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by H. F. Mattson, Jr.

Abstract. We simplify the proofs of four results in [3], restating two of them for greater clarity.

The main purpose of this note is to give a brief transparent proof of Theorem 7 of [3], the main upper bound of that paper. The secondary purpose is to give a more direct statement and proof of the integer programming determination of covering radius of [3].

Theorem 7 of [3] follows from a simple result in [2], which we state with the notation (for the linear code $A$)

$g(A)$ : = a generator matrix of $A$,
$t(A)$ : = the covering radius of $A$.

THEOREM 1 [2]. If $A$ is a code with generator matrix

$$g(A) = \begin{pmatrix} g(A_1) & * \\ 0 & g(A_0) \\ X & \bar{X} \end{pmatrix}$$

then $t(A) \leq t(A_0) + t(A_1)$.

To describe the codes $A_0$ and $A_1$: Pick any subset $X$ of coordinate-places of $A$. $A_1$ is the projection of $A$ on $X$; we get $A_0$ from the subcode $D$ of $A$ which vanishes on $X$ by projecting $D$ on $\bar{X}$. ($A_0 [A_1]$ is sometimes called a shortened [punctured] code of $A$.)

Before stating Theorem 2, let us agree that all codes $B, C$ are binary, linear, and have no coordinates identically 0. (The last need not be true of $C_0$.) We also need the following notation:
(2.1) \[ S_k := [2^k - 1, k] \text{ simplex code.} \]

(2.2) \( B \) denotes an \([n, k]\) code having in \( g(B) \) exactly \( m_i \geq 0 \) copies of column \( i \) of \( g(S_k) \) for \( i = 1, \ldots, 2^k - 1 \). Thus \( n = \sum m_i \).

(2.3) We often identify a vector in \( \mathbb{Z}_2^n \) with its support. In this note the support is a subset of the set of columns of \( S_k \), or a multisubset thereof. In that identification we may denote the weight of the vector \( x \) by \( |x| \), the cardinality of the support of \( x \). The columns of \( g(B) \) form a multisubset of the set of columns of \( g(S_k) \). The vector \((m_1, \ldots, m_{2^k - 1})\) of multiplicities of the columns is called the \textit{signature} of \( B \).

(3) The normalized covering radius \([3]\) of \( B \) is defined as
\[
\rho(B) := \rho(m_1, \ldots, m_{2^k - 1}) := t(B) - \sum_i \left\lfloor \frac{m_i}{2} \right\rfloor.
\]

The \textit{projective core} of \( B \) is the code \( C \) for which \( g(C) \) consists of the columns of \( g(B) \) without any repetitions. I.e., in the signature \((\ldots, \nu_i, \ldots)\) of \( C \), \( \nu_i = 1 \) if \( m_i > 0 \) and \( \nu_i = 0 \) if \( m_i = 0 \).

For any column \( Q \) of \( g(B) \) we define \( \eta := \eta_Q \) to be the total number of vectors \( \{P, Q, R\} \) of weight 3 in \( C^\perp \) for which \( m_P \) and \( m_R \) are odd. The vectors are denoted as in (2.3).

Before going on, we comment on (3). Recall from [1, II D] the definition of a concatenation \( A \) of the \([n_1, k_1]\) code \( A_1 \) and the \([n_2, k_2]\) code \( A_2 \), with \( k_1 \leq k_2 \). It has generator matrix
\[
g(A) = \begin{bmatrix} g(A_1) & | & g(A_2) \\ 0 & | & 0 \end{bmatrix},
\]
and its covering radius satisfies \( t(A) \geq t(A_1) + t(A_2) \) [1, II D]. We take \( A_2 \), say, to be the "even" part of the code \( B \). That is, write \( m_i = 2\mu_i + \epsilon_i \), where \( \epsilon_i = 0 \) or \( 1 \), and take \( A_1 \) and \( A_2 \) to have signatures \((\ldots, \epsilon_i, \ldots)\) and \((\ldots, 2\mu_i, \ldots)\), respectively. Then \( B \) is a
catenation of $A_1$ and $A_2$, and $t(B) \geq t(A_1) + t(A_2)$. From \cite{2, (11)} we get an immediate proof of Thm. 6 of \cite{3}: $t(A_2) = \sum \mu_i$, since the “double” of any code of length $\ell$ has covering radius $\ell$. Therefore, $t(B) \geq t(A_1) + \sum \mu_i$ and $\rho(B) \geq t(A_1)$. (This is Thm. 5 of \cite{3}.)

To state the result, choose any column $Q$ of $g(B)$. After row-operations (which do not change $B$ even though they permute the $m_i$'s) column $Q$ becomes simply $(10 \cdots 0)^\top$, and

\begin{equation}
\begin{array}{c|*{2}{c|c}}
\multicolumn{4}{c}{m_Q} \\
\hline
 11 \cdots 1 & & \\
0 & & g(B_0) \\
\end{array}
\end{equation}

where $B_0$ has signature $(m'_1, m'_2, \ldots, m'_{2^k-1-1})$.

**THEOREM 2** (\cite{3}). The normalized covering radius of $B$ satisfies

$$\rho(B) \leq \eta_Q + \rho(m'_1, \ldots, m'_{2^k-1-1}).$$

**Proof.** Since $B_1$ in \eqref{eq:4} is an $[m_Q, 1, m_Q]$ repetition code, $t(B_1) = \lfloor m_Q/2 \rfloor$. Thus, from Theorem 1,

\begin{equation}
t(B) \leq \lfloor m_Q/2 \rfloor + t(B_0).
\end{equation}

To express \eqref{eq:5} in terms of normalized covering radii, we subtract $\sum_i \lfloor m_i/2 \rfloor$ from both sides. We get

\begin{equation}
\rho(B) := t(B) - \sum_i \lfloor m_i/2 \rfloor \leq t(B_0) - \sum_{i \neq Q} \lfloor m_i/2 \rfloor.
\end{equation}

Each pair of columns $P$ and $R$ of $g(B)$ which agree except on their top coordinate have sum $Q$. That is, for some vector $N$, $P = (0, N)^\top$ and $R = (1, N)^\top$. Thus $m_P + m_R = m'_N$, and $\{P, Q, R\}$ is (the support of) a vector of weight 3 in $C^\perp$. We note that
unless \( m_P \) and \( m_R \) are odd, in which case the right-hand side of (7) must be decreased by 1. Thus (6) becomes

\[
\rho(B) \leq t(B_0) - \sum_j \left\lfloor \frac{m_j}{2} \right\rfloor + \eta. \quad \Box
\]

Remark. Theorem 1 allowed us to avoid the notion of "height" used in [3]. We have also restated the result by defining \( \eta \) not with finite geometry, as in [3], but in terms of the code. Except for this change of language the proof after (5) is similar to that of [3].

Finally, we simplify the integer programming determination [3, Thm. 1] of \( \rho(B) \) by eliminating "height" from the statement and proof.

In terms of (2), it is simple to see [1] that \( x \) is a coset leader of a code \( A \) iff

\[
\forall a \in A \quad 2|x \cap a| \leq |a|.
\]

Letting the \([n, k]\) code \( B \) have signature \((\cdots, m_i, \cdots)\), define [3,(5)] for any \( x \in \mathbb{Z}_2^n \),

\[
x := (x^{(1)}, \ldots, x^{(n)}),
\]

where \( x^{(i)} \) is the "sub" vector of the coordinates of \( x \) at the \( m_i \) places where column \( i \) appears in \( g(B) \). Define

\[
(9)
\]

\[
w_i(x) := wt(x^{(i)}).
\]

It follows that \( 0 \leq w_i(x) \leq m_i \) for all \( i \) and \( x \), and that \( wt(x) = \sum_i w_i(x) \).

We also project \( B \) onto the projective core \( C \) by the rule

\[
b = (\ldots, b^{(i)}, \ldots) \rightarrow (\ldots, c_i, \ldots) = c,
\]

where \( c_i = 1 \) iff \( b^{(i)} \neq 0 \). It follows that \( |b| = \sum_i c_i m_i \), where \( c_i \) is regarded as real 0 or 1.
Using (2.3) we calculate for any \( b \in B \) and any \( x \in \mathbb{Z}_2^n \)

\[
x \cap b = \bigcup_{i} x^{(i)} \cap b^{(i)} = \bigcup_{c_i=1} x^{(i)}.
\]

Hence

\[
|x \cap b| = \sum_{i} c_i w_i(x).
\]

Thus we see from (8) that \( x \) is a coset leader for \( B \) iff for all \( c = (\ldots, c_i, \ldots) \) in \( C \),

\[
\sum_{i} c_i w_i(x) \leq \frac{1}{2} \sum_{i} c_i m_i.
\]

Since the covering radius of \( B \) is the weight of a coset leader of maximum weight we have proved (cf. [3, Thm. 1])

**THEOREM 3.** The covering radius of \( B \) is the solution to the following integer programming problem:

Maximize \( W := w_1 + \cdots + w_{2^s-1} \) subject to the constraints

\[
w_i \in \mathbb{Z}, 0 \leq w_i \leq m_i
\]

and \( \sum_{i} c_i w_i \leq \frac{1}{2} \sum_{i} c_i m_i \) for all \( c = (c_i) \in C \).

**COROLLARY.** \( \rho(B) = \max W - \sum_{i} \left\lfloor \frac{m_i}{2} \right\rfloor \).

**References**

