	3.0708	3.0875	3.1042	3.1212	3.1383
0.995	3.2554	3.2723	3.2893	3.3064	3.3237	3.3411	3.3586	3.3762	3.3940	3.4118

$P^*\backslash\nu$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69
0.600	0.8962	0.9071	0.9180	0.9289	0.9399	0.9508	0.9617	0.9726	0.9835	0.9945
0.650	1.0389	1.0502	1.0614	1.0727	1.0840	1.0953	1.1065	1.1179	1.1292	1.1405
0.700	1.1894	1.2011	1.2127	1.2244	1.2360	1.2477	1.2594	1.2712	1.2829	1.2946
0.750	1.3521	1.3641	1.3762	1.3883	1.4004	1.4125	1.4247	1.4369	1.4491	1.4613
0.800	1.5334	1.5459	1.5585	1.5711	1.5837	1.5963	1.6090	1.6217	1.6344	1.6472
0.850	1.7451	1.7582	1.7713	1.7845	1.7977	1.8110	1.8242	1.8376	1.8509	1.8643
0.900	2.0120	2.0258	2.0397	2.0536	2.0676	2.0816	2.0957	2.1098	2.1240	2.1382
0.950	2.4085	2.4235	2.4385	2.4536	2.4688	2.4843	2.4993	2.5147	2.5301	2.5456
0.975	2.7533	2.7693	2.7854	2.8016	2.8178	2.8342	2.8506	2.8671	2.8837	2.9003
0.990	3.1554	3.1726	3.1900	3.2074	3.2250	3.2427	3.2604	3.2783	3.2962	3.3143
0.995	3.4298	3.4480	3.4662	3.4846	3.5031	3.5216	3.5404	3.5592	3.5776	3.5972

$P^*\backslash\nu$	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79
0.600	1.0054	1.0163	1.0273	1.0382	1.0492	1.0601	1.0711	1.0820	1.0930	1.1040
0.650	1.1518	1.1632	1.1745	1.1859	1.1972	1.2086	1.2200	1.2314	1.2428	1.2542
0.700	1.3064	1.3182	1.3300	1.3418	1.3536	1.3654	1.3773	1.3891	1.4010	1.4129
0.750	1.4735	1.4858	1.4980	1.5103	1.5227	1.5350	1.5473	1.5597	1.5721	1.5845
0.800	1.6599	1.6727	1.6856	1.6984	1.7113	1.7242	1.7372	1.7501	1.7630	1.7762
0.850	1.8777	1.8912	1.9047	1.9182	1.9307	1.9454	1.9590	1.9727	1.9864	2.0002
0.900	2.1525	2.1668	2.1812	2.1956	2.2100	2.2245	2.2391	2.2536	2.2683	2.2829
0.950	2.5612	2.5768	2.5925	2.6083	2.6241	2.6400	2.6559	2.6719	2.6880	2.7041
0.975	2.9171	2.9339	2.9508	2.9678	2.9848	3.0025	3.0192	3.0364	3.0538	3.0712
0.990	3.3325	3.3507	3.3693	3.3875	3.4061	3.4247	3.4435	3.4623	3.4813	3.5003
0.995	3.6163	3.6355	3.6550	3.6744	3.6941	3.7149	3.7337	3.7536	3.7736	3.7938

$P^*\backslash\nu$	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
0.600	1.1149	1.1259	1.1369	1.1479	1.1588	1.1698	1.1808	1.1918	1.2028	1.2138
0.650	1.2656	1.2770	1.2885	1.2999	1.3114	1.3228	1.3343	1.3458	1.3572	1.3687
0.700	1.4248	1.4367	1.4486	1.4606	1.4725	1.4845	1.4964	1.5084	1.5204	1.5325
0.750	1.5970	1.6094	1.6219	1.6344	1.6469	1.6594	1.6719	1.6845	1.6971	1.7097
0.800	1.7892	1.8023	1.8153	1.8285	1.8416	1.8548	1.8680	1.8812	1.8944	1.9077
0.850	2.0139	2.0277	2.0416	2.0554	2.0693	2.0833	2.0972	2.1112	2.1253	2.1393
0.900	2.2977	2.3124	2.3273	2.3420	2.3570	2.3719	2.3869	2.4019	2.4170	2.4321
0.950	2.7203	2.7365	2.7528	2.7692	2.7856	2.8020	2.8186	2.8352	2.8518	2.8685
0.975	3.0887	3.1063	3.1242	3.1417	3.1596	3.1773	3.1953	3.2133	3.2314	3.2495
0.990	3.5194	3.5386	3.5579	3.5773	3.5967	3.6162	3.6360	3.6556	3.6755	3.6954
0.995	3.8142	3.8344	3.8548	3.8754	3.8960	3.9168	3.9377	3.9586	3.9800	4.0008

$P^*\backslash\nu$	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.600	1.2248	1.2358	1.2468	1.2578	1.2689	1.2799	1.2909	1.3019	1.3129	1.3240
0.650	1.3802	1.3917	1.4032	1.4147	1.4263	1.4378	1.4493	1.4609	1.4724	1.4840
0.700	1.5445	1.5565	1.5686	1.5806	1.5927	1.6048	1.6169	1.6290	1.6411	1.6533
0.750	1.7223	1.7349	1.7476	1.7602	1.7729	1.7856	1.7983	1.8111	1.8238	1.8366
0.800	1.9210	1.9343	1.9476	1.9610	1.9743	1.9877	2.0012	2.0146	2.0281	2.0416
0.850	2.1534	2.1675	2.1817	2.1959	2.2101	2.2243	2.2386	2.2529	2.2672	2.2816
0.900	2.4472	2.4624	2.4776	2.4927	2.5081	2.5234	2.5388	2.5542	2.5697	2.5851
0.950	2.8852	2.9020	2.9189	2.9358	2.9528	2.9698	2.9868	3.0040	3.0211	3.0383
0.975	3.2677	3.2860	3.3044	3.3228	3.3413	3.3598	3.3784	3.3970	3.4158	3.4345
0.990	3.7154	3.7355	3.7556	3.7758	3.7961	3.8165	3.8369	3.8576	3.8782	3.8989
0.995	4.0221	4.0433	4.0649	4.0864	4.1080	4.1297	4.1515	4.1733	4.1947	4.2174

$P^*\backslash\nu$	0.991	0.992	0.993	0.994	0.995	0.996	0.997	0.998	0.999	1.000
0.600	1.3251	1.3262	1.3273	1.3284	1.3295	1.3306	1.3317	1.3328	1.3339	1.3350
0.650	1.4852	1.4863	1.4875	1.4886	1.4898	1.4909	1.4921	1.4933	1.4944	1.4956
0.700	1.6545	1.6557	1.6569	1.6581	1.6593	1.6606	1.6618	1.6630	1.6642	1.6654
0.750	1.8379	1.8392	1.8404	1.8417	1.8430	1.8443	1.8455	1.8468	1.8481	1.8494
0.800	2.0429	2.0443	2.0456	2.0470	2.0483	2.0497	2.0510	2.0524	2.0537	2.0551
0.850	2.2830	2.2844	2.2859	2.2873	2.2888	2.2902	2.2916	2.2931	2.2945	2.2960
0.900	2.5867	2.5882	2.5898	2.5913	2.5929	2.5944	2.5960	2.5976	2.5991	2.6007
0.950	3.0400	3.0418	3.0435	3.0452	3.0470	3.0487	3.0504	3.0521	3.0539	3.0556
0.975	3.4364	3.4383	3.4402	3.4421	3.4439	3.4458	3.4477	3.4493	3.4515	3.4534
0.990	3.9009	3.9030	3.9051	3.9072	3.9093	3.9113	3.9133	3.9155	3.9176	3.9196
0.995	4.2196	4.2217	4.2239	4.2262	4.2284	4.2306	4.2328	4.2351	4.2373	4.2395

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Table 6.1: $k = 5$

$P^*\backslash\nu$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
0.600	1.3350	1.4455	1.5564	1.6676	1.7791	1.8910	2.0032	2.1157	2.2286	2.3417
0.650	1.4956	1.6117	1.7284	1.8459	1.9640	2.0827	2.2019	2.3217	2.4420	2.5629
0.700	1.6654	1.7875	1.9107	2.0349	2.1600	2.2860	2.4129	2.5405	2.6687	2.7977
0.750	1.8494	1.9781	2.1083	2.2400	2.3729	2.5070	2.6421	2.7783	2.9153	3.0532
0.800	2.0551	2.1914	2.3297	2.4698	2.6115	2.7548	2.8994	3.0453	3.1922	3.3402
0.850	2.2960	2.4413	2.5892	2.7394	2.8917	3.0460	3.2018	3.3593	3.5181	3.6782
0.900	2.6007	2.7578	2.9181	3.0815	3.2475	3.4159	3.5864	3.7588	3.9330	4.1087
0.950	3.0556	3.2308	3.4103	3.5941	3.7812	3.9714	4.1645	4.3602	4.5580	4.7578
0.975	3.4534	3.6451	3.8422	4.0441	4.2504	4.4605	4.6741	4.8907	5.1100	5.3318
0.990	3.9196	4.1313	4.3497	4.5740	4.8035	5.0378	5.2762	5.5183	5.7636	6.0118
0.995	4.2395	4.4653	4.6987	4.9387	5.1846	5.4360	5.6919	5.9519	6.2173	6.4823

$P^*\backslash\nu$	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
0.600	2.4552	2.5690	2.6829	2.7972	2.9116	3.0262	3.1410	3.2560	3.3712	3.4864
0.650	2.6840	2.8056	2.9276	3.0499	3.1725	3.2953	3.4184	3.5418	3.6653	3.7891
0.700	2.9271	3.0572	3.1876	3.3186	3.4498	3.5815	3.7135	3.8458	3.9783	4.1111
0.750	3.1917	3.3310	3.4708	3.6112	3.7520	3.8933	4.0350	4.1771	4.3196	4.4623
0.800	3.4891	3.6388	3.7892	3.9403	4.0918	4.2443	4.3971	4.5503	4.7040	4.8580
0.850	3.8393	4.0015	4.1646	4.3285	4.4932	4.6585	4.8245	4.9910	5.1581	5.3256
0.900	4.2858	4.4641	4.6436	4.8241	5.0055	5.1878	5.3709	5.5546	5.7390	5.9240
0.950	4.9595	5.1628	5.3676	5.5737	5.7810	5.9893	6.1987	6.4090	6.6200	6.8319
0.975	5.5557	5.7816	6.0093	6.2385	6.4691	6.7008	6.9342	7.1684	7.4035	7.6395
0.990	6.2627	6.5159	6.7711	7.0283	7.2871	7.5475	7.8092	8.0721	8.3362	8.6013
0.995	6.7525	7.0249	7.2992	7.5759	7.8545	8.1347	8.4164	8.7001	8.9839	9.2692

$P^*\backslash\nu$	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
0.600	3.6018	3.7173	3.8329	3.9486	4.0644	4.1803	4.2962	4.4122	4.5283	4.6445
0.650	3.9130	4.0370	4.1612	4.2856	4.4101	4.5347	4.6594	4.7842	4.9091	5.0341
0.700	4.2441	4.3774	4.5108	4.6444	4.7782	4.9122	5.0462	5.1805	5.3148	5.4493
0.750	4.6054	4.7487	4.8922	5.0360	5.1800	5.3242	5.4686	5.6132	5.7579	5.9028
0.800	5.0124	5.1672	5.3223	5.4776	5.6332	5.7891	5.9452	6.1015	6.2579	6.4146
0.850	5.4936	5.6620	5.8308	5.9999	6.1693	6.3391	6.5091	6.6794	6.8499	7.0206
0.900	6.1096	6.2956	6.4821	6.6690	6.8563	7.0439	7.2319	7.4202	7.6088	7.7977
0.950	7.0443	7.2575	7.4712	7.6855	7.9002	8.1153	8.3310	8.5470	8.7633	8.9800
0.975	7.8763	8.1138	8.3521	8.5909	8.8303	9.0703	9.3107	9.5516	9.7929	10.0346
0.990	8.8673	9.1342	9.4022	9.6703	9.9393	10.2090	10.4791	10.7501	11.0213	11.2930
0.995	9.5558	9.8432	10.1314	10.4204	10.7101	11.0006	11.2919	11.5834	11.8740	12.1685

$P^*\backslash\nu$	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
0.600	4.7607	4.8770	4.9933	5.1096	5.2261	5.3425	5.4590	5.5756	5.6921	5.8088
0.650	5.1592	5.2844	5.4096	5.5349	5.6603	5.7858	5.9113	6.0368	6.1625	6.2881
0.700	5.5839	5.7186	5.8534	5.9882	6.1232	6.2583	6.3935	6.5287	6.6639	6.7993
0.750	6.0478	6.1930	6.3383	6.4837	6.6291	6.7748	6.9204	7.0662	7.2121	7.3581
0.800	6.5715	6.7285	6.8856	7.0429	7.2004	7.3579	7.5156	7.6734	7.8313	7.9893
0.850	7.1916	7.3627	7.5340	7.7054	7.8771	8.0489	8.2208	8.3927	8.5651	8.7374
0.900	7.9868	8.1762	8.3658	8.5555	8.7455	8.9357	9.1260	9.3165	9.5071	9.6979
0.950	9.1970	9.4144	9.6320	9.8497	10.0678	10.2861	10.5046	10.7232	10.9421	11.1612
0.975	10.2767	10.5190	10.7618	11.0048	11.2473	11.4916	11.7355	11.9794	12.2236	12.4680
0.990	11.5652	11.8376	12.1105	12.3837	12.6575	12.9312	13.2052	13.4797	13.7543	14.0292
0.995	12.4612	12.7549	13.0486	13.3422	13.6377	13.9327	14.2278	14.5234	14.8194	15.1153

$P^*\backslash\nu$	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
0.600	5.9254	6.0421	6.1588	6.2755	6.3923	6.5091	6.6259	6.7428	6.8596	6.9766
0.650	6.4138	6.5395	6.6654	6.7912	6.9171	7.0430	7.1690	7.2950	7.4210	7.5470
0.700	6.9347	7.0702	7.2057	7.3413	7.4769	7.6126	7.7483	7.8840	8.0198	8.1557
0.750	7.5041	7.6502	7.7964	7.9426	8.0890	8.2353	8.3818	8.5282	8.6748	8.8213
0.800	8.1474	8.3056	8.4638	8.6224	8.7806	8.9391	9.0976	9.2562	9.4149	9.5737
0.850	8.9098	9.0823	9.2550	9.4277	9.6005	9.7734	9.9464	10.1194	10.2926	10.4658
0.900	9.8888	10.0798	10.2710	10.4622	10.6536	10.8451	11.0367	11.2284	11.4201	11.6120
0.950	11.3804	11.5998	11.8193	12.0390	12.2588	12.4788	12.6988	12.9190	13.1393	13.3596
0.975	12.7128	12.9576	13.2026	13.4478	13.6931	13.9385	14.1841	14.4299	14.6757	14.9217
0.990	14.3043	14.5795	14.8551	15.1308	15.4065	15.6826	15.9589	16.2352	16.5116	16.7882
0.995	15.4116	15.7082	16.0050	16.3023	16.5992	16.8964	17.1940	17.4912	17.7895	18.0870

$P^*\backslash\nu$	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9
0.600	7.0934	7.2104	7.3273	7.4443	7.5613	7.6782	7.7953	7.9123	8.0293	8.1440
0.650	7.6731	7.7992	7.9253	8.0515	8.1777	8.3039	8.4301	8.5563	8.6826	8.8089
0.700	8.2916	8.4275	8.5634	8.6994	8.8354	8.9714	9.1075	9.2436	9.3797	9.5159
0.750	8.9679	9.1146	9.2613	9.4081	9.5548	9.7017	9.8485	9.9954	10.1423	10.2892
0.800	9.7325	9.8913	10.0502	10.2092	10.3681	10.5272	10.6863	10.8454	11.0046	11.1637
0.850	10.6391	10.8124	10.9858	11.1595	11.3328	11.5063	11.6800	11.8536	12.0273	12.2010
0.900	11.8039	11.9959	12.1880	12.3801	12.5723	12.7645	12.9569	13.1493	13.3418	13.5342
0.950	13.5801	13.8007	14.0214	14.2419	14.4629	14.6838	14.9049	15.1258	15.3470	15.5681
0.975	15.1677	15.4138	15.6605	15.9065	16.1525	16.3996	16.6462	16.8930	17.1398	17.3867
0.990	17.0647	17.3419	17.6189	17.8960	18.1732	18.4507	18.7279	19.0054	19.2831	19.5604
0.995	18.3856	18.6840	18.9825	19.2806	19.5793	19.8778	20.1769	20.4757	20.7748	21.0739

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Table 6.1: $k = 5$

$P^*\backslash\nu$	7.0	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
0.600	8.2635	8.3806	8.4976	8.6149	8.7319	8.8490	8.9661	9.0833	9.2005	9.3178
0.650	8.9352	9.0615	9.1878	9.3142	9.4405	9.5669	9.6934	9.8198	9.9462	10.0727
0.700	9.6520	9.7882	9.9244	10.0608	10.1969	10.3332	10.4698	10.6058	10.7422	10.8785
0.750	10.4363	10.5833	10.7304	10.8773	11.0245	11.1715	11.3187	11.4659	11.6131	11.7603
0.800	11.3230	11.4822	11.6415	11.8009	11.9603	12.1196	12.2791	12.4385	12.5980	12.7575
0.850	12.3748	12.5487	12.7225	12.8965	13.0704	13.2444	13.4184	13.5924	13.7665	13.9406
0.900	13.7280	13.9169	14.1120	14.3047	14.4975	14.6902	14.8831	15.0759	15.2688	15.4615
0.950	15.7894	16.0131	16.2319	16.4534	16.6749	16.8964	17.1179	17.3396	17.5612	17.7831
0.975	17.6336	17.8807	18.1278	18.3750	18.6222	18.8694	19.1167	19.3638	19.6116	19.8591
0.990	19.8383	20.1159	20.3940	20.6720	20.9502	21.2282	21.5062	21.7846	22.0628	22.3412
0.995	21.3745	21.6714	21.9717	22.2717	22.5711	22.8704	23.1698	23.4701	23.7671	24.0694

$P^*\backslash\nu$	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9
0.600	9.4348	9.5520	9.6692	9.7864	9.9036	10.0208	10.1381	10.2553	10.3725	10.4898
0.650	10.1991	10.3255	10.4520	10.5785	10.7050	10.8315	10.9581	11.0847	11.2111	11.3377
0.700	11.0149	11.1512	11.2877	11.4241	11.5605	11.6969	11.8334	11.9698	12.1064	12.2428
0.750	11.9075	12.0548	12.2020	12.3493	12.4966	12.6439	12.7912	12.9386	13.0859	13.2333
0.800	12.9170	13.0765	13.2361	13.3957	13.5553	13.7149	13.8746	14.0343	14.1939	14.3536
0.850	14.1147	14.2889	14.4631	14.6373	14.8116	14.9858	15.1601	15.3342	15.5088	15.6831
0.900	15.6547	15.8477	16.0407	16.2337	16.4268	16.6199	16.8130	17.0062	17.1994	17.3926
0.950	18.0046	18.2264	18.4483	18.6701	18.8920	19.1139	19.3359	19.5577	19.7799	20.0020
0.975	20.1066	20.3542	20.6019	20.8494	21.0968	21.3449	21.5927	21.8405	22.0887	22.3363
0.990	22.6197	22.8981	23.1765	23.4554	23.7339	24.0125	24.2913	24.5699	24.8487	25.1275
0.995	24.3694	24.6693	24.9681	25.2700	25.5699	25.8707	26.1703	26.4709	26.7711	27.0714

$P^*\backslash\nu$	9.0	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
0.600	10.6070	10.7243	10.8416	10.9590	11.0761	11.1934	11.3107	11.4280	11.5453	11.6628
0.650	11.4642	11.5907	11.7174	11.8440	11.9706	12.0971	12.2238	12.3504	12.4770	12.6036
0.700	12.3793	12.5158	12.6523	12.7889	12.9254	13.0619	13.1985	13.3351	13.4717	13.6083
0.750	13.3807	13.5281	13.6755	13.8230	13.9704	14.1179	14.2653	14.4128	14.5603	14.7078
0.800	14.5134	14.6731	14.8329	14.9926	15.1524	15.3122	15.4720	15.6319	15.7917	15.9515
0.850	15.8575	16.0319	16.2063	16.3807	16.5551	16.7296	16.9041	17.0787	17.2531	17.4277
0.900	17.5859	17.7791	17.9724	18.1657	18.3591	18.5525	18.7457	18.9392	19.1326	19.3260
0.950	20.2241	20.4461	20.6683	20.8905	21.1127	21.3349	21.5572	21.7794	22.0017	22.2240
0.975	22.5842	22.8323	23.0801	23.3283	23.5762	23.8211	24.0724	24.2944	24.5688	24.8169
0.990	25.4063	25.6853	25.9644	26.2432	26.5222	26.8012	27.0805	27.3595	27.6378	27.9169
0.995	27.3720	27.6723	27.9725	28.2729	28.5738	28.8712	29.1750	29.4754	29.7760	30.0766

$P^*\backslash\nu$	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0	55.0
0.600	11.7799	17.6496	23.5232	29.3985	35.2746	41.1512	47.0280	52.9050	58.7822	64.6596
0.650	12.7303	19.0676	25.4104	31.7555	38.1016	44.4484	50.7949	57.1431	63.4903	69.8385
0.700	13.7448	20.5819	27.4258	34.2727	41.1210	47.9700	54.8196	61.6698	68.5197	75.3701
0.750	14.8553	22.2396	29.6325	37.0287	44.4268	51.8259	59.2254	66.6257	74.0260	81.4267
0.800	16.1114	24.1152	32.1292	40.1475	48.1678	56.1893	64.2116	72.2342	80.2574	88.2805
0.850	17.6022	26.3422	35.0937	43.8505	52.6097	61.3710	70.1318	78.8938	87.6564	96.4192
0.900	19.5195	29.2064	38.9078	48.6151	58.3254	68.0356	77.7499	87.4634	97.1773	106.8917
0.950	22.4466	33.5810	44.7333	55.8928	67.0556	78.2205	89.3867	100.5537	111.7213	122.8895
0.975	25.0651	37.4964	49.9478	62.4075	74.8709	87.3368	99.8052	112.2722	124.7423	137.2114
0.990	28.1969	42.1800	56.1860	70.1971	84.2205	98.2432	112.2681	126.2919	140.3208	154.3459
0.995	30.3779	45.4443	60.5305	75.6298	90.7340	105.8409	120.9477	136.0589	151.1650	166.2756

$P^*\backslash\nu$	60.0	65.0	70.0	75.0	80.0	85.0	90.0	95.0	100.0
0.600	70.5368	76.4143	82.2917	88.1692	94.0467	99.9244	105.8014	111.6794	117.5561
0.650	76.1863	82.5344	88.8825	95.2306	101.5786	107.9267	114.2749	120.6231	126.9714
0.700	82.2206	89.0709	95.9218	102.7724	109.6231	116.4739	123.3248	130.1761	137.0266
0.750	88.8274	96.2283	103.6292	111.0304	118.4315	125.8330	133.2338	140.6342	148.0363
0.800	96.3042	104.3279	112.3517	120.3753	128.3995	136.4235	144.4478	152.4715	160.4959
0.850	105.1823	113.9456	122.7087	131.4724	140.2358	148.9997	157.7627	166.5266	175.2906
0.900	116.6063	126.3213	136.0359	145.7500	155.4662	165.1821	174.8965	184.6133	194.3284
0.950	134.0582	145.2272	156.3955	167.5640	178.7334	189.9023	201.0722	212.2417	223.4108
0.975	149.6815	162.1525	174.6223	187.0925	199.5615	212.0335	224.5051	236.9756	249.4471
0.990	168.3728	182.3973	196.4279	210.4536	224.4847	238.5115	252.5399	266.5668	280.5969
0.995	181.6895	196.5027	211.6263	226.7296	241.8283	256.9545	272.0672	287.1753	302.2903

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Table 6.1: $k = 6$

$P^*\backslash\nu$	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59
0.600	0.8493	0.8614	0.8735	0.8856	0.8978	0.9099	0.9220	0.9341	0.9463	0.9584
0.650	0.9884	1.0008	1.0132	1.0256	1.0380	1.0504	1.0628	1.0753	1.0877	1.1002
0.700	1.1350	1.1477	1.1604	1.1731	1.1858	1.1986	1.2113	1.2241	1.2369	1.2497
0.750	1.2933	1.3064	1.3194	1.3324	1.3455	1.3586	1.3718	1.3849	1.3981	1.4112
0.800	1.4698	1.4832	1.4966	1.5100	1.5235	1.5370	1.5505	1.5641	1.5777	1.5913
0.850	1.6756	1.6894	1.7033	1.7172	1.7312	1.7451	1.7592	1.7732	1.7873	1.8015
0.900	1.9348	1.9492	1.9637	1.9782	1.9927	2.0074	2.0220	2.0367	2.0515	2.0663
0.950	2.3195	2.3348	2.3502	2.3656	2.3811	2.3967	2.4123	2.4281	2.4438	2.4596
0.975	2.6537	2.6698	2.6860	2.7022	2.7186	2.7350	2.7515	2.7681	2.7848	2.8015
0.990	3.0429	3.0599	3.0771	3.0944	3.1117	3.1292	3.1468	3.1644	3.1822	3.2001
0.995	3.3083	3.3259	3.3438	3.3618	3.3799	3.3981	3.4164	3.4349	3.4534	3.4721

$P^*\backslash\nu$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69
0.600	0.9705	0.9827	0.9948	1.0070	1.0192	1.0313	1.0435	1.0557	1.0679	1.0800
0.650	1.1126	1.1251	1.1376	1.1501	1.1626	1.1751	1.1877	1.2002	1.2127	1.2253
0.700	1.2625	1.2754	1.2882	1.3011	1.3140	1.3269	1.3398	1.3527	1.3656	1.3791
0.750	1.4245	1.4377	1.4509	1.4642	1.4775	1.4908	1.5041	1.5175	1.5309	1.5443
0.800	1.6050	1.6187	1.6323	1.6461	1.6598	1.6736	1.6874	1.7013	1.7151	1.7290
0.850	1.8156	1.8298	1.8441	1.8584	1.8727	1.8870	1.9014	1.9158	1.9303	1.9448
0.900	2.0811	2.0960	2.1110	2.1259	2.1410	2.1561	2.1711	2.1864	2.2016	2.2168
0.950	2.4755	2.4914	2.5074	2.5235	2.5397	2.5559	2.5721	2.5885	2.6049	2.6213
0.975	2.8183	2.8353	2.8523	2.8693	2.8865	2.9038	2.9210	2.9384	2.9559	2.9734
0.990	3.2181	3.2362	3.2543	3.2726	3.2910	3.3095	3.3280	3.3467	3.3654	3.3843
0.995	3.4908	3.5098	3.5288	3.5480	3.5671	3.5864	3.6060	3.6255	3.6451	3.6650

$P^*\backslash\nu$	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79
0.600	1.0922	1.1044	1.1166	1.1288	1.1410	1.1532	1.1654	1.1777	1.1899	1.2021
0.650	1.2379	1.2504	1.2630	1.2756	1.2882	1.3008	1.3134	1.3260	1.3387	1.3513
0.700	1.3916	1.4045	1.4175	1.4305	1.4436	1.4566	1.4696	1.4827	1.4958	1.5089
0.750	1.5577	1.5711	1.5845	1.5980	1.6115	1.6250	1.6385	1.6521	1.6656	1.6792
0.800	1.7429	1.7569	1.7708	1.7848	1.7989	1.8129	1.8270	1.8411	1.8552	1.8693
0.850	1.9593	1.9738	1.9884	2.0031	2.0177	2.0324	2.0471	2.0619	2.0766	2.0915
0.900	2.2321	2.2475	2.2629	2.2783	2.2938	2.3093	2.3249	2.3405	2.3561	2.3718
0.950	2.6378	2.6544	2.6710	2.6877	2.7045	2.7213	2.7382	2.7551	2.7721	2.7891
0.975	2.9910	3.0087	3.0265	3.0443	3.0622	3.0802	3.0983	3.1164	3.1346	3.1529
0.990	3.4033	3.4222	3.4414	3.4606	3.4797	3.4994	3.5188	3.5384	3.5581	3.5779
0.995	3.6849	3.7046	3.7250	3.7452	3.7656	3.7860	3.8065	3.8272	3.8478	3.8687

$P^*\backslash\nu$	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
0.600	1.2143	1.2266	1.2388	1.2510	1.2633	1.2755	1.2878	1.3000	1.3123	1.3246
0.650	1.3640	1.3766	1.3893	1.4020	1.4146	1.4273	1.4400	1.4527	1.4654	1.4782
0.700	1.5220	1.5351	1.5482	1.5613	1.5745	1.5877	1.6008	1.6140	1.6272	1.6404
0.750	1.6928	1.7064	1.7201	1.7337	1.7474	1.7611	1.7748	1.7885	1.8022	1.8160
0.800	1.8835	1.8977	1.9119	1.9261	1.9404	1.9547	1.9690	1.9833	1.9977	2.0121
0.850	2.1063	2.1212	2.1361	2.1511	2.1660	2.1810	2.1961	2.2111	2.2262	2.2413
0.900	2.3875	2.4033	2.4189	2.4350	2.4509	2.4668	2.4828	2.4988	2.5148	2.5309
0.950	2.8062	2.8234	2.8406	2.8578	2.8752	2.8925	2.9099	2.9274	2.9450	2.9625
0.975	3.1712	3.1896	3.2081	3.2266	3.2452	3.2639	3.2827	3.3015	3.3204	3.3393
0.990	3.5975	3.6177	3.6377	3.6578	3.6780	3.6983	3.7187	3.7392	3.7596	3.7802
0.995	3.8896	3.9105	3.9318	3.9530	3.9743	3.9957	4.0174	4.0389	4.0605	4.0824

$P^*\backslash\nu$	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.600	1.3368	1.3491	1.3614	1.3736	1.3859	1.3982	1.4105	1.4228	1.4351	1.4474
0.650	1.4909	1.5036	1.5164	1.5291	1.5419	1.5546	1.5674	1.5802	1.5930	1.6058
0.700	1.6537	1.6669	1.6802	1.6934	1.7067	1.7200	1.7333	1.7466	1.7599	1.7732
0.750	1.8298	1.8436	1.8574	1.8712	1.8850	1.8989	1.9128	1.9267	1.9406	1.9545
0.800	2.0265	2.0409	2.0553	2.0698	2.0843	2.0988	2.1133	2.1279	2.1424	2.1570
0.850	2.2565	2.2716	2.2868	2.3021	2.3173	2.3326	2.3479	2.3633	2.3787	2.3941
0.900	2.5470	2.5632	2.5794	2.5956	2.6119	2.6282	2.6445	2.6609	2.6773	2.6938
0.950	2.9802	2.9978	3.0156	3.0334	3.0512	3.0691	3.0870	3.1050	3.1230	3.1411
0.975	3.3583	3.3774	3.3965	3.4157	3.4350	3.4543	3.4737	3.4931	3.5126	3.5322
0.990	3.8009	3.8217	3.8426	3.8635	3.8845	3.9056	3.9267	3.9480	3.9693	3.9907
0.995	4.1042	4.1262	4.1481	4.1704	4.1925	4.2150	4.2375	4.2600	4.2825	4.3052

$P^*\backslash\nu$	0.991	0.992	0.993	0.994	0.995	0.996	0.997	0.998	0.999	1.000
0.600	1.4486	1.4498	1.4511	1.4523	1.4535	1.4548	1.4560	1.4572	1.4584	1.4597
0.650	1.6071	1.6083	1.6096	1.6109	1.6122	1.6135	1.6147	1.6160	1.6173	1.6186
0.700	1.7746	1.7759	1.7772	1.7785	1.7799	1.7812	1.7825	1.7839	1.7852	1.7865
0.750	1.9559	1.9573	1.9587	1.9600	1.9614	1.9628	1.9642	1.9656	1.9670	1.9684
0.800	2.1585	2.1599	2.1614	2.1629	2.1643	2.1658	2.1672	2.1687	2.1702	2.1716
0.850	2.3956	2.3973	2.3987	2.4002	2.4018	2.4033	2.4049	2.4064	2.4079	2.4095
0.900	2.6954	2.6971	2.6987	2.7003	2.7020	2.7037	2.7053	2.7070	2.7086	2.7103
0.950	3.1429	3.1447	3.1465	3.1483	3.1502	3.1520	3.1538	3.1556	3.1574	3.1592
0.975	3.5341	3.5361	3.5381	3.5400	3.5420	3.5440	3.5459	3.5479	3.5499	3.5518
0.990	3.9927	3.9949	3.9971	3.9993	4.0014	4.0035	4.0055	4.0078	4.0100	4.0121
0.995	4.3075	4.3097	4.3121	4.3143	4.3178	4.3188	4.3211	4.3234	4.3255	4.3279

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Table 6.1: $k = 6$

$P^*\backslash\nu$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
0.600	1.4597	1.5829	1.7064	1.8303	1.9545	2.0791	2.2039	2.3290	2.4544	2.5801
0.650	1.6186	1.7470	1.8761	2.0058	2.1361	2.2669	2.3983	2.5302	2.6625	2.7952
0.700	1.7865	1.9205	2.0556	2.1916	2.3284	2.4661	2.6045	2.7437	2.8834	3.0238
0.750	1.9684	2.1086	2.2502	2.3931	2.5372	2.6824	2.8286	2.9757	3.1237	3.2724
0.800	2.1716	2.3189	2.4679	2.6187	2.7711	2.9249	3.0800	3.2362	3.3935	3.5518
0.850	2.4095	2.5651	2.7232	2.8835	3.0458	3.2098	3.3755	3.5427	3.7112	3.8809
0.900	2.7103	2.8769	3.0467	3.2193	3.3944	3.5719	3.7514	3.9329	4.1160	4.3007
0.950	3.1592	3.3429	3.5309	3.7227	3.9179	4.1162	4.3173	4.5209	4.7267	4.9345
0.975	3.5518	3.7511	3.9557	4.1650	4.3785	4.5959	4.8167	5.0406	5.2671	5.4960
0.990	4.0121	4.2305	4.4554	4.6861	4.9222	5.1629	5.4077	5.6566	5.9080	6.1623
0.995	4.3279	4.5601	4.7993	5.0453	5.2973	5.5544	5.8164	6.0823	6.3520	6.6250

$P^*\backslash\nu$	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
0.600	2.7060	2.8321	2.9584	3.0850	3.2117	3.3386	3.4656	3.5928	3.7201	3.8475
0.650	2.9284	3.0618	3.1956	3.3297	3.4641	3.5987	3.7335	3.8686	4.0039	4.1393
0.700	3.1646	3.3060	3.4478	3.5900	3.7325	3.8755	4.0187	4.1622	4.3060	4.4501
0.750	3.4218	3.5718	3.7224	3.8736	4.0252	4.1773	4.3298	4.4826	4.6358	4.7894
0.800	3.7109	3.8709	4.0316	4.1929	4.3548	4.5174	4.6804	4.8439	5.0078	5.1721
0.850	4.0518	4.2236	4.3963	4.5699	4.7442	4.9192	5.0949	5.2711	5.4478	5.6251
0.900	4.4867	4.6740	4.8624	5.0519	5.2423	5.4336	5.6256	5.8184	6.0118	6.2058
0.950	5.1442	5.3555	5.5682	5.7824	5.9978	6.2142	6.4316	6.6500	6.8692	7.0892
0.975	5.7273	5.9605	6.1954	6.4320	6.6700	6.9093	7.1497	7.3913	7.6339	7.8774
0.990	6.4202	6.6798	6.9417	7.2055	7.4712	7.7379	8.0063	8.2760	8.5468	8.8186
0.995	6.9008	7.1794	7.4601	7.7430	8.0278	8.3143	8.6022	8.8916	9.1799	9.4742

$P^*\backslash\nu$	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
0.600	3.9751	4.1028	4.2305	4.3584	4.4864	4.6144	4.7425	4.8707	4.9990	5.1273
0.650	4.2749	4.4107	4.5467	4.6827	4.8189	4.9553	5.0917	5.2282	5.3649	5.5016
0.700	4.5944	4.7389	4.8836	5.0284	5.1735	5.3187	5.4641	5.6096	5.7553	5.9010
0.750	4.9432	5.0973	5.2516	5.4062	5.5610	5.7160	5.8712	6.0266	6.1822	6.3379
0.800	5.3368	5.5018	5.6672	5.8328	5.9987	6.1649	6.3313	6.4979	6.6647	6.8317
0.850	5.8028	5.9809	6.1594	6.3382	6.5174	6.6969	6.8766	7.0567	7.2369	7.4175
0.900	6.4004	6.5954	6.7910	6.9870	7.1834	7.3801	7.5772	7.7746	7.9724	8.1704
0.950	7.3098	7.5311	7.7531	7.9754	8.1984	8.4218	8.6456	8.8698	9.0944	9.3193
0.975	8.1216	8.3666	8.6123	8.8584	9.1057	9.3531	9.6011	9.8496	10.0983	10.3477
0.990	9.0914	9.3651	9.6396	9.9149	10.1908	10.4673	10.7444	11.0220	11.3005	11.5790
0.995	9.7681	10.0606	10.3552	10.6507	10.9469	11.2437	11.5413	11.8395	12.1380	12.4372

$P^*\backslash\nu$	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
0.600	5.2557	5.3841	5.5126	5.6412	5.7698	5.8984	6.0271	6.1558	6.2846	6.4134
0.650	5.6385	5.7754	5.9124	6.0494	6.1866	6.3238	6.4610	6.5984	6.7357	6.8732
0.700	6.0469	6.1929	6.3390	6.4852	6.6315	6.7779	6.9243	7.0709	7.2175	7.3641
0.750	6.4937	6.6497	6.8059	6.9621	7.1184	7.2749	7.4315	7.5881	7.7448	7.9017
0.800	6.9989	7.1662	7.3338	7.5014	7.6692	7.8371	8.0052	8.1733	8.3416	8.5100
0.850	7.5982	7.7791	7.9602	8.1415	8.3229	8.5045	8.6863	8.8682	9.0502	9.2325
0.900	8.3686	8.5670	8.7658	8.9648	9.1639	9.3632	9.5627	9.7624	9.9622	10.1622
0.950	9.5446	9.7702	9.9960	10.2222	10.4485	10.6751	10.9019	11.1289	11.3561	11.5835
0.975	10.5974	10.8474	11.0977	11.3483	11.5992	11.8505	12.1019	12.3536	12.6054	12.8576
0.990	11.8580	12.1373	12.4171	12.6971	12.9777	13.2585	13.5396	13.8210	14.1025	14.3843
0.995	12.7368	13.0372	13.3380	13.6381	13.9400	14.2406	14.5424	14.8447	15.1471	15.4497

$P^*\backslash\nu$	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
0.600	6.5423	6.6712	6.8001	6.9290	7.0580	7.1870	7.3160	7.4450	7.5741	7.7032
0.650	7.0106	7.1482	7.2857	7.4234	7.5610	7.6986	7.8364	7.9742	8.1120	8.2498
0.700	7.5109	7.6577	7.8045	7.9509	8.0984	8.2454	8.3925	8.5396	8.6867	8.8339
0.750	8.0586	8.2156	8.3726	8.5297	8.6869	8.8442	9.0015	9.1589	9.3163	9.4738
0.800	8.6784	8.8470	9.0157	9.1844	9.3532	9.5221	9.6910	9.8601	10.0292	10.1983
0.850	9.4146	9.5970	9.7795	9.9621	10.1447	10.3275	10.5103	10.6932	10.8763	11.0593
0.900	10.3623	10.5624	10.7629	10.9634	11.1638	11.3646	11.5654	11.7663	11.9673	12.1684
0.950	11.8111	12.0388	12.2667	12.4947	12.7229	12.9511	13.1795	13.4081	13.6367	13.8654
0.975	13.1099	13.3623	13.6145	13.8679	14.1208	14.3739	14.6273	14.8801	15.1342	15.3878
0.990	14.6665	14.9487	15.2311	15.5138	15.7968	16.0798	16.3628	16.6463	16.9298	17.2133
0.995	15.7524	16.0557	16.3552	16.6628	16.9662	17.2703	17.5741	17.8784	18.1831	18.4875

$P^*\backslash\nu$	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9
0.600	7.8323	7.9615	8.0906	8.2198	8.3490	8.4782	8.6075	8.7367	8.8660	8.9952
0.650	8.3877	8.5256	8.6635	8.8014	8.9393	9.0774	9.2154	9.3535	9.4915	9.6296
0.700	8.9811	9.1284	9.2757	9.4230	9.5704	9.7178	9.8652	10.0127	10.1602	10.3077
0.750	9.6313	9.7888	9.9464	10.1041	10.2618	10.4195	10.5773	10.7351	10.8929	11.0508
0.800	10.3675	10.5368	10.7061	10.8755	11.0449	11.2143	11.3838	11.5534	11.7230	11.8926
0.850	11.2425	11.4257	11.6089	11.7922	11.9756	12.1591	12.3426	12.5261	12.7098	12.8934
0.900	12.3696	12.5708	12.7721	12.9735	13.1749	13.3765	13.5780	13.7797	13.9814	14.1831
0.950	14.0943	14.3232	14.5523	14.7814	15.0106	15.2398	15.4692	15.6987	15.9281	16.1577
0.975	15.6416	15.8955	16.1495	16.4036	16.6584	16.9144	17.1665	17.4209	17.6754	17.9300
0.990	17.4972	17.7811	18.0650	18.3503	18.6333	18.9177	19.2021	19.4867	19.7713	20.0560
0.995	18.7922	19.0970	19.4024	19.7071	20.0124	20.3177	20.6230	20.9288	21.2347	21.5402

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Table 6.1: $k = 6$

$P^*\backslash\nu$	7.0	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
0.600	9.1245	9.2539	9.3832	9.5125	9.6419	9.7712	9.9006	10.0299	10.1593	10.2887
0.650	9.7677	9.9058	10.0440	10.1821	10.3203	10.4585	10.5967	10.7349	10.8732	11.0114
0.700	10.4552	10.6028	10.7504	10.8980	11.0456	11.1933	11.3409	11.4886	11.6363	11.7840
0.750	11.2087	11.3666	11.5246	11.6826	11.8406	11.9986	12.1567	12.3148	12.4729	12.6310
0.800	12.0623	12.2320	12.4017	12.5715	12.7413	12.9111	13.0809	13.2508	13.4208	13.5907
0.850	13.0771	13.2608	13.4445	13.6284	13.8122	13.9961	14.1800	14.3639	14.5479	14.7319
0.900	14.3849	14.5868	14.7887	14.9906	15.1926	15.3947	15.5967	15.7989	16.0010	16.2030
0.950	16.3873	16.6171	16.8468	17.0767	17.3065	17.5364	17.7663	17.9964	18.2264	18.4566
0.975	18.1847	18.4396	18.6943	18.9491	19.2040	19.4591	19.7142	19.9693	20.2245	20.4796
0.990	20.3408	20.6257	20.9107	21.1956	21.4807	21.7658	22.0510	22.3363	22.6204	22.9069
0.995	21.8463	22.1520	22.4580	22.7643	23.0703	23.3763	23.6828	23.9892	24.2948	24.6018

$P^*\backslash\nu$	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9
0.600	10.4181	10.5476	10.6770	10.8064	10.9359	11.0653	11.1948	11.3228	11.4538	11.5832
0.650	11.1496	11.2879	11.4262	11.5645	11.7028	11.8412	11.9795	12.1178	12.2562	12.3946
0.700	11.9318	12.0795	12.2273	12.3751	12.5229	12.6707	12.8187	12.9664	13.1143	13.2622
0.750	12.7892	12.9473	13.1055	13.2637	13.4220	13.5802	13.7384	13.8967	14.0551	14.2134
0.800	13.7607	13.9307	14.1007	14.2707	14.4408	14.6109	14.7809	14.9511	15.1212	15.2914
0.850	14.9160	15.1001	15.2842	15.4683	15.6521	15.8366	16.0208	16.2050	16.3893	16.5736
0.900	16.4055	16.6077	16.8100	17.0123	17.2147	17.4171	17.6195	17.8234	18.0244	18.2269
0.950	18.6867	18.9169	19.1471	19.3774	19.6077	19.8380	20.0684	20.2988	20.5293	20.7597
0.975	20.7350	20.9903	21.2458	21.5012	21.7565	22.0121	22.2675	22.5231	22.7788	23.0345
0.990	23.1927	23.4779	23.8086	24.0496	24.3351	24.6208	24.9066	25.1925	25.4781	25.7644
0.995	24.9084	25.2150	25.5218	25.8288	26.1352	26.4426	26.7490	27.0561	27.3631	27.6715

$P^*\backslash\nu$	9.0	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
0.600	11.7127	11.8422	11.9717	12.1012	12.2307	12.3603	12.4898	12.6193	12.7489	12.8786
0.650	12.5329	12.6713	12.8097	12.9481	13.0865	13.2250	13.3634	13.5018	13.6402	13.7787
0.700	13.4101	13.5579	13.7059	13.8538	14.0017	14.1496	14.2976	14.4456	14.5935	14.7415
0.750	14.3717	14.5300	14.6884	14.8468	15.0051	15.1635	15.3220	15.4804	15.6388	15.7972
0.800	15.4615	15.6317	15.8020	15.9722	16.1424	16.3127	16.4830	16.6530	16.8235	16.9939
0.850	16.7579	16.9422	17.1265	17.3109	17.4952	17.6796	17.8640	18.0484	18.2329	18.4174
0.900	18.4294	18.6319	18.8345	19.0372	19.2397	19.4424	19.6450	19.8477	20.0504	20.2531
0.950	20.9902	21.2208	21.4513	21.6820	21.9126	22.1433	22.3739	22.6046	22.8354	23.0660
0.975	23.2902	23.5458	23.8012	24.0574	24.3132	24.5690	24.8249	25.0810	25.3366	25.5927
0.990	26.0500	26.3358	26.6220	26.9075	27.1941	27.4802	27.7660	28.0524	28.3386	28.6251
0.995	27.9772	28.2837	28.5912	28.8986	29.2060	29.5132	29.8204	30.1272	30.4347	30.7419

$P^*\backslash\nu$	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0	55.0
0.600	13.0080	19.4899	25.9762	32.4642	38.9532	45.4426	51.9324	58.4223	64.9124	71.4026
0.650	13.9172	20.8459	27.7807	34.7176	41.6558	48.5947	55.5343	62.4737	69.4135	76.3535
0.700	14.8895	22.2966	29.7110	37.1285	44.5475	51.9674	59.3878	66.8087	74.2299	81.6507
0.750	15.9557	23.8876	31.8286	39.7733	47.7200	55.6674	63.6156	71.5644	79.5133	87.4625
0.800	17.1642	25.6916	34.2299	42.7725	51.3173	59.8633	68.4102	76.9576	85.5053	94.0533
0.850	18.6018	27.8384	37.0877	46.3422	55.5991	64.8577	74.1171	83.3772	92.6376	101.8984
0.900	20.4559	30.6079	40.7751	50.9483	61.1246	71.3026	81.4814	91.6612	101.8417	112.0226
0.950	23.2969	34.8537	46.4290	58.0106	69.5971	81.1853	92.7747	104.3652	115.9562	127.5475
0.975	25.8487	38.6685	51.5091	64.3587	77.2118	90.0680	102.9250	115.7828	128.6418	141.5007
0.990	28.9111	43.2484	57.6089	71.9804	86.3538	100.7319	115.1107	129.4921	143.8110	158.2541
0.995	31.0495	46.4468	61.8567	77.3065	92.7396	108.1817	123.6269	139.0674	154.5122	169.9574

$P^*\backslash\nu$	60.0	65.0	70.0	75.0	80.0	85.0	90.0	95.0	100.0
0.600	77.8929	84.3833	90.8736	97.3641	103.8544	110.3450	116.8352	123.3262	129.8167
0.650	83.2935	90.2337	97.1740	104.1142	111.0545	117.9949	124.9354	131.8756	138.8163
0.700	89.0723	96.4940	103.9153	111.3369	118.7585	126.1802	133.6020	141.0238	148.4456
0.750	95.4119	103.3614	111.3111	119.2607	127.2105	135.1602	143.1101	151.0601	159.0100
0.800	102.6014	111.1500	119.6983	128.2470	136.7959	145.3443	153.8934	162.4402	170.9908
0.850	111.1595	120.4210	129.6820	138.9435	148.2052	157.4668	166.7266	175.9901	185.2515
0.900	122.2033	132.3845	142.5653	152.7473	162.9287	173.1105	183.2923	193.4742	203.6565
0.950	139.1397	150.7312	162.3246	173.9156	185.5080	197.1006	208.6930	220.2864	231.8789
0.975	154.3604	167.2209	180.0815	192.9412	205.8027	218.6637	231.5238	244.3845	257.2462
0.990	172.6329	187.0204	201.4026	215.7867	230.1686	244.5536	258.9359	273.3215	287.7073
0.995	185.4045	200.8508	216.2983	231.7451	247.1908	262.6365	278.0710	293.5309	308.9750

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Table 6.1: $k = 7$

$P^*\backslash\nu$	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59
0.600	0.8991	0.9122	0.9253	0.9384	0.9515	0.9646	0.9777	0.9909	1.0040	1.0171
0.650	1.0378	1.0512	1.0645	1.0779	1.0913	1.1047	1.1181	1.1315	1.1449	1.1584
0.700	1.1840	1.1977	1.2113	1.2250	1.2387	1.2524	1.2661	1.2799	1.2936	1.3074
0.750	1.3419	1.3559	1.3698	1.3838	1.3979	1.4119	1.4260	1.4400	1.4542	1.4683
0.800	1.5178	1.5322	1.5465	1.5609	1.5752	1.5897	1.6041	1.6186	1.6331	1.6476
0.850	1.7230	1.7378	1.7525	1.7673	1.7822	1.7971	1.8120	1.8269	1.8419	1.8569
0.900	1.9815	1.9967	2.0121	2.0274	2.0428	2.0583	2.0738	2.0894	2.1050	2.1206
0.950	2.3650	2.3811	2.3972	2.4135	2.4298	2.4461	2.4626	2.4791	2.4956	2.5122
0.975	2.6981	2.7149	2.7319	2.7489	2.7660	2.7832	2.8004	2.8178	2.8352	2.8527
0.990	3.0860	3.1037	3.1216	3.1396	3.1576	3.1758	3.1940	3.2124	3.2309	3.2494
0.995	3.3504	3.3689	3.3874	3.4061	3.4248	3.4436	3.4626	3.4818	3.5009	3.5202

$P^*\backslash\nu$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69
0.600	1.0303	1.0434	1.0566	1.0696	1.0828	1.0960	1.1092	1.1223	1.1355	1.1487
0.650	1.1718	1.1853	1.1987	1.2122	1.2257	1.2392	1.2527	1.2662	1.2797	1.2933
0.700	1.3211	1.3349	1.3487	1.3626	1.3764	1.3902	1.4041	1.4180	1.4319	1.4458
0.750	1.4824	1.4966	1.5108	1.5250	1.5392	1.5534	1.5677	1.5820	1.5963	1.6106
0.800	1.6622	1.6768	1.6914	1.7060	1.7207	1.7354	1.7501	1.7648	1.7796	1.7944
0.850	1.8720	1.8871	1.9022	1.9168	1.9325	1.9477	1.9630	1.9783	1.9936	2.0089
0.900	2.1363	2.1520	2.1678	2.1836	2.1995	2.2154	2.2314	2.2474	2.2634	2.2795
0.950	2.5289	2.5456	2.5624	2.5793	2.5962	2.6132	2.6302	2.6473	2.6644	2.6817
0.975	2.8702	2.8879	2.9056	2.9234	2.9412	2.9592	2.9772	2.9953	3.0135	3.0317
0.990	3.2679	3.2868	3.3056	3.3246	3.3437	3.3627	3.3819	3.4013	3.4206	3.4401
0.995	3.5395	3.5592	3.5788	3.5984	3.6183	3.6383	3.6584	3.6785	3.6988	3.7193

$P^*\backslash\nu$	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79
0.600	1.1619	1.1751	1.1883	1.2014	1.2147	1.2278	1.2411	1.2543	1.2675	1.2807
0.650	1.3068	1.3203	1.3339	1.3474	1.3610	1.3746	1.3882	1.4018	1.4154	1.4290
0.700	1.4597	1.4736	1.4876	1.5015	1.5155	1.5295	1.5435	1.5575	1.5715	1.5855
0.750	1.6249	1.6393	1.6536	1.6680	1.6824	1.6969	1.7113	1.7258	1.7403	1.7548
0.800	1.8092	1.8240	1.8389	1.8538	1.8687	1.8836	1.8986	1.9135	1.9285	1.9436
0.850	2.0243	2.0397	2.0552	2.0707	2.0862	2.1017	2.1173	2.1329	2.1485	2.1642
0.900	2.2956	2.3118	2.3280	2.3442	2.3605	2.3768	2.3932	2.4096	2.4261	2.4425
0.950	2.6989	2.7162	2.7336	2.7511	2.7686	2.7861	2.8037	2.8214	2.8391	2.8568
0.975	3.0500	3.0684	3.0868	3.1053	3.1239	3.1426	3.1613	3.1801	3.1995	3.2179
0.990	3.4597	3.4793	3.4991	3.5190	3.5389	3.5589	3.5788	3.5994	3.6195	3.6399
0.995	3.7397	3.7602	3.7810	3.8017	3.8226	3.8436	3.8647	3.8859	3.9071	3.9286

$P^*\backslash\nu$	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
0.600	1.2939	1.3071	1.3204	1.3336	1.3468	1.3601	1.3733	1.3866	1.3998	1.4131
0.650	1.4426	1.4562	1.4699	1.4835	1.4972	1.5108	1.5245	1.5382	1.5518	1.5655
0.700	1.5996	1.6136	1.6277	1.6418	1.6559	1.6700	1.6841	1.6982	1.7124	1.7265
0.750	1.7693	1.7838	1.7983	1.8129	1.8275	1.8421	1.8567	1.8713	1.8860	1.9007
0.800	1.9586	1.9737	1.9888	2.0039	2.0190	2.0342	2.0494	2.0646	2.0798	2.0951
0.850	2.1799	2.1956	2.2113	2.2271	2.2429	2.2588	2.2746	2.2905	2.3064	2.3224
0.900	2.4591	2.4756	2.4922	2.5089	2.5256	2.5423	2.5590	2.5758	2.5926	2.6097
0.950	2.8747	2.8925	2.9105	2.9284	2.9465	2.9645	2.9827	3.0009	3.0191	3.0374
0.975	3.2370	3.2560	3.2751	3.2944	3.3136	3.3329	3.3523	3.3718	3.3913	3.4109
0.990	3.6603	3.6809	3.7015	3.7222	3.7430	3.7639	3.7847	3.8058	3.8269	3.8480
0.995	3.9501	3.9716	3.9933	4.0150	4.0369	4.0590	4.0809	4.1031	4.1253	4.1476

$P^*\backslash\nu$	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.600	1.4263	1.4396	1.4529	1.4663	1.4794	1.4927	1.5060	1.5193	1.5325	1.5458
0.650	1.5793	1.5929	1.6066	1.6203	1.6341	1.6478	1.6615	1.6753	1.6890	1.7028
0.700	1.7407	1.7548	1.7690	1.7832	1.7974	1.8117	1.8259	1.8401	1.8544	1.8686
0.750	1.9153	1.9300	1.9447	1.9595	1.9742	1.9890	2.0037	2.0185	2.0333	2.0482
0.800	2.1103	2.1256	2.1410	2.1563	2.1716	2.1870	2.2024	2.2178	2.2333	2.2487
0.850	2.3384	2.3544	2.3704	2.3865	2.4026	2.4187	2.4348	2.4510	2.4672	2.4834
0.900	2.6264	2.6433	2.6603	2.6773	2.6944	2.7114	2.7286	2.7457	2.7629	2.7801
0.950	3.0557	3.0741	3.0925	3.1110	3.1295	3.1481	3.1667	3.1854	3.2041	3.2229
0.975	3.4305	3.4502	3.4700	3.4898	3.5097	3.5297	3.5500	3.5698	3.5899	3.6101
0.990	3.8693	3.8907	3.9120	3.9335	3.9551	3.9767	3.9985	4.0203	4.0421	4.0665
0.995	4.1699	4.1925	4.2151	4.2372	4.2605	4.2834	4.3063	4.3293	4.3525	4.3756

$P^*\backslash\nu$	0.991	0.992	0.993	0.994	0.995	0.996	0.997	0.998	0.999	1.000
0.600	1.5472	1.5485	1.5498	1.5511	1.5525	1.5538	1.5551	1.5565	1.5578	1.5591
0.650	1.7042	1.7055	1.7069	1.7083	1.7097	1.7110	1.7124	1.7138	1.7152	1.7165
0.700	1.8701	1.8715	1.8729	1.8744	1.8758	1.8772	1.8786	1.8801	1.8815	1.8829
0.750	2.0496	2.0511	2.0526	2.0541	2.0556	2.0571	2.0585	2.0600	2.0615	2.0630
0.800	2.2503	2.2518	2.2534	2.2549	2.2565	2.2580	2.2595	2.2611	2.2626	2.2642
0.850	2.4850	2.4866	2.4883	2.4899	2.4915	2.4931	2.4948	2.4964	2.4980	2.4996
0.900	2.7818	2.7835	2.7853	2.7870	2.7887	2.7904	2.7922	2.7939	2.7956	2.7973
0.950	3.2247	3.2266	3.2285	3.2305	3.2323	3.2342	3.2361	3.2379	3.2398	3.2417
0.975	3.6121	3.6141	3.6161	3.6182	3.6202	3.6222	3.6242	3.6263	3.6283	3.6303
0.990	4.0663	4.0685	4.0707	4.0729	4.0750	4.0772	4.0795	4.0817	4.0839	4.0860
0.995	4.3781	4.3803	4.3826	4.3848	4.3873	4.3896	4.3922	4.3942	4.3966	4.3990

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Table 6.1: $k = 7$

$P^*\backslash\nu$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
0.600	1.5591	1.6922	1.8257	1.9594	2.0935	2.2278	2.3624	2.4972	2.6323	2.7677
0.650	1.7165	1.8546	1.9932	2.1325	2.2723	2.4126	2.5533	2.6946	2.8362	2.9783
0.700	1.8829	2.0262	2.1705	2.3156	2.4616	2.6084	2.7558	2.9039	3.0527	3.2019
0.750	2.0630	2.2121	2.3626	2.5143	2.6671	2.8210	2.9758	3.1316	3.2881	3.4453
0.800	2.2642	2.4200	2.5775	2.7367	2.8974	3.0594	3.2227	3.3871	3.5526	3.7190
0.850	2.4996	2.6634	2.8295	2.9977	3.1678	3.3396	3.5130	3.6879	3.8642	4.0415
0.900	2.7973	2.9715	3.1487	3.3287	3.5111	3.6958	3.8825	4.0711	4.2613	4.4531
0.950	3.2417	3.4319	3.6268	3.8251	4.0268	4.2315	4.4390	4.6489	4.8612	5.0754
0.975	3.6303	3.8357	4.0463	4.2615	4.4809	4.7041	4.9307	5.1603	5.3926	5.6274
0.990	4.0860	4.3099	4.5403	4.7761	5.0175	5.2631	5.5131	5.7668	6.0238	6.2838
0.995	4.3990	4.6358	4.8801	5.1310	5.3848	5.6497	5.9163	6.1871	6.4617	6.7395

$P^*\backslash\nu$	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
0.600	2.9032	3.0390	3.1749	3.3110	3.4473	3.5838	3.7204	3.8571	3.9940	4.1310
0.650	3.1207	3.2634	3.4064	3.5497	3.6933	3.8372	3.9812	4.1255	4.2699	4.4146
0.700	3.3517	3.5019	3.6526	3.8037	3.9551	4.1069	4.2590	4.4114	4.5640	4.7169
0.750	3.6033	3.7618	3.9209	4.0806	4.2407	4.4012	4.5622	4.7236	4.8853	5.0473
0.800	3.8862	4.0543	4.2230	4.3925	4.5625	4.7331	4.9043	5.0759	5.2479	5.4203
0.850	4.2200	4.3995	4.5799	4.7611	4.9430	5.1257	5.3090	5.4929	5.6773	5.8622
0.900	4.6462	4.8407	5.0362	5.2328	5.4303	5.6287	5.8279	6.0278	6.2284	6.4296
0.950	5.2914	5.5091	5.7283	5.9489	6.1706	6.3935	6.6174	6.8422	7.0679	7.2943
0.975	5.8644	6.1034	6.3443	6.5867	6.8306	7.0757	7.3221	7.5696	7.8181	8.0676
0.990	6.5464	6.8114	7.0786	7.3476	7.6184	7.8907	8.1645	8.4398	8.7157	8.9930
0.995	7.0203	7.3036	7.5894	7.8773	8.1672	8.4587	8.7514	9.0460	9.3411	9.6384

$P^*\backslash\nu$	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
0.600	4.2680	4.4053	4.5426	4.6800	4.8175	4.9551	5.0927	5.2305	5.3683	5.5061
0.650	4.5594	4.7044	4.8496	4.9948	5.1403	5.2872	5.4315	5.5772	5.7231	5.8690
0.700	4.8701	5.0235	5.1770	5.3308	5.4847	5.6388	5.7930	5.9474	6.1019	6.2566
0.750	5.2096	5.3722	5.5351	5.6982	5.8615	6.0250	6.1888	6.3527	6.5167	6.6810
0.800	5.5931	5.7663	5.9397	6.1135	6.2876	6.4619	6.6364	6.8112	6.9861	7.1613
0.850	6.0476	6.2334	6.4196	6.6062	6.7931	6.9803	7.1677	7.3555	7.5435	7.7325
0.900	6.6313	6.8336	7.0364	7.2395	7.4431	7.6471	7.8514	8.0561	8.2610	8.4663
0.950	7.5214	7.7492	7.9776	8.2066	8.4361	8.6660	8.8964	9.1272	9.3584	9.5899
0.975	8.3178	8.5688	8.8205	9.0729	9.3258	9.5793	9.8334	10.0879	10.3428	10.5981
0.990	9.2713	9.5504	9.8304	10.1111	10.3924	10.6745	10.9570	11.2402	11.5239	11.8080
0.995	9.9364	10.2354	10.5351	10.8356	11.1373	11.4388	11.7419	12.0452	12.3489	12.6533

$P^*\backslash\nu$	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
0.600	5.6441	5.7821	5.9201	6.0582	6.1964	6.3346	6.4728	6.6111	6.7495	6.8878
0.650	6.0151	6.1612	6.3075	6.4537	6.6001	6.7466	6.8930	7.0396	7.1862	7.3329
0.700	6.4114	6.5663	6.7213	6.8764	7.0316	7.1868	7.3422	7.4976	7.6531	7.8087
0.750	6.8454	7.0099	7.1746	7.3394	7.5043	7.6692	7.8344	7.9996	8.1649	8.3303
0.800	7.3366	7.5122	7.6878	7.8637	8.0396	8.2158	8.3920	8.5683	8.7448	8.9214
0.850	7.9202	8.1089	8.2978	8.4868	8.6760	8.8654	9.0549	9.2446	9.4344	9.6243
0.900	8.6718	8.8775	9.0835	9.2897	9.4961	9.7027	9.9095	10.1165	10.3235	10.5308
0.950	9.8218	10.0539	10.2864	10.5191	10.7521	10.9853	11.2187	11.4523	11.6862	11.9202
0.975	10.8538	11.1099	11.3663	11.6233	11.8801	12.1373	12.3949	12.6527	12.9107	13.1689
0.990	12.0925	12.3774	12.6629	12.9486	13.2330	13.5210	13.8076	14.0946	14.3817	14.6693
0.995	12.9585	13.2633	13.5690	13.8750	14.1816	14.4882	14.7951	15.1026	15.4105	15.7178

$P^*\backslash\nu$	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
0.600	7.0263	7.1647	7.3032	7.4417	7.5802	7.7188	7.8574	7.9960	8.1346	8.2733
0.650	7.4796	7.6264	7.7732	7.9201	8.0670	8.2139	8.3609	8.5079	8.6550	8.8021
0.700	7.9644	8.1201	8.2758	8.4317	8.5876	8.7435	8.8995	9.0555	9.2116	9.3677
0.750	8.4958	8.6614	8.8270	8.9927	9.1587	9.3243	9.4902	9.6561	9.8221	9.9879
0.800	9.0980	9.2748	9.4517	9.6286	9.8056	9.9827	10.1599	10.3371	10.5144	10.6918
0.850	9.8144	10.0045	10.1948	10.3852	10.5757	10.7652	10.9569	11.1476	11.3384	11.5293
0.900	10.7383	10.9458	11.1535	11.3613	11.5692	11.7772	11.9853	12.1935	12.4018	12.6102
0.950	12.1544	12.3887	12.6232	12.8579	13.0927	13.3277	13.5628	13.7979	14.0332	14.2687
0.975	13.4276	13.6860	13.9448	14.2037	14.4629	14.7222	14.9815	15.2411	15.5009	15.7606
0.990	14.9569	15.2446	15.5328	15.8231	16.1096	16.3982	16.6870	16.9761	17.2646	17.5544
0.995	16.0265	16.3347	16.6435	16.9524	17.2607	17.5704	17.8798	18.0586	18.4992	18.8088

$P^*\backslash\nu$	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9
0.600	8.4120	8.5507	8.6895	8.8282	8.9670	9.1058	9.2446	9.3834	9.5223	9.6612
0.650	8.9492	9.0963	9.2435	9.3907	9.5380	9.6852	9.8325	9.9798	10.1271	10.2745
0.700	9.5239	9.6800	9.8363	9.9926	10.1489	10.3052	10.4615	10.6180	10.7744	10.9309
0.750	10.1543	10.3205	10.4866	10.6529	10.8192	10.9855	11.1519	11.3183	11.4847	11.6512
0.800	10.8692	11.0467	11.2243	11.4018	11.5795	11.7572	11.9349	12.1127	12.2906	12.4684
0.850	11.7203	11.9113	12.1024	12.2935	12.4847	12.6760	12.8673	13.0587	13.2501	13.4416
0.900	12.8187	13.0273	13.2361	13.4446	13.6534	13.8623	14.0713	14.2801	14.4892	14.6983
0.950	14.5042	14.7397	14.9755	15.2112	15.4472	15.6831	15.9191	16.1553	16.3914	16.6277
0.975	16.0206	16.2805	16.5408	16.8010	17.0614	17.3218	17.5823	17.8430	18.1038	18.3645
0.990	17.8438	18.1330	18.4229	18.7127	19.0036	19.2947	19.5826	19.8751	20.1641	20.4532
0.995	19.1187	19.4291	19.7395	20.0498	20.3606	20.6712	20.9817	21.2928	21.6031	21.9144

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Table 6.1: $k = 7$

$P^*\backslash\nu$	7.0	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
0.600	9.8000	9.9389	10.0778	10.2167	10.3557	10.4946	10.6336	10.7725	10.9115	11.0505
0.650	10.4218	10.5692	10.7166	10.8641	11.0115	11.1590	11.3065	11.4539	11.6014	11.7490
0.700	11.0870	11.2438	11.4004	11.5569	11.7135	11.8701	12.0267	12.1834	12.3400	12.4967
0.750	11.8177	11.9842	12.1508	12.3174	12.4840	12.6507	12.8173	12.9840	13.1507	13.3175
0.800	12.6463	12.8242	13.0022	13.1802	13.3583	13.5363	13.7144	13.8926	14.0707	14.2489
0.850	13.6331	13.8247	14.0162	14.2079	14.3996	14.5913	14.7830	14.9748	15.1666	15.3585
0.900	14.9074	15.1167	15.3259	15.5352	15.7446	15.9539	16.1637	16.3728	16.5823	16.7919
0.950	16.8641	17.1004	17.3369	17.5734	17.8100	18.0466	18.2833	18.5199	18.7567	18.9935
0.975	18.6252	18.8864	19.1472	19.4085	19.6694	19.9307	20.1919	20.4532	20.7146	20.9758
0.990	20.7440	21.0343	21.3248	21.6156	21.9063	22.1970	22.4878	22.7790	23.0699	23.3610
0.995	22.2258	22.5370	22.8484	23.1595	23.4715	23.7827	24.0947	24.4060	24.7181	25.0297

$P^*\backslash\nu$	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9
0.600	11.1895	11.3285	11.4676	11.6066	11.7456	11.8847	12.0237	12.1628	12.3018	12.4409
0.650	11.8965	12.0441	12.1916	12.3392	12.4868	12.6344	12.7820	12.9296	13.0773	13.2249
0.700	12.6534	12.8102	12.9668	13.1236	13.2803	13.4371	13.5939	13.7507	13.9075	14.0644
0.750	13.4842	13.6510	13.8178	13.9847	14.1515	14.3184	14.4853	14.6521	14.8191	14.9860
0.800	14.4271	14.6054	14.7837	14.9619	15.1402	15.3186	15.4969	15.6753	15.8537	16.0321
0.850	15.5504	15.7423	15.9342	16.1262	16.3182	16.5103	16.7023	16.8944	17.0865	17.2786
0.900	17.0017	17.2111	17.4208	17.6305	17.8402	18.0501	18.2598	18.4696	18.6795	18.8894
0.950	19.2303	19.4672	19.7042	19.9412	20.1782	20.4145	20.6524	20.8895	21.1266	21.3638
0.975	21.2375	21.4990	21.7606	22.0221	22.2839	22.5456	22.8072	23.0691	23.3308	23.5926
0.990	23.6521	23.9432	24.2343	24.5243	24.8172	25.1085	25.3999	25.6913	25.9829	26.2745
0.995	25.3420	25.6538	25.9657	26.2779	26.5897	26.9019	27.2146	27.5264	27.8384	28.1504

$P^*\backslash\nu$	9.0	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
0.600	12.5800	12.7191	12.8582	12.9973	13.1364	13.2756	13.4147	13.5538	13.6930	13.8321
0.650	13.3726	13.5202	13.6679	13.8156	13.9633	14.1110	14.2587	14.4064	14.5541	14.7019
0.700	14.2212	14.3781	14.5349	14.6918	14.8487	15.0056	15.1625	15.3194	15.4764	15.6333
0.750	15.1530	15.3200	15.4869	15.6539	15.8209	15.9879	16.1549	16.3220	16.4890	16.6561
0.800	16.2106	16.3890	16.5675	16.7460	16.9245	17.1028	17.2815	17.4601	17.6386	17.8173
0.850	17.4708	17.6629	17.8551	18.0473	18.2395	18.4317	18.6240	18.8163	19.0086	19.2009
0.900	19.0992	19.3091	19.5191	19.7290	19.9390	20.1490	20.3590	20.5691	20.7791	20.9892
0.950	21.6009	21.8383	22.0756	22.3129	22.5502	22.7876	23.0241	23.2623	23.4998	23.7372
0.975	23.8545	24.1164	24.3784	24.6403	24.9024	25.1643	25.4265	25.6886	25.9508	26.2128
0.990	26.5661	26.8576	27.1493	27.4410	27.7327	28.0245	28.3148	28.6084	28.9017	29.1922
0.995	28.4634	28.7759	29.0887	29.4012	29.7138	30.0261	30.3388	30.6517	30.9643	31.2768

$P^*\backslash\nu$	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0	55.0
0.600	13.9713	20.9334	27.9002	34.8688	41.8384	48.8085	55.7790	62.7497	69.7205	76.6915
0.650	14.8496	22.2430	29.6425	37.0446	44.4479	51.8520	59.2565	66.6612	74.0663	81.4715
0.700	15.7904	23.6458	31.5092	39.3758	47.2439	55.1129	62.9824	70.8525	78.7227	86.5931
0.750	16.8232	25.1868	33.5600	41.9368	50.3158	58.6957	67.0763	75.4576	83.8389	92.2206
0.800	17.9958	26.9368	35.8892	44.8457	53.8050	62.7655	71.7267	80.6884	89.6506	98.6130
0.850	19.3933	29.0232	38.6663	48.3147	57.9659	67.6185	77.2722	86.9256	96.5812	106.2361
0.900	21.1993	31.7207	42.2577	52.8008	63.3470	73.8952	84.4444	94.9949	105.5450	116.0957
0.950	23.9747	35.8682	47.7800	59.6996	71.6228	83.5485	95.4760	107.4030	119.3312	131.2600
0.975	26.4750	39.6059	52.7580	65.9186	79.0823	92.2495	105.4193	118.5896	131.7599	144.9310
0.990	29.4806	44.1054	58.7505	73.4047	88.0661	102.7283	117.3919	132.0576	146.7239	161.3901
0.995	31.5898	47.2534	62.9458	78.6470	94.3526	110.0631	125.7735	141.4828	157.2038	172.9129

$P^*\backslash\nu$	60.0	65.0	70.0	75.0	80.0	85.0	90.0	95.0	100.0
0.600	83.6614	90.6336	97.6048	104.5760	111.5473	118.5186	125.4898	132.4612	139.4324
0.650	88.8767	96.2821	103.6874	111.0931	118.4985	125.9041	133.3097	140.7152	148.1211
0.700	94.4643	102.3294	110.2052	118.0760	125.9469	133.8180	141.6885	149.5599	157.4310
0.750	100.6026	108.9845	117.3663	125.7486	134.1309	142.5131	150.8957	159.2761	167.6450
0.800	107.5758	116.5384	125.5014	134.4645	143.4275	152.3908	161.3540	170.3172	179.2960
0.850	115.8915	125.5469	135.2026	144.8581	154.5146	164.1700	173.8265	183.4821	193.1380
0.900	126.6469	137.1984	147.7492	158.3012	168.8534	179.4054	189.9575	200.5094	211.0612
0.950	143.1889	155.1183	167.0480	178.9777	190.9076	202.8376	214.7687	226.6978	238.6288
0.975	158.1025	171.2747	184.4458	197.6183	210.7918	223.9643	237.1357	250.3052	263.4825
0.990	176.0489	190.7309	205.3935	220.0622	234.7293	249.3983	264.0605	278.7341	293.4037
0.995	188.6238	204.3441	220.0590	235.7750	251.4904	267.2081	282.9240	298.6359	314.3516

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Table 6.1: $k = 8$

$P^*\backslash\nu$	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59
0.600	0.9404	0.9543	0.9682	0.9821	0.9960	1.0099	1.0239	1.0378	1.0518	1.0657
0.650	1.0787	1.0929	1.1070	1.1212	1.1354	1.1496	1.1638	1.1780	1.1923	1.2065
0.700	1.2246	1.2390	1.2535	1.2679	1.2824	1.2969	1.3114	1.3259	1.3405	1.3550
0.750	1.3821	1.3968	1.4116	1.4263	1.4411	1.4559	1.4708	1.4856	1.5005	1.5154
0.800	1.5576	1.5726	1.5877	1.6029	1.6180	1.6332	1.6484	1.6636	1.6789	1.6941
0.850	1.7623	1.7777	1.7932	1.8088	1.8243	1.8400	1.8556	1.8713	1.8870	1.9027
0.900	2.0200	2.0360	2.0520	2.0681	2.0842	2.1004	2.1166	2.1329	2.1492	2.1655
0.950	2.4025	2.4193	2.4361	2.4531	2.4700	2.4870	2.5041	2.5213	2.5385	2.5557
0.975	2.7348	2.7522	2.7698	2.7875	2.8052	2.8230	2.8409	2.8588	2.8768	2.8950
0.990	3.1216	3.1400	3.1584	3.1819	3.1956	3.2144	3.2332	3.2521	3.2712	3.2903
0.995	3.3854	3.4043	3.4235	3.4427	3.4621	3.4813	3.5009	3.5205	3.5402	3.5600

$P^*\backslash\nu$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69
0.600	1.0797	1.0936	1.1076	1.1216	1.1355	1.1495	1.1635	1.1775	1.1914	1.2054
0.650	1.2208	1.2350	1.2493	1.2636	1.2778	1.2921	1.3064	1.3207	1.3351	1.3494
0.700	1.3696	1.3841	1.3987	1.4133	1.4280	1.4426	1.4572	1.4719	1.4866	1.5012
0.750	1.5303	1.5452	1.5602	1.5751	1.5901	1.6051	1.6201	1.6352	1.6502	1.6653
0.800	1.7094	1.7246	1.7401	1.7555	1.7709	1.7863	1.8018	1.8173	1.8328	1.8483
0.850	1.9185	1.9343	1.9502	1.9660	1.9819	1.9978	2.0138	2.0298	2.0458	2.0619
0.900	2.1819	2.1983	2.2148	2.2313	2.2478	2.2644	2.2810	2.2977	2.3144	2.3312
0.950	2.5730	2.5904	2.6079	2.6253	2.6429	2.6605	2.6782	2.6958	2.7137	2.7315
0.975	2.9131	2.9314	2.9497	2.9680	2.9865	3.0050	3.0236	3.0423	3.0611	3.0799
0.990	3.3094	3.3288	3.3482	3.3676	3.3872	3.4069	3.4267	3.4464	3.4664	3.4864
0.995	3.5800	3.6001	3.6202	3.6404	3.6608	3.6813	3.7018	3.7225	3.7433	3.7641

$P^*\backslash\nu$	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79
0.600	1.2194	1.2334	1.2474	1.2615	1.2755	1.2895	1.3035	1.3175	1.3317	1.3456
0.650	1.3637	1.3781	1.3924	1.4068	1.4211	1.4355	1.4499	1.4643	1.4787	1.4931
0.700	1.5159	1.5306	1.5454	1.5601	1.5748	1.5896	1.6044	1.6192	1.6339	1.6488
0.750	1.6804	1.6956	1.7107	1.7258	1.7410	1.7561	1.7713	1.7866	1.8018	1.8170
0.800	1.8638	1.8794	1.8950	1.9106	1.9263	1.9419	1.9576	1.9733	1.9890	2.0048
0.850	2.0780	2.0941	2.1103	2.1264	2.1426	2.1589	2.1752	2.1915	2.2078	2.2241
0.900	2.3480	2.3648	2.3817	2.3986	2.4156	2.4326	2.4496	2.4667	2.4838	2.5009
0.950	2.7494	2.7673	2.7853	2.8033	2.8215	2.8396	2.8578	2.8761	2.8944	2.9128
0.975	3.0988	3.1177	3.1367	3.1558	3.1750	3.1942	3.2135	3.2329	3.2523	3.2718
0.990	3.5065	3.5267	3.5469	3.5673	3.5878	3.6083	3.6289	3.6496	3.6704	3.6913
0.995	3.7851	3.8062	3.8274	3.8486	3.8700	3.8914	3.9130	3.9347	3.9565	3.9783

$P^*\backslash\nu$	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
0.600	1.3596	1.3737	1.3877	1.4017	1.4158	1.4298	1.4439	1.4580	1.4720	1.4861
0.650	1.5075	1.5219	1.5363	1.5508	1.5652	1.5797	1.5941	1.6086	1.6230	1.6375
0.700	1.6636	1.6784	1.6932	1.7081	1.7230	1.7378	1.7527	1.7676	1.7825	1.7974
0.750	1.8323	1.8476	1.8629	1.8782	1.8935	1.9089	1.9242	1.9396	1.9550	1.9704
0.800	2.0206	2.0364	2.0522	2.0680	2.0839	2.0998	2.1157	2.1316	2.1475	2.1635
0.850	2.2405	2.2569	2.2734	2.2898	2.3064	2.3229	2.3394	2.3560	2.3726	2.3893
0.900	2.5181	2.5353	2.5526	2.5699	2.5871	2.6046	2.6220	2.6394	2.6569	2.6744
0.950	2.9312	2.9497	2.9682	2.9868	3.0054	3.0241	3.0428	3.0616	3.0804	3.0993
0.975	3.2913	3.3109	3.3306	3.3504	3.3702	3.3901	3.4099	3.4300	3.4500	3.4702
0.990	3.7121	3.7332	3.7543	3.7755	3.7969	3.8182	3.8396	3.8611	3.8827	3.9043
0.995	4.0002	4.0222	4.0444	4.0666	4.0889	4.1113	4.1338	4.1565	4.1791	4.2024

$P^*\backslash\nu$	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.600	1.5002	1.5142	1.5283	1.5424	1.5565	1.5706	1.5847	1.5987	1.6128	1.6269
0.650	1.6520	1.6665	1.6810	1.6955	1.7100	1.7245	1.7391	1.7536	1.7681	1.7827
0.700	1.8124	1.8273	1.8423	1.8572	1.8722	1.8872	1.9022	1.9172	1.9322	1.9472
0.750	1.9858	2.0013	2.0167	2.0322	2.0477	2.0632	2.0787	2.0942	2.1097	2.1253
0.800	2.1795	2.1955	2.2115	2.2276	2.2436	2.2597	2.2758	2.2919	2.3081	2.3242
0.850	2.4059	2.4226	2.4393	2.4561	2.4728	2.4896	2.5064	2.5233	2.5401	2.5570
0.900	2.6919	2.7095	2.7271	2.7447	2.7624	2.7801	2.7979	2.8157	2.8335	2.8513
0.950	3.1182	3.1371	3.1562	3.1752	3.1943	3.2135	3.2327	3.2519	3.2712	3.2905
0.975	3.4903	3.5106	3.5309	3.5512	3.5717	3.5921	3.6127	3.6332	3.6539	3.6746
0.990	3.9261	3.9485	3.9698	3.9919	4.0138	4.0359	4.0581	4.0803	4.1026	4.1250
0.995	4.2247	4.2477	4.2708	4.2938	4.3169	4.3402	4.3637	4.3872	4.4106	4.4339

$P^*\backslash\nu$	0.991	0.992	0.993	0.994	0.995	0.996	0.997	0.998	0.999	1.000
0.600	1.6284	1.6298	1.6312	1.6325	1.6340	1.6354	1.6368	1.6382	1.6396	1.6410
0.650	1.7841	1.7856	1.7870	1.7885	1.7900	1.7914	1.7929	1.7943	1.7958	1.7972
0.700	1.9487	1.9502	1.9517	1.9532	1.9547	1.9563	1.9578	1.9592	1.9608	1.9623
0.750	2.1269	2.1284	2.1300	2.1315	2.1331	2.1346	2.1362	2.1378	2.1393	2.1409
0.800	2.3259	2.3275	2.3291	2.3307	2.3323	2.3339	2.3355	2.3372	2.3388	2.3404
0.850	2.5587	2.5604	2.5621	2.5638	2.5655	2.5671	2.5689	2.5705	2.5722	2.5739
0.900	2.8531	2.8549	2.8567	2.8584	2.8602	2.8620	2.8638	2.8656	2.8674	2.8692
0.950	3.2925	3.2944	3.2963	3.2983	3.3002	3.3021	3.3041	3.3060	3.3079	3.3099
0.975	3.6767	3.6787	3.6808	3.6829	3.6850	3.6870	3.6891	3.6912	3.6933	3.6953
0.990	4.1273	4.1295	4.1317	4.1340	4.1363	4.1385	4.1407	4.1430	4.1452	4.1475
0.995	4.4364	4.4389	4.4413	4.4437	4.4461	4.4483	4.4509	4.4531	4.4555	4.4579

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Table 6.1: $k = 8$

$P^*\backslash\nu$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
0.600	1.6410	1.7822	1.9237	2.0655	2.2076	2.3499	2.4924	2.6352	2.7782	2.9214
0.650	1.7972	1.9431	2.0895	2.2365	2.3841	2.5321	2.6805	2.8294	2.9787	3.1283
0.700	1.9623	2.1131	2.2649	2.4176	2.5711	2.7252	2.8801	3.0355	3.1916	3.3482
0.750	2.1409	2.2973	2.4550	2.6139	2.7739	2.9349	3.0969	3.2596	3.4232	3.5875
0.800	2.3404	2.5032	2.6677	2.8338	3.0013	3.1701	3.3402	3.5113	3.6835	3.8566
0.850	2.5739	2.7444	2.9170	3.0917	3.2683	3.4466	3.6264	3.8076	3.9902	4.1739
0.900	2.8692	3.0496	3.2330	3.4190	3.6074	3.7981	3.9907	4.1852	4.3814	4.5791
0.950	3.3099	3.5060	3.7061	3.9099	4.1169	4.3270	4.5398	4.7551	4.9725	5.1921
0.975	3.6953	3.9059	4.1215	4.3415	4.5658	4.7939	5.0253	5.2598	5.4969	5.7366
0.990	4.1475	4.3758	4.6106	4.8509	5.0964	5.3466	5.6008	5.8589	6.1203	6.3847
0.995	4.4579	4.6991	4.9474	5.2023	5.4631	5.7291	5.9998	6.2746	6.5532	6.8351

$P^*\backslash\nu$	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
0.600	3.0648	3.2084	3.3522	3.4961	3.6403	3.7845	3.9289	4.0735	4.2182	4.3630
0.650	3.2783	3.4286	3.5792	3.7301	3.8812	4.0326	4.1842	4.3360	4.4881	4.6403
0.700	3.5052	3.6628	3.8207	3.9791	4.1378	4.2969	4.4562	4.6159	4.7758	4.9360
0.750	3.7524	3.9179	4.0841	4.2507	4.4178	4.5853	4.7533	4.9216	5.0903	5.2593
0.800	4.0305	4.2052	4.3807	4.5568	4.7335	4.9108	5.0886	5.2668	5.4456	5.6247
0.850	4.3587	4.5445	4.7312	4.9187	5.1070	5.2960	5.4856	5.6759	5.8666	6.0579
0.900	4.7781	4.9784	5.1798	5.3823	5.5857	5.7900	5.9951	6.2010	6.4075	6.6146
0.950	5.4135	5.6365	5.8610	6.0868	6.3139	6.5421	6.7714	7.0015	7.2326	7.4644
0.975	5.9784	6.2223	6.4679	6.7152	6.9640	7.2141	7.4654	7.7179	7.9712	8.2256
0.990	6.6516	6.9208	7.1925	7.4660	7.7410	8.0180	8.2963	8.5758	8.8565	9.1383
0.995	7.1199	7.4073	7.6973	7.9894	8.2832	8.5790	8.8764	9.1746	9.4747	9.7758

$P^*\backslash\nu$	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
0.600	4.5079	4.6529	4.7980	4.9432	5.0886	5.2339	5.3794	5.5250	5.6706	5.8163
0.650	4.7926	4.9452	5.0979	5.2507	5.4037	5.5568	5.7100	5.8633	6.0167	6.1702
0.700	5.0964	5.2571	5.4179	5.5789	5.7401	5.9015	6.0630	6.2247	6.3865	6.5485
0.750	5.4286	5.5982	5.7681	5.9382	6.1085	6.2794	6.4498	6.6207	6.7918	6.9630
0.800	5.8042	5.9840	6.1641	6.3446	6.5254	6.7063	6.8876	7.0691	7.2507	7.4326
0.850	6.2496	6.4417	6.6343	6.8272	7.0204	7.2140	7.4078	7.6020	7.7964	7.9910
0.900	6.8223	7.0305	7.2392	7.4483	7.6579	7.8678	8.0781	8.2887	8.4996	8.7109
0.950	7.6970	7.9301	8.1639	8.3983	8.6332	8.8684	9.1044	9.3406	9.5772	9.8142
0.975	8.4808	8.7368	8.9935	9.2509	9.5088	9.7673	10.0263	10.2858	10.5458	10.8062
0.990	9.4210	9.7046	9.9892	10.2744	10.5606	10.8470	11.1342	11.4220	11.7102	11.9992
0.995	10.0780	10.3811	10.6854	10.9902	11.2959	11.6022	11.9085	12.2168	12.5248	12.8338

$P^*\backslash\nu$	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
0.600	5.9620	6.1078	6.2537	6.3996	6.5456	6.6916	6.8377	6.9838	7.1300	7.2762
0.650	6.3238	6.4775	6.6313	6.7852	6.9391	7.0931	7.2472	7.4013	7.5555	7.7097
0.700	6.7106	6.8727	7.0350	7.1974	7.3599	7.5225	7.6852	7.8479	8.0107	8.1736
0.750	7.1344	7.3060	7.4777	7.6495	7.8214	7.9934	8.1656	8.3378	8.5102	8.6826
0.800	7.6147	7.7969	7.9793	8.1619	8.3446	8.5274	8.7104	8.8935	9.0767	9.2600
0.850	8.1858	8.3809	8.5762	8.7716	8.9672	9.1630	9.3589	9.5550	9.7512	9.9476
0.900	8.9224	9.1341	9.3461	9.5583	9.7707	9.9834	10.1962	10.4091	10.6222	10.8356
0.950	10.0515	10.2891	10.5271	10.7652	11.0037	11.2457	11.4813	11.7204	11.9597	12.1992
0.975	11.0670	11.3281	11.5896	11.8514	12.1134	12.3758	12.6384	12.9013	13.1644	13.4276
0.990	12.2882	12.5778	12.8677	13.1581	13.4488	13.7398	14.0310	14.3222	14.6144	14.9066
0.995	13.1429	13.4525	13.7624	14.0725	14.3836	14.6949	15.0064	15.3179	15.6299	15.9423

$P^*\backslash\nu$	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
0.600	7.4224	7.5687	7.7150	7.8614	8.0077	8.1541	8.3006	8.4470	8.5935	8.7400
0.650	7.8640	8.0183	8.1728	8.3272	8.4817	8.6362	8.7908	8.9454	9.1000	9.2547
0.700	8.3366	8.4996	8.6627	8.8258	8.9890	9.1523	9.3156	9.4790	9.6423	9.8058
0.750	8.8551	9.0277	9.2004	9.3732	9.5460	9.7179	9.8918	10.0648	10.2379	10.4110
0.800	9.4434	9.6270	9.8106	9.9942	10.1780	10.3619	10.5458	10.7298	10.9139	11.0980
0.850	10.1441	10.3407	10.5373	10.7341	10.9310	11.1280	11.3251	11.5228	11.7195	11.9168
0.900	11.0490	11.2626	11.4763	11.6901	11.9038	12.1181	12.3323	12.5466	12.7609	12.9754
0.950	12.4390	12.6788	12.9189	13.1590	13.3993	13.6398	13.8805	14.1211	14.3619	14.6029
0.975	13.6912	13.9549	14.2188	14.4829	14.7470	15.0114	15.2760	15.5407	15.8054	16.0703
0.990	15.1988	15.4913	15.7840	16.0771	16.3704	16.6635	16.9571	17.2506	17.5444	17.8383
0.995	16.2548	16.5676	16.8806	17.1940	17.5074	17.8209	18.1349	18.4489	18.7618	19.0771

$P^*\backslash\nu$	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9
0.600	8.8866	9.0331	9.1797	9.3263	9.4729	9.6195	9.7662	9.9128	10.0595	10.2062
0.650	9.4094	9.5641	9.7189	9.8737	10.0285	10.1834	10.3383	10.4931	10.6481	10.8030
0.700	9.9693	10.1328	10.2920	10.4600	10.6236	10.7873	10.9510	11.1147	11.2788	11.4423
0.750	10.5841	10.7574	10.9306	11.1039	11.2773	11.4507	11.6241	11.7975	11.9711	12.1446
0.800	11.2822	11.4665	11.6508	11.8351	12.0196	12.2040	12.3886	12.5731	12.7577	12.9423
0.850	12.1142	12.3117	12.5092	12.7068	12.9045	13.1022	13.3000	13.4978	13.6957	13.8936
0.900	13.1899	13.4045	13.6192	13.8340	14.0488	14.2638	14.4790	14.6938	14.9089	15.1240
0.950	14.8439	15.0851	15.3263	15.5675	15.8090	16.0505	16.2921	16.5338	16.7755	17.0173
0.975	16.3354	16.6006	16.8658	17.1314	17.3967	17.6623	17.9280	18.1937	18.4595	18.7255
0.990	18.1326	18.4266	18.7213	19.0155	19.3100	19.6047	19.8995	20.1942	20.4893	20.7867
0.995	19.3906	19.7062	20.0204	20.3363	20.6505	20.9657	21.2809	21.5975	21.9118	22.2260

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Table 6.1: $k = 8$

$P^*\backslash\nu$	7.0	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
0.600	10.3530	10.4997	10.6464	10.7932	10.9400	11.0868	11.2336	11.3804	11.5272	11.6741
0.650	10.9580	11.1129	11.2679	11.4230	11.5780	11.7331	11.8881	12.0432	12.1984	12.3535
0.700	11.6061	11.7700	11.9338	12.0977	12.2616	12.4256	12.5896	12.7535	12.9176	13.0815
0.750	12.3182	12.4918	12.6654	12.8391	13.0128	13.1865	13.3602	13.5340	13.7078	13.8815
0.800	13.1270	13.3117	13.4965	13.6813	13.8661	14.0510	14.2359	14.4208	14.6057	14.7907
0.850	14.0916	14.2896	14.4877	14.6858	14.8839	15.0821	15.2803	15.4785	15.6768	15.8752
0.900	15.3393	15.5545	15.7699	15.9853	16.2007	16.4161	16.6317	16.8472	17.0628	17.2784
0.950	17.2592	17.5011	17.7432	17.9851	18.2273	18.4694	18.7117	18.9539	19.1962	19.4386
0.975	18.9914	19.2575	19.5236	19.7897	20.0561	20.3224	20.5889	20.8553	21.1217	21.3884
0.990	21.0797	21.3746	21.6700	21.9653	22.2609	22.5563	22.8518	23.1475	23.4433	23.7389
0.995	22.5429	22.8583	23.1743	23.4894	23.8059	24.1219	24.4380	24.7543	25.0706	25.3864

$P^*\backslash\nu$	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9
0.600	11.8209	11.9678	12.1147	12.2615	12.4084	12.5553	12.7022	12.8492	12.9961	13.1430
0.650	12.5086	12.6638	12.8189	12.9741	13.1293	13.2845	13.4397	13.5950	13.7502	13.9056
0.700	13.2456	13.4096	13.5737	13.7378	13.9019	14.0660	14.2302	14.3943	14.5585	14.7227
0.750	14.0555	14.2293	14.4032	14.5771	14.7511	14.9250	15.0990	15.2730	15.4469	15.6210
0.800	14.9757	15.1608	15.3458	15.5309	15.7160	15.9011	16.0863	16.2715	16.4566	16.6419
0.850	16.0733	16.2721	16.4703	16.6687	16.8672	17.0657	17.2642	17.4628	17.6614	17.8600
0.900	17.4941	17.7098	17.9255	18.1413	18.3572	18.5730	18.7888	19.0048	19.2207	19.4367
0.950	19.6810	19.9235	20.1660	20.4085	20.6510	20.8937	21.1364	21.3790	21.6217	21.8644
0.975	21.6550	21.9216	22.1883	22.4551	22.7218	22.9888	23.2557	23.5225	23.7894	24.0565
0.990	24.0350	24.3308	24.6263	24.9226	25.2188	25.5148	25.8110	26.1068	26.4033	26.6998
0.995	25.7029	26.0195	26.3360	26.6526	26.9691	27.2857	27.6022	27.9188	28.2362	28.5530

$P^*\backslash\nu$	9.0	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
0.600	13.2899	13.4369	13.5838	13.7308	13.8778	14.0248	14.1717	14.3187	14.4657	14.6127
0.650	14.0607	14.2160	14.3713	14.5266	14.6819	14.8372	14.9925	15.1478	15.3031	15.4585
0.700	14.8869	15.0511	15.2153	15.3796	15.5438	15.7080	15.8724	16.0366	16.2009	16.3652
0.750	15.7950	15.9690	16.1431	16.3172	16.4913	16.6654	16.8395	17.0136	17.1878	17.3619
0.800	16.8271	17.0123	17.1976	17.3829	17.5682	17.7535	17.9388	18.1241	18.3095	18.4949
0.850	18.0586	18.2572	18.4559	18.6546	18.8533	19.0520	19.2507	19.4495	19.6482	19.8470
0.900	19.6526	19.8686	20.0846	20.3007	20.5168	20.7329	20.9490	21.1651	21.3821	21.5975
0.950	22.1072	22.3501	22.5929	22.8358	23.0789	23.3220	23.5646	23.8075	24.0506	24.2935
0.975	24.3235	24.5905	24.8576	25.1248	25.3919	25.6591	25.9264	26.1941	26.4610	26.7282
0.990	26.9960	27.2922	27.5856	27.8852	28.1812	28.4780	28.7747	29.0712	29.3679	29.6644
0.995	28.8692	29.1863	29.5034	29.8204	30.1375	30.4543	30.7712	31.0887	31.4050	31.7228

$P^*\backslash\nu$	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0	55.0
0.600	14.7598	22.1149	29.4749	36.8370	44.1999	51.5633	58.9274	66.2915	73.6558	81.0197
0.650	15.6138	23.3879	31.1684	38.9513	46.7360	54.5212	62.3069	70.0929	77.8791	85.6655
0.700	16.5295	24.7531	32.9848	41.2198	49.4564	57.6940	65.9322	74.1708	82.4096	90.6486
0.750	17.5360	26.2544	34.9826	43.7147	52.4489	61.1841	69.9201	78.6565	87.3931	96.1304
0.800	18.6803	27.9617	37.2549	46.5525	55.8525	65.1539	74.4561	83.7589	93.0622	102.3657
0.850	20.0458	30.0001	39.9679	49.9412	59.9173	69.8948	79.8735	89.8527	99.8321	109.8128
0.900	21.8137	32.6401	43.4828	54.3316	65.1836	76.0375	86.8927	97.7481	108.6049	119.4616
0.950	24.5366	36.7089	48.9002	61.0991	73.3020	85.5071	97.7135	109.9206	122.1290	134.3373
0.975	26.9956	40.3848	53.7955	67.2149	80.6361	94.0646	107.4928	120.9223	134.3508	147.7810
0.990	29.9612	44.8192	59.7014	74.5934	89.4905	104.3893	119.2926	134.1936	149.0950	164.0038
0.995	32.0400	47.9288	63.8430	79.7702	95.6989	111.6431	127.5666	143.5035	159.4421	175.3792

$P^*\backslash\nu$	60.0	65.0	70.0	75.0	80.0	85.0	90.0	95.0	100.0
0.600	88.3848	95.7507	103.1141	110.4787	117.8434	125.2082	132.5730	139.9378	147.3026
0.650	93.4521	101.2387	109.0253	116.8120	124.5988	132.3856	140.1725	147.9593	155.7462
0.700	98.8878	107.1272	115.3666	123.6061	131.8457	140.0854	148.3247	156.5646	164.8041
0.750	104.8676	113.6050	122.3424	131.0798	139.8177	148.5553	157.2930	166.0308	174.7684
0.800	111.6696	120.9733	130.2773	139.5815	148.8859	158.1908	167.4944	176.7987	186.1032
0.850	119.7928	129.7749	139.7543	149.7352	159.7159	169.6969	179.6781	189.6590	199.6402
0.900	130.3187	141.1761	152.0334	162.8909	173.7490	184.6069	195.4649	206.3227	217.1807
0.950	146.5462	158.7552	170.9642	183.1735	195.3836	207.5929	219.8027	232.0125	244.2231
0.975	161.2110	174.6425	188.0731	201.5054	214.9353	228.3674	241.7990	255.2312	268.6588
0.990	178.9081	193.8132	208.7188	223.6231	238.5291	253.4343	268.3416	283.2468	298.1543
0.995	191.3231	207.2974	223.1954	239.1366	255.0763	271.0169	286.9560	302.8967	318.8363

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Table 6.1: $k = 9$

$P^*\backslash\nu$	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59
0.600	0.9753	0.9899	1.0045	1.0191	1.0337	1.0483	1.0630	1.0776	1.0922	1.1069
0.650	1.1134	1.1282	1.1430	1.1579	1.1728	1.1877	1.2026	1.2174	1.2324	1.2473
0.700	1.2589	1.2740	1.2892	1.3043	1.3194	1.3346	1.3498	1.3649	1.3801	1.3954
0.750	1.4161	1.4315	1.4469	1.4623	1.4778	1.4932	1.5087	1.5242	1.5397	1.5553
0.800	1.5912	1.6069	1.6227	1.6384	1.6542	1.6700	1.6859	1.7017	1.7176	1.7335
0.850	1.7955	1.8116	1.8277	1.8439	1.8601	1.8763	1.8925	1.9088	1.9252	1.9415
0.900	2.0527	2.0693	2.0859	2.1026	2.1193	2.1361	2.1529	2.1697	2.1866	2.2035
0.950	2.4344	2.4517	2.4691	2.4866	2.5041	2.5217	2.5394	2.5571	2.5748	2.5926
0.975	2.7659	2.7839	2.8020	2.8202	2.8385	2.8568	2.8752	2.8937	2.9122	2.9308
0.990	3.1519	3.1708	3.1897	3.2088	3.2279	3.2472	3.2665	3.2859	3.3054	3.3250
0.995	3.4151	3.4345	3.4542	3.4738	3.4935	3.5135	3.5334	3.5535	3.5737	3.5940

$P^*\backslash\nu$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69
0.600	1.1215	1.1361	1.1508	1.1655	1.1801	1.1948	1.2094	1.2241	1.2388	1.2535
0.650	1.2622	1.2771	1.2921	1.3070	1.3220	1.3369	1.3519	1.3669	1.3819	1.3969
0.700	1.4106	1.4258	1.4411	1.4563	1.4716	1.4869	1.5022	1.5175	1.5329	1.5482
0.750	1.5708	1.5864	1.6020	1.6176	1.6332	1.6489	1.6646	1.6802	1.6960	1.7117
0.800	1.7495	1.7654	1.7814	1.7974	1.8134	1.8295	1.8456	1.8617	1.8778	1.8939
0.850	1.9579	1.9743	1.9908	2.0072	2.0237	2.0403	2.0569	2.0735	2.0901	2.1068
0.900	2.2205	2.2375	2.2546	2.2717	2.2888	2.3060	2.3232	2.3404	2.3577	2.3750
0.950	2.6105	2.6284	2.6464	2.6645	2.6825	2.7007	2.7189	2.7371	2.7554	2.7738
0.975	2.9495	2.9683	2.9871	3.0060	3.0247	3.0440	3.0631	3.0823	3.1015	3.1208
0.990	3.3448	3.3645	3.3843	3.4043	3.4243	3.4444	3.4646	3.4849	3.5053	3.5258
0.995	3.6143	3.6349	3.6555	3.6762	3.6970	3.7177	3.7388	3.7600	3.7812	3.8024

$P^*\backslash\nu$	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79
0.600	1.2682	1.2828	1.2975	1.3122	1.3269	1.3416	1.3563	1.3710	1.3858	1.4005
0.650	1.4119	1.4269	1.4419	1.4570	1.4720	1.4870	1.5021	1.5172	1.5322	1.5473
0.700	1.5636	1.5789	1.5943	1.6097	1.6251	1.6405	1.6559	1.6713	1.6868	1.7023
0.750	1.7274	1.7432	1.7589	1.7747	1.7905	1.8063	1.8222	1.8380	1.8539	1.8698
0.800	1.9101	1.9263	1.9425	1.9588	1.9750	1.9913	2.0076	2.0239	2.0403	2.0566
0.850	2.1234	2.1402	2.1569	2.1737	2.1905	2.2073	2.2242	2.2411	2.2580	2.2749
0.900	2.3924	2.4098	2.4272	2.4447	2.4622	2.4798	2.4974	2.5150	2.5327	2.5504
0.950	2.7922	2.8107	2.8292	2.8478	2.8664	2.8851	2.9038	2.9226	2.9414	2.9603
0.975	3.1402	3.1597	3.1792	3.1987	3.2184	3.2381	3.2579	3.2777	3.2976	3.3176
0.990	3.5463	3.5670	3.5877	3.6084	3.6294	3.6504	3.6711	3.6947	3.7138	3.7350
0.995	3.8238	3.8454	3.8669	3.8886	3.9105	3.9317	3.9543	3.9763	3.9983	4.0206

$P^*\backslash\nu$	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
0.600	1.4152	1.4299	1.4452	1.4594	1.4741	1.4889	1.5036	1.5183	1.5331	1.5478
0.650	1.5624	1.5775	1.5926	1.6077	1.6228	1.6379	1.6530	1.6681	1.6833	1.6984
0.700	1.7177	1.7332	1.7487	1.7642	1.7797	1.7952	1.8108	1.8263	1.8419	1.8574
0.750	1.8857	1.9016	1.9175	1.9334	1.9494	1.9654	1.9814	1.9974	2.0134	2.0294
0.800	2.0730	2.0894	2.1059	2.1223	2.1388	2.1553	2.1718	2.1883	2.2049	2.2214
0.850	2.2919	2.3089	2.3259	2.3430	2.3601	2.3772	2.3943	2.4115	2.4287	2.4459
0.900	2.5681	2.5859	2.6037	2.6216	2.6394	2.6574	2.6753	2.6933	2.7113	2.7294
0.950	2.9792	2.9982	3.0173	3.0363	3.0555	3.0746	3.0939	3.1131	3.1325	3.1518
0.975	3.3376	3.3576	3.3778	3.3980	3.4183	3.4387	3.4591	3.4795	3.5000	3.5205
0.990	3.7564	3.7779	3.7995	3.8210	3.8427	3.8645	3.8863	3.9082	3.9302	3.9523
0.995	4.0431	4.0655	4.0880	4.1109	4.1333	4.1561	4.1792	4.2021	4.2251	4.2482

$P^*\backslash\nu$	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.600	1.5626	1.5773	1.5921	1.6069	1.6216	1.6364	1.6512	1.6659	1.6807	1.6955
0.650	1.7136	1.7287	1.7439	1.7591	1.7742	1.7894	1.8046	1.8198	1.8350	1.8502
0.700	1.8730	1.8886	1.9042	1.9198	1.9354	1.9511	1.9667	1.9823	1.9980	2.0137
0.750	2.0455	2.0616	2.0776	2.0937	2.1098	2.1260	2.1421	2.1582	2.1744	2.1906
0.800	2.2380	2.2546	2.2713	2.2879	2.3046	2.3213	2.3380	2.3547	2.3714	2.3882
0.850	2.4631	2.4804	2.4977	2.5150	2.5323	2.5497	2.5671	2.5845	2.6019	2.6194
0.900	2.7475	2.7656	2.7837	2.8019	2.8201	2.8384	2.8567	2.8751	2.8933	2.9117
0.950	3.1712	3.1907	3.2102	3.2298	3.2494	3.2690	3.2887	3.3084	3.3282	3.3480
0.975	3.5412	3.5619	3.5827	3.6035	3.6244	3.6453	3.6663	3.6873	3.7084	3.7296
0.990	3.9748	3.9967	4.0190	4.0411	4.0638	4.0863	4.1088	4.1315	4.1543	4.1771
0.995	4.2715	4.2948	4.3182	4.3417	4.3653	4.3889	4.4127	4.4364	4.4603	4.4842

$P^*\backslash\nu$	0.991	0.992	0.993	0.994	0.995	0.996	0.997	0.998	0.999	1.000
0.600	1.6970	1.6985	1.6999	1.7014	1.7029	1.7044	1.7059	1.7073	1.7088	1.7103
0.650	1.8517	1.8533	1.8548	1.8563	1.8578	1.8593	1.8609	1.8624	1.8639	1.8654
0.700	2.0152	2.0168	2.0184	2.0199	2.0215	2.0231	2.0247	2.0262	2.0278	2.0294
0.750	2.1922	2.1938	2.1954	2.1971	2.1987	2.2003	2.2019	2.2035	2.2051	2.2068
0.800	2.3899	2.3915	2.3932	2.3949	2.3966	2.3983	2.3999	2.4016	2.4033	2.4050
0.850	2.6212	2.6229	2.6247	2.6264	2.6281	2.6299	2.6316	2.6334	2.6351	2.6369
0.900	2.9136	2.9154	2.9172	2.9191	2.9209	2.9228	2.9246	2.9264	2.9283	2.9301
0.950	3.3500	3.3519	3.3540	3.3559	3.3579	3.3599	3.3619	3.3639	3.3659	3.3679
0.975	3.7316	3.7326	3.7359	3.7380	3.7401	3.7423	3.7444	3.7465	3.7486	3.7508
0.990	4.1793	4.1816	4.1839	4.1862	4.1885	4.1908	4.1931	4.1953	4.1976	4.2000
0.995	4.4866	4.4890	4.4914	4.4938	4.4963	4.4987	4.5011	4.5035	4.5059	4.5083

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Table 6.1: $k = 9$

$P^*\backslash\nu$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
0.600	1.7103	1.8583	2.0065	2.1551	2.3038	2.4528	2.6021	2.7515	2.9012	3.0510
0.650	1.8654	2.0179	2.1709	2.3245	2.4785	2.6330	2.7880	2.9433	3.0990	3.2550
0.700	2.0294	2.1866	2.3448	2.5038	2.6635	2.8240	2.9850	3.1467	3.3090	3.4718
0.750	2.2068	2.3694	2.5332	2.6982	2.8643	3.0313	3.1992	3.3680	3.5375	3.7077
0.800	2.4050	2.5737	2.7441	2.9160	3.0893	3.2639	3.4397	3.6165	3.7944	3.9730
0.850	2.6369	2.8130	2.9912	3.1715	3.3535	3.5373	3.7225	3.9092	4.0971	4.2862
0.900	2.9301	3.1159	3.3045	3.4957	3.6892	3.8849	4.0827	4.2822	4.4834	4.6861
0.950	3.3679	3.5688	3.7736	3.9820	4.1937	4.4083	4.6257	4.8455	5.0676	5.2916
0.975	3.7508	3.9656	4.1856	4.4099	4.6384	4.8705	5.1061	5.3447	5.5860	5.8298
0.990	4.2000	4.4321	4.6706	4.9149	5.1641	5.4180	5.6760	5.9380	6.2028	6.4709
0.995	4.5083	4.7530	5.0049	5.2633	5.5269	5.7970	6.0712	6.3495	6.6316	6.9170

$P^*\backslash\nu$	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
0.600	3.2011	3.3513	3.5017	3.6523	3.8030	3.9539	4.1049	4.2560	4.4073	4.5587
0.650	3.4114	3.5681	3.7252	3.8824	4.0399	4.1977	4.3556	4.5139	4.6722	4.8308
0.700	3.6350	3.7987	3.9628	4.1274	4.2922	4.4574	4.6229	4.7888	4.9548	5.1212
0.750	3.8786	4.0501	4.2221	4.3946	4.5677	4.7411	4.9150	5.0892	5.2638	5.4388
0.800	4.1528	4.3331	4.5143	4.6960	4.8784	5.0613	5.2448	5.4287	5.6130	5.7978
0.850	4.4764	4.6676	4.8596	5.0525	5.2462	5.4405	5.6355	5.8311	6.0272	6.2239
0.900	4.8901	5.0955	5.3019	5.5094	5.7179	5.9271	6.1373	6.3481	6.5597	6.7718
0.950	5.5175	5.7450	5.9740	6.2044	6.4360	6.6688	6.9026	7.1373	7.3729	7.6093
0.975	6.0757	6.3237	6.5735	6.8260	7.0779	7.3322	7.5883	7.8443	8.1020	8.3606
0.990	6.7417	7.0147	7.2901	7.5674	7.8464	8.1269	8.4091	8.6924	8.9768	9.2626
0.995	7.2053	7.4963	7.7897	8.0853	8.3817	8.6820	8.9827	9.2851	9.5886	9.8935

$P^*\backslash\nu$	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
0.600	4.7102	4.8618	5.0135	5.1653	5.3172	5.4692	5.6213	5.7734	5.9256	6.0779
0.650	4.9896	5.1485	5.3075	5.4667	5.6261	5.7855	5.9451	6.1048	6.2646	6.4245
0.700	5.2877	5.4545	5.6215	5.7887	5.9560	6.1236	6.2913	6.4591	6.6271	6.7952
0.750	5.6139	5.7895	5.9653	6.1412	6.3175	6.4940	6.6706	6.8475	7.0245	7.2017
0.800	5.9830	6.1685	6.3543	6.5404	6.7268	6.9135	7.1004	7.2876	7.4750	7.6625
0.850	6.4210	6.6185	6.8164	7.0147	7.2134	7.4123	7.6116	7.8111	8.0109	8.2110
0.900	6.9846	7.1978	7.4116	7.6258	7.8404	8.0554	8.2708	8.4865	8.7025	8.9188
0.950	7.8464	8.0842	8.3230	8.5616	8.8011	9.0411	9.2816	9.5224	9.7637	10.0054
0.975	8.6200	8.8802	9.1412	9.4028	9.6650	9.9278	10.1911	10.4549	10.7192	10.9839
0.990	9.5492	9.8367	10.1252	10.4143	10.7041	10.9947	11.2858	11.5775	11.8697	12.1621
0.995	10.1996	10.5060	10.8137	11.1225	11.4319	11.7419	12.0527	12.3582	12.6758	12.9883

$P^*\backslash\nu$	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
0.600	6.2302	6.3826	6.5351	6.6876	6.8402	6.9928	7.1455	7.2982	7.4510	7.6038
0.650	6.5845	6.7446	6.9048	7.0650	7.2253	7.3857	7.5462	7.7067	7.8672	8.0279
0.700	6.9634	7.1318	7.3002	7.4688	7.6375	7.8062	7.9751	8.1440	8.3130	8.4820
0.750	7.3790	7.5565	7.7342	7.9119	8.0898	8.2678	8.4459	8.6241	8.8024	8.9808
0.800	7.8503	8.0383	8.2263	8.4146	8.6030	8.7915	8.9802	9.1690	9.3579	9.5470
0.850	8.4112	8.6117	8.8124	9.0133	9.2143	9.4155	9.6169	9.8185	10.0201	10.2219
0.900	9.1354	9.3522	9.5693	9.7866	10.0042	10.2219	10.4399	10.6579	10.8762	11.0946
0.950	10.2473	10.4896	10.7321	10.9750	11.2181	11.4615	11.7051	11.9489	12.1929	12.4371
0.975	11.2489	11.5144	11.7802	12.0463	12.3127	12.5793	12.8463	13.1135	13.3809	13.6486
0.990	12.4556	12.7490	13.0430	13.3373	13.6319	13.9267	14.2221	14.5177	14.8137	15.1095
0.995	13.3010	13.6142	13.9279	14.2415	14.5570	14.8718	15.1870	15.5303	15.8178	16.1344

$P^*\backslash\nu$	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
0.600	7.7566	7.9095	8.0624	8.2153	8.3683	8.5213	8.6744	8.8274	8.9805	9.1336
0.650	8.1886	8.3493	8.5101	8.6709	8.8318	8.9928	9.1537	9.3147	9.4758	9.6368
0.700	8.6512	8.8204	8.9897	9.1589	9.3284	9.4978	9.6673	9.8369	10.0065	10.1761
0.750	9.1592	9.3378	9.5164	9.6951	9.8739	10.0528	10.2317	10.4107	10.5897	10.7688
0.800	9.7361	9.9253	10.1146	10.3040	10.4936	10.6831	10.8728	11.0625	11.2523	11.4422
0.850	10.4239	10.6258	10.8280	11.0302	11.2326	11.4350	11.6376	11.8402	12.0429	12.2457
0.900	11.3131	11.5318	11.7506	11.9696	12.1887	12.4079	12.6272	12.8466	13.0661	13.2855
0.950	12.6815	12.9260	13.1708	13.4156	13.6606	13.9058	14.1510	14.3965	14.6420	14.8877
0.975	13.9164	14.1847	14.4537	14.7211	14.9897	15.2585	15.5273	15.7963	16.0656	16.3349
0.990	15.4059	15.7025	15.9993	16.2961	16.5932	16.8905	17.1880	17.4856	17.7835	18.0814
0.995	16.4505	16.7653	17.0840	17.4009	17.7181	18.0357	18.3533	18.6708	18.9886	19.3070

$P^*\backslash\nu$	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9
0.600	9.2868	9.4399	9.5931	9.7463	9.8996	10.0528	10.2061	10.3593	10.5127	10.6660
0.650	9.7979	9.9591	10.1202	10.2815	10.4427	10.6039	10.7652	10.9262	11.0879	11.2492
0.700	10.3458	10.5155	10.6853	10.8551	11.0249	11.1948	11.3647	11.5346	11.7046	11.8745
0.750	10.9479	11.1271	11.3063	11.4856	11.6649	11.8443	12.0237	12.2031	12.3826	12.5621
0.800	11.6321	11.8221	12.0122	12.2022	12.3924	12.5826	12.7729	12.9632	13.1535	13.3438
0.850	12.4485	12.6515	12.8544	13.0575	13.2607	13.4639	13.6671	13.8704	14.0737	14.2772
0.900	13.5054	13.7252	13.9450	14.1650	14.3850	14.6051	14.8252	15.0454	15.2656	15.4860
0.950	15.1334	15.3792	15.6252	15.8712	16.1174	16.3633	16.6088	16.8563	17.1028	17.3493
0.975	16.6041	16.8738	17.1434	17.4129	17.6830	17.9531	18.2253	18.4931	18.7634	19.0334
0.990	18.3795	18.6777	18.9761	19.2744	19.5731	19.8717	20.1705	20.4694	20.7683	21.0675
0.995	19.6253	19.9434	20.2619	20.5805	20.8993	21.2183	21.5372	21.8564	22.1851	22.4949

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Table 6.1: $k = 9$

$P^*\backslash\nu$	7.0	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
0.600	10.8193	10.9727	11.1260	11.2794	11.4328	11.5862	11.7396	11.8931	12.0465	12.2000
0.650	11.4106	11.5720	11.7335	11.8948	12.0563	12.2178	12.3793	12.5408	12.7023	12.8638
0.700	12.0446	12.2146	12.3847	12.5548	12.7249	12.8951	13.0652	13.2355	13.4056	13.5758
0.750	12.7417	12.9213	13.1009	13.2805	13.4602	13.6399	13.8197	13.9994	14.1792	14.3590
0.800	13.5344	13.7248	13.9153	14.1058	14.2964	14.4870	14.6776	14.8683	15.0590	15.2497
0.850	14.4806	14.6841	14.8876	15.0912	15.2948	15.4985	15.7022	15.9060	16.1097	16.3137
0.900	15.7064	15.9268	16.1473	16.3678	16.5884	16.8090	17.0297	17.2504	17.4712	17.6920
0.950	17.5959	17.8425	18.0892	18.3360	18.5828	18.8297	19.0767	19.3237	19.5706	19.8179
0.975	19.3039	19.5745	19.8450	20.1156	20.3863	20.6570	20.9277	21.1987	21.4695	21.7404
0.990	21.3667	21.6659	21.9639	22.2644	22.5644	22.8635	23.1631	23.4630	23.7626	24.0624
0.995	22.8142	23.1337	23.4534	23.7728	24.0925	24.4126	24.7323	25.0525	25.3717	25.6923

$P^*\backslash\nu$	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9
0.600	12.3535	12.5069	12.6606	12.8139	12.9674	13.1210	13.2745	13.4280	13.5816	13.7349
0.650	13.0254	13.1870	13.3485	13.5101	13.6718	13.8334	13.9950	14.1567	14.3184	14.4800
0.700	13.7461	13.9164	14.0866	14.2569	14.4273	14.5975	14.7679	14.9383	15.1087	15.2791
0.750	14.5388	14.7187	14.8986	15.0785	15.2584	15.4383	15.6183	15.7982	15.9782	16.1582
0.800	15.4405	15.6313	15.8221	16.0129	16.2038	16.3947	16.5856	16.7765	16.9674	17.1584
0.850	16.5174	16.7212	16.9251	17.1291	17.3329	17.5370	17.7410	17.9451	18.1491	18.3532
0.900	17.9128	18.1337	18.3546	18.5756	18.7965	19.0175	19.2386	19.4597	19.6808	19.9019
0.950	20.0650	20.3121	20.5594	20.8067	21.0540	21.3014	21.5487	21.7962	22.0436	22.2911
0.975	22.0114	22.2824	22.5535	22.8247	23.0959	23.3669	23.6385	23.9097	24.1811	24.4525
0.990	24.3623	24.6624	24.9623	25.2651	25.5623	25.8625	26.1626	26.4629	26.7630	27.0633
0.995	26.0122	26.3332	26.6534	26.9735	27.2952	27.6139	27.9348	28.2553	28.5760	28.8963

$P^*\backslash\nu$	9.0	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
0.600	13.8887	14.0423	14.1958	14.3494	14.5030	14.6566	14.8102	14.9639	15.1174	15.2711
0.650	14.6417	14.8034	14.9651	15.1268	15.2885	15.4502	15.6120	15.7737	15.9355	16.0972
0.700	15.4495	15.6199	15.7903	15.9608	16.1312	16.3017	16.4722	16.6427	16.8132	16.9837
0.750	16.3383	16.5187	16.6984	16.8784	17.0585	17.2386	17.4187	17.5989	17.7790	17.9592
0.800	17.3494	17.5404	17.7314	17.9225	18.1135	18.3046	18.4957	18.6868	18.8779	19.0691
0.850	18.5573	18.7615	18.9656	19.1697	19.3740	19.5782	19.7824	19.9867	20.1909	20.3952
0.900	20.1230	20.3442	20.5654	20.7867	21.0079	21.2292	21.4505	21.6718	21.8932	22.1145
0.950	22.5386	22.7862	23.0338	23.2814	23.5291	23.7767	24.0244	24.2721	24.5199	24.7668
0.975	24.7240	24.9954	25.2668	25.5382	25.8100	26.0814	26.3536	26.6248	26.8965	27.1682
0.990	27.3640	27.6650	27.9647	28.2651	28.5655	28.8659	29.1666	29.4674	29.7681	30.0685
0.995	29.2168	29.5382	29.8585	30.1795	30.5001	30.8212	31.1388	31.4632	31.7823	32.1051

$P^*\backslash\nu$	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0	55.0
0.600	15.4247	23.1114	30.8031	38.4968	46.1915	53.8869	61.5826	69.2786	76.9747	84.6709
0.650	16.2590	24.3544	32.4566	40.5615	48.6676	56.7747	64.8821	72.9900	81.0980	89.2062
0.700	17.1542	25.6888	34.2318	42.7781	51.3262	59.8753	68.4254	76.9750	85.5254	94.0760
0.750	18.1393	27.1578	36.1863	45.2190	54.2537	63.2896	72.3262	81.3632	90.4008	99.4385
0.800	19.2602	28.8300	38.4117	47.9982	57.5872	67.1774	76.7686	86.3601	95.9525	105.5449
0.850	20.5995	30.8289	41.0722	51.3211	61.5729	71.8263	82.0805	92.3357	102.5913	112.8470
0.900	22.3359	33.4219	44.5239	55.6325	66.7446	77.8586	88.9727	100.0889	111.2060	122.3225
0.950	25.0155	37.4253	49.8545	62.2916	74.7327	87.1759	99.6191	112.0665	124.5126	136.9592
0.975	27.4400	41.0497	54.6811	68.3214	81.9659	95.6137	109.2621	122.9127	136.5641	150.2141
0.990	30.3685	45.4300	60.5151	75.6103	90.7100	105.8128	120.1172	136.0255	151.1310	166.2383
0.995	32.4259	48.4961	64.6116	80.7288	96.8466	112.9752	129.1033	145.2319	161.3623	177.4930

$P^*\backslash\nu$	60.0	65.0	70.0	75.0	80.0	85.0	90.0	95.0	100.0
0.600	92.3674	100.0638	107.7602	115.4568	123.1533	130.8483	138.5467	146.2432	153.9394
0.650	97.3143	105.4230	113.5308	121.6401	129.7487	137.8573	145.9660	154.0746	162.1855
0.700	102.6267	111.1775	119.7284	128.2794	136.8306	145.3816	153.9328	162.4840	171.0353
0.750	108.4764	117.5144	126.5526	135.5907	144.6291	153.6674	162.7060	171.7444	180.7827
0.800	115.1376	124.7303	134.3234	143.9165	153.5097	163.1030	172.6962	182.2897	191.8830
0.850	123.1032	133.3597	143.6160	153.8726	164.1292	174.3863	184.6428	194.8997	205.1569
0.900	133.4397	144.5575	155.6748	166.7920	177.9099	189.0278	200.1457	211.2639	222.3816
0.950	149.4064	161.8537	174.2986	186.7489	199.1971	211.6449	224.0933	236.5416	248.9891
0.975	163.8658	177.5177	191.1698	204.8227	218.4752	232.1278	245.7807	259.4334	273.0867
0.990	181.3462	196.4561	211.5620	226.6702	241.7804	256.8905	271.9995	287.1075	302.2152
0.995	193.6407	209.7521	225.8848	242.0140	258.1461	274.2779	290.4100	306.5424	322.6680

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Table 6.1: $k = 10$

$P^*\backslash\nu$	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59
0.600	1.0054	1.0206	1.0358	1.0510	1.0663	1.0815	1.0967	1.1119	1.1271	1.1424
0.650	1.1433	1.1587	1.1741	1.1896	1.2051	1.2205	1.2360	1.2515	1.2670	1.2825
0.700	1.2886	1.3043	1.3200	1.3357	1.3514	1.3671	1.3829	1.3986	1.4144	1.4302
0.750	1.4455	1.4615	1.4774	1.4934	1.5094	1.5255	1.5415	1.5575	1.5737	1.5898
0.800	1.6203	1.6366	1.6529	1.6692	1.6855	1.7019	1.7183	1.7347	1.7511	1.7676
0.850	1.8242	1.8409	1.8575	1.8742	1.8909	1.9077	1.9245	1.9413	1.9582	1.9751
0.900	2.0810	2.0981	2.1152	2.1324	2.1497	2.1669	2.1843	2.2016	2.2190	2.2364
0.950	2.4620	2.4798	2.4977	2.5157	2.5337	2.5518	2.5699	2.5881	2.6064	2.6245
0.975	2.7929	2.8114	2.8300	2.8486	2.8673	2.8861	2.9050	2.9239	2.9429	2.9620
0.990	3.1781	3.1975	3.2169	3.2364	3.2559	3.2756	3.2953	3.3152	3.3351	3.3551
0.995	3.4409	3.4607	3.4808	3.5008	3.5210	3.5413	3.5617	3.5822	3.6028	3.6234

$P^*\backslash\nu$	0.60	0.61	0.62	0.63	0.64	0.65	0.66	0.67	0.68	0.69
0.600	1.1576	1.1729	1.1881	1.2034	1.2186	1.2339	1.2491	1.2644	1.2797	1.2949
0.650	1.2980	1.3135	1.3290	1.3445	1.3601	1.3756	1.3912	1.4068	1.4223	1.4379
0.700	1.4460	1.4618	1.4776	1.4935	1.5093	1.5252	1.5411	1.5569	1.5728	1.5887
0.750	1.6059	1.6220	1.6382	1.6543	1.6705	1.6867	1.7029	1.7192	1.7354	1.7517
0.800	1.7841	1.8006	1.8171	1.8336	1.8502	1.8668	1.8834	1.9001	1.9167	1.9334
0.850	1.9920	2.0089	2.0259	2.0429	2.0599	2.0770	2.0941	2.1112	2.1284	2.1456
0.900	2.2539	2.2714	2.2890	2.3066	2.3242	2.3419	2.3596	2.3774	2.3951	2.4130
0.950	2.6429	2.6613	2.6798	2.6983	2.7168	2.7355	2.7541	2.7729	2.7916	2.8105
0.975	2.9811	3.0003	3.0196	3.0390	3.0583	3.0778	3.0973	3.1170	3.1366	3.1564
0.990	3.3752	3.3954	3.4157	3.4361	3.4565	3.4771	3.4977	3.5183	3.5391	3.5600
0.995	3.6443	3.6651	3.6861	3.7072	3.7284	3.7497	3.7710	3.7925	3.8141	3.8358

$P^*\backslash\nu$	0.70	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79
0.600	1.3102	1.3255	1.3408	1.3561	1.3713	1.3866	1.4019	1.4172	1.4325	1.4479
0.650	1.4535	1.4691	1.4847	1.5003	1.5159	1.5315	1.5472	1.5628	1.5784	1.5941
0.700	1.6047	1.6206	1.6365	1.6525	1.6685	1.6844	1.7004	1.7164	1.7324	1.7483
0.750	1.7680	1.7843	1.8006	1.8170	1.8333	1.8496	1.8661	1.8825	1.8989	1.9153
0.800	1.9501	1.9668	1.9836	2.0004	2.0172	2.0340	2.0508	2.0677	2.0845	2.1014
0.850	2.1627	2.1800	2.1973	2.2146	2.2319	2.2492	2.2666	2.2840	2.3014	2.3189
0.900	2.4308	2.4487	2.4667	2.4847	2.5026	2.5207	2.5388	2.5569	2.5750	2.5932
0.950	2.8293	2.8483	2.8672	2.8863	2.9054	2.9245	2.9437	2.9629	2.9822	3.0015
0.975	3.1762	3.1960	3.2160	3.2360	3.2560	3.2762	3.2964	3.3166	3.3369	3.3573
0.990	3.5809	3.6020	3.6231	3.6443	3.6656	3.6869	3.7084	3.7299	3.7514	3.7731
0.995	3.8573	3.8794	3.9014	3.9234	3.9456	3.9678	3.9901	4.0125	4.0350	4.0576

$P^*\backslash\nu$	0.80	0.81	0.82	0.83	0.84	0.85	0.86	0.87	0.88	0.89
0.600	1.4632	1.4785	1.4938	1.5091	1.5244	1.5398	1.5551	1.5704	1.5858	1.6011
0.650	1.6098	1.6254	1.6411	1.6568	1.6725	1.6882	1.7039	1.7196	1.7353	1.7510
0.700	1.7645	1.7805	1.7966	1.8126	1.8287	1.8448	1.8609	1.8770	1.8931	1.9093
0.750	1.9318	1.9482	1.9647	1.9812	1.9977	2.0142	2.0307	2.0473	2.0638	2.0804
0.800	2.1184	2.1353	2.1523	2.1692	2.1862	2.2033	2.2203	2.2373	2.2544	2.2715
0.850	2.3364	2.3539	2.3714	2.3890	2.4066	2.4242	2.4418	2.4595	2.4772	2.4949
0.900	2.6115	2.6297	2.6480	2.6663	2.6847	2.7031	2.7215	2.7400	2.7585	2.7770
0.950	3.0209	3.0403	3.0598	3.0793	3.0989	3.1185	3.1382	3.1579	3.1776	3.1974
0.975	3.3778	3.3982	3.4188	3.4394	3.4601	3.4809	3.5017	3.5225	3.5435	3.5644
0.990	3.7949	3.8167	3.8386	3.8606	3.8829	3.9048	3.9270	3.9493	3.9716	3.9941
0.995	4.0825	4.1031	4.1260	4.1489	4.1720	4.1951	4.2184	4.2417	4.2651	4.2886

$P^*\backslash\nu$	0.90	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99
0.600	1.6164	1.6318	1.6471	1.6625	1.6778	1.6932	1.7086	1.7239	1.7393	1.7546
0.650	1.7667	1.7824	1.7982	1.8139	1.8296	1.8454	1.8612	1.8769	1.8927	1.9085
0.700	1.9254	1.9415	1.9577	1.9739	1.9900	2.0062	2.0224	2.0386	2.0548	2.0711
0.750	2.0970	2.1136	2.1302	2.1469	2.1635	2.1802	2.1969	2.2136	2.2302	2.2470
0.800	2.2886	2.3058	2.3229	2.3401	2.3573	2.3745	2.3917	2.4089	2.4262	2.4435
0.850	2.5126	2.5304	2.5482	2.5660	2.5838	2.6017	2.6196	2.6375	2.6554	2.6734
0.900	2.7956	2.8142	2.8328	2.8515	2.8701	2.8889	2.9076	2.9265	2.9452	2.9641
0.950	3.2173	3.2372	3.2571	3.2771	3.2972	3.3172	3.3373	3.3575	3.3777	3.3979
0.975	3.5855	3.6066	3.6277	3.6489	3.6701	3.6915	3.7129	3.7343	3.7558	3.7773
0.990	4.0167	4.0391	4.0618	4.0845	4.1073	4.1302	4.1531	4.1761	4.1992	4.2224
0.995	4.3122	4.3357	4.3595	4.3833	4.4072	4.4312	4.4552	4.4794	4.5038	4.5280

$P^*\backslash\nu$	0.991	0.992	0.993	0.994	0.995	0.996	0.997	0.998	0.999	1.000
0.600	1.7562	1.7577	1.7593	1.7608	1.7623	1.7639	1.7654	1.7669	1.7685	1.7700
0.650	1.9101	1.9117	1.9132	1.9148	1.9164	1.9180	1.9195	1.9211	1.9227	1.9243
0.700	2.0727	2.0743	2.0759	2.0775	2.0792	2.0808	2.0824	2.0840	2.0857	2.0873
0.750	2.2486	2.2503	2.2520	2.2537	2.2553	2.2570	2.2587	2.2603	2.2620	2.2637
0.800	2.4452	2.4469	2.4486	2.4504	2.4521	2.4538	2.4556	2.4573	2.4590	2.4608
0.850	2.6752	2.6770	2.6788	2.6806	2.6824	2.6842	2.6860	2.6878	2.6896	2.6914
0.900	2.9659	2.9678	2.9697	2.9716	2.9735	2.9754	2.9773	2.9792	2.9810	2.9829
0.950	3.3999	3.4020	3.4040	3.4060	3.4080	3.4101	3.4121	3.4141	3.4162	3.4182
0.975	3.7795	3.7816	3.7838	3.7859	3.7881	3.7903	3.7924	3.7946	3.7967	3.7989
0.990	4.2247	4.2270	4.2294	4.2316	4.2340	4.2363	4.2386	4.2409	4.2432	4.2455
0.995	4.5304	4.5328	4.5352	4.5376	4.5402	4.5425	4.5450	4.5474	4.5498	4.5523

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Table 6.1: $k = 10$

$P^*\backslash\nu$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
0.600	1.7700	1.9239	2.0780	2.2323	2.3869	2.5417	2.6966	2.8519	3.0073	3.1628
0.650	1.9243	2.0824	2.2411	2.4003	2.5600	2.7201	2.8806	3.0415	3.2028	3.3644
0.700	2.0873	2.2501	2.4137	2.5782	2.7434	2.9092	3.0757	3.2428	3.4104	3.5785
0.750	2.2637	2.4317	2.6008	2.7711	2.9424	3.1146	3.2877	3.4617	3.6363	3.8117
0.800	2.4608	2.6346	2.8101	2.9871	3.1654	3.3450	3.5257	3.7075	3.8903	4.0740
0.850	2.6914	2.8724	3.0548	3.2405	3.4273	3.6158	3.8058	3.9971	4.1898	4.3835
0.900	2.9829	3.1733	3.3665	3.5621	3.7602	3.9603	4.1625	4.3664	4.5719	4.7790
0.950	3.4182	3.6233	3.8322	4.0447	4.2604	4.4790	4.7004	4.9241	5.1501	5.3781
0.975	3.7989	4.0176	4.2413	4.4693	4.7015	4.9373	5.1765	5.4186	5.6635	5.9109
0.990	4.2455	4.4813	4.7231	4.9706	5.2231	5.4802	5.7415	6.0065	6.2748	6.5461
0.995	4.5523	4.8002	5.0552	5.3167	5.5839	5.8564	6.1336	6.4149	6.6999	6.9884

$P^*\backslash\nu$	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
0.600	3.3186	3.4745	3.6307	3.7869	3.9433	4.0999	4.2566	4.4134	4.5704	4.7275
0.650	3.5263	3.6885	3.8510	4.0138	4.1769	4.3401	4.5036	4.6673	4.8312	4.9953
0.700	3.7471	3.9161	4.0856	4.2554	4.4256	4.5961	4.7659	4.9381	5.1095	5.2811
0.750	3.9877	4.1643	4.3414	4.5191	4.6972	4.8758	5.0548	5.2342	5.4139	5.5939
0.800	4.2586	4.4439	4.6299	4.8165	5.0038	5.1916	5.3800	5.5688	5.7580	5.9477
0.850	4.5784	4.7742	4.9709	5.1685	5.3668	5.5658	5.7655	5.9657	6.1664	6.3677
0.900	4.9874	5.1970	5.4079	5.6197	5.8325	6.0462	6.2607	6.4759	6.6918	6.9083
0.950	5.6079	5.8394	6.0723	6.3067	6.5423	6.7790	7.0167	7.2554	7.4950	7.7354
0.975	6.1605	6.4121	6.6655	6.9206	7.1772	7.4351	7.6943	7.9543	8.2159	8.4782
0.990	6.8201	7.0966	7.3751	7.6557	7.9380	8.2220	8.5075	8.7944	9.0820	9.3710
0.995	7.2798	7.5739	7.8705	8.1691	8.4698	8.7720	9.0764	9.3815	9.6883	9.9961

$P^*\backslash\nu$	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
0.600	4.8847	5.0419	5.1993	5.3568	5.5144	5.6721	5.8298	5.9876	6.1455	6.3035
0.650	5.1595	5.3239	5.4885	5.6532	5.8181	5.9830	6.1481	6.3133	6.4786	6.6440
0.700	5.4530	5.6251	5.7974	5.9699	6.1426	6.3155	6.4885	6.6616	6.8349	7.0084
0.750	5.7743	5.9549	6.1358	6.3169	6.4983	6.6799	6.8617	7.0436	7.2258	7.4081
0.800	6.1378	6.3282	6.5189	6.7099	6.9013	7.0929	7.2851	7.4768	7.6691	7.8616
0.850	6.5695	6.7717	6.9743	7.1773	7.3806	7.5842	7.7882	7.9924	8.1969	8.4016
0.900	7.1254	7.3431	7.5612	7.7798	7.9988	8.2182	8.4380	8.6580	8.8786	9.0993
0.950	7.9765	8.2183	8.4607	8.7037	8.9476	9.1913	9.4357	9.6806	9.9259	10.1716
0.975	8.7411	9.0052	9.2699	9.5352	9.8011	10.0676	10.3348	10.6023	10.8703	11.1387
0.990	9.6611	9.9520	10.2437	10.5363	10.8296	11.1236	11.4181	11.7132	12.0088	12.3049
0.995	10.3053	10.6153	10.9268	11.2378	11.5532	11.8625	12.1779	12.4924	12.8079	13.1232

$P^*\backslash\nu$	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9
0.600	6.4615	6.6196	6.7778	6.9360	7.0942	7.2526	7.4109	7.5693	7.7278	7.8863
0.650	6.8095	6.9751	7.1408	7.3065	7.4724	7.6383	7.8042	7.9703	8.1364	8.3025
0.700	7.1819	7.3556	7.5294	7.7033	7.8773	8.0513	8.2255	8.3998	8.5741	8.7485
0.750	7.5906	7.7732	7.9560	8.1389	8.3219	8.5050	8.6882	8.8716	9.0550	9.2386
0.800	8.0542	8.2471	8.4402	8.6333	8.8267	9.0202	9.2138	9.4075	9.6014	9.7954
0.850	8.6066	8.8118	9.0172	9.2227	9.4285	9.6344	9.8405	10.0467	10.2531	10.4596
0.900	9.3203	9.5417	9.7631	9.9848	10.2068	10.4290	10.6513	10.8739	11.0965	11.3193
0.950	10.4176	10.6639	10.9106	11.1575	11.4055	11.6521	11.8998	12.1476	12.3957	12.6440
0.975	11.4075	11.6767	11.9462	12.2161	12.4863	12.7567	13.0275	13.2984	13.5696	13.8410
0.990	12.6015	12.8985	13.1958	13.4937	13.7917	14.0901	14.3890	14.6880	14.9873	15.2870
0.995	13.4394	13.7559	14.0729	14.3904	14.7084	15.0264	15.3445	15.6638	15.9827	16.3020

$P^*\backslash\nu$	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
0.600	8.0448	8.2034	8.3620	8.5206	8.6793	8.8380	8.9967	9.1555	9.3143	9.4731
0.650	8.4687	8.6350	8.8013	8.9676	9.1341	9.3004	9.4670	9.6335	9.8001	9.9667
0.700	8.9230	9.0975	9.2721	9.4468	9.6216	9.7965	9.9712	10.1461	10.3210	10.4960
0.750	9.4222	9.6059	9.7897	9.9736	10.1575	10.3415	10.5256	10.7097	10.8939	11.0781
0.800	9.9894	10.1836	10.3779	10.5722	10.7667	10.9613	11.1558	11.3506	11.5453	11.7401
0.850	10.6663	10.8730	11.0799	11.2869	11.4939	11.7011	11.9084	12.1157	12.3232	12.5307
0.900	11.5425	11.7656	11.9889	12.2123	12.4358	12.6595	12.8832	13.1071	13.3311	13.5551
0.950	12.8924	13.1411	13.3899	13.6388	13.8879	14.1371	14.3865	14.6360	14.8857	15.1355
0.975	14.1127	14.3845	14.6566	14.9287	15.2011	15.4737	15.7464	16.0192	16.2922	16.5653
0.990	15.5866	15.8865	16.1849	16.4872	16.7879	17.0887	17.3895	17.6907	17.9920	18.2934
0.995	16.6219	16.9417	17.2596	17.5819	17.8984	18.2231	18.5440	18.8647	19.1852	19.5079

$P^*\backslash\nu$	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9
0.600	9.6319	9.7908	9.9497	10.1086	10.2675	10.4265	10.5854	10.7444	10.9034	11.0624
0.650	10.1333	10.3000	10.4667	10.6334	10.8002	10.9670	11.1338	11.3006	11.4675	11.6343
0.700	10.6710	10.8461	11.0212	11.1964	11.3715	11.5467	11.7220	11.8973	12.0726	12.2480
0.750	11.2624	11.4468	11.6312	11.8156	12.0001	12.1846	12.3692	12.5539	12.7385	12.9232
0.800	11.9350	12.1300	12.3250	12.5200	12.7152	12.9103	13.1056	13.3008	13.4961	13.6915
0.850	12.7383	12.9460	13.1537	13.3615	13.5694	13.7773	13.9853	14.1934	14.4014	14.6096
0.900	13.7793	14.0035	14.2279	14.4522	14.6767	14.9013	15.1259	15.3506	15.5753	15.8001
0.950	15.3852	15.6352	15.8852	16.1354	16.3856	16.6359	16.8862	17.1369	17.3874	17.6380
0.975	16.8385	17.1118	17.3852	17.6589	17.9325	18.2061	18.4801	18.7541	19.0275	19.3022
0.990	18.5950	18.8968	19.1975	19.5005	19.8027	20.1047	20.4070	20.7095	21.0118	21.3145
0.995	19.8293	20.1509	20.4727	20.7945	21.1165	21.4389	21.7623	22.0835	22.4060	22.7288

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Table 6.1: $k = 10$

$P^*\backslash\nu$	7.0	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
0.600	11.2215	11.3805	11.5396	11.6987	11.8578	12.0169	12.1760	12.3352	12.4943	12.6535
0.650	11.8012	11.9682	12.1351	12.3021	12.4691	12.6361	12.8031	12.9702	13.1372	13.3043
0.700	12.4233	12.5987	12.7742	12.9496	13.1251	13.3006	13.4762	13.6517	13.8273	14.0029
0.750	13.1079	13.2927	13.4775	13.6623	13.8471	14.0320	14.2169	14.4018	14.5867	14.7717
0.800	13.8869	14.0823	14.2778	14.4733	14.6689	14.8644	15.0601	15.2557	15.4514	15.6471
0.850	14.8178	15.0260	15.2344	15.4426	15.6510	15.8595	16.0679	16.2764	16.4849	16.6935
0.900	16.0250	16.2499	16.4749	16.6998	16.9249	17.1501	17.3752	17.6004	17.8256	18.0509
0.950	17.8888	18.1395	18.3903	18.6412	18.8915	19.1432	19.3943	19.6454	19.8965	20.1477
0.975	19.5764	19.8507	20.1250	20.3994	20.6740	20.9484	21.2230	21.4976	21.7731	22.0472
0.990	21.6172	21.9199	22.2228	22.5259	22.8284	23.1316	23.4356	23.7383	24.0414	24.3447
0.995	23.0513	23.3739	23.6970	24.0175	24.3434	24.6689	24.9896	25.3124	25.6359	25.9596

$P^*\backslash\nu$	8.0	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9
0.600	12.8127	12.9719	13.1311	13.2903	13.4495	13.6087	13.7679	13.9272	14.0865	14.2457
0.650	13.4714	13.6387	13.8056	13.9728	14.1399	14.3071	14.4743	14.6414	14.8086	14.9759
0.700	14.1784	14.3541	14.5297	14.7054	14.8811	15.0568	15.2325	15.4082	15.5840	15.7597
0.750	14.9568	15.1418	15.3269	15.5119	15.6970	15.8822	16.0673	16.2524	16.4377	16.6228
0.800	15.8428	16.0386	16.2344	16.4302	16.6260	16.8219	17.0178	17.2137	17.4097	17.6056
0.850	16.9020	17.1107	17.3194	17.5280	17.7367	17.9455	18.1542	18.3630	18.5719	18.7807
0.900	18.2763	18.5016	18.7270	18.9525	19.1779	19.4027	19.6290	19.8545	20.0801	20.3057
0.950	20.3990	20.6503	20.9020	21.1530	21.4045	21.6560	21.9074	22.1590	22.4106	22.6622
0.975	22.3218	22.5968	22.8718	23.1471	23.4217	23.6969	23.9719	24.2471	24.5223	24.7975
0.990	24.6480	24.9509	25.2549	25.5586	25.8622	26.1659	26.4694	26.7734	27.0769	27.3811
0.995	26.2829	26.6063	26.9303	27.2537	27.5776	27.9009	28.2252	28.5497	28.8730	29.1965

$P^*\backslash\nu$	9.0	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
0.600	14.4050	14.5643	14.7236	14.8828	15.0422	15.2015	15.3608	15.5201	15.6795	15.8388
0.650	15.1431	15.3103	15.4776	15.6448	15.8121	15.9793	16.1466	16.3139	16.4812	16.6485
0.700	15.9355	16.1113	16.2871	16.4629	16.6387	16.8145	16.9904	17.1663	17.3421	17.5180
0.750	16.8080	16.9932	17.1785	17.3637	17.5490	17.7343	17.9196	18.1049	18.2902	18.4756
0.800	17.8015	17.9975	18.1935	18.3895	18.5856	18.7816	18.9777	19.1738	19.3699	19.5661
0.850	18.9896	19.1985	19.4074	19.6163	19.8253	20.0342	20.2432	20.4522	20.6613	20.8703
0.900	20.5314	20.7570	20.9827	21.2085	21.4342	21.6600	21.8858	22.1116	22.3375	22.5633
0.950	22.9139	23.1655	23.4172	23.6690	23.9203	24.1726	24.4244	24.6762	24.9281	25.1800
0.975	25.0727	25.3481	25.6234	25.8988	26.1738	26.4496	26.7251	27.0005	27.2761	27.5516
0.990	27.6840	27.9888	28.2928	28.5971	28.9005	29.2047	29.5089	29.8131	30.1173	30.4214
0.995	29.5210	29.8441	30.1688	30.4931	30.8175	31.1416	31.4657	31.7885	32.1136	32.4384

$P^*\backslash\nu$	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0	50.0	55.0
0.600	15.9981	23.9706	31.9482	39.9280	47.9088	55.8902	63.8720	71.8541	79.8364	87.8188
0.650	16.8158	25.1887	33.5683	41.9508	50.3347	58.7194	67.1046	75.4900	83.8759	92.2619
0.700	17.6939	26.4972	35.3090	44.1244	52.9414	61.7595	70.5783	79.3974	88.2168	97.0365
0.750	18.6609	27.9389	37.2271	46.5197	55.8142	65.1101	74.4066	83.7037	93.0013	102.2989
0.800	19.7622	29.5815	39.4131	49.2496	59.0882	68.9288	78.7700	88.6120	98.4547	108.2969
0.850	21.0794	31.5472	42.0292	52.5172	63.0077	73.5000	83.9934	94.4874	104.9821	115.4766
0.900	22.7892	34.1001	45.4277	56.7619	68.0995	79.4390	90.7797	102.1210	113.4630	124.8057
0.950	25.4319	38.0485	50.6847	63.3290	75.9771	88.6277	101.2797	113.9324	126.5861	139.2398
0.975	27.8271	41.6287	55.4527	69.2855	83.1229	96.9628	110.8046	124.6472	138.4906	152.3343
0.990	30.7252	45.9625	61.2246	76.5051	91.7743	107.0542	122.3350	137.6206	152.9038	168.1886
0.995	32.7630	49.0112	65.2821	81.5694	97.8577	114.1514	130.4398	146.7427	163.0384	179.3376

$P^*\backslash\nu$	60.0	65.0	70.0	75.0	80.0	85.0	90.0	95.0	100.0
0.600	95.8015	103.7839	111.7665	119.7492	127.7319	135.7147	143.6973	151.6805	159.6630
0.650	100.6480	109.0341	117.4204	125.8069	134.1932	142.5795	150.9661	159.3525	167.7391
0.700	105.8563	114.6763	123.4963	132.3164	141.1366	149.9572	158.7770	167.5975	176.4177
0.750	111.5967	120.8948	130.1928	139.4911	148.7894	158.0877	167.3861	176.6844	185.9831
0.800	118.1397	127.9825	137.8258	147.6685	157.5122	167.3556	177.1990	187.0427	196.8859
0.850	125.9714	136.4673	146.9628	157.4585	167.9542	178.4501	188.9458	199.4418	209.9381
0.900	136.1483	147.4914	158.8347	170.1779	181.5214	192.8648	204.2086	215.5527	226.8979
0.950	151.8941	164.5490	177.2041	189.8590	202.5141	215.1692	227.8248	240.4807	253.1396
0.975	166.1791	180.0206	193.8681	207.7128	221.5578	235.4031	249.2499	263.0946	276.9396
0.990	183.4737	198.7598	214.0436	229.3243	244.6169	259.9029	275.1880	290.4758	303.7891
0.995	195.6373	211.9324	228.2317	244.5305	260.8250	277.1320	293.4318	309.7339	326.0315

6.0.2 Matlab Code: SolveForH1

```
%Conducts a Binary search for the value of h that
%solve the integral equation.
%MinDeriv is the minimum derivative value
%MaxDeriv is the Maximum derivative value
%k is the number of populations
%PStar is P^*
function H=SolveForH1(MinDeriv,MaxDeriv,k,PStar)
    a=MinDeriv./MaxDeriv;
    LB=0;
    UB=5/a+5;
    HB=(UB+LB)./2;
    tol=.00000001
    PStarH=quadgk(@(z)NormIntegrander1(z,HB,a,k), 5/a 5, HB);
    while abs(PStar PStarH)>=tol
            if PStar<PStarH
                    UB=HB;
            else
                    LB=HB;
            end
            HBPrev=HB;
            HB=(UB+LB)./2;
            PStarH=quadgk(@(z)NormIntegrander1
                                    (z,HB,a,k), 5/a 5, HB);
    end
    H=HBPrev;

function Prob=NormIntegrander1(z,h,a,k)
        Prob1=(2*pi)^( .5)*exp( .5*z.^2);
        Prob2=(1 .5*erfc(a.*( z+h)./sqrt(2))).^(k 1);
        Prob=Prob1.*Prob2;
```

6.0.3 Matlab Code: SolveForH2

```
%Conducts a Binary search for the value of h that
%solve the integral equation.
%Derivatives is a 1xk row vector of derivatives
%k is the number of populations
%PStar is P^*
 function H=SolveForH2(Derivatives,PStar)
    Derivatives=sort(Derivatives);
    k=length(Derivatives);
    a=Derivatives(1,1)/Derivatives(1,k);
    LB=0;
    UB=5/a+5;
    HB=(UB+LB)./2;
    tol=.00000001;
    PStarH=quadgk(@(z)NormIntegrander2
                               (z,HB,Derivatives,k), 5/a 5, HB);
    while abs(PStar PStarH)>=tol
            if PStar<PStarH
                    UB=HB;
            else
                    LB=HB;
            end
            HBPrev=HB;
            HB=(UB+LB)./2;
            PStarH=quadgk(@(z)NormIntegrander2
                               (z,HB,Derivatives,k), 5/a 5, HB);
    end
    H=HBPrev;

function Prob=NormIntegrander2(z,h,Derivatives,k)
        Prob=(2*pi)^( .5)*exp( .5*z.^2);
        Prob1=1;
        a=Derivatives(1,1);
        for j=2:k
                b=a./Derivatives(1,j);
            Prob2=1 .5*erfc(b.*( z+h)./sqrt(2));
            Prob1=Prob1.*Prob2;
        end
        Prob=Prob1.*Prob;
```

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