

Syracuse University

**SURFACE**

---

Center for Policy Research

Maxwell School of Citizenship and Public  
Affairs

---

2007

## Asymptotic Properties of Estimators for the Linear Panel Regression Model with Individual Effects and Serially Correlated Errors: The Case of Stationary and Non-Stationary Regressors and Residuals

Badi H. Baltagi

*Syracuse University*, [bbaltagi@maxwell.syr.edu](mailto:bbaltagi@maxwell.syr.edu)

Chihwa Kao

*Syracuse University*, [cdkao@maxwell.syr.edu](mailto:cdkao@maxwell.syr.edu)

Long Liu

*Syracuse University*

Follow this and additional works at: <https://surface.syr.edu/cpr>



Part of the [Mathematics Commons](#)

---

### Recommended Citation

Baltagi, Badi H.; Kao, Chihwa; and Liu, Long, "Asymptotic Properties of Estimators for the Linear Panel Regression Model with Individual Effects and Serially Correlated Errors: The Case of Stationary and Non-Stationary Regressors and Residuals" (2007). *Center for Policy Research*. 72.

<https://surface.syr.edu/cpr/72>

This Working Paper is brought to you for free and open access by the Maxwell School of Citizenship and Public Affairs at SURFACE. It has been accepted for inclusion in Center for Policy Research by an authorized administrator of SURFACE. For more information, please contact [surface@syr.edu](mailto:surface@syr.edu).

**Center for Policy Research  
Working Paper No. 93**

**ASYMPTOTIC PROPERTIES OF ESTIMATORS FOR  
THE LINEAR PANEL REGRESSION MODEL WITH  
INDIVIDUAL EFFECTS AND SERIALY CORRELATED  
ERRORS: THE CASE OF STATIONARY AND  
NON-STATIONARY REGRESSORS AND RESIDUALS**

**Badi H. Baltagi, Chihwa Kao, and Long Liu**

**Center for Policy Research  
Maxwell School of Citizenship and Public Affairs  
Syracuse University  
426 Eggers Hall  
Syracuse, New York 13244-1020  
(315) 443-3114 | Fax (315) 443-1081  
e-mail: [ctrpol@syr.edu](mailto:ctrpol@syr.edu)**

**April 2007**

**\$5.00**

Up-to-date information about CPR's research projects and other activities is available from our World Wide Web site at [www-cpr.maxwell.syr.edu](http://www-cpr.maxwell.syr.edu). All recent working papers and Policy Briefs can be read and/or printed from there as well.

# CENTER FOR POLICY RESEARCH – Spring 2007

**Timothy Smeeding, Director**  
**Professor of Economics & Public Administration**

## Associate Directors

Margaret Austin  
Associate Director  
Budget and Administration

Douglas Wolf  
Professor of Public Administration  
Associate Director, Aging Studies Program

John Yinger  
Professor of Economics and Public Administration  
Associate Director, Metropolitan Studies Program

## SENIOR RESEARCH ASSOCIATES

Badi Baltagi..... Economics	Len Lopoo ..... Public Administration
Pablo Beramendi..... Political Science	Jerry Miner ..... Economics
Dan Black..... Economics	Jan Ondrich..... Economics
William Duncombe ..... Public Administration	John Palmer ..... Public Administration
Gary Engelhardt ..... Economics	Lori Ploutz-Snyder..... Exercise Science
Deborah Freund ..... Public Administration	David Popp..... Public Administration
Madonna Harrington Meyer ..... Sociology	Grant Reeher ..... Political Science
Christine Himes..... Sociology	Christopher Rohlfs ..... Economics
William C. Horrow ..... Economics	Stuart Rosenthal ..... Economics
Duke Kao ..... Economics	Ross Rubenstein..... Public Administration
Eric Kingson ..... Social Work	Margaret Usdansky ..... Sociology
Thomas Kniesner ..... Economics	Michael Wasylenko ..... Economics
Jeffrey Kubik ..... Economics	Janet Wilmoth ..... Sociology
Andrew London..... Sociology	

## GRADUATE ASSOCIATES

Javier Baez ..... Economics	Sung Hyo Hong..... Economics
Sonali Ballal ..... Public Administration	Joseph Marchand..... Economics
Jesse Bricker ..... Economics	Neelakshi Medhi ..... Social Science
Maria Brown..... Social Science	Larry Miller ..... Public Administration
Mike Eriksen ..... Economics	Wendy Parker ..... Sociology
Qu Feng..... Economics	Emily Pas ..... Economics
Katie Fitzpatrick..... Economics	Shawn Rohlin..... Economics
Alexandre Genest ..... Public Administration	Cynthia Searcy..... Public Administration
Julie Anna Golebiewski..... Economics	Jeff Thompson ..... Economics
Nadia Greenhalgh-Stanley ..... Economics	Coady Wing ..... Public Administration
Tamara Hafner..... Public Administration	Daniel Yanulavich ..... Public Administration
Yue Hu ..... Economics	Ryan Yeung ..... Public Administration

## STAFF

Kelly Bogart .....Administrative Secretary  
Martha Bonney.. ..Publications/Events Coordinator  
Karen Cimilluca.....Administrative Secretary  
Kitty Nasto.....Administrative Secretary

Candi Patterson.....Computer Consultant  
Mary Santy.....Administrative Secretary  
Tammy Tanner.....Librarian/Office Coordinator

## Abstract

This paper studies the asymptotic properties of standard panel data estimators in a simple panel regression model with error component disturbances. Both the regressor and the remainder disturbance term are assumed to be autoregressive and possibly non-stationary. Asymptotic distributions are derived for the standard panel data estimators including ordinary least squares, fixed effects, first-difference, and generalized least squares (GLS) estimators when both  $T$  and  $n$  are large. We show that all the estimators have asymptotic normal distributions and have different convergence rates dependent on the non-stationarity of the regressors and the remainder disturbances. We show using Monte Carlo experiments that the loss in efficiency of the OLS, FE and FD estimators relative to true GLS can be substantial.

Key Words: Panel Data, OLS, Fixed-Effects, First-Difference, GLS  
JEL classification: C33

# Asymptotic Properties of Estimators for the Linear Panel Regression Model with Individual effects and Serially Correlated Errors: The Case of Stationary and Non-Stationary Regressors and Residuals

Badi H. Baltagi\*, Chihwa Kao†, Long Liu‡  
Syracuse University

April 22, 2007

## Abstract

This paper studies the asymptotic properties of standard panel data estimators in a simple panel regression model with error component disturbances. Both the regressor and the remainder disturbance term are assumed to be autoregressive and possibly non-stationary. Asymptotic distributions are derived for the standard panel data estimators including ordinary least squares, fixed effects, first-difference, and generalized least squares (GLS) estimators when both  $T$  and  $n$  are large. We show that all the estimators have asymptotic normal distributions and have different convergence rates dependent on the non-stationarity of the regressors and the remainder disturbances. We show using Monte Carlo experiments that the loss in efficiency of the OLS, FE and FD estimators relative to true GLS can be substantial.

Key Words: *Panel Data, OLS, Fixed-Effects, First-Difference, GLS.*

## 1 Introduction

Econometricians have long been concerned with conditions under which the ordinary least squares (OLS) estimator is asymptotically efficient. The standard textbook result is that, under a general variance-covariance

---

\*Address correspondence to: Badi H. Baltagi, Center for Policy Research, 426 Eggers Hall, Syracuse University, Syracuse, NY 13244-1020; e-mail: bbaltagi@maxwell.syr.edu.

†Chihwa Kao, Center for Policy Research, 426 Eggers Hall, Syracuse University, Syracuse, NY 13244-1020; e-mail: cd-kao@maxwell.syr.edu.

‡Long Liu, Economics Department, 110 Eggers Hall, Syracuse University, Syracuse, NY 13244-1020; e-mail: loliu@maxwell.syr.edu.

structure on the disturbances, the OLS estimator is less efficient than generalized least squares (GLS). This is well documented for the case of stationary autoregressive disturbances and stationary regressors. However, Phillips and Park (1988) showed that in a regression with integrated regressors, OLS and GLS are asymptotically equivalent.

Recently, Choi (1999) studied the limiting distributions of the fixed effects (FE), GLS, and within-GLS estimators for a panel data regression model with autoregressive disturbances, while Choi (2002) extended this work to instrumental variables (IV) estimation. Phillips and Moon (1999) presented a fundamental framework for studying sequential and joint limit theories in nonstationary panel data analysis, while Kao (1999) studied the asymptotic properties of the FE estimator of a spurious regression and proposed residual-based tests for panel cointegration. See Baltagi and Kao (2000), Choi (2006) and Breitung and Pesaran (2006) for recent surveys of this rapidly growing subject. In an early finding, Baltagi and Krämer (1997) showed the equivalence of the GLS and FE estimators in a simple panel data regression with time trend as a regressor. Kao and Emerson (2004a, 2004b) extended Baltagi and Krämer to a model with serially correlated remainder errors. Kao and Emerson showed that the FE estimator is asymptotically equivalent to GLS when the error term is  $I(0)$ ; but that GLS is more efficient than FE when the error term is  $I(1)$ . It is known that the panel time trend can be seen as a special case of the panel regression with a non-zero drift  $I(1)$  regressor.

This paper extends the literature by studying the asymptotic properties of OLS, FE, first difference (FD) and GLS in the panel regression with an autocorrelated regressor and an autocorrelated remainder error (both of which can be stationary or nonstationary). We show that when the error term is  $I(0)$  and the regressor is  $I(1)$ , the FE estimator is asymptotically equivalent to the GLS estimator and OLS is less efficient than GLS (due to a slower convergence speed). However, when the error term and the regressor are  $I(1)$ , GLS is more efficient than the FE estimator since GLS is  $\sqrt{nT}$  consistent, while FE is  $\sqrt{n}$  consistent. This implies that GLS is the preferred estimator under both cases (i.e., regression error is either  $I(0)$  or  $I(1)$ ).

All asymptotic results in this paper assume that  $T \rightarrow \infty$  followed by  $n \rightarrow \infty$ . We use  $(n, T) \xrightarrow{\text{seq}} \infty$  to denote this sequential limit. We write the integral  $\int_0^1 W(s)ds$  as  $\int W$  and  $\tilde{W}$  as  $W - \int W$  when there is no ambiguity over limits.  $\Rightarrow$  to denote weak convergence,  $\equiv$  to denote equivalence in distribution,  $\xrightarrow{p}$  to denote convergence in probability,  $[x]$  to denote the largest integer  $\leq x$ ,  $I(0)$  and  $I(1)$  to signify a time series that is integrated of order zero and one, respectively, and  $BM(\Omega)$  to denote Brownian motion with covariance matrix  $\Omega$ . All proofs are collected in an appendix available upon request from the authors.

## 2 The Model and Assumptions

Consider the following panel regression:

$$y_{it} = \alpha + x_{it}\beta + u_{it}, \quad i = 1, \dots, n, \quad t = 1, \dots, T \quad (1)$$

where  $u_{it} = \mu_i + \nu_{it}$ , and  $\alpha$  and  $\beta$  are scalars. For simplicity, we consider the case of one regressor, but our results can be extended to the multiple regressor case. We assume that  $\mu_i \sim iid(0, \sigma_\mu^2)$  and  $\{\nu_{it}\}$  is AR(1)

$$\nu_{it} = \rho\nu_{it-1} + e_{it}, \quad |\rho| \leq 1 \quad (2)$$

where  $e_{it}$  is a white noise process with variance  $\varpi_e^2$ .

Let  $x_{it}$  be also an AR(1) such that

$$x_{it} = \lambda x_{it-1} + \varepsilon_{it}, \quad |\lambda| \leq 1 \quad (3)$$

where  $\varepsilon_{it}$  is a white noise process with variance  $\varpi_\varepsilon^2$ . In this paper, we assume that

$$E(\mu_i | x_{it}) = 0. \quad (4)$$

The initialization of this system is  $y_{i1} = x_{i1} = O_p(1)$  for all  $i$ . Our interest is in the estimation of the common slope  $\beta$ . This paper shows that the asymptotic properties of OLS, FE, FD, and GLS estimators depend crucially on the serial correlation properties of  $x_{it}$  and  $v_{it}$ . When  $y_{it}$  and  $x_{it}$  are both  $I(1)$  but  $v_{it}$  is  $I(0)$ , equation (1) is a panel cointegrated model. On the other hand, when  $v_{it}$  is  $I(1)$  and  $y_{it}$  and  $x_{it}$  are both  $I(1)$ , equation (1) is a panel spurious model. FE estimators for panel cointegrated and panel spurious models have been discussed in Phillips and Moon (1999) and Kao (1999). The case of a panel time trend model,  $x_{it} = t$ , has been studied by Baltagi and Krämer (1997) and Kao and Emerson (2004a, 2004b).

Next, we characterize the innovation vector  $\mathbf{w}_{it} = (e_{it}, \varepsilon_{it})'$ . We assume that  $\mathbf{w}_{it}$  is a linear process that satisfies the following assumption:

**Assumption 1** For each  $i$ , we assume:

1.  $\mathbf{w}_{it} = \Pi(L)\boldsymbol{\eta}_{it} = \sum_{j=0}^{\infty} \Pi_j \boldsymbol{\eta}_{it-j}$ ,  $\sum_{j=0}^{\infty} j^a \|\Pi_j\| < \infty$ ,  $|\Pi(1)| \neq 0$  for some  $a > 1$ .
2. For a given  $i$ ,  $\boldsymbol{\eta}_{it}$  is i.i.d. with zero mean and variance-covariance matrix  $\Xi$ , and finite fourth order cumulants.

**Assumption 2** We assume  $\boldsymbol{\eta}_{it}$  and  $\boldsymbol{\eta}_{jt}$  are independent for  $i \neq j$ . That is we assume cross-sectional independence for our model.

Assumption 1 implies that the partial sum process  $\frac{1}{\sqrt{T}} \sum_{t=1}^{[Tr]} \mathbf{w}_{it}$  satisfies the following multivariate invariance principle:

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{[Tr]} \mathbf{w}_{it} \Rightarrow \mathbf{B}_i(r) = \mathbf{B}\mathbf{M}_i(\Omega) \text{ as } T \rightarrow \infty \text{ for all } i, \quad (5)$$

where

$$\mathbf{B}_i = \begin{bmatrix} B_{ei} \\ B_{\varepsilon i} \end{bmatrix}.$$

The long-run  $2 \times 2$  covariance matrix of  $\{\mathbf{w}_{it}\}$  is given by

$$\begin{aligned} \Omega &= \sum_{j=-\infty}^{\infty} E(\mathbf{w}_{ij} \mathbf{w}'_{i0}) \\ &= \Pi(1) \Xi \Pi(1)' \\ &= \begin{bmatrix} \varpi_e^2 & \varpi_{e\varepsilon} \\ \varpi_{e\varepsilon} & \varpi_\varepsilon^2 \end{bmatrix}. \end{aligned}$$

The long-run covariance matrix can be decomposed into  $\Omega = \Sigma + 2\Gamma$ , where

$$\Gamma = \sum_{j=1}^{\infty} E(\mathbf{w}_{ij} \mathbf{w}'_{i0}) = \begin{bmatrix} \gamma_e^2 & \gamma_{e\varepsilon} \\ \gamma_{e\varepsilon} & \gamma_\varepsilon^2 \end{bmatrix} \quad (6)$$

and

$$\Sigma = E(\mathbf{w}_{i0} \mathbf{w}'_{i0}) = \begin{bmatrix} \sigma_e^2 & \sigma_{e\varepsilon} \\ \sigma_{e\varepsilon} & \sigma_\varepsilon^2 \end{bmatrix}. \quad (7)$$

Assuming  $\varpi_\varepsilon^2$  is non-zero, we define

$$\varpi_{e,\varepsilon} = \varpi_e^2 - \frac{\varpi_{e\varepsilon}^2}{\varpi_\varepsilon^2}. \quad (8)$$

Then,  $B_i$  can be rewritten as

$$\mathbf{B}_i = \begin{bmatrix} B_{ei} \\ B_{\varepsilon i} \end{bmatrix} = \begin{bmatrix} \varpi_{e,\varepsilon} & \varpi_{e\varepsilon}/\varpi_\varepsilon \\ 0 & \varpi_\varepsilon \end{bmatrix} \begin{bmatrix} V_i \\ W_i \end{bmatrix} \quad (9)$$

where  $\begin{bmatrix} V_i \\ W_i \end{bmatrix} = \mathbf{B}\mathbf{M}(I)$  is a standardized Brownian motion. Define the one-sided long-run covariance

$$\begin{aligned} \Delta &= \Sigma + \Gamma \\ &= \sum_{j=0}^{\infty} E(\mathbf{w}_{ij} \mathbf{w}'_{i0}) \end{aligned}$$

with

$$\Delta = \begin{bmatrix} \delta_e^2 & \delta_{e\varepsilon} \\ \delta_{e\varepsilon} & \delta_\varepsilon^2 \end{bmatrix}.$$



The assumption of constant variances/covariances across  $i$ , such as in  $\Omega$ ,  $\Sigma$ , and  $\Gamma$  is used to simplify the notation. It can be extended into the case where different variances are allowed for different  $i$  at the expense of more complicated notation.

### 3 OLS Estimator

The OLS estimator of  $\beta$  is given by

$$\widehat{\beta}_{OLS} = \frac{\sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x})(y_{it} - \bar{y})}{\sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x})^2} \quad (10)$$

where  $\bar{x} = \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T x_{it}$  and  $\bar{y} = \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T y_{it}$ .

**Theorem 1** *Under Assumptions 1 – 2, we obtain the following results:*

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

- (a)  $\widehat{\beta}_{OLS} - \beta \xrightarrow{p} \frac{1-\lambda^2}{\sigma_\varepsilon^2} \left[ \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it}\nu_{it}) \right]$ ,

- (b)  $\sqrt{nT} \left( \widehat{\beta}_{OLS} - \beta - \tau_{1NT}^{OLS} \right) \Rightarrow N(0, \kappa_1^{OLS})$ ,

where  $\tau_{1NT}^{OLS} = \frac{\lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it}\nu_{it})}{\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x})^2}$ ,  $\kappa_1^{OLS} = \frac{1}{\sigma_\varepsilon^4} \left[ \sigma_\mu^2 \varpi_\varepsilon^2 + \frac{(1-\lambda^2)^2}{(1-\rho\lambda)^2} [\psi_{00} + \sum_{r=1}^{\infty} \lambda^{2r} \psi_{0r} + \sum_{r=1}^{\infty} \rho^{2r} \psi_{r0}] \right]$ ,  
 $\psi_{0r} = E(\varepsilon_{i(t-r)}^2 e_{it}^2)$ ,  $\psi_{r0} = E(\varepsilon_{it}^2 e_{i(t-r)}^2)$ , and  $\psi_{00} = E(\varepsilon_{it}^2 e_{it}^2)$ .

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

- (a)  $\widehat{\beta}_{OLS} - \beta \xrightarrow{p} \frac{(1+\lambda)(-\frac{1}{2}\varpi_{e\varepsilon} + \delta_{e\varepsilon})}{\sigma_\varepsilon^2}$

- (b)  $\sqrt{n} \left( \widehat{\beta}_{OLS} - \beta - \tau_{2nT}^{OLS} \right) \Rightarrow N(0, \kappa_2^{OLS})$ ,

where  $\tau_{2nT}^{OLS} = \frac{(\frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T \nu_{i(t-1)} e_{it}) \frac{\varpi_{e\varepsilon}}{\sigma_\varepsilon^2} + \delta_{e\varepsilon}}{(1-\lambda) \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x})^2}$ ,  $\kappa_2^{OLS} = \frac{(1+\lambda)^2 \varpi_{\varepsilon, \varepsilon} \varpi_\varepsilon^2}{2\sigma_\varepsilon^4}$ .

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

- (a)  $\sqrt{T} \left( \widehat{\beta}_{OLS} - \beta \right) \xrightarrow{p} 0$ ,

- (b)  $\sqrt{nT} \left( \widehat{\beta}_{OLS} - \beta \right) \Rightarrow N(0, \kappa_3^{OLS})$ ,

where  $\kappa_3^{OLS} = \frac{4\sigma_\mu^2}{3\varpi_\varepsilon^2}$ .

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$\begin{aligned}
& \text{(a) } \widehat{\beta}_{OLS} - \beta \xrightarrow{p} \frac{2\delta_{\varepsilon\varepsilon}}{\varpi_{\varepsilon}^2}, \\
& \text{(b) } \sqrt{n} \left( \widehat{\beta}_{OLS} - \beta - \tau_{4NT}^{OLS} \right) \Rightarrow N \left( 0, \kappa_4^{OLS} \right), \\
& \text{where } \tau_{4NT}^{OLS} = \frac{\left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T x_{i(t-1)} \varepsilon_{it} \right) \frac{\varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon}}{\varpi_{\varepsilon}^2}}{\frac{1}{n} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i)^2}, \quad \kappa_4^{OLS} = \frac{2\varpi_{e,\varepsilon}}{3\varpi_{\varepsilon}^2}.
\end{aligned}$$

It is important to note that  $\varpi_{e,\varepsilon}/\varpi_{\varepsilon}^2$  can be seen as the long-run signal-to-noise ratio. The OLS estimator ignores the individual effects in the disturbance term. Thus, the variance of  $\mu_i$ , i.e.,  $\sigma_{\mu}^2$  might appear in the variance-covariance matrix of  $\widehat{\beta}_{OLS}$  depending on the case considered. In case 1, both  $\mu_i$  and  $\nu_{it}$  affect the variance of  $\widehat{\beta}_{OLS}$ . In cases 2 and 4,  $\nu_{it}$  dominates  $\mu_i$ . In case 3,  $\mu_i$  dominates  $\nu_{it}$  and hence the convergence speed is  $\sqrt{nT}$ , which differs from the  $T$ -asymptotics in the panel cointegration literature. Also the asymptotic normality of the OLS estimator comes naturally. When summing across  $i$ , the nonstandard asymptotic distribution due to unit root in the time dimension, such as for cases 2-4, is smoothed out.

**Corollary 1** *When  $E(e_{it}\varepsilon_{i(t+k)}) = 0$  for all  $i$  and  $k$ , under the assumptions in Theorem 1, then*

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{OLS} - \beta \right) \Rightarrow N \left( 0, \frac{\sigma_{\varepsilon}^2}{\sigma_{\varepsilon}^2} + \frac{(1-\lambda^2)^2 [\psi_{00} + \sum_{r=1}^{\infty} \lambda^{2r} \psi_{0r} + \sum_{r=1}^{\infty} \rho^{2r} \psi_{r0}] }{(1-\rho\lambda)^2 \sigma_{\varepsilon}^4} \right).$$

$$\text{When } \varepsilon_{it} \text{ and } e_{it} \text{ are independent, } \sqrt{nT} \left( \widehat{\beta}_{OLS} - \beta \right) \Rightarrow N \left( 0, \frac{\sigma_{\mu}^2}{\sigma_{\varepsilon}^2} + \frac{(1+\rho\lambda)(1-\lambda^2)\sigma_{\varepsilon}^2}{(1-\rho\lambda)(1-\rho^2)\sigma_{\varepsilon}^2} \right).$$

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$\sqrt{n} \left( \widehat{\beta}_{OLS} - \beta \right) \Rightarrow N \left( 0, \frac{(1-\lambda)^2 \sigma_{\varepsilon}^2}{2\sigma_{\varepsilon}^2} \right),$$

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{OLS} - \beta \right) \Rightarrow N \left( 0, \frac{4\sigma_{\mu}^2}{3\sigma_{\varepsilon}^2} \right),$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$\sqrt{n} \left( \widehat{\beta}_{OLS} - \beta \right) \Rightarrow N \left( 0, \frac{2\sigma_{\varepsilon}^2}{3\sigma_{\varepsilon}^2} \right).$$

Corollary 1 follows directly from Theorem 1.

## 4 FE Estimator

The Fixed-Effects estimator of  $\beta$  is given by

$$\widehat{\beta}_{FE} = \frac{\sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) (y_{it} - \bar{y}_i)}{\sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i)^2}, \quad (11)$$

where  $\bar{x}_i = \frac{1}{T} \sum_{t=1}^T x_{it}$  and  $\bar{y}_i = \frac{1}{T} \sum_{t=1}^T y_{it}$ .

**Theorem 2** Under Assumptions 1 – 2, we have the following results:

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$(a) \widehat{\beta}_{FE} - \beta \xrightarrow{p} \frac{1-\lambda^2}{\sigma_\varepsilon^2} \left[ \lim_{nT} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it}\nu_{it}) \right],$$

$$(b) \sqrt{nT} \left( \widehat{\beta}_{FE} - \beta - \tau_{1nT}^{FE} \right) \Rightarrow N(0, \kappa_1^{FE}),$$

$$\text{where } \tau_{1nT}^{FE} = \frac{\lim_{nT} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it}\nu_{it})}{\frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i)^2}, \kappa_1^{FE} = \frac{(1-\lambda^2)^2 [\psi_{00} + \sum_{r=1}^{\infty} \lambda^{2r} \psi_{0r} + \sum_{r=1}^{\infty} \rho^{2r} \psi_{r0}]}{(1-\rho\lambda)^2 \sigma_\varepsilon^4}, \psi_{0r} = E(\varepsilon_{i(t-r)}^2 e_{it}^2),$$

$$\psi_{r0} = E(\varepsilon_{it}^2 e_{i(t-r)}^2), \text{ and } \psi_{00} = E(\varepsilon_{it}^2 e_{it}^2).$$

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$(a) \widehat{\beta}_{FE} - \beta \xrightarrow{p} \frac{(1+\lambda)(-\frac{1}{2}\varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon})}{\sigma_\varepsilon^2},$$

$$(b) \sqrt{n} \left( \widehat{\beta}_{FE} - \beta - \tau_{2nT}^{FE} \right) \Rightarrow N(0, \kappa_2^{FE}),$$

$$\text{where } \tau_{2nT}^{FE} = \frac{(\frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T (\nu_{it} - \bar{\nu}_i) e_{it}) \frac{\varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon}}{\sigma_\varepsilon^2}}{(1-\lambda) \frac{1}{n} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i)^2} \text{ and } \kappa_2^{FE} = \frac{(1-\lambda)^2 \varpi_{\varepsilon,\varepsilon} \varpi_\varepsilon^2}{6\sigma_\varepsilon^4}.$$

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

$$(a) T \left( \widehat{\beta}_{FE} - \beta \right) \xrightarrow{p} \frac{-3\varpi_{\varepsilon\varepsilon} + 6\delta_{\varepsilon\varepsilon}}{(1-\rho)\varpi_\varepsilon^2},$$

$$(b) \sqrt{nT} \left( \widehat{\beta}_{FE} - \beta \right) - \sqrt{nT} \tau_{3nT}^{FE} \Rightarrow N(0, \kappa_3^{FE}),$$

$$\text{where } \tau_{3nT}^{FE} = \frac{(\frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) \varepsilon_{it}) \frac{\varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon}}{\sigma_\varepsilon^2}}{(1-\rho) \frac{1}{n} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i)^2} \text{ and } \kappa_3^{FE} = \frac{6\varpi_{\varepsilon,\varepsilon}}{(1-\rho)^2 \varpi_\varepsilon^2}.$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$(a) \widehat{\beta}_{FE} - \beta \xrightarrow{p} \frac{\varpi_{\varepsilon\varepsilon} + 6\delta_{\varepsilon\varepsilon}}{\varpi_\varepsilon^2},$$

$$(b) \sqrt{n} \left( \widehat{\beta}_{FE} - \beta - \tau_{4nT}^{FE} \right) \Rightarrow N(0, \kappa_4^{FE}),$$

$$\text{where } \tau_{4nT}^{FE} = \frac{(\frac{1}{n} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i)^2) \frac{\varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon}}{\sigma_\varepsilon^2}}{\frac{1}{n} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i)^2} \text{ and } \kappa_4^{FE} = \frac{2\varpi_{\varepsilon,\varepsilon}}{5\varpi_\varepsilon^2}.$$

Note  $\varpi_{\varepsilon\varepsilon}$  is due to the endogeneity of the regressor  $x_{it}$ , and  $\delta_{\varepsilon\varepsilon}$  is due to serial correlation. Because  $u_{it} - \bar{u}_i = \nu_{it} - \bar{\nu}_i$ , the individual effect  $\mu_i$  is eliminated for each individual.

**Corollary 2** When  $E(\varepsilon_{it}\varepsilon_{i(t+k)}) = 0$  for all  $i$  and  $k$ , under the same conditions as for Theorem 2, then

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{FE} - \beta \right) \Rightarrow N \left( 0, \frac{(1-\lambda^2)^2 [\psi_{00} + \sum_{r=1}^{\infty} \lambda^{2r} \psi_{0r} + \sum_{r=1}^{\infty} \rho^{2r} \psi_{r0}]}{(1-\rho\lambda)^2 \sigma_\varepsilon^4} \right).$$

$$\text{If } \varepsilon_{it} \text{ and } e_{it} \text{ are independent, } \sqrt{nT} \left( \widehat{\beta}_{FE} - \beta \right) \Rightarrow N \left( 0, \frac{(1+\rho\lambda)(1-\lambda^2)\sigma_\varepsilon^2}{(1-\rho\lambda)(1-\rho^2)\sigma_\varepsilon^2} \right).$$

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$\sqrt{n} \left( \widehat{\beta}_{FE} - \beta \right) \Rightarrow N \left( 0, \frac{(1-\lambda)^2 \sigma_\varepsilon^2}{6\sigma_\varepsilon^2} \right).$$

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{FE} - \beta \right) \Rightarrow N \left( 0, \frac{6\sigma_\varepsilon^2}{(1-\rho)^2 \sigma_\varepsilon^2} \right).$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$\sqrt{n} \left( \widehat{\beta}_{FE} - \beta \right) \Rightarrow N \left( 0, \frac{2\sigma_\varepsilon^2}{5\sigma_\varepsilon^2} \right).$$

Corollary 2 follows directly from Theorem 2. Note that case 1 is the textbook result under the assumptions of stationarity of the regressor and the disturbance term. Case 2 is new. Case 3 is discussed by Phillips and Moon (1999) and Kao and Chiang (2000). Case 4 is discussed in Kao (1999).

## 5 FD Estimator

The First-difference estimator of  $\beta$  is given by

$$\widehat{\beta}_{FD} = \frac{\sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})(y_{it} - y_{it-1})}{\sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2}. \quad (12)$$

**Theorem 3** *Under Assumptions 1 – 2, we obtain the following results:*

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$(a) \widehat{\beta}_{FD} - \beta \xrightarrow{p} \frac{\lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E[(x_{it} - x_{it-1})(\nu_{it} - \nu_{it-1})]}{\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2 + 2\lambda - 1)\gamma_\varepsilon^2}{1-\lambda}},$$

$$(b) \sqrt{nT} \left( \widehat{\beta}_{FD} - \beta - \tau_{1nT}^{FD} \right) \Rightarrow N(0, \kappa_1^{FD}), \text{ where}$$

$$\tau_{1nT}^{FD} = \frac{\lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E[(x_{it} - x_{it-1})(\nu_{it} - \nu_{it-1})]}{\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2},$$

and

$$\kappa_1^{FD} = \frac{(2 - \lambda - \rho)^2 \psi_{00} + \sum_{r=1}^{\infty} (-\rho^{r-1} + 2\rho^r - \rho^{r+1})^2 \psi_{0r} + \sum_{r=1}^{\infty} (-\lambda^{r-1} + 2\lambda^r - \lambda^{r+1})^2 \psi_{r0}}{(1 - \rho\lambda)^2 \left( \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2 + 2\lambda - 1)\gamma_\varepsilon^2}{1-\lambda} \right)^2}.$$

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$(a) \widehat{\beta}_{FD} - \beta \xrightarrow{p} \frac{\lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E[(\varepsilon_{it} + (\lambda - 1)x_{it-1})e_{it}]}{\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2 + 2\lambda - 1)\gamma_\varepsilon^2}{1-\lambda}},$$

(b)  $\sqrt{nT} \left( \widehat{\beta}_{FE} - \beta - \tau_{2nT}^{FD} \right) \Rightarrow N(0, \kappa_2^{FD})$ , where

$$\tau_{2nT}^{FD} = \frac{\lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E[(\varepsilon_{it} + (\lambda - 1)x_{it-1})e_{it}]}{\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2},$$

and

$$\kappa_2^{FD} = \frac{2\psi_{00}}{(1 + \lambda) \left( \frac{2\sigma_\varepsilon^2}{1 + \lambda} + \frac{2(-2\lambda^2 + 2\lambda - 1)\gamma_\varepsilon^2}{1 - \lambda} \right)^2}.$$

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

(a)  $\widehat{\beta}_{FD} - \beta \xrightarrow{p} \frac{1}{\sigma_\varepsilon^2} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E[\varepsilon_{it}(\nu_{it} - \nu_{it-1})]$ ,

(b)  $\sqrt{nT} \left( \widehat{\beta}_{FD} - \beta - \tau_{3nT}^{FD} \right) \Rightarrow N(0, \Pi_3^{FD})$ ,

$$\text{where } \tau_{3nT}^{FD} = \frac{\lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E[\varepsilon_{it}(\nu_{it} - \nu_{it-1})]}{\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2} \text{ and } \kappa_3^{FD} = \frac{2\psi_{00}}{(1 + \rho)\sigma_\varepsilon^4}.$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

(a)  $\widehat{\beta}_{FD} - \beta \xrightarrow{p} \frac{\sigma_{\varepsilon\varepsilon}}{\sigma_\varepsilon^2}$ ,

(b)  $\sqrt{nT} \left( \widehat{\beta}_{FD} - \beta - \tau_{4nT}^{FD} \right) \Rightarrow N(0, \kappa_4^{FD})$ ,

$$\text{where } \tau_{4nT}^{FD} = \frac{\sigma_{\varepsilon\varepsilon}}{\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2} \text{ and } \kappa_4^{FD} = \frac{\varpi_{e,\varepsilon}\varpi_\varepsilon^2}{\sigma_\varepsilon^4}.$$

Similar to the FE estimator, the individual effect  $\mu_i$  is also eliminated by the FD estimator because  $u_{it} - u_{it-1} = \nu_{it} - \nu_{it-1}$ . In cases 2 and 4,  $\rho = 1$ , and the FD estimator is asymptotically equivalent to the GLS estimator because both methods transform the disturbance from  $I(1)$  into  $I(0)$ . Actually, the FD estimator is mathematically the same as the GLS estimator except for the omission of the first observation for each individual.

**Corollary 3** *When  $E(e_{it}\varepsilon_{i(t+k)}) = 0$  for all  $i$  and  $k$ , under the same conditions as for Theorem 3, then*

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{FD} - \beta \right) \Rightarrow N\left(0, \frac{(1+\lambda)^2 \left[ (2-\lambda-\rho)^2 \psi_{00} + \sum_{r=1}^{\infty} (-\rho^{r-1} + 2\rho^r - \rho^{r+1})^2 \psi_{0r} + \sum_{r=1}^{\infty} (-\lambda^{r-1} + 2\lambda^r - \lambda^{r+1})^2 \psi_{r0} \right]}{4(1-\rho\lambda)^2 \sigma_\varepsilon^4}\right).$$

$$\text{If } \varepsilon_{it} \text{ and } e_{it} \text{ are independent, } \sqrt{nT} \left( \widehat{\beta}_{FD} - \beta \right) \Rightarrow N\left(0, \frac{(1+\lambda)^2 \left[ (2-\rho-\lambda)^2 + \frac{(1-\rho)^3}{1+\rho} + \frac{(1-\lambda)^3}{1+\lambda} \right] \sigma_\varepsilon^2}{4(1-\rho\lambda)^2 \sigma_\varepsilon^2}\right).$$

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{FD} - \beta \right) \Rightarrow N\left(0, \frac{(1+\lambda)\psi_{00}}{2\sigma_\varepsilon^4}\right).$$

$$\text{If } \varepsilon_{it} \text{ and } e_{it} \text{ are independent, } \sqrt{nT} \left( \widehat{\beta}_{FE} - \beta \right) \Rightarrow N\left(0, \frac{(1+\lambda)\sigma_\varepsilon^2}{2\sigma_\varepsilon^2}\right).$$

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{FD} - \beta \right) \Rightarrow N \left( 0, \frac{2\psi_{00}}{(1+\rho)\sigma_\varepsilon^2} \right).$$

$$\text{If } \varepsilon_{it} \text{ and } e_{it} \text{ are independent, } \sqrt{nT} \left( \widehat{\beta}_{FD} - \beta \right) \Rightarrow N \left( 0, \frac{2\sigma_\varepsilon^2}{(1+\rho)\sigma_\varepsilon^2} \right).$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{FD} - \beta \right) \Rightarrow N \left( 0, \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2} \right).$$

Corollary 3 follows directly from Theorem 3.

## 6 GLS Estimator

Let us rewrite equation (1) in vector form

$$\mathbf{y} = \alpha \boldsymbol{\iota}_{nT} + \mathbf{x}\beta + \mathbf{u}$$

where  $\mathbf{y}$  is  $nT \times 1$ ,  $\mathbf{x}$  is a vector of  $x_{it}$  of dimension  $nT \times 1$ ,  $\boldsymbol{\iota}_{nT}$  is a vector of ones of dimension  $nT$ . and  $\mathbf{u}$  is  $nT \times 1$ . As shown in the Appendix,

$$\widehat{\beta}_{GLS} = \left[ \mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} (\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT})^{-1} \boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x} \right]^{-1} \left[ \mathbf{x}'\Phi^{-1}\mathbf{y} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} (\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT})^{-1} \boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{y} \right] \quad (13)$$

and

$$\widehat{\beta}_{GLS} - \beta = \left[ \mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} (\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT})^{-1} \boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x} \right]^{-1} \left[ \mathbf{x}'\Phi^{-1}\mathbf{u} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} (\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT})^{-1} \boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{u} \right],$$

where  $\Phi = E(\mathbf{u}\mathbf{u}')$ .

One can decompose the variance-covariance matrix into

$$\Phi = E(\mathbf{u}\mathbf{u}') = \sigma_\mu^2 (I_n \otimes \boldsymbol{\iota}_T \boldsymbol{\iota}'_T) + \varpi_\varepsilon^2 (I_n \otimes \mathbf{A})$$

where  $\boldsymbol{\iota}_T$  is a vector of ones of dimension  $T$ .  $\mathbf{A}$  is the variance-covariance matrix of  $\nu_{it}$ ,

$$\mathbf{A} = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{bmatrix}$$

when  $|\rho| < 1$  and

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & T \end{bmatrix}$$

when  $\rho = 1$ . Thus, it can be shown that

$$\Phi^{-1} = I_n \otimes \left[ \frac{1}{\varpi_e^2} \left( \mathbf{A}^{-1} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{A}^{-1} \boldsymbol{\iota}_T \boldsymbol{\iota}_T' \mathbf{A}^{-1} \right) \right]$$

where  $\theta = \boldsymbol{\iota}_T' \mathbf{A}^{-1} \boldsymbol{\iota}_T$ .

When  $|\rho| < 1$ , this estimation is equivalent to the Prais-Winsten transformation method suggested by Baltagi and Li (1991). One can easily verify that  $\mathbf{A}^{-1} = \mathbf{C}\mathbf{C}'$ , where

$$\mathbf{C} = \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -\rho & 1 & 0 \\ 0 & 0 & 0 & 0 & -\rho & 1 \end{bmatrix}$$

is the Prais-Winsten transformation matrix as in Baltagi and Li (1991).

Thus, we have the following theorem:

**Theorem 4** *Under Assumptions 1 – 2, we obtain the following results:*

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$(a) \left( \widehat{\beta}_{GLS} - \beta \right) \xrightarrow{p} \frac{\lim \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}]}{\frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho)\gamma_\varepsilon^2}{1-\lambda}}$$

$$(b) \sqrt{nT} \left( \widehat{\beta}_{GLS} - \beta - \tau_{1nT}^{GLS} \right) \Rightarrow N(0, \kappa_1^{GLS}),$$

$$\text{where } \tau_{1nT}^{GLS} = \frac{\lim \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}]}{\sigma_\varepsilon^2 \frac{1}{nT} X' \Phi^{-1} X}, \kappa_1^{GLS} = \frac{(1-2\rho\lambda+\rho^2)\psi_{00}}{(1-\lambda^2) \left[ \frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho)\gamma_\varepsilon^2}{1-\lambda} \right]^2}$$

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$(a) \left( \widehat{\beta}_{GLS} - \beta \right) \xrightarrow{p} \frac{\lim \frac{1}{n} \frac{1}{T} \sum_{i=1}^n \sum_{t=1}^T E[(x_{it} - x_{it-1})e_{it}]}{\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2 + 2\lambda - 1)\gamma_\varepsilon^2}{1-\lambda}},$$

$$(b) \sqrt{nT} \left( \widehat{\beta}_{GLS} - \beta - \tau_{2nT}^{GLS} \right) \Rightarrow N \left( 0, \kappa_2^{GLS} \right),$$

$$\text{where } \tau_{2nT}^{GLS} = \frac{\lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - x_{it-1})e_{it}]}{\sigma_\varepsilon^2 \frac{1}{nT} X' \Phi^{-1} X}, \kappa_2^{GLS} = \frac{2\psi_{00}}{(1+\lambda) \left[ \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2 + 2\lambda - 1)\gamma_\varepsilon^2}{1-\lambda} \right]^2}$$

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

$$(a) T \left( \widehat{\beta}_{GLS} - \beta \right) \xrightarrow{p} \frac{-3\varpi_{\varepsilon\varepsilon} + 6\gamma_{\varepsilon\varepsilon} + \frac{6}{1-\rho}\sigma_{\varepsilon\varepsilon}}{(1-\rho)\varpi_\varepsilon^2},$$

$$(b) \sqrt{nT} \left( \widehat{\beta}_{GLS} - \beta \right) - \sqrt{nT}\tau_{3nT}^{GLS} \Rightarrow N \left( 0, \kappa_3^{GLS} \right),$$

$$\text{where } \tau_{3nT}^{GLS} = \frac{\frac{1}{n} \sum_{i=1}^n (1-\rho) \left( \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) \varepsilon_{it} \right) \frac{\varpi_{\varepsilon\varepsilon}}{\varpi_\varepsilon^2} + \gamma_{\varepsilon\varepsilon} + \frac{\sigma_{\varepsilon\varepsilon}}{1-\rho} \right)}{\sigma_\varepsilon^2 \frac{1}{nT} X' \Phi^{-1} X}, \kappa_3^{GLS} = \frac{6\varpi_{\varepsilon\varepsilon}}{(1-\rho)^2 \varpi_\varepsilon^2}.$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$(a) \sqrt{T} \left( \widehat{\beta}_{GLS} - \beta \right) \xrightarrow{p} \frac{\sigma_{\varepsilon\varepsilon}}{\sigma_\varepsilon^2},$$

$$(b) \sqrt{nT} \left( \widehat{\beta}_{GLS} - \beta \right) - \sqrt{nT}\tau_{4nT}^{GLS} \Rightarrow N \left( 0, \kappa_4^{GLS} \right),$$

$$\text{where } \tau_{4nT}^{GLS} = \frac{\sqrt{n}\sigma_{\varepsilon\varepsilon}}{\sigma_\varepsilon^2 \frac{1}{nT} X' \Phi^{-1} X}, \kappa_4^{GLS} = \frac{\varpi_\varepsilon^2 \varpi_\varepsilon^2}{\sigma_\varepsilon^4}.$$

It is well known that the random effects model imposes the critical assumption that  $\mu_i$  needs to be independent of  $x_{it}$ . It is worth pointing out that this assumption is only needed for the case  $\rho < 1$ . When  $\rho = 1$ , the GLS transformation is identical to the first-difference estimation except for the first observation of each individual. The Cochrane-Orcutt (CO) procedure of GLS transformation ignores the information contained in the first observation. Hence, if one use Cochrane-Orcutt (CO) procedure, the GLS estimation will be the same as the FD estimation and  $\mu_i$  need not to be independent of  $x_{it}$  when  $\rho = 1$ . When  $\rho < 1$ ,  $E(\mu_i | x_{it}) = 0$  is required, otherwise  $\widehat{\beta}_{GLS}$  would be biased and inconsistent. In this case, one may use the within or first-difference transformation to wipe out  $\mu_i$  and then run GLS estimation. The Within-GLS or FD-GLS estimators can be shown asymptotically as efficient as GLS estimator. However, this is beyond the scope of this paper and can be left as a further extension. The following corollary follows directly from Theorem 4.

**Corollary 4** *When  $E(e_{it}\varepsilon_{i(t+k)}) = 0$  for all  $i$  and  $k$ , under the same conditions as for Theorem 4, then*

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{GLS} - \beta \right) \Rightarrow N \left( 0, \frac{(1-\lambda^2)\psi_{00}}{(1-2\rho\lambda + \rho^2)\sigma_\varepsilon^4} \right).$$

$$\text{If } \varepsilon_{it} \text{ and } e_{it} \text{ are independent, } \sqrt{nT} \left( \widehat{\beta}_{GLS} - \beta \right) \Rightarrow N \left( 0, \frac{(1-\lambda^2)\sigma_\varepsilon^2}{(1-2\rho\lambda + \rho^2)\sigma_\varepsilon^2} \right).$$



2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{GLS} - \beta \right) \Rightarrow N \left( 0, \frac{(1+\lambda)\psi_{00}}{2\sigma_\varepsilon^4} \right).$$

If  $\varepsilon_{it}$  and  $e_{it}$  are independent,  $\sqrt{nT} \left( \widehat{\beta}_{GLS} - \beta \right) \Rightarrow N \left( 0, \frac{(1+\lambda)\sigma_\varepsilon^2}{2\sigma_\varepsilon^2} \right)$ .

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{GLS} - \beta \right) \Rightarrow N \left( 0, \frac{6\sigma_\varepsilon^2}{(1-\rho)^2\sigma_\varepsilon^2} \right).$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$\sqrt{nT} \left( \widehat{\beta}_{GLS} - \beta \right) \Rightarrow N \left( 0, \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2} \right).$$

Case 1 is the textbook result. Case 3 is discussed in Choi (1999). Cases 2 and 4 are new.

## 7 Feasible GLS Estimator

It is clear that the GLS estimator in Section 6 is not feasible. In this section, we discuss feasible GLS estimation. Assuming  $E(e_{it}\varepsilon_{i(t+k)}) = 0$  for all  $i$  and  $k$  and  $\varepsilon_{it}$  and  $e_{it}$  are independent, a feasible GLS estimator can be calculated by estimating the autocorrelation coefficient  $\rho$  and the variance components  $\sigma_\mu^2$  and  $\sigma_\varepsilon^2$ . To estimate these parameters, we take the following steps.

First, retrieve the residual estimator  $\widehat{v}_{it}$  from the FE regression in (1). Now  $\rho$  can be estimated as the correlation between  $\widehat{v}_{it}$  and  $\widehat{v}_{it-1}$ , i.e.,

$$\widehat{\rho} = \frac{\sum_{i=1}^n \sum_{t=2}^T (\widehat{v}_{it} - \bar{\widehat{v}}) (\widehat{v}_{it-1} - \bar{\widehat{v}})}{\sqrt{\sum_{i=1}^n \sum_{t=2}^T (\widehat{v}_{it-1} - \bar{\widehat{v}})^2} \sqrt{\sum_{i=1}^n \sum_{t=2}^T (\widehat{v}_{it-1} - \bar{\widehat{v}})^2}}, \quad (14)$$

where  $\bar{\widehat{v}}$  is the sample average of  $\widehat{v}_{it}$ . Alternatively, as suggested by Baltagi and Li (1991), one can estimate  $\rho$  by

$$\widehat{\rho} = \frac{\sum_{i=1}^n \sum_{t=2}^T \widehat{v}_{it} \widehat{v}_{it-1}}{\sum_{i=1}^n \sum_{t=2}^T (\widehat{v}_{it-1})^2}.$$

Baltagi and Li (1997) suggests another consistent estimator  $\widehat{\rho} = (\widetilde{Q}_1 - \widetilde{Q}_2) / (\widetilde{Q}_0 - \widetilde{Q}_1)$ , where  $\widetilde{Q}_s = \sum_{i=1}^n \sum_{t=s+1}^T \widehat{u}_{it} \widehat{u}_{it-s} / n(T-s)$ . We choose the correlation coefficient estimator because it ensures that  $\widehat{\rho}$  is always between 0 and 1. It can be shown that  $\widehat{\rho}$  in (14) is a consistent estimator of  $\rho$  by using the Theorem 2, i.e.,  $\widehat{\rho} \xrightarrow{P} \rho$  if  $|\rho| < 1$ .

Next, using the FE residuals  $\widehat{v}_{it}$  and the estimate of the autocorrelation coefficient  $\widehat{\rho}$ , we can get  $\widehat{e}_{it}$ . Therefore  $\sigma_\varepsilon^2$  can be estimated by  $\widehat{\sigma}_\varepsilon^2 = \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \widehat{e}_{it}^2$ . Also,  $\sigma_\mu^2$  can be estimated using  $\widehat{\sigma}_\mu^2 =$

$\frac{1}{nT} \sum_{i=1}^N \sum_{t=1}^T (\hat{u}_{it}^2 - \hat{\nu}_{it}^2)$ , where  $\hat{u}_{it}$  denote the OLS residuals from equation (10).  $\hat{\sigma}_e^2$  and  $\hat{\sigma}_\mu^2$  are consistent estimators for  $\sigma_e^2$  and  $\sigma_\mu^2$  respectively, i.e.,  $\hat{\sigma}_e^2 \xrightarrow{p} \sigma_e^2$ ,  $\hat{\sigma}_\mu^2 \xrightarrow{p} \sigma_\mu^2$  if  $|\rho| < 1$ . These variance components can be estimated by using the variance decomposition and the Prais-Winsten (PW) transformation suggested by Baltagi and Li (1991). Alternatively, one can also use the Cochrane-Orcutt (CO) procedure, which ignores the information contained in the first observation. As suggested by Maeshiro (1976), Beach and MacKinnon (1978) and Park and Mitchell (1980), estimation using the PW transformation is more efficient than using the CO procedure when the regressors are trended.

When the assumptions of corollary 4 hold, one can show that feasible GLS has the same asymptotic distribution as true GLS. Define  $\phi = (\rho, \sigma_\mu^2, \sigma_e^2)$  and  $\hat{\phi}$  is its corresponding estimator. Then  $\Phi = \Phi(\phi)$ . Further define that  $G_k(\phi) = \partial\Phi^{-1}(\phi)/\partial\phi_k$ , where  $k = 1, 2, 3$ . For example, in case 1, a Taylor's series expansion as in Fuller and Battese (1973) gives

$$\begin{aligned}
& \sqrt{nT}(\hat{\tau}_{GLS} - \tau) \\
&= \left( \frac{Z'\Phi^{-1}(\hat{\phi})Z}{nT} \right)^{-1} \left( \frac{Z'\Phi^{-1}(\hat{\phi})u}{\sqrt{nT}} \right) \\
&= \left( \frac{Z'\Phi^{-1}(\phi)Z}{nT} \right)^{-1} \left( \frac{Z'\Phi^{-1}(\phi)u}{\sqrt{nT}} \right) + \sum_{k=1}^3 \left[ \left( \frac{Z'\Phi^{-1}(\phi^*)Z}{nT} \right)^{-1} \left( \frac{Z'G_k(\phi^*)u}{\sqrt{nT}} \right) \right. \\
&\quad \left. - \left( \frac{Z'\Phi^{-1}(\phi^*)Z}{nT} \right)^{-1} \left( \frac{Z'G_k(\phi^*)Z}{nT} \right) \left( \frac{Z'\Phi^{-1}(\phi^*)Z}{nT} \right)^{-1} \left( \frac{Z'\Phi^{-1}(\phi^*)u}{\sqrt{nT}} \right) \right] (\hat{\phi} - \phi) \\
&= \left( \frac{Z'\Phi^{-1}(\phi)Z}{nT} \right)^{-1} \left( \frac{Z'\Phi^{-1}(\phi)u}{\sqrt{nT}} \right) + o_p(1)
\end{aligned}$$

where  $\phi^*$  lies between  $\hat{\phi}$  and  $\phi$ , hence  $\phi^* \xrightarrow{p} \phi$ . The last equal sign holds if  $\frac{Z'\Phi^{-1}(\phi)Z}{nT} = O_p(1)$ ,  $\frac{Z'G_k(\phi^*)Z}{nT} = O_p(1)$ ,  $\frac{Z'\Phi^{-1}(\phi^*)u}{\sqrt{nT}} = O_p(1)$  and  $\frac{Z'G_k(\phi^*)u}{\sqrt{nT}} = O_p(1)$ . This follows using similar arguments in the proofs of the Theorems above. The proofs are similar for the remaining three cases and are omitted to save space.

## 8 Efficiency Comparisons

This section summarizes the relative efficiency of OLS, FE, GLS and FD estimators. First, the speed of convergence for the different cases considered are summarized as follows:

	OLS	FE	FD	GLS
Case 1: $ \rho  < 1$ and $ \lambda  < 1$	$\sqrt{nT}$	$\sqrt{nT}$	$\sqrt{nT}$	$\sqrt{nT}$
Case 2: $\rho = 1$ and $ \lambda  < 1$	$\sqrt{n}$	$\sqrt{n}$	$\sqrt{nT}$	$\sqrt{nT}$
Case 3: $ \rho  < 1$ and $\lambda = 1$	$\sqrt{nT}$	$\sqrt{nT}$	$\sqrt{nT}$	$\sqrt{nT}$
Case 4: $\rho = 1$ and $\lambda = 1$	$\sqrt{n}$	$\sqrt{n}$	$\sqrt{nT}$	$\sqrt{nT}$

In case 1, the four estimators have the same convergence speed of  $\sqrt{nT}$ . The efficiency of the OLS estimator is hard to compare with the remaining estimators because OLS does not difference out  $\mu_i$ , and as a result its variance still contains  $\sigma_\mu^2$ . That GLS is more efficient than FE and FD is evident from the Gauss-Markov theorem. Since these estimators all converge at same rate  $\sqrt{nT}$ , we plot the relative efficiency of the FE and FD estimators with respect to true GLS in Figure 1 and 2. The relative efficiency of the FE estimator with respect to true GLS is given by

$$\begin{aligned} \text{var}(\widehat{\beta}_{FE}) / \text{var}(\widehat{\beta}_{GLS}) &= \frac{(1 + \rho\lambda)(1 - \lambda^2)\sigma_\varepsilon^2}{(1 - \rho\lambda)(1 - \rho^2)\sigma_\varepsilon^2} / \frac{(1 - \lambda^2)\sigma_\varepsilon^2}{(1 - 2\rho\lambda + \rho^2)\sigma_\varepsilon^2} \\ &= \frac{(1 + \rho\lambda)(1 - 2\rho\lambda + \rho^2)}{(1 - \rho\lambda)(1 - \rho^2)}. \end{aligned}$$

The relative efficiency of the FD estimator with respect to true GLS is given by

$$\begin{aligned} \text{var}(\widehat{\beta}_{FD}) / \text{var}(\widehat{\beta}_{GLS}) &= \frac{(1 + \lambda)^2 \left[ (2 - \rho - \lambda)^2 + \frac{(1-\rho)^3}{1+\rho} + \frac{(1-\lambda)^3}{1+\lambda} \right] \sigma_\varepsilon^2}{4(1 - \rho\lambda)^2 \sigma_\varepsilon^2} / \frac{(1 - \lambda^2)\sigma_\varepsilon^2}{(1 - 2\rho\lambda + \rho^2)\sigma_\varepsilon^2} \\ &= \frac{(1 + \lambda)(1 - 2\rho\lambda + \rho^2) \left[ (2 - \rho - \lambda)^2 + \frac{(1-\rho)^3}{1+\rho} + \frac{(1-\lambda)^3}{1+\lambda} \right]}{4(1 - \lambda)(1 - \rho\lambda)^2}. \end{aligned}$$

One can easily verify that both relative efficiencies are larger or equal to 1. Comparing the GLS estimator with the FE and FD estimators, the relative efficiency depends on the values of  $\rho$  and  $\lambda$ . As shown in Figure 1 and 2, when  $\rho$  is small, the FE estimator performs well in terms of relative efficiency with respect to true GLS. When  $\rho$  is large, the FD estimators performs well in terms of relative efficiency with respect to true GLS.

In case 2, the disturbance is  $I(1)$  but the regressor is  $I(0)$ . The noise is strong so that it dominates the signal. In the time series case, the OLS estimator is not consistent. After double smoothing using panel data, the asymptotic distribution becomes normal and the convergence speed is  $\sqrt{n}$ . GLS estimation, however, transforms the disturbance into  $I(0)$ . Therefore the convergence speed is  $\sqrt{nT}$ . When the disturbance is  $I(1)$ , first-difference estimation will be the same as GLS except for the first observation. Hence it is also  $\sqrt{nT}$  consistent.

In case 3, the disturbance is  $I(0)$  but the regressor is  $I(1)$ . This is the cointegration case. The cointegration literature shows that the GLS estimators is  $T$  consistent in time series models. In the panel data model, both GLS and FE are  $\sqrt{nT}$  consistent.

In case 4, both the disturbance and the regressor are  $I(1)$ . This is the spurious regression case. As shown in Kao (1999), the FE estimator is  $\sqrt{n}$  consistent. For the same reason given in case 2, first-differencing transforms the disturbance term from  $I(1)$  to  $I(0)$ . Therefore, the convergence speed of both the GLS or the FD estimators is  $\sqrt{nT}$ .

In case 3, the FE estimator is more efficient than the FD estimator when  $v_{it}$  are stationary, including the special case when  $v_{it}$  are serially uncorrelated. In cases 2 and 4, the FD estimator is more efficient when  $v_{it}$  follows a random walk. These results verify the conclusion in Wooldridge (2002). However, in case 1, when  $\rho$  is large, even though  $v_{it}$  does not follow a random walk, the FD estimator is still more efficient than the FE estimator.

## 9 Monte Carlo Simulation

This section reports the results of Monte Carlo experiments designed to investigate the finite sample relative efficiency of the OLS, FE, FD, GLS-CO, GLS-PW estimators with respect to true GLS. The model is generated by

$$y_{it} = x_{it}\beta + \mu_i + v_{it}, \quad i = 1, \dots, n, \quad t = 1, \dots, T, \quad (15)$$

with  $\beta = 10$ ,  $\mu_i \stackrel{iid}{\sim} N(0, 5)$  and  $v_{it}$  and  $x_{it}$  follow an AR(1) process given in (2) and (3), respectively with  $\rho$  and  $\lambda$  varying over the range (0, 0.2, 0.4, 0.6, 0.8, 0.9, 1) and  $\sigma_\varepsilon^2 = \sigma_e^2 = 5$ . The sample sizes  $n$  and  $T$  are varied over the range (20, 40, 60, 120, 240). For each experiment, we perform 10,000 replications. For each replication we estimate the model using OLS, FE, FD, GLS-CO, GLS-PW and true GLS. Even with this modest design we had 1225 experiments. GAUSS for Windows 6.0 was used to perform the simulations. Random numbers for  $\mu_i$  and  $\varepsilon_{it}$  were generated by the GAUSS procedure RNDNS. We generated  $n(T+1000)$  random numbers and then split them into  $n$  series so that each series had the same mean and variance. The first 1,000 observations were discarded for each series.

Tables 1-3 give the relative mean square error (MSE) of each estimator of  $\beta$  with respect to true GLS for various values of  $\rho$ ,  $\lambda$ ,  $n$ , and  $T$ . We only report 3 Tables to give a flavour of the results, the rest are available upon request from the authors. Several conclusions emerge from these results. First, the true GLS estimator is the most efficient one in terms of mean squared error. Its efficiency gain over the OLS estimator is enormous particularly when  $\rho$  and/or  $\lambda$  is large. Second, the FE estimator is less efficient than true GLS,

but more efficient than the feasible GLS estimator when  $\rho = 0$ . However, when  $\rho$  increases, the feasible GLS estimator quickly becomes more efficient than the FE estimator. Third, the FD estimator is also less efficient than true GLS. When  $\rho$  increases, the FD estimator becomes as efficient as the GLS estimator. Interestingly, the FD estimator behaves poorly when  $\lambda$  is close to 1 but  $\rho$  is small. Fourth, the feasible GLS estimator is slightly less efficient than the true GLS estimator and beats OLS, FE and FD as long as  $\rho > 0.2$ . In summary, our simulation results show that the feasible GLS estimator performs well, and is second best only to true GLS when  $\rho > 0.2$

## 10 Conclusion

In this paper, we compared the efficiency of OLS, FE, FD, and GLS estimators in panel models with  $I(0)$  and  $I(1)$  regressor and regression error. When the regression error is  $I(0)$  and the regressor is  $I(1)$  and hence the model is cointegrated, both the FE and GLS estimators are asymptotically efficient. When the regression error is  $I(1)$  and the regressor is  $I(1)$  and hence the model is spurious, the FE and GLS estimators are  $\sqrt{n}$  and  $\sqrt{nT}$  consistent, respectively. This implies that GLS is the preferred estimator as far as the regression error specification is concerned since GLS converges at as good or better rate in both cases (i.e., regression error is either  $I(0)$  or  $I(1)$ ).

## References

- [1] Baltagi, B. (2005), *Econometric Analysis of Panel Data*, New York, Wiley.
- [2] Baltagi, B., and Kao, C. (2000), “Nonstationary Panels, Cointegration in Panels and Dynamic Panels: A Survey”, *Advances in Econometrics*, 15, 7-51.
- [3] Baltagi, B. H., and Krämer, W. (1997), 97.2.1. A Simple Linear Trend Model with Error Components, *Econometric Theory*, 13, 463-463.
- [4] Baltagi, B. H., and Li, Q. (1991), “A Transformation That Will Circumvent the Problem of Autocorrelation in an Error Component Model,” *Journal of Econometrics*, 52, 371-380.
- [5] Baltagi, B. H., and Li, Q. (1997), “Monte Carlo Results on Pure and Pretest Estimators of an Error Component Model with Autocorrelated Disturbances,” *Annales D'Économie et de Statistique*, 48, 69-82.
- [6] Beach, B. M., and MacKinnon, J. G. (1978), “A Maximum Likelihood Procedure for Regression with Autocorrelation Errors,” *Econometrica*, 46, 51-58.
- [7] Breitung, J., and Pesaran, M. H. (2005), “Unit Roots and Cointegration in Panels,” forthcoming in L. Matyas, and P. Sevestre, *The Econometrics of Panel Data* (Third Edition), Kluwer Academic Publishers..
- [8] Choi, I. (1999), “Asymptotic Analysis of a Nonstationary Error Component Model,” Working paper, Kookmin University, Korea.
- [9] Choi, I. (2002), “Instrumental Variables Estimation of a Nearly Nonstationary, Heterogeneous Error Component Model,” *Journal of Econometrics*, 109, 1-32.
- [10] Choi, I. (2006), “Nonstationary Panels,” *Palgrave Handbooks of Econometrics*, Vol. 1, 511-539. Palgrave Macmillan: New York.
- [11] Choi, C., Hu, L., and Ogaki, M. (2004), “A Spurious Regression Approach to Estimating Structural Parameters”, Working paper, Ohio State University, US.
- [12] Fuller, W., and Battese, G. E. (1973), “Transformations for Estimation of Linear Models with Nested-Error Structure,” *Journal of the American Statistical Association*, 68, 626-632.
- [13] Hsiao, C. (1986), *Analysis of Panel Data*, Cambridge, Cambridge University Press.

- [14] Kao, C. (1999), "Spurious Regression and Residual-Based Tests for Cointegration in Panel Data," *Journal of Econometrics*, 90, 1-44.
- [15] Kao, C., and Chiang, M-H. (2000), "On the Estimation and Inference of a Cointegrated Regression in Panel Data," *Advances in Econometrics*, 15, 179-222.
- [16] Kao, C., and Emerson, J. (2002a), "Testing for Structural Change of a Time Trend Regression in Panel Data: Part I," *Journal of Propagations in Probability and Statistics*, 2, 57-75.
- [17] Kao, C., and Emerson, J. (2002b), "Testing for Structural Change of a Time Trend Regression in Panel Data: Part II," *Journal of Propagations in Probability and Statistics*, 2, 207-250.
- [18] Maeshiro, A. (1976), "Autoregressive Transformation, Trended Independent Variables and Autocorrelated Disturbance Terms," *The Review of Economics and Statistics*, 58, 497-500.
- [19] Park, S. J., and Mitchell, B. M. (1980), "Estimating the Autocorrelated Error Model with Trended Data," *Journal of Econometrics*, 13, 185-201.
- [20] Phillips, P. C. B., (1986), "Understanding spurious regressions in econometrics," *Journal of Econometrics*, 33, 311-340.
- [21] Phillips, P. C. B., and Moon, H., (1999), "Linear Regression Limit Theory for Nonstationary Panel Data," *Econometrica*, 67, 1057-1111.
- [22] Phillips, P. C. B., and Park, J. Y. (1988), "Asymptotic Equivalence of Ordinary Least Squares and Generalized Least Squares in Regressions with Integrated Regressors," *Journal of the American Statistical Association*, 83, 111-115.
- [23] Summers, R., and Heston, A. (1991), "The Penn World Table; An Expanded Set of International Comparisons 1950-1988," *Quarterly Journal of Economics*, 106, 327-368.
- [24] Wooldridge, J. (2002), *Econometric Analysis of Cross Section and Panel Data*, Cambridge, MIT Press.

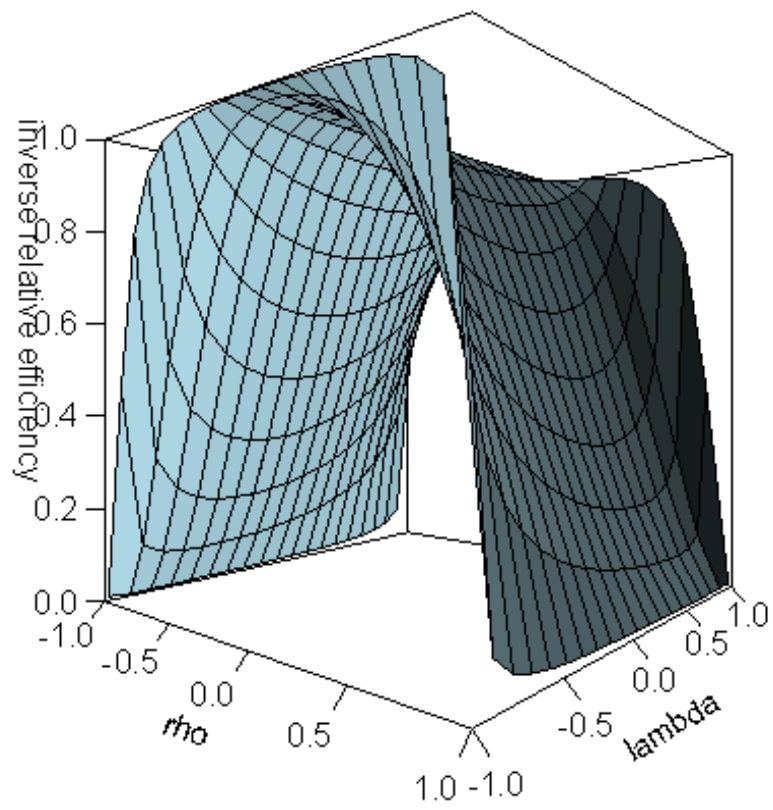


Figure 1: Relative Efficiency of GLS to FE Estimator



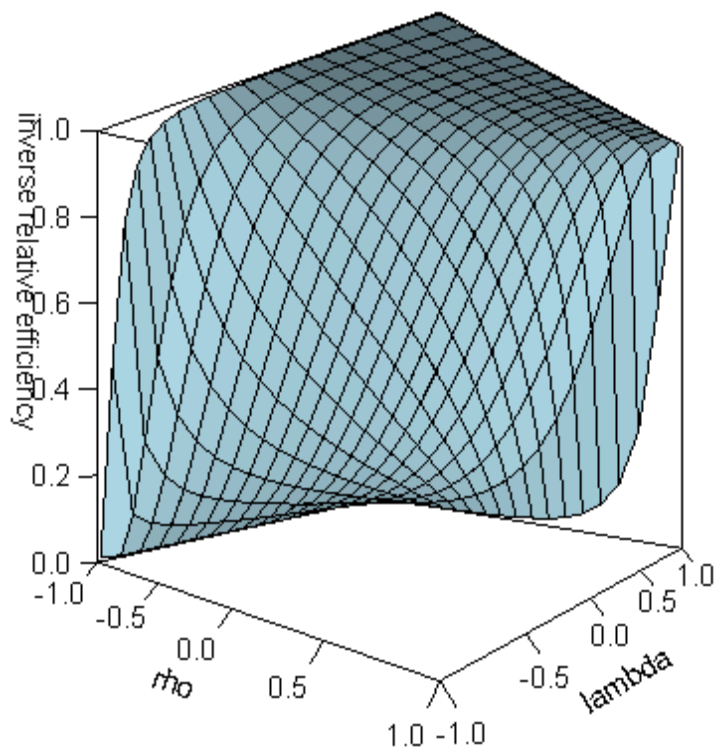


Figure 2: Relative Efficiency of GLS to FD Estimator

Table 1: Relative Efficiencies of Standard Panel Data Estimators (N = 40, T= 20)

$\rho$		$\lambda$						
		0	0.2	0.4	0.6	0.8	0.9	1
OLS	0	1.915	2.310	2.897	3.838	5.305	5.799	5.211
	0.2	2.077	2.404	2.845	3.496	4.436	4.668	4.065
	0.4	2.512	2.815	3.154	3.565	4.032	3.994	3.273
	0.6	3.427	3.793	4.108	4.340	4.346	3.956	2.906
	0.8	5.784	6.476	7.017	7.260	6.745	5.653	3.506
	0.9	9.257	10.529	11.590	12.198	11.475	9.528	5.451
	1	22.685	26.255	29.502	31.992	31.608	27.087	15.126
FE	0	1.002	1.003	1.005	1.009	1.024	1.048	1.102
	0.2	1.073	1.075	1.073	1.071	1.076	1.098	1.161
	0.4	1.315	1.332	1.332	1.311	1.278	1.281	1.330
	0.6	1.838	1.915	1.949	1.918	1.811	1.755	1.733
	0.8	2.970	3.223	3.397	3.426	3.198	2.972	2.650
	0.9	4.066	4.505	4.843	4.976	4.678	4.294	3.619
	1	7.306	8.326	9.222	9.843	9.767	9.330	8.142
FD	0	1.486	1.699	2.021	2.561	3.601	4.594	7.548
	0.2	1.223	1.329	1.494	1.780	2.355	2.934	4.722
	0.4	1.095	1.141	1.215	1.350	1.642	1.964	3.022
	0.6	1.037	1.053	1.079	1.129	1.253	1.413	1.991
	0.8	1.013	1.016	1.022	1.032	1.064	1.117	1.342
	0.9	1.006	1.007	1.009	1.011	1.020	1.036	1.117
	1	1.016	1.019	1.021	1.022	1.023	1.023	1.024
GLS-PW	0	1.272	1.357	1.456	1.565	1.648	1.652	1.625
	0.2	1.120	1.168	1.232	1.316	1.410	1.440	1.453
	0.4	1.043	1.063	1.092	1.137	1.201	1.234	1.270
	0.6	1.012	1.017	1.025	1.039	1.065	1.085	1.119
	0.8	1.002	1.002	1.003	1.004	1.008	1.014	1.027
	0.9	1.000	1.000	1.000	1.000	1.002	1.006	1.021
	1	1.016	1.018	1.020	1.024	1.031	1.045	1.111

Notes: (a) Relative mean square error with respect to the true GLS. (b) 10,000 replications.  $\sigma_\mu^2 = \sigma_\epsilon^2 = 5$ .

Table 2: Relative Efficiencies of Standard Panel Data Estimators (N = 60, T= 60)

$\rho$		$\lambda$						
		0	0.2	0.4	0.6	0.8	0.9	1
OLS	0	1.963	2.414	3.123	4.441	7.826	12.201	13.958
	0.2	2.101	2.467	2.989	3.881	6.062	8.836	9.628
	0.4	2.523	2.858	3.249	3.796	4.980	6.446	6.329
	0.6	3.445	3.848	4.207	4.509	4.844	5.209	4.077
	0.8	6.008	6.802	7.438	7.754	7.290	6.375	3.150
	0.9	10.598	12.223	13.622	14.515	13.850	11.702	4.367
	1	61.715	73.063	84.083	94.300	100.976	97.379	41.327
FE	0	1.000	1.000	1.000	1.000	1.002	1.005	1.037
	0.2	1.081	1.083	1.078	1.064	1.039	1.028	1.062
	0.4	1.359	1.379	1.371	1.323	1.219	1.151	1.149
	0.6	2.010	2.110	2.147	2.073	1.802	1.576	1.415
	0.8	3.871	4.291	4.601	4.681	4.180	3.485	2.511
	0.9	6.686	7.647	8.490	9.050	8.675	7.464	4.664
	1	20.601	24.289	27.902	31.303	33.476	32.129	23.459
FD	0	1.485	1.714	2.079	2.779	4.672	7.440	21.119
	0.2	1.211	1.322	1.505	1.864	2.861	4.347	11.967
	0.4	1.082	1.128	1.206	1.367	1.834	2.559	6.559
	0.6	1.026	1.041	1.067	1.121	1.292	1.579	3.449
	0.8	1.007	1.009	1.014	1.023	1.056	1.123	1.768
	0.9	1.002	1.003	1.004	1.007	1.014	1.030	1.267
	1	1.006	1.007	1.008	1.009	1.010	1.010	1.010
GLS-PW	0	1.293	1.392	1.512	1.660	1.820	1.834	1.667
	0.2	1.126	1.181	1.258	1.376	1.564	1.663	1.612
	0.4	1.043	1.066	1.101	1.162	1.291	1.399	1.455
	0.6	1.011	1.017	1.026	1.045	1.095	1.155	1.246
	0.8	1.001	1.002	1.003	1.005	1.010	1.021	1.062
	0.9	1.000	1.000	1.000	1.000	1.001	1.002	1.012
	1	1.005	1.006	1.007	1.008	1.009	1.010	1.046

Notes: (a) Relative mean square error with respect to the true GLS. (b) 10,000 replications.  $\sigma_\mu^2 = \sigma_\epsilon^2 = 5$ .

Table 3: Relative Efficiencies of Standard Panel Data Estimators (N = 240, T= 60)

$\rho$		$\lambda$						
		0	0.2	0.4	0.6	0.8	0.9	1
OLS	0	1.903	2.330	3.022	4.305	7.364	11.040	12.103
	0.2	2.042	2.390	2.895	3.771	5.770	8.095	8.331
	0.4	2.462	2.783	3.152	3.690	4.802	6.021	5.490
	0.6	3.385	3.783	4.110	4.392	4.731	5.003	3.598
	0.8	5.877	6.679	7.269	7.545	7.169	6.308	2.932
	0.9	10.283	11.912	13.242	14.094	13.687	11.809	4.236
	1	66.063	78.409	89.876	100.477	108.863	107.078	43.764
FE	0	1.000	1.001	1.001	1.002	1.006	1.011	1.024
	0.2	1.085	1.088	1.082	1.073	1.065	1.060	1.054
	0.4	1.369	1.395	1.384	1.344	1.274	1.219	1.147
	0.6	2.036	2.152	2.182	2.113	1.912	1.718	1.433
	0.8	3.850	4.301	4.598	4.683	4.352	3.789	2.601
	0.9	6.426	7.390	8.173	8.699	8.595	7.716	4.808
	1	19.538	23.082	26.375	29.349	31.405	30.910	26.383
FD	0	1.471	1.687	2.051	2.733	4.389	6.646	18.377
	0.2	1.210	1.312	1.495	1.853	2.748	3.973	10.465
	0.4	1.085	1.125	1.203	1.368	1.803	2.409	5.791
	0.6	1.030	1.041	1.065	1.123	1.289	1.536	3.103
	0.8	1.009	1.011	1.014	1.024	1.059	1.120	1.675
	0.9	1.004	1.004	1.005	1.007	1.015	1.030	1.241
	1	0.999	0.998	0.997	0.996	0.998	0.998	1.000
GLS-PW	0	1.290	1.384	1.508	1.647	1.714	1.640	1.527
	0.2	1.128	1.179	1.258	1.374	1.504	1.523	1.473
	0.4	1.047	1.066	1.102	1.165	1.268	1.321	1.336
	0.6	1.014	1.018	1.027	1.047	1.091	1.125	1.163
	0.8	1.003	1.003	1.004	1.006	1.013	1.019	1.033
	0.9	1.001	1.001	1.001	1.001	1.003	1.004	1.008
	1	0.999	0.998	0.997	0.997	0.999	1.002	1.045

Notes: (a) Relative mean square error with respect to the true GLS. (b) 10,000 replications.  $\sigma_\mu^2 = \sigma_\epsilon^2 = 5$ .

## Appendix

### A Proof of Theorem 1

The following lemmas are needed to prove Theorem 1. All limits are taken as  $T \rightarrow \infty$  and followed by  $n \rightarrow \infty$  sequentially,  $(n, T) \xrightarrow{\text{seq}} \infty$ .

**Lemma 1** *If Assumptions 1 – 2 hold, then*

1. If  $|\lambda| < 1$ ,  $\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x})^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}$ .
2. If  $\lambda = 1$ ,  $\frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x})^2 \xrightarrow{p} \frac{\varpi_\varepsilon^2}{2}$ .

**Proof.** Consider (1). For a fixed  $n$ , it is clear to see that

$$\begin{aligned} & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x})^2 \\ &= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it}^2 \right] - (\bar{x})^2 \\ & \xrightarrow{p} \frac{1}{n} \sum_{i=1}^n \left[ \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)} \right] - \frac{1}{n} \sum_{i=1}^n E(x_{it}) \\ &= \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)} \end{aligned}$$

as  $T \rightarrow \infty$  because

$$\begin{aligned} E(x_{it}^2) &= E \left( \sum_{j=0}^{\infty} \lambda^j \varepsilon_{i(t-j)} \right)^2 \\ &= \sum_{j=0}^{\infty} E(\lambda^j \varepsilon_{i(t-j)})^2 + \sum_{j=0}^{\infty} \sum_{k=0, k \neq j}^{\infty} E(\lambda^{j+k} \varepsilon_{i(t-j)} \varepsilon_{i(t-k)}) \\ &= \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)} \end{aligned}$$

and  $\bar{x} = \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T x_{it} \xrightarrow{p} \frac{1}{n} \sum_{i=1}^n E(x_{it}) = 0$  as  $T \rightarrow \infty$ . Note  $E(x_{it}) = 0$  for all  $i$  and  $t$  since there is no non-zero drift in (??). Then obviously,

$$\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x})^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}$$

holds for all  $n$  and hence it holds for a large  $n$  as well. This proves (1).

Next we consider (2). Similarly for a fixed  $n$

$$\begin{aligned}
& \frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x})^2 \\
&= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T^2} \sum_{t=1}^T x_{it}^2 \right] - \frac{1}{T} \bar{x}^2 \\
&= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T \left( \frac{x_{it}}{\sqrt{T}} \right)^2 \right] - \left[ \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T} \sum_{t=1}^T \frac{x_{it}}{\sqrt{T}} \right) \right]^2 \\
&\Rightarrow \frac{1}{n} \sum_{i=1}^n \left( \varpi_\varepsilon^2 \int W_i^2 \right) - \left[ \frac{1}{n} \sum_{i=1}^n \varpi_\varepsilon \int W_i \right]^2
\end{aligned}$$

as  $T \rightarrow \infty$  because  $\frac{1}{T} \sum_{t=1}^T \left( \frac{x_{it}}{\sqrt{T}} \right)^2 \Rightarrow \varpi_\varepsilon^2 \int W_i^2$  and  $\frac{1}{T} \sum_{t=1}^T \frac{x_{it}}{\sqrt{T}} \Rightarrow \varpi_\varepsilon \int W_i$ .

Then

$$\frac{1}{n} \sum_{i=1}^n \left( \varpi_\varepsilon^2 \int W_i^2 \right) - \left[ \frac{1}{n} \sum_{i=1}^n \varpi_\varepsilon \int W_i \right]^2 \xrightarrow{p} \frac{\varpi_\varepsilon^2}{2}.$$

as  $n \rightarrow \infty$  by a law of large numbers (LLN). This is because

$$E \int W_i^2 = \frac{1}{2}$$

and

$$E \int W_i = 0.$$

This proves (2). ■

**Lemma 2** *If Assumptions 1 – 3 hold, then*

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$\text{(a) } \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \xrightarrow{p} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it})$$

(b)

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} - \sqrt{n} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \\
&\Rightarrow N \left( 0, \frac{\sigma_\mu^2 \varpi_\varepsilon^2}{(1-\lambda)^2} + \frac{\psi_{00} + \sum_{r=1}^{\infty} \lambda^{2r} \psi_{0r} + \sum_{r=1}^{\infty} \rho^{2r} \psi_{r0}}{(1-\rho\lambda)^2} \right)
\end{aligned}$$

where  $\psi_{0r} = E\left(\varepsilon_{i(t-r)}^2 e_{it}^2\right)$ ,  $\psi_{r0} = E\left(\varepsilon_{it}^2 e_{i(t-r)}^2\right)$ ,  $\psi_{00} = E\left(\varepsilon_{it}^2 e_{it}^2\right)$ .

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$\begin{aligned} \text{(a)} \quad & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \xrightarrow{p} \frac{-\frac{1}{2}\varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon}}{1-\lambda} \\ \text{(b)} \quad & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} - \sqrt{n} \left[ \frac{\left(\frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T \varpi_{\varepsilon\varepsilon} \nu_{i(t-1)} e_{it}\right) + \delta_{\varepsilon\varepsilon}}{1-\lambda} \right] \Rightarrow \\ & N\left(0, \frac{\varpi_{\varepsilon.\varepsilon} \varpi_{\varepsilon}^2}{2(1-\lambda)^2}\right). \end{aligned}$$

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

$$\begin{aligned} \text{(a)} \quad & \frac{1}{nT^{3/2}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \xrightarrow{p} 0 \\ \text{(b)} \quad & \frac{1}{\sqrt{nT^{3/2}}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \Rightarrow N\left(0, \frac{\sigma_{\mu}^2 \varpi_{\varepsilon}^2}{3}\right). \end{aligned}$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$\begin{aligned} \text{(a)} \quad & \frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \xrightarrow{p} \delta_{\varepsilon\varepsilon} + \frac{\varpi_{\varepsilon\varepsilon}}{2} \\ \text{(b)} \quad & \frac{1}{\sqrt{nT^2}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} - \sqrt{n} \left[ \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T x_{i(t-1)} \varepsilon_{it}\right) \frac{\varpi_{\varepsilon\varepsilon}}{\varpi_{\varepsilon}^2} + \delta_{\varepsilon\varepsilon} \right] \Rightarrow \\ & N\left(0, \frac{\varpi_{\varepsilon.\varepsilon} \varpi_{\varepsilon}^2}{6}\right). \end{aligned}$$

**Proof. :**

Consider (1). For part (a), we note

$$\begin{aligned} & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \\ = & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) (\mu_i + \nu_{it}) \\ = & \frac{1}{\sqrt{T}} \frac{1}{n} \sum_{i=1}^n \left[ \left(\frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it}\right) \mu_i \right] - \frac{1}{\sqrt{T}} \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it}\right) \left(\frac{1}{n} \sum_{i=1}^n \mu_i\right) \\ & + \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right] - \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T x_{it}\right) \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T \nu_{it}\right) \\ = & I - II + III - IV. \end{aligned}$$

Consider  $I$ . It is easy to see that for a fixed  $n$ ,

$$\frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \mu_i \right] = \frac{1}{n} \sum_{i=1}^n Z_i \mu_i$$

as  $T \rightarrow \infty$  where

$$\frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \Rightarrow Z_i \sim N \left( 0, \frac{\varpi_\varepsilon^2}{(1-\lambda)^2} \right)$$

by a central limit theorem (CLT) since  $E(x_{it}) = 0$ . Then

$$\frac{1}{n} \sum_{i=1}^n Z_i \mu_i = o_p(1)$$

as  $n \rightarrow \infty$  by a LLN and the assumption that  $\mu_i$  and  $x_{it}$  are uncorrected as in (??). Hence  $I = \frac{1}{\sqrt{T}} o_p(1)$ . Consider  $II$ . For a fixed  $n$ ,

$$\begin{aligned} & \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \mu_i \right) \\ &= \left( \frac{1}{n} \sum_{i=1}^n Z_i \right) \left( \frac{1}{n} \sum_{i=1}^n \mu_i \right) \end{aligned}$$

as  $T \rightarrow \infty$ . Then clearly

$$\left( \frac{1}{n} \sum_{i=1}^n Z_i \right) \left( \frac{1}{n} \sum_{i=1}^n \mu_i \right) \xrightarrow{p} 0$$

as  $n \rightarrow \infty$  because  $E(\mu_i) = 0$ . This proves that

$$II = \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \mu_i \right) = o_p(1).$$

Next we consider  $III$ . Clearly

$$\frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right] = \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) + o_p(1)$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . Also it is easy to see that

$$IV = o_p(1).$$

Collecting  $I - IV$  we then prove (a).



For part (b), for a fixed  $n$ ,

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \\
= & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) (\mu_i + \nu_{it}) - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \mu_i \right] - \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \mu_i \right) \\
& + \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \nu_{it} \right) - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \right] \\
& - \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) \\
= & I - II + III - IV.
\end{aligned}$$

For  $I$ ,

$$I = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \mu_i \right] \Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i \mu_i$$

as  $T \rightarrow \infty$ , and

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i \mu_i \Rightarrow N \left( 0, \frac{\sigma_\mu^2 \omega_\varepsilon^2}{(1-\lambda)^2} \right)$$

as  $n \rightarrow \infty$  by a CLT. For  $III$ ,

$$\begin{aligned}
& \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \nu_{it} \right) - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \\
= & \sqrt{nT} \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \left( x_{it} \nu_{it} - \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \right) \\
\Rightarrow & N \left( 0, \frac{\psi_{i00} + \sum_{r=1}^{\infty} \lambda^{2r} \psi_{i0r} + \sum_{r=1}^{\infty} \rho^{2r} \psi_{ir0}}{(1-\rho\lambda)^2} \right)
\end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ , which is based on Lemma A0 in Choi(1999). It is easy to see that

$$II = \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \mu_i \right) = o_p(1) O_p(1).$$

$$IV = \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) = \frac{1}{\sqrt{T}} o_p(1) O_p(1) = o_p(1).$$

Hence, we have

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \\ \Rightarrow & N \left( 0, \frac{\sigma_\mu^2 \varpi_\varepsilon^2}{(1-\lambda)^2} + \frac{\psi_{00} + \sum_{r=1}^{\infty} \lambda^{2r} \psi_{0r} + \sum_{r=1}^{\infty} \rho^{2r} \psi_{r0}}{(1-\rho\lambda)^2} \right) \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . This proves (b).

Consider (2). For part (a), note

$$\begin{aligned} & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \\ = & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) (\mu_i + \nu_{it}) \\ = & \frac{1}{\sqrt{T}} \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \mu_i \right] - \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \mu_i \right) \\ & + \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right] - \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T \frac{\nu_{it}}{\sqrt{T}} \right) \\ = & I - II + III - IV. \end{aligned}$$

First we consider *III*. Note

$$\frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \Rightarrow \frac{1}{1-\lambda} \left[ \varpi_e \varpi_{\varepsilon,e}^{1/2} \left( \int V_i dW_i \right) + \varpi_{e\varepsilon} \left( \int V_i dV_i \right) + \delta_{e\varepsilon} \right]$$

as  $T \rightarrow \infty$ . The above is taken from Lemma 1(a) in Kao and Chiang (2000).

Then

$$\begin{aligned} III &= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right] \\ &= \frac{1}{n} \sum_{i=1}^n \frac{1}{1-\lambda} \left[ \varpi_e \varpi_{\varepsilon,e}^{1/2} \left( \int V_i dW_i \right) + \varpi_{e\varepsilon} \left( \int V_i dV_i \right) + \delta_{e\varepsilon} \right] \\ &\xrightarrow{p} \frac{-\frac{1}{2} \varpi_{e\varepsilon} + \delta_{e\varepsilon}}{1-\lambda} \end{aligned}$$

as  $n \rightarrow \infty$  because  $E(\int V_i dW_i) = 0$  and  $E(\int V_i dV_i) = -\frac{1}{2}$ . It is clear to

see that

$$I = \frac{1}{\sqrt{T}} \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \mu_i \right] = \frac{1}{\sqrt{T}} o_p(1),$$

$$II = \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \mu_i \right) = \frac{1}{\sqrt{T}} o_p(1)$$

and

$$IV = \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T \frac{\nu_{it}}{\sqrt{T}} \right) = o_p(1)$$

because  $\frac{1}{T^{3/2}} \sum_{t=1}^T \nu_{it} \Rightarrow \int V_i$  and  $E(\int V_i) = 0$ . This proves (a).

For part (b),

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \\ &= \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) (\mu_i + \nu_{it}) \\ &= \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \mu_i \right] - \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \mu_i \right) \\ & \quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right] - \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T \nu_{it} \right) \\ &= I - II + III - IV. \end{aligned}$$

First consider *III*,

$$\begin{aligned} & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right] - \sqrt{n} \left[ \frac{\left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T \varpi_{e\varepsilon} \nu_{i,t-1} e_{it} \right) + \delta_{e\varepsilon}}{1-\lambda} \right] \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{1-\lambda} \left[ \varpi_{e\varepsilon} \varpi_{\varepsilon.e}^{1/2} \left( \int V_i dW_i \right) + \varpi_{e\varepsilon} \left( \int V_i dV_i \right) + \delta_{e\varepsilon} \right] \\ & \quad - \sqrt{n} \left[ \frac{\frac{1}{n} \sum_{i=1}^n (\varpi_{e\varepsilon} \int V_i dV_i) + \delta_{e\varepsilon}}{1-\lambda} \right] + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{1-\lambda} \left[ \varpi_{e\varepsilon} \varpi_{\varepsilon.e}^{1/2} \left( \int V_i dW_i \right) \right] + o_p(1) \\ &\Rightarrow N \left( 0, \frac{\varpi_{\varepsilon.e} \varpi_e^2}{2(1-\lambda)^2} \right), \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . It is easy to see that

$$I = \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \mu_i \right] = \frac{1}{\sqrt{T}} O_p(1),$$

$$II = \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \mu_i \right) = \frac{1}{\sqrt{T}} o_p(1) O_p(1)$$

and

$$IV = \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T \nu_{it} \right) = o_p(1) O_p(1).$$

Hence, we have

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} - \sqrt{n} \left[ \frac{\left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T \varpi_{e\varepsilon} \nu_{i(t-1)} e_{it} \right) + \delta_{e\varepsilon}}{1 - \lambda} \right] \\ \Rightarrow & N \left( 0, \frac{\varpi_{\varepsilon, e} \varpi_e^2}{2(1 - \lambda)^2} \right), \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

Consider (3). For part (a), it is easy to see that

$$\begin{aligned} & \frac{1}{nT^{3/2}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \\ = & \frac{1}{nT^{3/2}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) (\mu_i + \nu_{it}) \\ = & \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \mu_i \right] - \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \mu_i \right) \\ & + \frac{1}{\sqrt{T}} \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right) \right] - \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) \\ = & I - II + III - IV. \end{aligned}$$

For a fixed  $n$ ,

$$\begin{aligned} I &= \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \mu_i \right] \\ &\Rightarrow \frac{1}{n} \sum_{i=1}^n \left[ \left( \varpi_\varepsilon \int W_i \right) \mu_i \right] \end{aligned}$$

as  $T \rightarrow \infty$  by a CLT. As  $n \rightarrow \infty$  by a LLN and the assumption that  $\mu_i$  and  $x_{it}$  are uncorrected as in (??), we have  $I = o_p(1)$ . It is easy to show that

$$II = \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \mu_i \right) = o_p(1) O_p(1),$$

$$III = \frac{1}{\sqrt{T}} \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right) \right] = \frac{1}{\sqrt{T}} o_p(1)$$

and

$$IV = \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) = \frac{1}{\sqrt{T}} o_p(1) o_p(1)$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$  because  $\frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \Rightarrow \varpi_\varepsilon \int W_i$ ,  $E[\varpi_\varepsilon \int W_i] = 0$ ,  $E[\mu_i] = 0$ ,  $\frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \Rightarrow \varpi_\varepsilon \varpi_e \int W_i dV_i$ ,  $E[\varpi_\varepsilon \varpi_e \int W_i dV_i] = 0$ , and  $\frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \Rightarrow N\left(0, \frac{\varpi_e^2}{(1-\rho)^2}\right)$ . Hence, we have

$$\frac{1}{nT^{3/2}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \xrightarrow{P} 0$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

For part (b), note that

$$\begin{aligned} & \frac{1}{n^{1/2} T^{3/2}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \\ &= \frac{1}{n^{1/2} T^{3/2}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) (\mu_i + \nu_{it}) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \mu_i \right] - \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \mu_i \right) \\ & \quad + \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right] - \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) \\ &= I - II + III - IV. \end{aligned}$$

For a fixed  $n$ ,

$$\begin{aligned} I &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \mu_i \right] \\ &\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \varpi_\varepsilon \int W_i \right) \mu_i \right] \end{aligned}$$

as  $T \rightarrow \infty$ , and

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \varpi_\varepsilon \int W_i \right) \mu_i \right] \Rightarrow N\left(0, \frac{\sigma_\mu^2 \varpi_\varepsilon^2}{3}\right)$$

as  $n \rightarrow \infty$  by a CLT with  $E[\varpi_\varepsilon \int W_i] = 0$  and  $Var[\varpi_\varepsilon \int W_i] = \frac{1}{3}\varpi_\varepsilon^2$ . It is easy to show that

$$II = \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \mu_i \right) = o_p(1) O_p(1),$$

$$III = \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right] = \frac{1}{\sqrt{T}} O_p(1),$$

and

$$IV = \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) = \frac{1}{\sqrt{T}} o_p(1) O_p(1)$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . Hence, we have

$$\frac{1}{n^{1/2} T^{3/2}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \Rightarrow N\left(0, \frac{\sigma_\mu^2 \varpi_\varepsilon^2}{3}\right)$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

Consider (4). For part (a), it is easy to see that

$$\begin{aligned} & \frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \\ &= \frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) (\mu_i + \nu_{it}) \\ &= \frac{1}{\sqrt{T}} \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \mu_i \right] - \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \mu_i \right) \\ & \quad + \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T^2} \sum_{t=1}^T x_{it} \nu_{it} \right] - \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T \nu_{it} \right) \\ &= I - II + III - IV. \end{aligned}$$

For a fixed  $n$ ,

$$\begin{aligned} III &= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T^2} \sum_{t=1}^T x_{it} \nu_{it} \right] \\ &\Rightarrow \frac{1}{n} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \int W_i V_i + \varpi_{\varepsilon e} \left( \int W_i^2 \right) + \delta_{\varepsilon e} \right] \end{aligned}$$

as  $T \rightarrow \infty$  by a CLT. We then have

$$\frac{1}{n} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \int W_i V_i + \varpi_{\varepsilon e} \left( \int W_i^2 \right) + \delta_{\varepsilon e} \right] \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}.$$

as  $n \rightarrow \infty$ . So

$$III \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . Again

$$I = \frac{1}{\sqrt{T}} \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \mu_i \right] = \frac{1}{\sqrt{T}} o_p(1)$$

$$II = \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \mu_i \right) = \frac{1}{\sqrt{T}} o_p(1) o_p(1)$$

and

$$IV = \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T \nu_{it} \right) = o_p(1) o_p(1)$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . Hence, we have

$$\frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

For part (b),

$$\begin{aligned} & \frac{1}{\sqrt{n}T^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} \\ &= \frac{1}{\sqrt{n}T^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) (\mu_i + \nu_{it}) \\ &= \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \mu_i \right] - \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \mu_i \right) \\ & \quad + \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T x_{it} \nu_{it} - \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T \nu_{it} \right) \\ &= I - II + III - IV. \end{aligned}$$

For a fixed  $n$ ,

$$\begin{aligned} III &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T x_{it} \nu_{it} \\ &\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \int W_i V_i + \varpi_{\varepsilon e} \left( \int W_i^2 \right) + \delta_{\varepsilon e} \right] \end{aligned}$$

as  $T \rightarrow \infty$  by a CLT. As  $(n, T) \xrightarrow{\text{seq}} \infty$ , we have

$$\begin{aligned} &\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T x_{it} \nu_{it} - \sqrt{n} \left[ \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T x_{i(t-1)} \varepsilon_{it} \right) \frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2} + \delta_{\varepsilon e} \right] \\ &\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \int W_i V_i + \varpi_{\varepsilon e} \left( \int W_i^2 \right) + \delta_{\varepsilon e} \right] - \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \varpi_{\varepsilon e} \left( \int W_i^2 \right) + \delta_{\varepsilon e} \right] \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \int W_i V_i \right] \\ &\Rightarrow N \left( 0, \frac{\varpi_{e,\varepsilon} \varpi_\varepsilon^2}{6} \right). \end{aligned}$$

Also it is easy to see that

$$\begin{aligned} I &= \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \mu_i \right] = \frac{1}{\sqrt{T}} O_p(1), \\ II &= \frac{1}{\sqrt{T}} \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \mu_i \right) = \frac{1}{\sqrt{T}} o_p(1) O_p(1) \end{aligned}$$

and

$$IV = \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^{3/2}} \sum_{t=1}^T \nu_{it} \right) = o_p(1) O_p(1)$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . Hence, we have

$$\frac{1}{\sqrt{n} T^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}) u_{it} - \sqrt{n} \left[ \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T x_{i(t-1)} \varepsilon_{it} \right) \frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2} + \delta_{\varepsilon e} \right] \Rightarrow N \left( 0, \frac{\varpi_{e,\varepsilon} \varpi_\varepsilon^2}{6} \right),$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . ■

## B Proof of Theorem 1

**Proof.** The proof is straightforward by using lemmas 1 and 2. ■



## C Proof of Theorem 2

The following lemmas will be used to prove Theorem 2.

**Lemma 3** *If Assumptions 1 – 2 hold, then*

1. If  $|\lambda| < 1$ ,  $\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i)^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}$ .
2. If  $\lambda = 1$ ,  $\frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i)^2 \xrightarrow{p} \frac{\varpi_\varepsilon^2}{6}$ .

**Proof.** Consider (1). For a fixed  $n$ , it is clear to see that

$$\begin{aligned} & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i)^2 \\ &= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it}^2 \right] - \frac{1}{n} \sum_{i=1}^n [\bar{x}_i^2] \\ &\xrightarrow{p} \frac{1}{n} \sum_{i=1}^n \left[ \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)} \right] \\ &= \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)} \end{aligned}$$

as  $T \rightarrow \infty$  because  $\bar{x}_i = \frac{1}{T} \sum_{t=1}^T x_{it} \xrightarrow{p} E(x_{it}) = 0$ . Hence,

$$\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i)^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}$$

holds for all  $n$  and hence it holds for a large  $n$  as well. This proves (1).

Next we consider (2). Note for a fixed  $n$

$$\begin{aligned} & \frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i)^2 \\ &= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i)^2 \right] \\ &\Rightarrow \frac{1}{n} \sum_{i=1}^n \left( \varpi_\varepsilon^2 \int \tilde{W}_i^2 \right) \end{aligned}$$

as  $T \rightarrow \infty$ . As  $n \rightarrow \infty$ ,

$$\frac{1}{n} \sum_{i=1}^n \left( \varpi_\varepsilon^2 \int \tilde{W}_i^2 \right) \xrightarrow{p} \frac{\varpi_\varepsilon^2}{6}.$$

by a LLN since

$$E \left( \int \tilde{W}_i^2 \right) = \frac{1}{6}.$$

Hence,

$$\frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i)^2 \xrightarrow{p} \frac{\varpi_\varepsilon^2}{6}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . This proves (2). ■

**Lemma 4** *If Assumptions 1 – 2 hold, then*

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$\begin{aligned} \text{(a)} \quad & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) u_{it} \xrightarrow{p} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \\ \text{(b)} \quad & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) u_{it} - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \\ & \Rightarrow N \left( 0, \frac{\psi_{00} + \sum_{r=1}^{\infty} \lambda^{2r} \psi_{0r} + \sum_{r=1}^{\infty} \rho^{2r} \psi_{r0}}{(1-\rho\lambda)^2} \right) \\ & \text{where } \psi_{0r} = E(\varepsilon_{t-r}^2 e_t^2), \psi_{r0} = E(\varepsilon_t^2 e_{t-r}^2), \text{ and } \psi_{00} = E(\varepsilon_t^2 e_t^2). \end{aligned}$$

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$\begin{aligned} \text{(a)} \quad & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) u_{it} \xrightarrow{p} \frac{-\frac{1}{2} \varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon}}{1-\lambda} \\ \text{(b)} \quad & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} - \sqrt{n} \left[ \frac{\left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T (\nu_{it} - \bar{\nu}_i) e_{it} \right) \frac{\varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon}}{\varpi_\varepsilon^2}}{1-\lambda} \right] \Rightarrow \\ & N \left( 0, \frac{\varpi_{\varepsilon,\varepsilon} \varpi_\varepsilon^2}{6(1-\lambda)^2} \right). \end{aligned}$$

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

$$\begin{aligned} \text{(a)} \quad & \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) u_{it} \Rightarrow \frac{1}{1-\rho} \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int \tilde{W}_i dV_i \right) + \varpi_{\varepsilon\varepsilon} \left( \int \tilde{W}_i dW_i' \right) + \delta_{\varepsilon\varepsilon} \right], \\ \text{(b)} \quad & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) u_{it} \xrightarrow{p} \frac{-\frac{1}{2} \varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon}}{1-\rho}, \\ \text{(c)} \quad & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} - \sqrt{n} \left[ \frac{\left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) \varepsilon_{it} \right) \frac{\varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon}}{\varpi_\varepsilon^2}}{1-\rho} \right] \Rightarrow \\ & N \left( 0, \frac{\varpi_{e,\varepsilon} \varpi_\varepsilon^2}{6(1-\rho)^2} \right). \end{aligned}$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$\begin{aligned} \text{(a)} \quad & \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i) u_{it} \Rightarrow \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int \tilde{W}_i \tilde{V}_i \right) + \varpi_{\varepsilon\varepsilon} \left( \int \tilde{W}_i^2 \right) + \delta_{\varepsilon\varepsilon}, \\ \text{(b)} \quad & \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i) u_{it} \right] \xrightarrow{p} \frac{\varpi_{\varepsilon\varepsilon}}{6} + \delta_{\varepsilon\varepsilon}, \end{aligned}$$

$$(c) \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i) u_{it} \right] - \sqrt{n} \left[ \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i)^2 \right) \frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^2} + \delta_{\varepsilon e} \right] \Rightarrow N \left( 0, \frac{\varpi_{\varepsilon e} \varpi_{\varepsilon}^2}{90} \right).$$

**Proof.**

Consider (1). For part (a), note that

$$\begin{aligned} & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \\ &= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right] - \frac{1}{T} \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) \right] \end{aligned}$$

Because

$$\frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} \nu_{it} \right] = \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) + o_p(1)$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . Also it is easy to see that

$$\frac{1}{T} \frac{1}{n} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) \right] = o_p(1).$$

Hence, we have

$$\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} = \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) + o_p(1).$$

This proves (a).

For part (b), for a fixed  $n$ ,

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \nu_{it} \right) - \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) \right] \end{aligned}$$

Because

$$\begin{aligned} & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \nu_{it} \right) - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \\ & \Rightarrow N \left( 0, \frac{\psi_{00} + \sum_{r=1}^{\infty} \lambda^{2r} \psi_{0r} + \sum_{r=1}^{\infty} \rho^{2r} \psi_{r0}}{(1 - \rho\lambda)^2} \right) \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ , where  $\psi_{0r} = E(\varepsilon_{t-r}^2 e_t^2)$ ,  $\psi_{r0} = E(\varepsilon_t^2 e_{t-r}^2)$ ,  $\psi_{00} = E(\varepsilon_t^2 e_t^2)$ .

Also it is easy to see that

$$\frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) \right] = o_p(1)$$

Hence, we have

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \\ = & \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \nu_{it} \right) - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} \nu_{it}) \right] \\ & - \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T x_{it} \right) \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T \nu_{it} \right) \right] \\ \Rightarrow & N \left( 0, \frac{\psi_{00} + \sum_{r=1}^{\infty} \lambda^{2r} \psi_{0r} + \sum_{r=1}^{\infty} \rho^{2r} \psi_{r0}}{(1 - \rho\lambda)^2} \right). \end{aligned}$$

Consider (2). For part (a), for a fixed  $n$ , note that

$$\begin{aligned} & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \\ = & \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} (\nu_{it} - \bar{\nu}_i) \right] \\ \Rightarrow & \frac{1}{n} \sum_{i=1}^n \left[ \frac{\varpi_e \varpi_{\varepsilon, e}^{1/2} \left( \int \tilde{V}_i dW_i \right) + \varpi_{e\varepsilon} \left( \int \tilde{V}_i dV_i \right) + \delta_{e\varepsilon}}{1 - \lambda} \right] \end{aligned}$$

as  $T \rightarrow \infty$ , by a CLT because

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \left[ \frac{\varpi_e \varpi_{\varepsilon, e}^{1/2} \left( \int \tilde{V}_i dW_i \right) + \varpi_{e\varepsilon} \left( \int \tilde{V}_i dV_i \right) + \delta_{e\varepsilon}}{1 - \lambda} \right] \\ & \xrightarrow{p} \frac{-\frac{1}{2} \varpi_{e\varepsilon} + \delta_{e\varepsilon}}{1 - \lambda}, \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . Hence, we have

$$\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \xrightarrow{p} \frac{-\frac{1}{2} \varpi_{e\varepsilon} + \delta_{e\varepsilon}}{1 - \lambda}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

For part (b), for a fixed  $n$ ,

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it} (\nu_{it} - \bar{\nu}_i) \right] \\
&\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{\varpi_e \varpi_{\varepsilon,e}^{1/2} \left( \int \tilde{V}_i dW_i \right) + \varpi_{e\varepsilon} \left( \int \tilde{V}_i dV_i \right) + \delta_{e\varepsilon}}{1 - \lambda} \right]
\end{aligned}$$

as  $T \rightarrow \infty$ , by a CLT. Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} - \sqrt{n} \left[ \frac{\left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T (\nu_{it} - \bar{\nu}_i) e_{it} \right) \frac{\varpi_{e\varepsilon}}{\varpi_e^2} + \delta_{e\varepsilon}}{1 - \lambda} \right] \\
&\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{\varpi_e \varpi_{\varepsilon,e}^{1/2} \left( \int \tilde{V}_i dW_i \right) + \varpi_{e\varepsilon} \left( \int \tilde{V}_i dV_i \right) + \delta_{e\varepsilon}}{1 - \lambda} \right] - \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{\varpi_{e\varepsilon} \left( \int \tilde{V}_i dV_i \right) + \delta_{e\varepsilon}}{1 - \lambda} \right] \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{\varpi_e \varpi_{\varepsilon,e}^{1/2} \left( \int \tilde{V}_i dW_i \right)}{1 - \lambda} \right] \\
&\Rightarrow N \left( 0, \frac{\varpi_{\varepsilon,e} \varpi_e^2}{6(1 - \lambda)^2} \right).
\end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

Consider (3). For part (a), note that for a fixed  $n$ ,

$$\begin{aligned}
& \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \\
&= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \right] \\
&\Rightarrow \frac{1}{n} \sum_{i=1}^n \left[ \frac{\varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int \tilde{W}_i dV_i \right) + \varpi_{e\varepsilon} \left( \int \tilde{W}_i dW_i \right) + \delta_{e\varepsilon}}{1 - \rho} \right]
\end{aligned}$$

as  $T \rightarrow \infty$ , by a central limit theorem. And

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \left[ \frac{\varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int \tilde{W}_i dV_i \right) + \varpi_{e\varepsilon} \left( \int \tilde{W}_i dW_i \right) + \delta_{e\varepsilon}}{1 - \rho} \right] \\
&\xrightarrow{p} \frac{-\frac{1}{2} \varpi_{e\varepsilon} + \delta_{e\varepsilon}}{1 - \rho},
\end{aligned}$$

as  $n \rightarrow \infty$ . Hence, we have

$$\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \xrightarrow{p} \frac{-\frac{1}{2}\varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon}}{1 - \rho}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

For part (b), for a fixed  $n$ ,

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \right] \\ &\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{\varpi_{\varepsilon} \varpi_{e,\varepsilon}^{1/2} \left( \int \widetilde{W}_i dV_i \right) + \varpi_{\varepsilon\varepsilon} \left( \int \widetilde{W}_i dW_i \right) + \delta_{\varepsilon\varepsilon}}{1 - \rho} \right], \end{aligned}$$

as  $T \rightarrow \infty$ , by a central limit theorem. Hence, we have

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} - \sqrt{n} \left[ \frac{\left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) \varepsilon_{it} \right) \frac{\varpi_{\varepsilon\varepsilon}}{\varpi_{\varepsilon}^2} + \delta_{\varepsilon\varepsilon}}{1 - \rho} \right] \\ &\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{\varpi_{\varepsilon} \varpi_{e,\varepsilon}^{1/2} \left( \int \widetilde{W}_i dV_i \right) + \varpi_{\varepsilon\varepsilon} \left( \int \widetilde{W}_i dW_i \right) + \delta_{\varepsilon\varepsilon}}{1 - \rho} \right] \\ &\quad - \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{\varpi_{\varepsilon\varepsilon} \left( \int \widetilde{W}_i dW_i \right) + \delta_{\varepsilon\varepsilon}}{1 - \rho} \right] \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{\varpi_{\varepsilon} \varpi_{e,\varepsilon}^{1/2} \left( \int \widetilde{W}_i dV_i \right)}{1 - \rho} \right] \\ &\Rightarrow N \left( 0, \frac{\varpi_{e,\varepsilon} \varpi_{\varepsilon}^2}{6(1 - \rho)^2} \right). \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

If  $\rho = 1$  and  $\lambda = 1$ ,

Consider (4). For part (a), note that

$$\begin{aligned}
& \frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \\
&= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \right] \\
&\Rightarrow \frac{1}{n} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int \widetilde{W}_i \tilde{V}_i \right) + \left( \int \widetilde{W}_i^2 \right) \varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon} \right]
\end{aligned}$$

as  $T \rightarrow \infty$ , by a central limit theorem. And

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int \widetilde{W}_i \tilde{V}_i \right) + \left( \int \widetilde{W}_i^2 \right) \varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon} \right] \\
& \xrightarrow{p} \frac{\varpi_{\varepsilon\varepsilon}}{6} + \delta_{\varepsilon\varepsilon}
\end{aligned}$$

as  $n \rightarrow \infty$ . Hence, we have

$$\frac{1}{nT^2} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \xrightarrow{p} \frac{\varpi_{\varepsilon\varepsilon}}{6} + \delta_{\varepsilon\varepsilon}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

For part (b), for a fixed  $n$ ,

$$\begin{aligned}
& \frac{1}{\sqrt{nT^2}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} \right] \\
&\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int \widetilde{W}_i \tilde{V}_i \right) + \left( \int \widetilde{W}_i^2 \right) \varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon} \right],
\end{aligned}$$

as  $T \rightarrow \infty$ , by a central limit theorem. Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT^2}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i) \nu_{it} - \sqrt{n} \left[ \left( \frac{1}{n} \sum_{i=1}^n \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \bar{x}_i)^2 \right) \frac{\varpi_{\varepsilon\varepsilon}}{\varpi_\varepsilon^2} + \delta_{\varepsilon\varepsilon} \right] \\
&\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int \widetilde{W}_i \tilde{V}_i \right) + \left( \int \widetilde{W}_i^2 \right) \varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon} \right] - \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \int \widetilde{W}_i^2 \right) \varpi_{\varepsilon\varepsilon} + \delta_{\varepsilon\varepsilon} \right] \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int \widetilde{W}_i \tilde{V}_i \right) \right] \\
&\Rightarrow N \left( 0, \frac{\varpi_{e,\varepsilon} \varpi_\varepsilon^2}{90} \right).
\end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . ■

## D Proof of Theorem 2

**Proof.** The proof of Theorem 2 is straightforward with above lemmas. ■

## E Proof of Theorem 3

The following lemmas will be used to prove Theorem 3.

**Lemma 5** *If Assumptions 1 – 2 hold, then*

1. If  $|\lambda| < 1$ ,  $\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2 \xrightarrow{p} \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2+2\lambda-1)\gamma_\varepsilon^2}{1-\lambda}$ ,
2. If  $\lambda = 1$ ,  $\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2 \xrightarrow{p} \sigma_\varepsilon^2$ .

**Proof.** Consider (1). If  $|\lambda| < 1$ , for a fixed  $n$ , it is clear to see that

$$\begin{aligned}
& \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2 \\
&= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - x_{it-1})^2 \right] \\
&= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T x_{it}^2 + \frac{1}{T} \sum_{t=1}^T x_{it-1}^2 - \frac{2}{T} \sum_{t=1}^T x_{it}x_{it-1} \right] \\
&\xrightarrow{p} \frac{1}{n} \sum_{i=1}^n \left[ 2 \left( \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{1-\lambda} \right) - 2 \left[ \lambda \left( \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{1-\lambda} \right) + \frac{\gamma_\varepsilon^2}{1-\lambda} \right] \right] \\
&= \frac{1}{n} \sum_{i=1}^n \left[ \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2+2\lambda-1)\gamma_\varepsilon^2}{1-\lambda} \right] \\
&\xrightarrow{p} \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2+2\lambda-1)\gamma_\varepsilon^2}{1-\lambda}
\end{aligned}$$

as  $T \rightarrow \infty$  because  $\frac{1}{T} \sum_{t=1}^T x_{it}^2 \xrightarrow{p} E(x_{it}^2) = \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}$  and  $\frac{1}{T} \sum_{t=1}^T x_{it}x_{it-1} \xrightarrow{p} E(x_{it}x_{it-1}) = \lambda \left( \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{1-\lambda} \right) + \frac{\gamma_\varepsilon^2}{1-\lambda}$ . Hence,

$$\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - \bar{x}_i)^2 \xrightarrow{p} \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2+2\lambda-1)\gamma_\varepsilon^2}{1-\lambda}$$

holds for all  $n$  and hence it holds for a large  $n$  as well. This proves (1).



Next we consider (2). If  $\lambda = 1$ ,

$$\begin{aligned} & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2 \\ &= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - x_{it-1})^2 \right] \\ &= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2 \right] \xrightarrow{p} \sigma_\varepsilon^2 \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . This proves (2). ■

**Lemma 6** *If Assumptions 1 – 2 hold, then*

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$\begin{aligned} \text{(a)} \quad & \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \right] \xrightarrow{p} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \\ \text{(b)} \quad & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \right] - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \\ & \Rightarrow N \left( 0, \frac{(2-\lambda-\rho)^2 \psi_{00} + \sum_{r=1}^{\infty} (-\rho^{r-1} + 2\rho^r - \rho^{r+1})^2 \psi_{0r} + \sum_{r=1}^{\infty} (-\lambda^{r-1} + 2\lambda^r - \lambda^{r+1})^2 \psi_{r0}}{(1-\rho\lambda)^2} \right) \\ & \text{where } \psi_{0r} = E(\varepsilon_{t-r}^2 e_t^2), \psi_{r0} = E(\varepsilon_t^2 e_{t-r}^2), \psi_{00} = E(\varepsilon_t^2 e_t^2). \end{aligned}$$

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$\begin{aligned} \text{(a)} \quad & \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \right] \xrightarrow{p} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E (\varepsilon_{it} + (\lambda - 1) x_{it-1}) e_{it} \\ \text{(b)} \quad & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \right] - \sqrt{n} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E (\varepsilon_{it} + (\lambda - 1) x_{it-1}) e_{it} \Rightarrow \\ & N \left( 0, \frac{2\psi_{00}}{1+\lambda} \right), \text{ where } \psi_{00} = E(\varepsilon_t^2 e_t^2) \end{aligned}$$

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

$$\begin{aligned} \text{(a)} \quad & \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \right] \xrightarrow{p} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \varepsilon_{it} (\nu_{it} - \nu_{it-1}) \\ \text{(b)} \quad & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \right] - \sqrt{n} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \varepsilon_{it} (\nu_{it} - \nu_{it-1}) \Rightarrow \\ & N \left( 0, \frac{2\psi_{00}}{1+\rho} \right), \text{ where } \psi_{00} = E(\varepsilon_t^2 e_t^2). \end{aligned}$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$\text{(a)} \quad \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \right] \xrightarrow{p} \sigma_{\varepsilon e}$$

$$(b) \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \right] - \sqrt{nT} \sigma_{\varepsilon\varepsilon} \Rightarrow N(0, \varpi_{\varepsilon}^2 \varpi_{\varepsilon}^2).$$

**Proof.** Consider (1). For part (a), it is easy to see that

$$\begin{aligned} & \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \\ & \xrightarrow{p} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

For part (b), note that

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^T (\varepsilon_{it} + (\lambda - 1) x_{it-1}) (e_{it} + (\rho - 1) \nu_{it-1}) \right] \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^T (\varepsilon_{it} + (\lambda - 1) \varepsilon_{it-1} + \lambda(\lambda - 1) \varepsilon_{it-2} + \lambda^2(\lambda - 1) \varepsilon_{it-3} + \dots) \right. \\ & \quad \left. (e_{it} + (\rho - 1) e_{it-1} + \rho(\rho - 1) e_{it-2} + \rho^2(\rho - 1) e_{it-3} + \dots) \right] \end{aligned}$$

Hence, we have

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^T (\varepsilon_{it} + (\lambda - 1) \varepsilon_{it-1} + \lambda(\lambda - 1) \varepsilon_{it-2} + \lambda^2(\lambda - 1) \varepsilon_{it-3} + \dots) \right. \\ & \quad \left. (e_{it} + (\rho - 1) e_{it-1} + \rho(\rho - 1) e_{it-2} + \rho^2(\rho - 1) e_{it-3} + \dots) \right] \\ & \quad - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \\ & \Rightarrow N \left( 0, \frac{(2 - \lambda - \rho)^2 \psi_{00} + \sum_{r=1}^{\infty} (-\rho^{r-1} + 2\rho^r - \rho^{r+1})^2 \psi_{0r} + \sum_{r=1}^{\infty} (-\lambda^{r-1} + 2\lambda^r - \lambda^{r+1})^2 \psi_{r0}}{(1 - \rho\lambda)^2} \right) \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ , where  $\psi_{0r} = E(\varepsilon_{t-r}^2 e_t^2)$ ,  $\psi_{r0} = E(\varepsilon_t^2 e_{t-r}^2)$ ,  $\psi_{00} = E(\varepsilon_t^2 e_t^2)$ .

Consider (2). For part (a), it is easy to see that

$$\begin{aligned}
& \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \\
&= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (\varepsilon_{it} + (\lambda - 1) x_{it-1}) e_{it} \right] \\
&\xrightarrow{p} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(\varepsilon_{it} + (\lambda - 1) x_{it-1}) e_{it}
\end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

For part (b), using Lemma A0 in Choi (1999), we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) - \sqrt{n} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(\varepsilon_{it} + (\lambda - 1) x_{it-1}) e_{it} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T (\varepsilon_{it} + (\lambda - 1) x_{it-1}) e_{it} \right] - \sqrt{n} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(\varepsilon_{it} + (\lambda - 1) x_{it-1}) e_{it} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T [(\varepsilon_{it} + (\lambda - 1) \varepsilon_{it-1} + \lambda(\lambda - 1) \varepsilon_{it-2} + \lambda^2(\lambda - 1) \varepsilon_{it-3} + \dots) e_{it}] \right] \\
&\quad - \sqrt{n} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T E(\varepsilon_{it} + (\lambda - 1) x_{it-1}) e_{it} \\
&\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ N \left( 0, \frac{2\psi_{00}}{1 + \lambda} \right) \right] = N \left( 0, \frac{2\psi_{00}}{1 + \lambda} \right)
\end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ , where  $\psi_{00} = E(\varepsilon_t^2 e_t^2)$ .

Consider (3). For part (a), it is easy to see that

$$\begin{aligned}
& \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \\
&= \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} (\nu_{it} - \nu_{it-1}) \right] \\
&\xrightarrow{p} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \varepsilon_{it} (\nu_{it} - \nu_{it-1})
\end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

For part (b),

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) - \sqrt{n} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \varepsilon_{it} (\nu_{it} - \nu_{it-1}) \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} (\nu_{it} - \nu_{it-1}) \right] - \sqrt{n} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \varepsilon_{it} (\nu_{it} - \nu_{it-1}) \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} (e_{it} + (\rho - 1) \nu_{it-1}) \right] - \sqrt{n} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \varepsilon_{it} (\nu_{it} - \nu_{it-1}) \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T [\varepsilon_{it} (e_{it} + (\rho - 1) e_{it-1} + \rho(\rho - 1) e_{it-2} + \rho^2(\rho - 1) e_{it-3} + \dots)] \right] \\
& - \sqrt{n} \lim \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T \varepsilon_{it} (\nu_{it} - \nu_{it-1}) \\
\Rightarrow & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ N \left( 0, \frac{2\psi_{00}}{1 + \rho} \right) \right] = N \left( 0, \frac{2\psi_{00}}{1 + \rho} \right)
\end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ , where  $\psi_{00} = E(\varepsilon_t^2 e_t^2)$ .

Consider (4). For part (a),

$$\begin{aligned}
& \frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \\
= & \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} e_{it} \right] \\
& \xrightarrow{p} \sigma_{\varepsilon e}
\end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

For part (b), note that

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1}) (\nu_{it} - \nu_{it-1}) \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} e_{it} \right]
\end{aligned}$$

For a fixed  $n$ ,

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})(\nu_{it} - \nu_{it-1}) - \sqrt{nT}\sigma_{\varepsilon\varepsilon} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} e_{it} \right] - \sqrt{nT}\sigma_{\varepsilon\varepsilon} \\
&\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i
\end{aligned}$$

as  $T \rightarrow \infty$ , by a central limit theorem, where  $Z_i \sim N(0, \varpi_e^2 \varpi_\varepsilon^2)$ .

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i \Rightarrow N(0, \varpi_e^2 \varpi_\varepsilon^2)$$

as  $n \rightarrow \infty$ . Hence, we have

$$\frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})(\nu_{it} - \nu_{it-1}) - \sqrt{nT}\sigma_{\varepsilon\varepsilon} \Rightarrow N(0, \varpi_e^2 \varpi_\varepsilon^2)$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . ■

**Proof of Theorem 3:**

**Proof.** By  $\hat{\beta}_{FD} - \beta = \frac{\sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})(\nu_{it} - \nu_{it-1})}{\sum_{i=1}^n \sum_{t=1}^T (x_{it} - x_{it-1})^2}$ , the proof of Theorem 3 is straightforward with above lemmas. ■

## F Proof of Theorem 4

Define  $\mathbf{z} = [\boldsymbol{\nu}_{nT}, \mathbf{x}]$ , then

$$\begin{aligned}
\begin{pmatrix} \hat{\alpha}_{GLS} \\ \hat{\beta}_{GLS} \end{pmatrix} &= (\mathbf{z}'\Phi^{-1}\mathbf{z})^{-1} (\mathbf{z}'\Phi^{-1}\mathbf{y}) \\
&= \left( \begin{bmatrix} \boldsymbol{\nu}'_{nT} \\ \mathbf{x}' \end{bmatrix} \Phi^{-1} \begin{bmatrix} \boldsymbol{\nu}_{nT} & \mathbf{x} \end{bmatrix} \right)^{-1} \left( \begin{bmatrix} \boldsymbol{\nu}'_{nT} \\ \mathbf{x}' \end{bmatrix} \Phi^{-1} \mathbf{y} \right) \\
&= \begin{bmatrix} \boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT} & \boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{x} \\ \mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT} & \mathbf{x}' \Phi^{-1} \mathbf{x} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{y} \\ \mathbf{x}' \Phi^{-1} \mathbf{y} \end{bmatrix} \\
&= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{y} \\ \mathbf{x}' \Phi^{-1} \mathbf{y} \end{bmatrix} \\
&= \begin{bmatrix} F_{11} \boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{y} + F_{12} \mathbf{x}' \Phi^{-1} \mathbf{y} \\ F_{21} \boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{y} + F_{22} \mathbf{x}' \Phi^{-1} \mathbf{y} \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
F_{11} &= \left[ \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} - \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{x} (\mathbf{x}' \Phi^{-1} \mathbf{x})^{-1} \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \right]^{-1}, \\
F_{12} &= - \left( \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} \right)^{-1} \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{x} \left[ \mathbf{x}' \Phi^{-1} \mathbf{x} - \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \left( \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} \right)^{-1} \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{x} \right]^{-1}, \\
F_{21} &= - \left[ \mathbf{x}' \Phi^{-1} \mathbf{x} - \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \left( \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} \right)^{-1} \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{x} \right]^{-1} \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \left( \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} \right)^{-1}, \\
F_{22} &= \left[ \mathbf{x}' \Phi^{-1} \mathbf{x} - \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \left( \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} \right)^{-1} \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{x} \right]^{-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\widehat{\beta}_{GLS} &= F_{21} \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{y} + F_{22} \mathbf{x}' \Phi^{-1} \mathbf{y} \\
&= \left[ \mathbf{x}' \Phi^{-1} \mathbf{x} - \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \left( \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} \right)^{-1} \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{x} \right]^{-1} \left[ \mathbf{x}' \Phi^{-1} \mathbf{y} - \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \left( \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} \right)^{-1} \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{y} \right]
\end{aligned}$$

and

$$\widehat{\beta}_{GLS} - \beta = G_1^{-1} G_2,$$

where  $G_1 = \mathbf{x}' \Phi^{-1} \mathbf{x} - \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \left( \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} \right)^{-1} \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{x}$  and  $G_2 = \mathbf{x}' \Phi^{-1} \mathbf{u} - \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \left( \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} \right)^{-1} \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{u}$ .

With the definition of  $\Phi$ , we have

$$\begin{aligned}
\mathbf{x}' \Phi^{-1} \mathbf{x} &= \frac{1}{\varpi_e^2} \sum_{i=1}^n \left( \mathbf{x}'_i \mathbf{A}_i^{-1} \mathbf{x}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{x}'_i \mathbf{A}_i^{-1} \boldsymbol{\iota}_T \boldsymbol{\iota}'_T \mathbf{A}_i^{-1} \mathbf{x}_i \right) \\
\mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} &= \frac{1}{\varpi_e^2} \sum_{i=1}^n \left( \mathbf{x}'_i \mathbf{A}_i^{-1} \boldsymbol{\iota}_T - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{x}'_i \mathbf{A}_i^{-1} \boldsymbol{\iota}_T \boldsymbol{\iota}'_T \mathbf{A}_i^{-1} \mathbf{x}_i \right) \\
&= \frac{1}{\varpi_e^2} \sum_{i=1}^n \left( \mathbf{x}'_i \mathbf{A}_i^{-1} \boldsymbol{\iota}_T - \frac{\theta \sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{x}'_i \mathbf{A}_i^{-1} \boldsymbol{\iota}_T \right) \\
&= \frac{1}{\varpi_e^2 + \theta \sigma_\mu^2} \sum_{i=1}^n \left( \mathbf{x}'_i \mathbf{A}_i^{-1} \boldsymbol{\iota}_T \right),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT} &= \frac{1}{\varpi_e^2} \sum_{i=1}^n \left( \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_T - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_T \right) \\
&= \frac{1}{\varpi_e^2} \sum_{i=1}^n \left( \theta - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \theta^2 \right) \\
&= \frac{1}{\varpi_e^2} n \theta \left( 1 - \frac{\theta \sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \right) \\
&= \frac{n \theta}{\varpi_e^2 + \theta \sigma_\mu^2},
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}' \Phi^{-1} \mathbf{u} &= \frac{1}{\varpi_e^2} \sum_{i=1}^n \left( \mathbf{x}'_i \mathbf{A}^{-1} \mathbf{u}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \mathbf{u}_i \right) \\
&= \frac{1}{\varpi_e^2} \sum_{i=1}^n \left[ \left( \mu_i \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T - \frac{\mu_i \sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_T \right) + \left( \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \right) \right] \\
&= \frac{1}{\varpi_e^2} \sum_{i=1}^n \left[ \left( \mu_i \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T - \frac{\mu_i \theta \sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \right) + \left( \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \right) \right] \\
&= \frac{1}{\varpi_e^2} \sum_{i=1}^n \left[ \frac{\mu_i \sigma_e^2}{\sigma_e^2 + \theta \sigma_\mu^2} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T + \left( \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \right) \right],
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u} &= \frac{1}{\sigma_e^2} \sum_{i=1}^n \left( \boldsymbol{\nu}'_T \mathbf{A}^{-1} \mathbf{u}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \mathbf{u}_i \right) \\
&= \frac{1}{\sigma_e^2} \sum_{i=1}^n \left[ \left( \mu_i \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_T - \frac{\mu_i \sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_T \right) + \left( \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \right) \right] \\
&= \frac{1}{\sigma_e^2} \sum_{i=1}^n \left[ \left( \mu_i \theta - \frac{\mu_i \theta^2 \sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \right) + \left( \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i - \frac{\theta \sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \right) \right] \\
&= \frac{1}{\sigma_e^2} \sum_{i=1}^n \left[ \frac{\mu_i \theta \varpi_e^2}{\varpi_e^2 + \theta \sigma_\mu^2} + \frac{\theta \varpi_e^2}{\varpi_e^2 + \theta \sigma_\mu^2} \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \right] \\
&= \sum_{i=1}^n \left[ \frac{\theta}{\varpi_e^2 + \theta \sigma_\mu^2} (\mu_i + \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i) \right]
\end{aligned}$$

because  $\mathbf{u}_i = \mu_i \boldsymbol{\nu}_T + \boldsymbol{\nu}_i$ .

$$\text{When } |\rho| < 1, \mathbf{A} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1} \\ \rho & 1 & \rho & \cdots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1 \end{bmatrix}, \mathbf{A}^{-1} = \begin{bmatrix} 1 & -\rho & 0 & 0 & \cdots & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & \cdots & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 \\ 0 & 0 & 0 & \cdots & 0 & -\rho \end{bmatrix}$$

It can be shown  $\mathbf{A}^{-1} = \mathbf{C}\mathbf{C}'$ , where  $\mathbf{C} = \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -\rho & 1 & 0 \\ 0 & 0 & 0 & 0 & -\rho & 1 \end{bmatrix}$

is the Prais-Winsten (PW) transformation matrix suggested in Baltagi and Li (1991). Hence

$$\mathbf{C}\mathbf{x}_i = \begin{bmatrix} \sqrt{1-\rho^2}x'_{i1} \\ x'_{i2} - \rho x'_{i1} \\ x'_{i3} - \rho x'_{i2} \\ \vdots \\ x'_{iT-1} - \rho x'_{iT-2} \\ x'_{iT} - \rho x'_{iT-1} \end{bmatrix}, \quad \mathbf{C}\boldsymbol{\nu}_i = \begin{bmatrix} \sqrt{1-\rho^2}\nu'_{i1} \\ \nu'_{i2} - \rho\nu'_{i1} \\ \nu'_{i3} - \rho\nu'_{i2} \\ \vdots \\ \nu'_{iT-1} - \rho\nu'_{iT-2} \\ \nu'_{iT} - \rho\nu'_{iT-1} \end{bmatrix} = \begin{bmatrix} \sqrt{1-\rho^2}e'_{i1} \\ e'_{i2} \\ e'_{i3} \\ \vdots \\ e'_{iT-1} \\ e'_{iT} \end{bmatrix},$$

$$\mathbf{C}\boldsymbol{\iota}_T = \begin{bmatrix} \sqrt{1-\rho^2} \\ 1-\rho \\ 1-\rho \\ \vdots \\ 1-\rho \\ 1-\rho \end{bmatrix}. \quad \text{And}$$

$$\mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i = \mathbf{x}'_i \mathbf{C}' \mathbf{C} \mathbf{x}_i \approx \sum_{t=1}^T (x_{it} - \rho x_{it-1})^2$$

$$\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i = \mathbf{x}'_i \mathbf{C}' \mathbf{C} \boldsymbol{\nu}_i \approx \sum_{t=1}^T (x_{it} - \rho x_{it-1}) e_{it}$$

$$\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T = \mathbf{x}'_i \mathbf{C}' \mathbf{C} \boldsymbol{\iota}_T \approx (1-\rho) \sum_{t=1}^T (x_{it} - \rho x_{it-1})$$

$$\boldsymbol{\iota}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i = \boldsymbol{\iota}'_T \mathbf{C}' \mathbf{C} \boldsymbol{\nu}_i \approx (1-\rho) \sum_{t=1}^T e_{it}$$

$$\theta = \boldsymbol{\iota}'_T \mathbf{A}^{-1} \boldsymbol{\iota}_T = \boldsymbol{\iota}'_T \mathbf{C}' \mathbf{C} \boldsymbol{\iota}_T \approx \sum_{t=1}^T (1-\rho)^2 = O\left((1-\rho)^2 T\right).$$

When  $\rho = 1$ ,  $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & T \end{bmatrix}$ ,  $\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$ .



It can be shown  $\mathbf{A}^{-1} = \mathbf{C}\mathbf{C}'$ , where  $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$ .

Hence  $\mathbf{C}\mathbf{x}_i = \begin{bmatrix} x'_{i1} \\ x'_{i2} - x'_{i1} \\ x'_{i3} - x'_{i2} \\ \vdots \\ x'_{iT-1} - x'_{iT-2} \\ x'_{iT} - x'_{iT-1} \end{bmatrix}$ ,  $\mathbf{C}\boldsymbol{\nu}_i = \begin{bmatrix} \nu'_{i1} \\ \nu'_{i2} - \nu'_{i1} \\ \nu'_{i3} - \nu'_{i2} \\ \vdots \\ \nu'_{iT-1} - \nu'_{iT-2} \\ \nu'_{iT} - \nu'_{iT-1} \end{bmatrix} = \begin{bmatrix} e'_{i1} \\ e'_{i2} \\ e'_{i3} \\ \vdots \\ e'_{iT-1} \\ e'_{iT} \end{bmatrix}$ ,  $\mathbf{C}\boldsymbol{\nu}_T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$ .

And

$$\mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i = \mathbf{x}'_i \mathbf{C}' \mathbf{C} \mathbf{x}_i \approx \sum_{t=1}^T (x_{it} - x_{it-1})^2$$

$$\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i = \mathbf{x}'_i \mathbf{C}' \mathbf{C} \boldsymbol{\nu}_i \approx \sum_{t=1}^T (x_{it} - x_{it-1}) e_{it}$$

$$\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T = \mathbf{x}'_i \mathbf{C}' \mathbf{C} \boldsymbol{\nu}_T = x'_{i1}$$

$$\boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i = \boldsymbol{\nu}'_T \mathbf{C}' \mathbf{C} \boldsymbol{\nu}_i = \nu'_{i1}$$

$$\theta = \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_T = \boldsymbol{\nu}'_T \mathbf{C}' \mathbf{C} \boldsymbol{\nu}_T = 1.$$

The following lemmas will be used to prove Theorem 4.

**Lemma 7** *If Assumptions 1 – 2 hold, then*

1. If  $|\rho| < 1$  and  $|\lambda| < 1$ ,

$$(a) \frac{1}{nT} G_1 \xrightarrow{p} \frac{1}{\varpi_e^2} \left[ \frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho)\gamma_\varepsilon^2}{1-\lambda} \right],$$

$$(b) \frac{1}{nT} G_2 \xrightarrow{p} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}],$$

$$(c) \frac{1}{\sqrt{nT}} G_2 - \sqrt{nT} \frac{1}{\varpi_\varepsilon^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E [(x_{it} - \rho x_{it-1}) e_{it}] \Rightarrow \frac{1}{\varpi_\varepsilon^2} N \left( 0, \frac{(1-2\rho\lambda+\rho^2)\psi_{00}}{1-\lambda^2} \right).$$

2. If  $\rho = 1$  and  $|\lambda| < 1$ ,

$$(a) \frac{1}{nT} G_1 \xrightarrow{p} \frac{1}{\varpi_\varepsilon^2} \left[ \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2+2\lambda-1)\gamma_\varepsilon^2}{1-\lambda} \right],$$

$$(b) \frac{1}{nT} G_2 \xrightarrow{p} \frac{1}{\varpi_\varepsilon^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E [(x_{it} - x_{it-1}) e_{it}],$$

$$(c) \frac{1}{\sqrt{nT}} G_2 - \sqrt{nT} \frac{1}{\varpi_\varepsilon^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E [(x_{it} - x_{it-1}) e_{it}] \Rightarrow \frac{1}{\varpi_\varepsilon^2} N \left( 0, \frac{2\psi_{00}}{1+\lambda} \right).$$

3. If  $|\rho| < 1$  and  $\lambda = 1$ ,

$$(a) \frac{1}{nT^2} G_1 \xrightarrow{p} \frac{(1-\rho)^2 \varpi_\varepsilon^2}{6\varpi_\varepsilon^2},$$

$$(b) \frac{1}{nT} G_2 \xrightarrow{p} \frac{1}{\varpi_\varepsilon^2} \left[ (1-\rho) \left[ -\frac{1}{2} \varpi_{\varepsilon\varepsilon} + \gamma_{\varepsilon\varepsilon} \right] + \sigma_{\varepsilon\varepsilon} \right],$$

$$(c) \frac{1}{\sqrt{nT}} G_2 - \frac{1}{\varpi_\varepsilon^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n (1-\rho) \left[ \left( \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) \varepsilon_{it} \right) \frac{\varpi_{\varepsilon\varepsilon}}{\varpi_\varepsilon^2} + \gamma_{\varepsilon\varepsilon} + \frac{\sigma_{\varepsilon\varepsilon}}{1-\rho} \right] \Rightarrow N \left( 0, \frac{(1-\rho)^2 \varpi_\varepsilon^2 \varpi_{\varepsilon\varepsilon}}{6\varpi_\varepsilon^4} \right),$$

4. If  $\rho = 1$  and  $\lambda = 1$ ,

$$(a) \frac{1}{nT} G_1 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{\varpi_\varepsilon^2},$$

$$(b) \frac{1}{nT} G_2 \xrightarrow{p} \frac{\sigma_{\varepsilon\varepsilon}}{\varpi_\varepsilon^2},$$

$$(c) \frac{1}{\sqrt{nT}} G_2 - \sqrt{nT} \frac{\sigma_{\varepsilon\varepsilon}}{\varpi_\varepsilon^2} \Rightarrow \frac{1}{\varpi_\varepsilon^2} N \left( 0, \varpi_\varepsilon^2 \varpi_\varepsilon^2 \right).$$

**Proof.**

Note that

$$\frac{1}{nT} G_1 = \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} - \frac{1}{T} \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \left( \frac{\boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{x}}{n}$$

First consider

$$\begin{aligned} & \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} \\ &= \frac{1}{\varpi_\varepsilon^2} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i - \frac{T\sigma_\mu^2}{\varpi_\varepsilon^2 + \theta\sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T}{T} \frac{\boldsymbol{\iota}'_T \mathbf{A}^{-1} \mathbf{x}_i}{T} \right) \end{aligned}$$

For a fixed  $n$ ,

$$\begin{aligned}
& \frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i \\
&= \frac{1}{T} \sum_{t=1}^T (x_{it} - \rho x_{it-1})^2 \\
&= \frac{1}{T} \sum_{t=1}^T x_{it}^2 + \rho^2 \frac{1}{T} \sum_{t=1}^T x_{it-1}^2 - \rho \frac{1}{T} \sum_{t=1}^T 2x_{it-1}x_{it} \\
&\xrightarrow{p} (1 + \rho^2) \left( \frac{\sigma_\varepsilon^2}{1 - \lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{1 - \lambda} \right) - 2\rho \left[ \lambda \left( \frac{\sigma_\varepsilon^2}{1 - \lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{1 - \lambda} \right) + \frac{\gamma_\varepsilon^2}{1 - \lambda} \right] \\
&= \frac{(1 - 2\rho\lambda + \rho^2) \sigma_\varepsilon^2}{1 - \lambda^2} + \frac{2(\lambda - 2\rho\lambda^2 + \rho^2\lambda - \rho) \gamma_\varepsilon^2}{1 - \lambda}
\end{aligned}$$

by a CLT and also

$$\frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T = (1 - \rho) \frac{1}{T} \sum_{t=1}^T (x_{it} - \rho x_{it-1}) \xrightarrow{p} 0$$

as  $T \rightarrow \infty$ . So, with  $\theta = O\left((1 - \rho)^2 T\right)$  when  $|\rho| < 1$ ,

$$\begin{aligned}
& \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} \\
&= \frac{1}{\varpi_e^2 n} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i - \frac{T\sigma_\mu^2}{\varpi_e^2 + \theta\sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{T} \frac{\boldsymbol{\nu}'_T \mathbf{A}^{-1} \mathbf{x}_i}{T} \right) \\
&\xrightarrow{p} \frac{1}{\varpi_e^2 n} \frac{1}{n} \sum_{i=1}^n \left[ \frac{(1 - 2\rho\lambda + \rho^2) \sigma_\varepsilon^2}{1 - \lambda^2} + \frac{2(\lambda - 2\rho\lambda^2 + \rho^2\lambda - \rho) \gamma_\varepsilon^2}{1 - \lambda} \right] \\
&= \frac{1}{\varpi_e^2} \left[ \frac{(1 - 2\rho\lambda + \rho^2) \sigma_\varepsilon^2}{1 - \lambda^2} + \frac{2(\lambda - 2\rho\lambda^2 + \rho^2\lambda - \rho) \gamma_\varepsilon^2}{1 - \lambda} \right]
\end{aligned}$$

holds for all  $n$  and hence it holds for a large  $n$  as well.

Also because

$$\frac{1}{n} \mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT} = \frac{1}{n} \sum_{i=1}^n \frac{1}{\sigma_e^2 + \theta\sigma_\mu^2} (\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T) \xrightarrow{p} 0$$

because  $\theta = O(T)$  and  $\frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \xrightarrow{p} 0$ . And

$$\frac{1}{n} \boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT} = \frac{1}{n} \frac{n\theta}{\varpi_e^2 + \theta\sigma_\mu^2} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned} \frac{1}{nT}G_1 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \left( \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x}}{n} \\ &\xrightarrow{p} \frac{1}{\varpi_e^2} \left[ \frac{(1-2\rho\lambda+\rho^2)\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho)\gamma_\varepsilon^2}{1-\lambda} \right] \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

Note that

$$\frac{1}{nT}G_2 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \left( \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{u}}{nT}$$

First consider

$$\begin{aligned} &\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\ &= \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \left( \frac{\varpi_e^2 \mu_i T}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T}{T} \right) + \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i}{T} - \frac{\sigma_\mu^2 T}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T}{T} \frac{\boldsymbol{\iota}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{T} \right) \right]. \end{aligned}$$

Because  $\frac{1}{T}\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T \xrightarrow{p} 0$ , which has been proved in part (a), and

$$\frac{1}{T}\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i = \frac{1}{T} \sum_{t=1}^T (x_{it} - \rho x_{it-1}) e_{it} \xrightarrow{p} \lim \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}],$$

and

$$\frac{1}{T}\boldsymbol{\iota}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i = (1-\rho) \frac{1}{T} \sum_{t=1}^T e_{it} \xrightarrow{p} 0$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . Hence,

$$\begin{aligned} &\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\ &= \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{T} \left( \frac{\varpi_e^2 \mu_i T}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T}{T} \right) + \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i}{T} - \frac{\sigma_\mu^2 T}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T}{T} \frac{\boldsymbol{\iota}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{T} \right) \right] \\ &\xrightarrow{p} \frac{1}{T} o(1) + \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \\ &= \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

Also because

$$\frac{1}{nT} \boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{u} = \frac{1}{n} \sum_{i=1}^n \frac{\theta}{\sigma_e^2 + \theta \sigma_\mu^2} \left( \frac{\mu_i}{T} + \frac{\boldsymbol{\iota}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{T} \right) \xrightarrow{p} 0$$

and  $\frac{1}{n} \mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT} \xrightarrow{p} 0$  and  $\frac{1}{n} \boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$ , which are proved in part (a).

Hence, we have

$$\begin{aligned} \frac{1}{nT} G_2 &= \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \left( \frac{\boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{u}}{nT} \\ &\xrightarrow{p} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

Note that

$$\frac{1}{\sqrt{nT}} G_2 = \frac{1}{\sqrt{nT}} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \left( \frac{\boldsymbol{\iota}'_{nT} \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}'_{nT} \Phi^{-1} \mathbf{u}}{\sqrt{nT}}$$

First consider

$$\begin{aligned} &\sqrt{nT} \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} \\ &= \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \left( \frac{\varpi_e^2 \mu_i T}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T}{T} \right) + \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} - \frac{\sigma_\mu^2 T}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T}{T} \frac{\boldsymbol{\iota}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \right) \right] \end{aligned}$$

For a fixed  $n$ ,  $\frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T \xrightarrow{p} 0$ , which has been proved in part (a), and

$$\begin{aligned} &\sqrt{T} \left[ \frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i - \lim \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \right] \\ &= \sqrt{T} \left[ \frac{1}{T} \sum_{t=1}^T (x_{it} - \rho x_{it-1}) e_{it} - \lim \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \right] \\ &= \sqrt{T} \left[ \frac{1}{T} \sum_{t=1}^T (\varepsilon_{it} + (\lambda - \rho) x_{it-1}) e_{it} - \lim \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \right] \\ &= \sqrt{T} \left\{ \frac{1}{T} \sum_{t=1}^T [(\varepsilon_{it} + (\lambda - \rho) \varepsilon_{it-1} + \lambda(\lambda - \rho) \varepsilon_{it-2} + \lambda^2(\lambda - \rho) \varepsilon_{it-3} + \dots) e_{it}] \right. \\ &\quad \left. - \lim \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \right\} \\ &= N \left( 0, \frac{(1 - 2\rho\lambda + \rho^2) \psi_{00}}{1 - \lambda^2} \right) \end{aligned}$$

where  $\psi_{00} = E(\varepsilon_t^2 e_t^2)$  and

$$\frac{1}{\sqrt{T}} \iota_T' \mathbf{A}^{-1} \boldsymbol{\nu}_i = (1 - \rho) \frac{1}{\sqrt{T}} \sum_{t=1}^T e_{it} \Rightarrow (1 - \rho) \varpi_e N(0, 1)$$

as  $T \rightarrow \infty$ . Hence,

$$\begin{aligned} & \sqrt{nT} \left[ \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \right] \\ &= \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \left( \frac{\varpi_e^2 \mu_i T}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}_i' \mathbf{A}^{-1} \boldsymbol{\nu}_i}{T} \right) + \left( \frac{\mathbf{x}_i' \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} - \frac{\sigma_\mu^2 T}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}_i' \mathbf{A}^{-1} \boldsymbol{\nu}_i}{T} \frac{\iota_T' \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \right) \right] \\ & \quad - \frac{1}{\varpi_e^2} \sqrt{nT} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \\ & \Rightarrow \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n N \left( 0, \frac{(1 - 2\rho\lambda + \rho^2) \psi_{00}}{1 - \lambda^2} \right) \\ & = \frac{1}{\varpi_e^2} N \left( 0, \frac{(1 - 2\rho\lambda + \rho^2) \psi_{00}}{1 - \lambda^2} \right) \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

Also consider that

$$\begin{aligned} \frac{1}{\sqrt{nT}} \iota_{nT}' \Phi^{-1} \mathbf{u} &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{\sqrt{T}} \frac{\theta}{\sigma_e^2 + \theta \sigma_\mu^2} (\mu_i + \iota_T' \mathbf{A}^{-1} \boldsymbol{\nu}_i) \\ &= \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\theta}{\sigma_e^2 + \theta \sigma_\mu^2} \mu_i + \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\theta}{\sigma_e^2 + \theta \sigma_\mu^2} \frac{\iota_T' \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \end{aligned}$$

It is easy to see that

$$\frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\theta}{\sigma_e^2 + \theta \sigma_\mu^2} \mu_i = \frac{1}{\sqrt{T}} o_p(1)$$

Also, for a fixed  $n$ ,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\theta}{\sigma_e^2 + \theta \sigma_\mu^2} \frac{\iota_T' \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{(1 - \rho) \varpi_e}{\sigma_\mu^2} N(0, 1) \right]$$

as  $T \rightarrow \infty$  because  $\frac{1}{\sqrt{T}} \iota_T' \mathbf{A}^{-1} \boldsymbol{\nu}_i \Rightarrow (1 - \rho) \varpi_e N(0, 1)$ , which has been proved in part (e). Also,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{(1 - \rho) \varpi_e}{\sigma_\mu^2} N(0, 1) \right] \Rightarrow \frac{(1 - \rho) \varpi_e}{\sigma_\mu^2} N(0, 1)$$

as  $n \rightarrow \infty$ . Therefore,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\theta}{\sigma_e^2 + \theta \sigma_\mu^2} \frac{\boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \Rightarrow \frac{(1-\rho) \varpi_e}{\sigma_\mu^2} N(0, 1)$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ . So,

$$\frac{1}{\sqrt{nT}} \boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u} \Rightarrow \frac{(1-\rho) \varpi_e}{\sigma_\mu^2} N(0, 1)$$

Hence, we have

$$\begin{aligned} & \frac{1}{\sqrt{nT}} G_2 - \sqrt{nT} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \\ &= \frac{1}{\sqrt{nT}} \mathbf{x}' \Phi^{-1} \mathbf{u} - \sqrt{nT} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \\ & \quad - \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u}}{\sqrt{nT}} \\ & \Rightarrow \frac{1}{\sigma_e^2} N \left( 0, \frac{(1-2\rho\lambda + \rho^2) \psi_{00}}{1-\lambda^2} \right). \end{aligned}$$

Note that

$$\frac{1}{nT} G_1 = \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} - \frac{1}{T} \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{x}}{n}$$

First consider that

$$\begin{aligned} & \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} \\ &= \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} \frac{\boldsymbol{\nu}'_T \mathbf{A}^{-1} \mathbf{x}_i}{\sqrt{T}} \right) \end{aligned}$$

for a fixed  $n$ ,

$$\begin{aligned} & \frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i \\ &= \frac{1}{T} \sum_{t=1}^T (x_{it} - x_{it-1})^2 \\ &= \frac{1}{T} \sum_{t=1}^T x_{it}^2 + \frac{1}{T} \sum_{t=1}^T x_{it-1}^2 - \frac{1}{T} \sum_{t=1}^T 2x_{it-1} x_{it} \\ & \xrightarrow{p} 2 \left( \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{1-\lambda} \right) - 2 \left[ \lambda \left( \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{1-\lambda} \right) + \frac{\gamma_\varepsilon^2}{1-\lambda} \right] \\ &= \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2 + 2\lambda - 1)\gamma_\varepsilon^2}{1-\lambda} \end{aligned}$$

and

$$\frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T = \frac{1}{\sqrt{T}} x_{i1} \xrightarrow{p} 0,$$

as  $T \rightarrow \infty$ . Hence

$$\begin{aligned} & \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} \\ &= \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} \frac{\boldsymbol{\nu}'_T \mathbf{A}^{-1} \mathbf{x}_i}{\sqrt{T}} \right) \\ & \xrightarrow{p} \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left[ \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2 + 2\lambda - 1)\gamma_\varepsilon^2}{1-\lambda} \right] \\ &= \frac{1}{\varpi_e^2} \left[ \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2 + 2\lambda - 1)\gamma_\varepsilon^2}{1-\lambda} \right] \end{aligned}$$

holds for all  $n$  and hence it holds for a large  $n$  as well.

Also because that

$$\frac{1}{n} \mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT} = \frac{1}{n} \sum_{i=1}^n \frac{1}{\sigma_e^2 + \theta \sigma_\mu^2} (X'_i A^{-1} \boldsymbol{\nu}_T) \xrightarrow{p} 0$$

since  $\theta = 1$ , and

$$\frac{1}{n} \boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT} = \frac{1}{n} \frac{n\theta}{\varpi_e^2 + \theta \sigma_\mu^2} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned} \frac{1}{nT} G_1 &= \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} - \frac{1}{T} \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{x}}{n} \\ & \xrightarrow{p} \frac{1}{\varpi_e^2} \left[ \frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2(-2\lambda^2 + 2\lambda - 1)\gamma_\varepsilon^2}{1-\lambda} \right] \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

Note that

$$\frac{1}{nT} G_2 = \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u}}{nT}$$

First consider that

$$\begin{aligned} & \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} \\ &= \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \left( \frac{\varpi_e^2 \mu_i}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} \right) + \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} \frac{\boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \right) \right] \end{aligned}$$



For a fixed  $n$ , because  $\frac{1}{\sqrt{T}}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_T \xrightarrow{p} 0$  which is proved in part (a), and

$$\frac{1}{T}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{T}\sum_{t=1}^T(x_{it} - x_{it-1})e_{it} \xrightarrow{p} \lim \frac{1}{T}\sum_{t=1}^T E[(x_{it} - x_{it-1})e_{it}],$$

and

$$\frac{1}{\sqrt{T}}\boldsymbol{\nu}'_T\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{\sqrt{T}}\boldsymbol{\nu}'_{i1} \xrightarrow{p} 0,$$

as  $T \rightarrow \infty$ . Hence,

$$\begin{aligned} & \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\ = & \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \left( \frac{\varpi_e^2 \mu_i}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} \right) + \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} \frac{\boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \right) \right] \\ & \xrightarrow{p} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - x_{it-1})e_{it}] \end{aligned}$$

as  $n \rightarrow \infty$ .

Also because that

$$\begin{aligned} & \frac{1}{n}\boldsymbol{\nu}'_{nT}\Phi^{-1}\mathbf{u} \\ = & \frac{1}{n} \sum_{i=1}^n \frac{\theta}{\varpi_e^2 + \theta \sigma_\mu^2} (\mu_i + \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i) \\ = & \frac{1}{n} \sum_{i=1}^n \frac{1}{\varpi_e^2 + \sigma_\mu^2} (\mu_i + \boldsymbol{\nu}'_{i1}) \\ & \xrightarrow{p} 0 \end{aligned}$$

and  $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\nu}_{nT} \xrightarrow{p} 0$  and  $\frac{1}{n}\boldsymbol{\nu}'_{nT}\Phi^{-1}\boldsymbol{\nu}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$ , which are proved in part (a).

Hence, we have

$$\begin{aligned} \frac{1}{nT}G_2 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT}\Phi^{-1}\boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT}\Phi^{-1}\mathbf{u}}{nT} \\ & \xrightarrow{p} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - x_{it-1})e_{it}] \end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

Note that

$$\frac{1}{nT}G_2 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT}\Phi^{-1}\boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT}\Phi^{-1}\mathbf{u}}{nT}$$

First consider that

$$\begin{aligned} & \sqrt{nT} \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} \\ = & \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \left( \frac{\varpi_e^2 \mu_i}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} \right) + \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \right) \right] \end{aligned}$$

For a fixed  $n$ , because  $\frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \xrightarrow{p} 0$  which is proved in part (a), and

$$\begin{aligned} & \sqrt{T} \left( \frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i - \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T E[(x_{it} - x_{it-1}) e_{it}] \right) \\ = & \frac{1}{\sqrt{T}} \sum_{t=1}^T (x_{it} - x_{it-1}) e_{it} - \sqrt{T} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T E[(x_{it} - x_{it-1}) e_{it}] \\ = & \frac{1}{\sqrt{T}} \sum_{t=1}^T (\varepsilon_{it} + (\lambda - 1) x_{it-1}) e_{it} - \sqrt{T} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T E[(x_{it} - x_{it-1}) e_{it}] \\ = & \frac{1}{\sqrt{T}} \sum_{t=1}^T [(\varepsilon_{it} + (\lambda - 1) \varepsilon_{it-1} + \lambda(\lambda - 1) \varepsilon_{it-2} + \lambda^2(\lambda - 1) \varepsilon_{it-3} + \dots) e_{it}] \\ & - \sqrt{T} \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T E[(x_{it} - x_{it-1}) e_{it}] \\ \Rightarrow & N\left(0, \frac{2\psi_{00}}{1 + \lambda}\right) \end{aligned}$$

where  $\psi_{00} = E(\varepsilon_t^2 e_t^2)$ , and

$$\frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i = \frac{1}{\sqrt{T}} x_{i1} \nu'_{i1} \xrightarrow{p} 0,$$

as  $T \rightarrow \infty$ . Hence,

$$\begin{aligned}
& \sqrt{nT} \left( \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - x_{it-1}) e_{it}] \right) \\
&= \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{\varpi_e^2 \mu_i}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} \right) \\
&\quad + \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \right) \\
&\quad - \sqrt{nT} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - x_{it-1}) e_{it}] \\
&\Rightarrow o_p(1) + \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ N \left( 0, \frac{2\psi_{00}}{1 + \lambda} \right) - o_p(1) \right] \\
&\Rightarrow \frac{1}{\varpi_e^2} N \left( 0, \frac{2\psi_{00}}{1 + \lambda} \right)
\end{aligned}$$

as  $(n, T) \xrightarrow{\text{seq}} \infty$ .

Also because that

$$\begin{aligned}
& \frac{1}{\sqrt{n}} \boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\theta}{\varpi_e^2 + \theta \sigma_\mu^2} (\mu_i + \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i) \\
&\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{\varpi_e^2 + \sigma_\mu^2} (\mu_i + \nu'_{i1}) \\
&\Rightarrow \frac{1}{\varpi_e^2 + \sigma_\mu^2} N(0, 1)
\end{aligned}$$

and also because  $\frac{1}{n} \mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT} \xrightarrow{p} 0$  and  $\frac{1}{n} \boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$ , which are proved in part (a). Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} G_2 - \sqrt{nT} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - x_{it-1}) e_{it}] \\
&= \frac{1}{\sqrt{nT}} \mathbf{x}' \Phi^{-1} \mathbf{u} - \sqrt{nT} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^n \frac{1}{T} \sum_{t=1}^T E[(x_{it} - \rho x_{it-1}) e_{it}] \\
&\quad - \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u}}{\sqrt{nT}} \\
&\Rightarrow \frac{1}{\varpi_e^2} N \left( 0, \frac{2\psi_{00}}{1 + \lambda} \right).
\end{aligned}$$

Note that

$$\frac{1}{nT^2}G_1 = \frac{1}{nT^2}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T} \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}} \left( \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x}}{n\sqrt{T}}$$

First consider that

$$\frac{1}{nT^2}\mathbf{x}'\Phi^{-1}\mathbf{x} = \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T^2} \mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i - \frac{T\sigma_\mu^2}{\varpi_e^2 + \theta\sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T}{T\sqrt{T}} \frac{\boldsymbol{\iota}'_T \mathbf{A}^{-1} \mathbf{x}_i}{T\sqrt{T}} \right)$$

For a fixed  $n$ , because  $\theta = O\left((1-\rho)^2 T\right)$  when  $|\rho| < 1$  and

$$\begin{aligned} & \frac{1}{T^2} \mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i \\ &= \frac{1}{T^2} \sum_{t=1}^T (x_{it} - \rho x_{it-1})^2 \\ &= \frac{1}{T^2} \sum_{t=1}^T ((1-\rho)x_{it-1} + \varepsilon_{it})^2 \\ &= (1-\rho)^2 \frac{1}{T^2} \sum_{t=1}^T x_{it-1}^2 + \frac{1}{T} \left[ (1-\rho) \frac{1}{T} \sum_{t=1}^T 2x_{it-1}\varepsilon_{it} + \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2 \right] \\ &\Rightarrow (1-\rho)^2 \varpi_e^2 \int W_i^2 + \frac{1}{T} o_p(1), \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{T\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\iota}_T \\ &= (1-\rho) \frac{1}{T\sqrt{T}} \sum_{t=1}^T (x_{it} - \rho x_{it-1}) \\ &= (1-\rho) \frac{1}{T\sqrt{T}} \sum_{t=1}^T ((1-\rho)x_{it-1} + \varepsilon_{it}) \\ &= (1-\rho)^2 \frac{1}{T\sqrt{T}} \sum_{t=1}^T x_{it-1} + \frac{1}{\sqrt{T}} \left[ (1-\rho) \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} \right] \\ &\Rightarrow (1-\rho)^2 \varpi_e \int W_i + \frac{1}{\sqrt{T}} o_p(1), \end{aligned}$$

as  $T \rightarrow \infty$ . Hence

$$\begin{aligned} \frac{1}{nT^2} \mathbf{x}' \Phi^{-1} \mathbf{x} &= \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T^2} \mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i - \frac{T\sigma_\mu^2}{\varpi_e^2 + \theta\sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{T\sqrt{T}} \frac{\boldsymbol{\nu}'_T \mathbf{A}^{-1} \mathbf{x}_i}{T\sqrt{T}} \right) \\ &\stackrel{p}{\rightarrow} \frac{1}{\varpi_e^2} E \left[ (1-\rho)^2 \varpi_\varepsilon^2 \int W_i^2 - (1-\rho)^2 \varpi_\varepsilon^2 \left( \int W_i \right)^2 \right] \\ &= \frac{(1-\rho)^2 \varpi_\varepsilon^2}{6\varpi_e^2} \end{aligned}$$

as  $n \rightarrow \infty$ .

And because

$$\begin{aligned} \frac{1}{n\sqrt{T}} \mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT} &= \frac{1}{n} \sum_{i=1}^n \frac{T}{\varpi_e^2 + \theta\sigma_\mu^2} \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{T\sqrt{T}} \right) \\ &\Rightarrow \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{\sigma_\mu^2} (1-\rho)^2 \varpi_\varepsilon \int W_i \right] \\ &\stackrel{p}{\rightarrow} 0 \end{aligned}$$

because  $\theta = O(T)$  and  $\frac{1}{T\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \Rightarrow (1-\rho)^2 \varpi_\varepsilon \int W_i + o_p(1)$ , which is proved in (a). And because

$$\frac{1}{n} \boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT} = \frac{1}{n} \frac{n\theta}{\varpi_e^2 + \theta\sigma_\mu^2} \stackrel{p}{\rightarrow} \frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned} \frac{1}{nT^2} G_1 &= \frac{1}{nT^2} \mathbf{x}' \Phi^{-1} \mathbf{x} - \frac{1}{T} \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n\sqrt{T}} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{x}}{n\sqrt{T}} \\ &\stackrel{p}{\rightarrow} \frac{(1-\rho)^2 \varpi_\varepsilon^2}{6\varpi_e^2} \end{aligned}$$

Note that

$$\frac{1}{nT} G_2 = \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n\sqrt{T}} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u}}{n\sqrt{T}}$$

First consider that

$$\begin{aligned} &\frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} \\ &= \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \left( \frac{\varpi_e^2 \mu_i T}{\varpi_e^2 + \theta\sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{T\sqrt{T}} \right) + \left( \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i}{T} - \frac{\sigma_\mu^2 T}{\varpi_e^2 + \theta\sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{T\sqrt{T}} \frac{\boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \right) \right] \end{aligned}$$

For a fixed  $n$ , because  $\frac{1}{T\sqrt{T}}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_T \Rightarrow (1-\rho)^2 \varpi_\varepsilon \int W_i + o_p(1)$ , which is proved in (a) and

$$\begin{aligned}
& \frac{1}{T}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_i \\
&= \frac{1}{T}\sum_{t=1}^T(x_{it}-\rho x_{it-1})e_{it} \\
&= \frac{1}{T}\sum_{t=1}^T((1-\rho)x_{it-1}+\varepsilon_{it})e_{it} \\
&= (1-\rho)\frac{1}{T}\sum_{t=1}^T(x_{it-1}e_{it})+\frac{1}{T}\sum_{t=1}^T(\varepsilon_{it}e_{it}) \\
&\Rightarrow (1-\rho)\left[\varpi_\varepsilon\varpi_{e,\varepsilon}^{1/2}\left(\int W_i dV_i\right)+\varpi_{\varepsilon e}\left(\int W_i dW_i\right)+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e},
\end{aligned}$$

and

$$\frac{1}{\sqrt{T}}\boldsymbol{\nu}'_T\mathbf{A}^{-1}\boldsymbol{\nu}_i=(1-\rho)\frac{1}{\sqrt{T}}\sum_{t=1}^Te_{it}\Rightarrow(1-\rho)\varpi_e V_i(1),$$

as  $T \rightarrow \infty$ . Hence,

$$\begin{aligned}
& \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
&= \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^n\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_T}{T\sqrt{T}}\right)+\left(\frac{\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2 T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_T}{T\sqrt{T}}\frac{\boldsymbol{\nu}'_T\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right] \\
&\Rightarrow \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^n\left\{(1-\rho)\left[\varpi_\varepsilon\varpi_{e,\varepsilon}^{1/2}\left(\int W_i dV_i\right)+\varpi_{\varepsilon e}\left(\int W_i dW_i\right)+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e}\right. \\
&\quad \left.-\frac{1}{(1-\rho)^2}(1-\rho)^3\left[\varpi_\varepsilon\varpi_{e,\varepsilon}^{1/2}\left(\int W_i\right)V_i(1)+\varpi_{\varepsilon e}\left(\int W_i\right)W_i(1)\right]\right\}+o_p(1) \\
&= \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^n\left\{(1-\rho)\left[\varpi_\varepsilon\varpi_{e,\varepsilon}^{1/2}\left(\int \tilde{W}_i dV_i\right)+\varpi_{\varepsilon e}\left(\int \tilde{W}_i dW_i\right)+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e}\right\} \\
&\xrightarrow{p}\frac{1}{\varpi_e^2}\left[(1-\rho)\left[-\frac{1}{2}\varpi_{\varepsilon e}+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e}\right]
\end{aligned}$$

as  $n \rightarrow \infty$ .

And because

$$\begin{aligned}
\frac{1}{n\sqrt{T}}\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{u} &= \frac{1}{n}\sum_{i=1}^n\frac{1}{\sqrt{T}}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}(\mu_i+\boldsymbol{\iota}'_T\mathbf{A}^{-1}\boldsymbol{\nu}_i) \\
&= \frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^n\left[\left(\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\right)\mu_i\right]+\frac{1}{n}\sum_{i=1}^n\left[\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\frac{\boldsymbol{\iota}'_T\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right] \\
&\Rightarrow o_p(1)+\frac{1}{n}\sum_{i=1}^n\left[\frac{(1-\rho)\varpi_e}{\sigma_\mu^2}V_i(1)\right] \\
&\xrightarrow{p} 0
\end{aligned}$$

and also because  $\frac{1}{n\sqrt{T}}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}0$  and  $\frac{1}{n}\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$ , which are proved in part (a). Hence, we have

$$\begin{aligned}
\frac{1}{nT}G_2 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{u}}{n\sqrt{T}} \\
&\xrightarrow{p}\frac{1}{\varpi_e^2}\left[(1-\rho)\left[-\frac{1}{2}\varpi_{\varepsilon\varepsilon}+\gamma_{\varepsilon\varepsilon}\right]+\sigma_{\varepsilon\varepsilon}\right]
\end{aligned}$$

Note that

$$\frac{1}{\sqrt{nT}}G_2-\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^n(1-\rho)\left[\left(\frac{1}{T}\sum_{t=1}^T(x_{it}-\bar{x}_i)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon\varepsilon}}{\varpi_e^2}+\gamma_{\varepsilon\varepsilon}+\frac{\sigma_{\varepsilon\varepsilon}}{1-\rho}\right]$$

First consider that

$$\begin{aligned}
&\frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
&= \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^n\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_iT}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\right)+\left(\frac{\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{1}{T^2}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}'_T\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]
\end{aligned}$$

For a fixed  $n$ , because  $\frac{1}{T\sqrt{T}}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T\Rightarrow(1-\rho)^2\varpi_\varepsilon\int W_i+o_p(1)$ , which is proved

in (a) and

$$\begin{aligned}
& \frac{1}{T^2} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \\
&= \frac{1}{T^2} \left[ (1-\rho) \sum_{t=1}^T (x_{it} - \rho x_{it-1}) \right] \left[ (1-\rho) \sum_{t=1}^T e_{it} \right] \\
&= (1-\rho)^2 \frac{1}{T^2} \left[ \sum_{t=1}^T ((1-\rho) x_{it-1} + \varepsilon_{it}) \right] \left[ \sum_{t=1}^T e_{it} \right] \\
&= (1-\rho)^3 \left[ \frac{1}{T\sqrt{T}} \sum_{t=1}^T x_{it-1} \right] \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^T e_{it} \right] + (1-\rho)^2 \frac{1}{\sqrt{T}} \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} \right] \left[ \frac{1}{T} \sum_{t=1}^T e_{it} \right] \\
&\Rightarrow (1-\rho)^3 \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int W_i \right) V_i(1) + \varpi_{\varepsilon\varepsilon} \left( \int W_i \right) W_i(1) \right] + \frac{1}{\sqrt{T}} o_p(1)
\end{aligned}$$

as  $T \rightarrow \infty$ . Hence,

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n (1-\rho) \left[ \left( \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) \varepsilon_{it} \right) \frac{\varpi_{\varepsilon\varepsilon}}{\varpi_e^2} + \gamma_{\varepsilon\varepsilon} + \frac{\sigma_{\varepsilon\varepsilon}}{1-\rho} \right] \\
&\Rightarrow \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ (1-\rho) \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int_i \tilde{W} dV_i \right) + \varpi_{\varepsilon\varepsilon} \left( \int_i \tilde{W} dW_i \right) \right] + \sigma_{\varepsilon\varepsilon} \right] \\
&\quad - \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n (1-\rho) \left[ \varpi_{\varepsilon\varepsilon} \left( \int_i \tilde{W} dW_i \right) + \gamma_{\varepsilon\varepsilon} + \frac{\sigma_{\varepsilon\varepsilon}}{1-\rho} \right] \\
&= \frac{1-\rho}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \varpi_\varepsilon \varpi_{e,\varepsilon}^{1/2} \left( \int_i \tilde{W} dV_i \right) \right] \\
&\Rightarrow N \left( 0, \frac{(1-\rho)^2 \varpi_\varepsilon^2 \varpi_{e,\varepsilon}}{6\varpi_e^4} \right)
\end{aligned}$$

as  $n \rightarrow \infty$ .

And because

$$\begin{aligned}
\frac{1}{\sqrt{nT}} \boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u} &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{\sqrt{T}} \frac{\theta}{\varpi_e^2 + \theta\sigma_\mu^2} (\mu_i + \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i) \\
&= \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\theta}{\varpi_e^2 + \theta\sigma_\mu^2} \mu_i + \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\theta}{\varpi_e^2 + \theta\sigma_\mu^2} \frac{\boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \\
&\Rightarrow o_p(1) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{(1-\rho) \varpi_e}{\sigma_\mu^2} V_i(1) \right] \\
&\Rightarrow \frac{(1-\rho) \varpi_e}{\sigma_\mu^2} N(0, 1)
\end{aligned}$$



and also because  $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} 0$  and  $\frac{1}{n}\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$ , which are proved in part (a). Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}G_2 - \frac{1}{\varpi_\varepsilon^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n (1-\rho) \left[ \left( \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) \varepsilon_{it} \right) \frac{\varpi_{\varepsilon\varepsilon}}{\varpi_\varepsilon^2} + \gamma_{\varepsilon\varepsilon} + \frac{\sigma_{\varepsilon\varepsilon}}{1-\rho} \right] \\
= & \frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{\varpi_\varepsilon^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n (1-\rho) \left[ \left( \frac{1}{T} \sum_{t=1}^T (x_{it} - \bar{x}_i) \varepsilon_{it} \right) \frac{\varpi_{\varepsilon\varepsilon}}{\varpi_\varepsilon^2} + \gamma_{\varepsilon\varepsilon} + \frac{\sigma_{\varepsilon\varepsilon}}{1-\rho} \right] \\
& - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}} \left( \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{u}}{\sqrt{nT}} \\
\Rightarrow & N \left( 0, \frac{(1-\rho)^2 \varpi_\varepsilon^2 \varpi_{\varepsilon\varepsilon}}{6\varpi_\varepsilon^4} \right).
\end{aligned}$$

Note that

$$\frac{1}{nT}G_1 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T} \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \left( \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x}}{n}$$

First consider that

$$\begin{aligned}
& \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\
= & \frac{1}{\sigma_\varepsilon^2} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T}\mathbf{x}'_i\mathbf{A}^{-1}\mathbf{x}_i - \frac{\sigma_\mu^2}{\sigma_\varepsilon^2 + \theta\sigma_\mu^2} \frac{\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}} \frac{\boldsymbol{\iota}'_T\mathbf{A}^{-1}\mathbf{x}_i}{\sqrt{T}} \right)
\end{aligned}$$

For a fixed  $n$ , because

$$\begin{aligned}
& \frac{1}{T}\mathbf{x}'_i\mathbf{A}^{-1}\mathbf{x}_i \\
= & \frac{1}{T} \sum_{t=1}^T (x_{it} - x_{it-1})^2 \\
= & \frac{1}{T} \sum_{t=1}^T \varepsilon_{it}^2 \\
\Rightarrow & \sigma_\varepsilon^2,
\end{aligned}$$

and

$$\frac{1}{\sqrt{T}}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T = \frac{1}{\sqrt{T}}x_{i1} \xrightarrow{p} 0,$$

as  $T \rightarrow \infty$ . Hence,

$$\begin{aligned} & \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} \\ &= \frac{1}{\sigma_e^2} \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \mathbf{x}_i - \frac{\sigma_\mu^2}{\sigma_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} \frac{\boldsymbol{\nu}'_T \mathbf{A}^{-1} \mathbf{x}_i}{\sqrt{T}} \right) \\ & \xrightarrow{p} \frac{\sigma_\varepsilon^2}{\varpi_e^2}. \end{aligned}$$

as  $n \rightarrow \infty$ .

Also because

$$\begin{aligned} \frac{1}{n} \mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT} &= \frac{1}{n} \sum_{i=1}^n \frac{1}{\varpi_e^2 + \theta \sigma_\mu^2} (\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T) \\ &\Rightarrow \frac{1}{n} \sum_{i=1}^n \frac{1}{\varpi_e^2 + \sigma_\mu^2} x_{i1} \\ &\xrightarrow{p} 0 \end{aligned}$$

because  $\theta = 1$ . And

$$\frac{1}{n} \boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT} = \frac{1}{n} \frac{n\theta}{\varpi_e^2 + \theta \sigma_\mu^2} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned} \frac{1}{nT} G_1 &= \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} - \frac{1}{T} \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{x}}{n} \\ &\xrightarrow{p} \frac{\sigma_\varepsilon^2}{\varpi_e^2} \end{aligned}$$

Note that

$$\frac{1}{nT} G_2 = \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{1}{T} \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u}}{n}$$

First consider that

$$\begin{aligned} & \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} \\ &= \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \frac{\varpi_e^2 \mu_i}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} + \frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i - \frac{1}{\sqrt{T}} \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \right] \end{aligned}$$

For a fixed  $n$ , because  $\frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T = \frac{1}{\sqrt{T}} x_{i1} \xrightarrow{p} 0$ , which is proved in part (a)

and

$$\frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i = \frac{1}{T} \sum_{t=1}^T \varepsilon_{it} e_{it} \xrightarrow{p} \sigma_{\varepsilon\varepsilon}$$

and

$$\frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i = \frac{1}{\sqrt{T}} x_{i1} \nu'_{i1} \xrightarrow{p} 0,$$

as  $T \rightarrow \infty$ . Hence,

$$\begin{aligned} & \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} \\ = & \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^n \left[ \frac{1}{\sqrt{T}} \frac{\varpi_e^2 \mu_i}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} + \frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i - \frac{1}{\sqrt{T}} \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \right] \\ & \xrightarrow{p} \frac{\sigma_{\varepsilon\varepsilon}}{\varpi_e^2} \end{aligned}$$

as  $n \rightarrow \infty$ .

And because

$$\begin{aligned} & \frac{1}{n} \boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u} \\ = & \frac{1}{n} \sum_{i=1}^n \frac{\theta}{\varpi_e^2 + \theta \sigma_\mu^2} (\mu_i + \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i) \\ = & \frac{1}{n} \sum_{i=1}^n \frac{1}{\varpi_e^2 + \sigma_\mu^2} (\mu_i + \nu'_{i1}) \\ & \xrightarrow{p} 0 \end{aligned}$$

and also because  $\frac{1}{n} \mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT} \xrightarrow{p} 0$  and  $\frac{1}{n} \boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$ , which are proved in part (a). Hence, we have

$$\begin{aligned} \frac{1}{nT} G_2 &= \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{1}{T} \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u}}{n} \\ & \xrightarrow{p} \frac{\sigma_{\varepsilon\varepsilon}}{\varpi_e^2} \end{aligned}$$

Note that

$$\frac{1}{\sqrt{nT}} G_2 = \frac{1}{\sqrt{nT}} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{1}{\sqrt{T}} \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \left( \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u}}{\sqrt{n}}$$

First consider that

$$\begin{aligned} & \sqrt{nT} \left( \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{\sigma_{\varepsilon\varepsilon}}{\varpi_e^2} \right) \\ = & \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{\varpi_e^2 \mu_i T}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} + \frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \right] - \sqrt{nT} \frac{\sigma_{\varepsilon\varepsilon}}{\varpi_e^2} \end{aligned}$$

For a fixed  $n$ , because  $\frac{1}{\sqrt{T}}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_T = \frac{1}{\sqrt{T}}x_{i1} \xrightarrow{p} 0$ , which is proved in part (a) and

$$\begin{aligned} & \sqrt{T} \left( \frac{1}{T} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i - \sigma_{\varepsilon\varepsilon} \right) \\ &= \frac{1}{\sqrt{T}} \sum_{t=1}^T (x_{it} - x_{it-1}) e_{it} + o(1) - \sqrt{T} \sigma_{\varepsilon\varepsilon} \\ &= \frac{1}{\sqrt{T}} \sum_{t=1}^T \varepsilon_{it} e_{it} + o(1) - \sqrt{T} \sigma_{\varepsilon\varepsilon} \\ &\Rightarrow N(0, \varpi_e^2 \varpi_\varepsilon^2), \end{aligned}$$

and  $\frac{1}{\sqrt{T}}\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_T\boldsymbol{\nu}'_T\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{\sqrt{T}}x_{i1}\nu'_{i1} \xrightarrow{p} 0$ , which is proved in part (a), as  $T \rightarrow \infty$ . Hence,

$$\begin{aligned} & \sqrt{nT} \left( \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{u} - \frac{\sigma_{\varepsilon\varepsilon}}{\varpi_e^2} \right) \\ &= \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[ \frac{\varpi_e^2 \mu_i T}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T}{\sqrt{T}} + \frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{1}{\sqrt{T}} \mathbf{x}'_i \mathbf{A}^{-1} \boldsymbol{\nu}_T \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i \right] - \sqrt{nT} \frac{\sigma_{\varepsilon\varepsilon}}{\varpi_e^2} \\ &\Rightarrow \frac{1}{\varpi_e^2} \frac{1}{\sqrt{n}} \sum_{i=1}^n [o_p(1) + N(0, \varpi_e^2 \varpi_\varepsilon^2) - o_p(1)] \\ &\Rightarrow \frac{1}{\varpi_e^2} N(0, \varpi_e^2 \varpi_\varepsilon^2) \end{aligned}$$

as  $n \rightarrow \infty$ .

Also because

$$\begin{aligned} & \frac{1}{\sqrt{n}} \boldsymbol{\nu}'_{nT} \Phi^{-1} \mathbf{u} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\theta}{\varpi_e^2 + \theta \sigma_\mu^2} (\mu_i + \boldsymbol{\nu}'_T \mathbf{A}^{-1} \boldsymbol{\nu}_i) \\ &\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{\varpi_e^2 + \sigma_\mu^2} (\mu_i + \nu'_{i1}) \\ &\Rightarrow \frac{\varpi_e + \sigma_\mu}{\sigma_\mu^2} N(0, 1) \end{aligned}$$

and also because  $\frac{1}{n} \mathbf{x}' \Phi^{-1} \boldsymbol{\nu}_{nT} \xrightarrow{p} 0$  and  $\frac{1}{n} \boldsymbol{\nu}'_{nT} \Phi^{-1} \boldsymbol{\nu}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$ , which are proved in

part (a). Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}G_2 - \sqrt{nT}\frac{\sigma_{\varepsilon\varepsilon}}{\varpi_e^2} \\
&= \left[ \frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u} - \sqrt{nT}\frac{\sigma_{\varepsilon\varepsilon}}{\varpi_e^2} \right] \\
&\quad - \frac{1}{\sqrt{T}}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \left( \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{u}}{\sqrt{n}} \\
&\Rightarrow \frac{1}{\varpi_e^2}N(0, \varpi_e^2\varpi_\varepsilon^2)
\end{aligned}$$

■

**Proof of Theorem 4:**

**Proof.** By  $\widehat{\beta}_{GLS} - \beta = G_1^{-1}G_2 = \left[ \mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} (\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT})^{-1} \boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x} \right]^{-1}$   
 $\left[ \mathbf{x}'\Phi^{-1}\mathbf{u} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} (\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT})^{-1} \boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{u} \right]$ , the proof of Theorem 4 is straightforward with above lemmas. ■