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A Characteristic Free Tilting Bundle for Grassmannians

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A CHARACTERISTIC FREE TILTING BUNDLE FOR GRASSMANNIANS

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Abstract. We construct a characteristic free tilting bundle on Grassmannians.

1. Introduction

Throughout K is a field of arbitrary characteristic. Let X be a smooth algebraic variety over K. A vector bundle $\mathcal{T}$ on X is called a *tilting bundle* if it satisfies the following two conditions.

(1) $\mathcal{T}$ classically generates the bounded derived category of coherent sheaves $\mathcal{D}^b(\mathrm{coh}(X))$. In other words, the smallest thick subcategory of $\mathcal{D}^b(\mathrm{coh}(X))$ containing $\mathcal{T}$ is $\mathcal{D}^b(\mathrm{coh}(X))$ itself.
(2) $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}, \mathcal{T}[i]) = 0$ for $i \neq 0$.

When K is a field of characteristic zero, Kapranov [7] constructs a tilting bundle on the Grassmannian variety $\mathbb{G} = \mathrm{Grass}(l, F) \cong \mathrm{Grass}(l, m)$ of l-dimensional subspaces of an m-dimensional K-vector space F as follows: we have a tautological exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow F^\vee \otimes_K \mathcal{O}_{\mathbb{G}} \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{1.1}$$

of vector bundles on $\mathbb{G}$. For a partition α write L_α for the associated Schur functor [9].

Theorem ([7]). *Let $B_{u,v}$ be the set of partitions with at most u rows and at most v columns. Then the vector bundle*

$$\mathcal{T}_K = \bigoplus_{\alpha \in B_{l,m-l}} L_\alpha \mathcal{Q}$$

is a tilting bundle on $\mathbb{G} = \mathrm{Grass}(l, F)$.

For fields K of positive characteristic p, Kaneda [6] shows that $\mathcal{T}_K$ remains tilting as long as $p \geqslant m - 1$. However $\mathcal{T}_K$ fails to be tilting in very small characteristics. See Example 4.3 below.

In this note we give a tilting bundle on $\mathbb{G}$ which exists in arbitrary characteristic.

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Main Theorem. *Define a vector bundle on* $\mathbb{G} = \operatorname{Grass}(l, F) \cong \operatorname{Grass}(l, m)$ *by*

$$\mathcal{T} = \bigoplus_{(u)} \bigwedge^{u_1} \mathcal{Q} \otimes \cdots \otimes \bigwedge^{u_{m-l}} \mathcal{Q}\,, \tag{1.2}$$

where the direct sum is over all sequences $l \geqslant u_1 \geqslant \cdots \geqslant u_{m-l} \geqslant 1$. *Then* $\mathcal{T}$ *is a tilting bundle on* $\mathbb{G}$.

In characteristic zero we recover Kapranov's theorem by working out the tensor products in (1.2) using Pieri's formula. Of course in contrast to Kapranov's tilting bundle, our tilting bundle is far from being multiplicity-free. In Remark 4.4 below we give some comments on the situation in characteristic p.

The proof of our main theorem depends on the following vanishing result which we will also use in [2].

Proposition 1.1. *Let* $u_1, \dots, u_{m-l}$ *be non-negative integers with* $u_j \geqslant 0$ *for all* j. *Then for all* $i > 0$ *and any partition* γ *one has*

$$H^i \left(\mathbb{G}, (\bigwedge^{u_1} \mathcal{Q})^\vee \otimes \cdots \otimes (\bigwedge^{u_{m-l}} \mathcal{Q})^\vee \otimes L_\gamma \mathcal{Q}\right) = 0\,.$$

The proof of Proposition 1.1 is a consequence of Kempf's vanishing theorem (see Theorem 3.1 below) and the vanishing of the cohomology of certain line bundles on projective space.

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3. Preliminaries

Throughout we use [5] as a convenient reference for facts about algebraic groups. If $H \subset G$ is an inclusion of algebraic groups over the ground field K, then the restriction functor from rational G-modules to rational H-modules has a right adjoint denoted by ind_H^G ([5, I.3.3]). Its right derived functors are denoted by $R^i \operatorname{ind}_H^G$. For an inclusion of groups $K \subset H \subset G$ and M a rational K-representation there is a spectral sequence [5, I.4.5(c)]

$$E_2^{pq} \colon R^p \operatorname{ind}_H^G R^q \operatorname{ind}_K^H M \Longrightarrow R^{p+q} \operatorname{ind}_K^G M\,. \tag{3.1}$$

If G/H is a scheme and V is a finite-dimensional representation then $\mathcal{L}_{G/H}(V)$ is by definition the G-equivariant vector bundle on G/H given by the sections of $(G \times V)/H$. The functor $\mathcal{L}_{G/H}(-)$ defines an equivalence between the finite-dimensional H-representations and the G-equivariant vector bundles on G/H. The inverse of this functor is given by taking the fiber in $[H]$.

If G/H is a scheme then $R^n \operatorname{ind}_H^G$ may be computed as [5, Prop. I.5.12]

$$R^n \operatorname{ind}_H^G M = H^n(G/H, \mathcal{L}_{G/H}(M))\,. \tag{3.2}$$

We now assume that G is a split reductive group with a given split maximal torus and Borel $T \subset B \subset G$. We let $X(T)$ be the character group of T and we identify the elements of $X(T)$ with the one-dimensional representations of T. The set of roots (the weights of $\operatorname{Lie} G$) is denoted by R. We have $R = R^- \coprod R^+$ where the negative roots R^- represent the roots of $\operatorname{Lie} B$. For $\alpha \in R$ we denote the corresponding coroot in $Y(T) = \operatorname{Hom}(X(T), \mathbb{Z})$ [5, II.1.3] by $\alpha^\vee$. The natural pairing between $X(T)$ and $Y(T)$ is denoted by $\langle -, - \rangle$. A weight $\lambda \in X(T)$ is dominant if $\langle \lambda, \alpha^\vee \rangle \geqslant 0$

for all positive roots α. The set of dominant weights is denoted by $X(T)_+$. For other unexplained terminology and notations we refer to [5].

The following is the celebrated Kempf vanishing result ([8], see also [5, II.4.5]).

Theorem 3.1. *If $\lambda \in X(T)_+$ then $R^i \operatorname{ind}_B^G \lambda = H^i(G/B, \mathcal{L}_{G/B}(\lambda))$ vanishes for all strictly positive i.*

4. Application to Grassmanians

We stick to the notation already introduced in the introduction. We will identify $\mathbb{G} = \operatorname{Grass}(l, F)$ with $\operatorname{Grass}(m-l, F^\vee)$ via the correspondence $(V \subset F) \mapsto ((F/V)^\vee \subset F^\vee)$.

For convenience we choose a basis for $(f_i)_{i=1,\dots,m}$ for F and a corresponding dual basis $(f_i^*)_i$ for $F^\vee$. We view $\mathbb{G}$ as homogeneous space G/P with $G = \mathrm{GL}(m)$ and $P \subset G$ the parabolic subgroup stabilizing the point $(W \subset F^\vee) \in \mathbb{G}$ where $W = \sum_{i=l+1}^m K f_i^*$. We let T and B be respectively the diagonal matrices and the lower triangular matrices in G. We identify $X(T)$ and $Y(T)$ with $\mathbb{Z}^m$ (with ε_i being the i^{th} natural basis element). Here $\sum a_i\varepsilon_i$ corresponds to the character $\operatorname{diag}(z_1,\dots,z_m) \mapsto z_1^{a_1}\cdots z_m^{a_m}$. Under this identification roots and coroots coincide and are given by $\varepsilon_i - \varepsilon_j$, $i \neq j$, a root being positive if $i < j$. The pairing between $X(T)$ and $Y(T)$ is the standard Euclidean scalar product and hence $X(T)_+ = \{\sum_i a_i\varepsilon_i \mid a_i \geqslant a_j \text{ for } i \leqslant j\}$.

Let $H = G_1 \times G_2 = \mathrm{GL}(l) \times \mathrm{GL}(m-l) \subset \mathrm{GL}(m)$ be the Levi-subgroup of P containing T. We put $B_i = B \cap G_i$, $T_i = T \cap G_i$.

For use in the proof below we fix an additional parabolic P° in G given by the stabilizer of the flag $(\sum_{i\geqslant p} K f_i^*)_{p=1,\dots,l}$. We let $G^\circ = \mathrm{GL}(m-l+1) \subset P^\circ \subset G = \mathrm{GL}(m)$ be the lower right $(m-l+1 \times m-l+1)$-block in $\mathrm{GL}(m)$. We put $T^\circ = T \cap G^\circ$, $B^\circ = B \cap G^\circ$. I.e. B° is the set of lower triangular matrices in G° and T° is the set of diagonal matrices.

We also recall the following.

Proposition 4.1 ([4, §4,§4.8][9, (4.1.10)])**.** *Let $\gamma = [\gamma_1,\dots,\gamma_m]$ be a partition (i.e. $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_m \geqslant 0$) and let $\gamma' = \sum_i \gamma_i\varepsilon_i$ be the corresponding weight. Then*

$$L_\gamma(F^\vee) = \operatorname{ind}_B^G \gamma' .$$

Proof of Proposition 1.1. Using the identity

$$(\textstyle\bigwedge^u \mathcal{Q})^\vee = \textstyle\bigwedge^{l-u} \mathcal{Q} \otimes_K \left(\textstyle\bigwedge^l \mathcal{Q}\right)^\vee$$

and the characteristic free version of the Littlewood-Richardson rule (see Theorem 4.2 below) we reduce immediately to the case $u_1 = \cdots = u_{m-l} = l$. The tautological exact sequence (1.1) allows us to write

$$\left(\textstyle\bigwedge^l \mathcal{Q}\right)^\vee = \textstyle\bigwedge^m F \otimes_K \textstyle\bigwedge^{m-l} \mathcal{R} .$$

Thus we need to prove that

$$L_\gamma \mathcal{Q} \otimes \textstyle\bigwedge^{m-l}\mathcal{R} \otimes \cdots \otimes \textstyle\bigwedge^{m-l}\mathcal{R}$$

(with $m-l$ factors of $\bigwedge^{m-l}\mathcal{R}$) has vanishing higher cohomology. Using (3.2) we see that we must prove that for $i > 0$ we have

$$R^i \operatorname{ind}_P^G \left(L_\gamma \mathcal{Q}_x \otimes \textstyle\bigwedge^{m-l}\mathcal{R}_x \otimes \cdots \otimes \textstyle\bigwedge^{m-l}\mathcal{R}_x\right) = 0 , \tag{4.1}$$

where $x = [P] \in G/P = \mathbb{G}$. Since $\mathcal{Q}$ has rank l, we may assume that γ has at most l entries. As above we write $\gamma' = \sum_{i=1}^{l} \gamma_i \varepsilon_i \in X(T_1)$ for the corresponding weight. Let $\sigma \in X(T_2)$ be given by $(m-l)\sum_{i=l+1}^{m} \varepsilon_i$ and put $\overline{\gamma} = \gamma' + \sigma \in X(T)$.

As $P/B \cong (G_1 \times G_2)/(B_1 \times B_2)$ we have

$$\begin{aligned} L_\gamma \mathcal{Q}_x \otimes \textstyle\bigwedge^{m-l} \mathcal{R}_x \otimes \cdots \otimes \textstyle\bigwedge^{m-l} \mathcal{R}_x &= \operatorname{ind}_{B_1}^{G_1} \gamma' \otimes \operatorname{ind}_{B_2}^{G_2} \sigma \\ &= \operatorname{ind}_B^P \overline{\gamma} \end{aligned}$$

The positive roots of G_1 are of the form $\varepsilon_i - \varepsilon_j$ with $i < j$ and $1 \leqslant i, j \leqslant l$. Similarly the positive roots of G_2 are of the form $\varepsilon_i - \varepsilon_j$ with $i < j$ and $l+1 \leqslant i, j \leqslant m-l$.

It follows that $\overline{\gamma}$ is dominant when viewed as a weight for T considered as a maximal torus in $H = G_1 \times G_2$. So Kempf vanishing implies that $R^i \operatorname{ind}_B^P \overline{\gamma} = R^i \operatorname{ind}_{B_1 \times B_2}^{G_1 \times G_2} \overline{\gamma} = 0$ for all $i > 0$.

Thus the spectral sequence (3.1) degenerates and we obtain

$$R^i \operatorname{ind}_P^G \left(L_\gamma \mathcal{Q}_x \otimes \textstyle\bigwedge^{m-l} \mathcal{R}_x \otimes \cdots \otimes \textstyle\bigwedge^{m-l} \mathcal{R}_x \right) = R^i \operatorname{ind}_B^G \overline{\gamma}\,. \tag{4.2}$$

Thus if $\overline{\gamma}$ is dominant (i.e. $\gamma_l \geqslant m-l$) then the desired vanishing (4.1) follows by invoking Kempf vanishing again.

Assume then that $\overline{\gamma}$ is not dominant, i.e. $0 \leqslant \gamma_l < m-l$. We claim that $R^i \operatorname{ind}_B^{P^\circ} \overline{\gamma} = 0$ for all i. Then by the spectral sequence (3.1) applied to $B \subset P^\circ \subset G$ we obtain that $R^i \operatorname{ind}_B^G \overline{\gamma} = 0$ for all i.

To prove the claim we note that $P^\circ/B \cong G^\circ/B^\circ$ and hence $R^i \operatorname{ind}_B^{P^\circ} \overline{\gamma} = R^i \operatorname{ind}_{B^\circ}^{G^\circ}(\overline{\gamma} \mid T^\circ)$. In other words we have reduced ourselves to the case $l = 1$ (replacing m by $m - l + 1$).

So now we assume $l = 1$. Thus $\mathbb{G} = \mathbb{P}^{m-1}$. The partition γ consists of a single entry γ_1 and $\sigma = \sum_{i=2}^{m}(m-1)\varepsilon_i$. Under the assumption $\gamma_1 < m-1$ we have to prove $R^i \operatorname{ind}_B^G \overline{\gamma} = 0$ for all i. Applying (4.2) in reverse this means we have to prove that

$$\mathcal{Q}^{\otimes \gamma_1} \otimes \left(\textstyle\bigwedge^{m-1} \mathcal{R} \right)^{\otimes m-1}$$

has vanishing cohomology on $\mathbb{P}^{m-1}$.

We now observe $\mathcal{Q} = \mathcal{O}_{\mathbb{P}^{m-1}}(1)$ and since

$$\mathcal{R} = \ker(\mathcal{O}_{\mathbb{P}^{m-1}}^m \to \mathcal{O}_{\mathbb{P}^{m-1}}(1))$$

we also have

$$\textstyle\bigwedge^{m-1} \mathcal{R} = \mathcal{O}_{\mathbb{P}^{m-1}}(-1)$$

so that

$$\mathcal{Q}^{\otimes \gamma_1} \otimes \textstyle\bigwedge^{m-1} \mathcal{R}^{\otimes m-1} = \mathcal{O}_{\mathbb{P}^{m-1}}(-m+1+\gamma_1)$$

It is standard that this line bundle has vanishing cohomology when $\gamma_1 < m-1$. □

We have used the following result.

Theorem 4.2 (Boffi [1])**.** *Let α and β be arbitrary partitions and E a K-vector space. There is a natural ("good") filtration on $L_\alpha E \otimes_K L_\beta E$ whose associated graded object is a direct sum of Schur functors $L_\gamma E$. The γ that appear, and their multiplicities, can be computed using the usual Littlewood-Richardson rule.*

In a good filtration as above, we may assume by [5, II.4.16, Remark (4)] that the $L_\gamma E$ which appear are in decreasing order for the lexicographic ordering on partitions, that is, the largest γ appear on top.

Proof of the Main Theorem. The main thing to prove is that $\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T},\mathcal{T}) = 0$ for $i \neq 0$. It follows from the usual spectral sequence argument that $\operatorname{Ext}^i_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T},\mathcal{T})$ is the i^{th} cohomology of $\mathscr{H}om_{\mathcal{O}_{\mathbb{G}}}(\mathcal{T},\mathcal{T}) = \mathcal{T}^\vee \otimes \mathcal{T}$. Applying Theorem 4.2 we see that it suffices to prove that $\mathcal{T}^\vee \otimes L_\gamma \mathcal{Q}$ has vanishing higher cohomology whenever γ is a partition with at most l rows. This is the content of Proposition 1.1.

Kapranov's resolution of the diagonal argument implies that $\mathcal{T}_K$ still classically generates $D^b(\operatorname{coh}(\mathbb{G}))$ [6, §4]. Thus we must show that $L_\alpha \mathcal{Q}$ for $\alpha \in B_{l,m-l}$ is in the thick subcategory $\mathcal{C}$ generated by $\mathcal{T}$. Assume this is not the case and let α be minimal for the lexicographic ordering on partitions such that $L_\alpha \mathcal{Q}$ is *not* in $\mathcal{C}$.

Let $\overline{\alpha}$ be the dual partition and consider $\mathcal{U} = \bigwedge^{\overline{\alpha}_1} \mathcal{Q} \otimes \cdots \otimes \bigwedge^{\overline{\alpha}_l} \mathcal{Q}$. By Theorem 4.2 and the comment following, $\mathcal{U}$ maps surjectively to $L_\alpha \mathcal{Q}$ and the kernel is an extension of various $L_\beta \mathcal{Q}$ with $\beta < \alpha$ (Pieri's formula, which is a special case of the Littlewood-Richardson rule, implies that $L_\alpha \mathcal{Q}$ appears with multiplicity one in $\mathcal{U}$). By the hypotheses all such $L_\beta \mathcal{Q}$ are in $\mathcal{C}$. Since $\mathcal{U}$ is in $\mathcal{C}$ as well we obtain that $L_\alpha \mathcal{Q}$ is in $\mathcal{C}$, which is a contradiction.

□

Example 4.3. Assume that K has characteristic 2 and put $\mathbb{G} = \operatorname{Grass}(2,4)$. Then the short exact sequence

$$0 \longrightarrow \textstyle\bigwedge^2 \mathcal{Q} \longrightarrow \mathcal{Q} \otimes \mathcal{Q} \longrightarrow S_2 \mathcal{Q} \longrightarrow 0 \tag{4.3}$$

is non-split. In particular $\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{G}}}(S_2\mathcal{Q}, \bigwedge^2 \mathcal{Q}) \neq 0$, so that $S_2\mathcal{Q}$ and $\bigwedge^2 \mathcal{Q}$ are not common direct summands of a tilting bundle on $\mathbb{G}$.

To see that (4.3) is not split, tensor with $(\bigwedge^2 \mathcal{Q})^\vee$ to obtain the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{G}} \longrightarrow \mathscr{E}nd\,(\mathcal{Q}) \longrightarrow (\textstyle\bigwedge^2 \mathcal{Q})^\vee \otimes S_2\mathcal{Q} \longrightarrow 0 \tag{4.4}$$

where the leftmost map is the obvious one. Any splitting of the inclusion $\mathcal{O}_{\mathbb{G}} \longrightarrow \mathscr{E}nd\,(\mathcal{Q})$ is of the form $\operatorname{Tr}(a-)$, where Tr is the reduced trace and a is an element of $\operatorname{End}(\mathcal{Q})$ such that $\operatorname{Tr}(a) = 1$. Hence it is sufficient to prove that $\operatorname{End}(\mathcal{Q}) = K$ since in that case we have $\operatorname{Tr}(a) = 0$ for any $a \in \operatorname{End}(\mathcal{Q})$.

By (the proof of) Proposition 1.1 we have $H^i(\mathbb{G}, (\bigwedge^2 \mathcal{Q})^\vee \otimes S_2\mathcal{Q}) = 0$ for all i (observe that if we go through to the proof we obtain a situation where $\overline{\gamma}$ is not dominant, whence all cohomology vanishes) and of course we also have $H^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}) = K$. Applying $H^0(\mathbb{G}, -)$ to (4.4) thus shows $\operatorname{End}(\mathcal{Q}) = K$.

Remark 4.4. By [3, Lemma (3.4)] (at least when K is algebraically closed) we obtain the following more economical tilting bundle for $\mathbb{G}$

$$\mathcal{T}^\circ = \bigoplus_{\alpha \in B_{l,m-l}} \mathcal{L}_{\mathbb{G}}(M(\alpha))\,,$$

where $M(\alpha)$ is the tilting $\operatorname{GL}(l)$-representation with highest weight α. Note however that the character of $M(\alpha)$ strongly depends on the characteristic. Hence so does the nature of $\mathcal{T}^\circ$.

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