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Abstract:

Parametric stochastic frontier models yield firm-level conditional distributions of inefficiency that are truncated normal. Given these distributions, how should one assess and rank firm-level efficiency? This study compares the techniques of estimating a) the conditional mean of inefficiency and b) probabilities that firms are most or least efficient. Monte Carlo experiments suggest that the efficiency probabilities are more reliable in terms of mean absolute percent error when inefficiency has large variation across firms. Along the way we tackle some interesting problems associated with simulating and assessing estimator performance in the stochastic frontier environment.

JEL Code: C12, C16, C44, D24

Key Words: Truncated Normal, Stochastic Frontier, Efficiency, Multivariate Probabilities

A Monte Carlo Study of Efficiency Estimates from Frontier Models

1 Introduction

A broad class of fully-parametric stochastic frontier models represent production or cost functions as composed-error regressions and imply that firm-level production or cost efficiency can be characterized as a truncated (at zero) normal distribution. Whether cross-sectional or panel data, cost frontier or production frontier, time-invariant or time-varying efficiency, parametric stochastic frontier models yield inefficiency distributions that are truncated normal. See, for example, Jondrow et al. (1982), Battese and Coelli (1988), Kumbhakar (1990), Battese and Coelli (1992), Cuesta (2000), and Greene (2005). After estimating the cost or production function for a sample of firms, parametric assumptions on the composed error are typically used to calculate the mean and variance of normal distributions, which (when truncated at zero) represent the conditional distributions of inefficiency for each firm. There are currently two very different frequentist approaches used to assess the efficiency of individual firms and create an efficiency ranking based on these distributions.[1] The traditional approach of calculating and ranking the conditional means of the truncated distributions is due to Jondrow et al. (1982) and Battese and Coelli (1988). These are absolute estimates of efficiency that, when ranked, reveal information on relative magnitudes of realizations from the truncated normal distributions. Recently, Horrace (2005) calculates probabilities on relative efficiency that allow statements to be made on which firm (in the sample) is most

[1] There is also a Bayesian inference literature for the stochastic frontier model. The techniques either directly or indirectly provide inference on relative ranks using Bayesian sampling techniques and are a viable alternative to the results presented here. For example, see Fernandez at al. (2002), Tsionas (2002), Kim and Schmidt (2000), and Koop et al. (1997).

or least efficient. That is, the approach yields statements like, "firm j is most (least) efficient relative to the rest with probability 0.3." Horrace claims that these efficiency probabilities are more meaningful than the traditional rankings of conditional means in the sense that they better summarize information on the relative rankings of the firms from the inefficiency distributions. In particular, they more accurately and more completely quantify the information on realizations from these distributions. The purpose of this study is assess the validity of this claim via simulation. We find that the probabilities are a more precise summary of the efficiency information revealed by the distributions.

If parametric frontier models are a correct representation of the data generation mechanism, then all that these models truly identify are the *distributions* of inefficiency and not estimates of realizations of inefficiency themselves. With these distributions in-hand it is then a question of how best to report the information they contain. Using the conditional mean of the truncated normal distribution as a point estimate of (in)efficiency is potentially misleading, since a firm's (in)efficiency is *not* a parameter *per se*.[2] Even more to the point, comparing firms by ranking these conditional means compounds the opportunity for misinterpretation, because the true efficiency differences across firms may not equal the differences of the conditional means in any particular sample. This problem was originally addressed by Horrace and Schmidt (1996), who calculate confidence intervals (percentiles) from the truncated distributions, and by Bera and Sharma (1999), who calculate the conditional vari-

[2] If there were sample realizations of technical inefficiency for each firm, we would naturally estimate some conditional mean. Here, however, the conditional mean estimate is derived directly from moment conditions imposed on the estimation problem itself and is, therefore, an artifact of the specification, not a "result" of the empirical exercise.

ances of the distributions. Even then, confidence intervals and conditional variances do not account for the multiplicity implied by the joint inferential statement that firm A is better than B, *and* better than C, *and* better than D, *etc.*[3] Finally, the conditional means are often interpreted as a measure of *absolute* efficiency, based on an out-of-sample standard, but this interpretation would be wrong if the most efficient firm in the population were actually *part* of the sample. Indeed, the idea of ranking efficiency necessarily implies a concern about *relative* efficiency, so approaching this with an absolute measure seems misguided.

The efficiency probabilities avoid all the aforementioned difficulties. They recognize that the point of interest is not ranked parameters but ranked potential realizations from estimated distributions. They implicitly account for the variability of inefficiency and, indeed, *all* the moments of the distributions.[4] They account for the multiplicity in the efficiency rank statement by assigning probabilities to *joint* statements on efficiency differences. Finally, they are statements on *relative* (not absolute) differences, which is the correct comparison for a within sample ranking. The only apparent shortcoming of these probabilities is computational cost; the conditional means involve simple algebra, while the probabilities require numerically calculating a probability integral.

This paper uses Monte Carlo methods to compare the precision of the conditional means and the probability statements. Since the two techniques and their units of measure are very different, we employ the unitless mean absolute percentage error (MAPE) to make compar-

[3] This has been accomplished in the semi-parametric, fixed-effect specification of the stochastic frontier, using the theory of multiple comparisons. See Horrace and Schmidt (2000).

[4] For example, one might suspect that the skew of a truncated distribution is as important as the mean and the variance in understanding distributional shape.

isons. The simulations also present several complications that underscore the difficulties of efficiency estimation, in general, and that provide insights into the inherent differences of the two estimation approaches. These are discussed in the sequel. We find that efficiency probabilities are more reliable when the variance of technical inefficiency is large; this is the "usual" case in the sense that it is the only time when estimation of inefficiency is at all precise and when it may be even warranted. In addition to the MAPE results, we present mean squared error (MSE) and bias calculations to examine the effects of changes in the variance parameters and sample sizes on the performance of each estimator (in isolation). We also demonstrate that relative efficiency probabilities can be made for any *subset* of the firms in the sample, where the subset might be selected based on some additional criterion which does not enter into the frontier estimation. (In fact, we use this technique to simplify our Monte Carlo study when the number of firms is exceedingly large.) The next section reviews the stochastic frontier model and defines the estimates to be studied, including the new subset probabilities. Section 3 contains the Monte Carlo study, and section 4 provides a final discussion of the results and concludes.

2 Efficiency Estimation

The parametric stochastic frontier model was introduced simultaneously by Aigner, Lovell and Schmidt (1977) and Meeusen and van den Broeck (1977). Since then, there have been many re-formulations of the basic model. For example, consider the standard linear frontier specification for panel data with time-invariant efficiency:

$$y_{jt} = x_{jt}'\beta + v_{jt} \pm u_j,\ j = 1, ..., n,\ t = 1, ..., T, \tag{1}$$

where y_{jt} is productive output or cost for firm j in period t; x_{jt} is a vector of production or cost inputs and β is an unknown parameter vector. The $v_{jt} \in \mathbb{R}$ are random variables representing shocks to the frontier. Let v_{jt} have an *iid* zero-mean normal distribution with variance σ_v^2. The $u_j \in \mathbb{R}_+$ are random variables representing productive or cost inefficiency, added to the cost function representation or subtracted from the production function representation. Let u_j have a distribution that is the absolute value of an *iid* zero-mean normal random variable with variance σ_u^2 (a half-normal distribution). Additionally, let the x_{jt}, v_{jt} and u_j be independent across j and across t. There are more flexible parameterizations of the linear model. For example, Kumbhakar (1990), Battese and Coelli (1992), and Cuesta (2000) consider forms of time-varying efficiency, u_{jt}. Greene (2005) considers an extremely flexible model that incorporates firm level heterogeneity in addition to the usual error components. Our selection of the more simple model in equation 1 is merely to parallel the model and discussions in Horrace (2005) and should not be construed as a limitation on the applicability of the results that follow. In fact, the inferential procedures detailed herein apply in time-varying efficiency models, in Greene (2005), or in *any* frontier model where the conditional distribution of efficiency is truncated normal (including the case where the *unconditional* distribution of efficiency is exponential). Per Jondrow et al. (1982), the distribution of u_{jt} conditional on $\epsilon_{jt} = v_{jt} \pm u_{jt}$ is a $N(\mu_{*j}, \sigma_*^2)$ random variable truncated below zero. Per

Battese and Coelli (1988), the μ_{*j} and σ_*^2 are:

$$\mu_{*j} = \pm\frac{\sigma_u^2 \overline{\epsilon}_j}{\sigma_u^2 + \frac{\sigma_v^2}{T}},\ j = 1, ..., n; \tag{2}$$

and

$$\sigma_*^2 = \frac{\sigma_u^2 \sigma_v^2}{T\sigma_u^2 + \sigma_v^2}, \tag{3}$$

where $\overline{\epsilon}_j = T^{-1}\sum_{t=1}^{T}\epsilon_{jt}$. (The right-hand side of equation 2 is "+" for the cost frontier or "$-$" for the production frontier) Parametric estimation usually proceeds by corrected GLS or MLE [e.g. Horrace and Schmidt (1996) for details], yielding estimates $\widehat{\beta}$, $\widehat{\sigma}_u^2$, and $\widehat{\sigma}_v^2$. Then, defining $e_{jt} = y_{jt} - x_{jt}'\widehat{\beta}$, "estimation" of μ_{*j} and σ_*^2 follows by substituting e_{jt} for ϵ_{jt}, $\widehat{\sigma}_u^2$ for σ_u^2, and $\widehat{\sigma}_v^2$ for σ_v^2 in equations 2 and 3. Then, for a log-production function, the usual measure of technical efficiency based on a $N(\widehat{\mu}_{*j}, \widehat{\sigma}_*^2)$ assumption is:

$$\widehat{\theta}_j = E(\exp\{-u_j\}|\overline{e}_j) = \exp\{-\widehat{\mu}_{*j} + \frac{1}{2}\widehat{\sigma}_*^2\}\frac{1-\Phi\left(\widehat{\sigma}_* - \frac{\widehat{\mu}_{*j}}{\widehat{\sigma}_*}\right)}{1-\Phi\left(\frac{-\widehat{\mu}_{*j}}{\widehat{\sigma}_*}\right)},\ j = 1, ..., n. \tag{4}$$

This is the sample equivalent of $\theta_j = E(\exp\{-u_j\}|\overline{\epsilon}_j)$, assuming that substitution of e_{jt} for ϵ_{jt} does not change the shape of the conditional distribution (or at least asymptotically). Horrace (2005) argues that the point estimate in 4 is "misleading." Granted the shape of the conditional distribution is truncated normal, but it is unrealistic to think that the first moment of an asymmetric, truncated distribution can summarize its entire probabilistic nature. Illustration of this point is the essence of the contributions of Horrace and Schmidt (1996) and Bera and Sharma (1999): the first moment does not adequately summarize efficiency, so one should also quantify the second moment by constructing confidence intervals (Horrace and Schmidt, 1996) or calculating the variance of the truncated distributions (Bera and Sharma, 1999). Ideally, one might calculate higher moments as well, particularly odd moments, which affect the probability of extreme realizations of inefficiency in clear ways. This suggests that the point estimate, $\widehat{\theta}_j$, does not adequately account for (or inform our understanding of) the varying shape of the conditional distribution of u across firms.

Horrace (2005) addresses these shortcomings in $\widehat{\theta}_j$ by calculating multivariate probabilities conditional on ϵ, given that the distribution of u_j is truncated (at zero) normal. These probabilities are:

$$P^j_{\max} = \Pr\{u_j < u_i \ \forall \ i \neq j\} \ j = 1, ...n, \tag{5}$$

$$P^j_{\min} = \Pr\{u_j > u_i \ \forall \ i \neq j\} \ j = 1, ...n, \tag{6}$$

Notice that there is room for confusion in the notation. The "max" notation in $P^j_{\max}$ is intended to represent the fact that j is "maximally efficient", which happens to coincide with u_j being *minimal* ($u_j < u_i \ \forall \ i \neq j$ in a probabilistic sense). The "max" notation should *not* be confused with "maximal u_j" , which is synonymous with "minimal efficiency". Similarly, the "min" notation in $P^j_{\min}$ represents the fact that j is "minimally efficient" ($u_j > u_i \ \forall \ i \neq j$ in a probabilistic sense). Specifically, the probabilities are given by:

$$P^j_{\max} = \int_0^\infty f_{u_j}(u) \prod_{i \neq j}^{n} [1 - F_{u_i}(u)] du,$$

and

$$P^j_{\min} = \int_0^\infty f_{u_j}(u) \prod_{i \neq j}^{n} F_{u_i}(u) du,$$

where $f_{u_j}(u)$ and $F_{u_j}(u)$ are the probability function and the cumulative distribution function of a $N(\mu_{*j}, \sigma_*^2)$ distribution truncated at zero, respectively. That is,

$$f_{u_j}(u) = \frac{(2\pi\sigma_*^2)^{-1/2}\exp\{-\frac{(u-\mu_{*j})^2}{2\sigma_*^2}\}}{1-\Phi(-\mu_{*j}/\sigma_*)},$$

and

$$F_{u_j}(u) = \frac{\Phi(\{u-\mu_{*j}\}/\sigma_*) - \Phi(-\mu_{*j}/\sigma_*)}{1-\Phi(-\mu_{*j}/\sigma_*)},$$

$u \in \mathbb{R}_+$, where Φ is the cumulative distribution function of the standard normal. The probabilities in equations 5 and 6 condense all the information on the relative differences of the distributions of efficiency into a single statement and also account for the multiplicity of the probability statement on maximal (minimal) efficiency, which the conditional mean and conditional variance cannot. In particular, they more adequately capture the effect of the shape of the distribution on the magnitude of a firm's realization of u than the point estimates $\widehat{\theta}_j$. Estimates of the probabilities, $\widehat{P}^j_{\max}$ and $\widehat{P}^j_{\min}$ follow by substituting estimates $\widehat{\mu}_{*j}$ and $\widehat{\sigma}_*$ into equations 5 and 6.

A useful feature of these probabilities is that they are statements of *relative* efficiency (efficiency relative to a within sample standard), whereas the typical efficiency measure, $\widehat{\theta}_j$, is a measure of *absolute* efficiency (efficiency relative to an unobserved population standard). Relative efficiency is often empirically relevant, as when the research question is about the most or least efficient firms within an industry. In addition, one may be interested in understanding relative performance among a subset of the sample of firms $j = 1, ...n$, based on a certain information criteria or decision rule. For example, one may be interested in estimating a cost function for a sample of 500 banks, but then only calculating probabilities of maximal cost efficiency for a subset of the banks with large assets. That is, one may be interested in how only the largest banks perform relative to one another, conditional on a common cost function for all banks. The probabilities $P^j_{\max}$ and $P^j_{\min}$ will change as the cardinality of and the membership within this subset changes. Let $N = \{1, ..., n\}$ be the set of all firm indices in the sample, and let the subset of interest be $J_\Omega \subset N$, based on some external information or decision rule Ω. Then the probabilities in equations 5 and 6 become:

$$P^j_{\Omega\max} = \int_0^\infty f_{u_j}(u) \prod_{i \neq j,\ i \in J_\Omega} [1 - F_{u_i}(u)] du, \tag{7}$$

and

$$P^j_{\Omega\min} = \int_0^\infty f_{u_j}(u) \prod_{i \neq j,\ i \in J_\Omega} F_{u_i}(u) du, \tag{8}$$

for all $j \in J_\Omega$. These will be different, in general, than the probabilities $P^j_{\max}$ and $P^j_{\min}$ of Horrace (2005). In fact, the probabilities in equations 5 and 6 are a special case of equations 7 and 8 when $J_\Omega = N$. If Ω is empirically relevant, then probabilities like $P^j_{\Omega\max}$ $(j \in J_\Omega)$ may be more useful than $P^j_{\max}$ $(j \in N)$. Also, experiments on the effects of different Ω and J_Ω on the probabilities in equations 7 and 8 may be of particular interest to empiricists. These types of experiments flow more naturally from relative efficiency measures like the probabilities in equations 7 and 8 than they do from absolute efficiency measures like $\widehat{\theta}$ in equation 4.

The next section examines the small and large sample performance of the estimates of $P^j_{\max}$, $P^j_{\min}$ and θ_j via Monte Carlo analysis. For each estimate we calculate MSE and bias for various sample sizes, (n, T), and various selections of σ^2_u/σ^2_v. Reliability comparisons across the different measures are made using the unitless MAPE.

3 Monte Carlo Experiment

The specification used for the experiment is the production function:

$$y_{jt} = \beta_0 + \beta_1 x_{jt} + v_{jt} - u_j. \tag{9}$$

The regressor x is needed for sampling variability to have a noticeable impact on the estimates $\widehat{P}^j_{\max}$ and $\widehat{P}^j_{\min}$. In a model with only a constant term, sampling variability in the estimation of β_0 alone would simply shift all the e_{jt} up or down by the same amount, and the $\widehat{\mu}_{*j}$ would all undergo an identical transformation from their true values. Hence, the difference between any two $\widehat{\mu}_{*j}$ and $\widehat{\mu}_{*i}$ would be unchanged, and only sampling variability in the estimate $\widehat{\sigma}_*$ would affect $\widehat{P}^j_{\Omega\max}$, and $\widehat{P}^j_{\Omega\min}$. This is due to the "relative nature" of the efficiency probability estimates; the absolute estimates $\widehat{\theta}_j$ are immune to this complication and could be analyzed without including a regressor in the specification.

Following Olsen, Schmidt and Waldman (1980), we fix the variance of the composed error term to $\sigma^2_\epsilon = \sigma^2_u + \sigma^2_v = 1$. Hence, the individual variances of v_{jt} and u_j may be characterized by a single parameter—we use the ratio $\gamma = \sigma^2_u/\sigma^2_v$. However, unlike the estimates in Olsen, Schmidt and Waldman (1980), the $\widehat{\theta}_j$, $\widehat{P}^j_{\max}$, and $\widehat{P}^j_{\min}$ are more complicated transformations of the data, so we cannot say immediately what the effect of changes in σ^2_ϵ would be.[5]

While we estimate the production function in equation 9 for the entire sample, we only estimate the various efficiency measures for a subset of five randomly chosen firms. This is done primarily for ease of computation of $P^j_{\max}$ and $P^j_{\min}$, which involve integration over a product of functions, one for each firm in the comparison group. In essence, we calcu-

[5] This is particularly difficult to predict for the efficiency probabilities.

late $\widehat{P}^{j}_{\Omega\max}$ and $\widehat{P}^{j}_{\Omega\min}$ for $j \in J_\Omega$ where Ω is the rule "randomly select five firms from N." Consequently, we only calculate five values of $\widehat{\theta}_j$, $j \in J_\Omega$ in each simulation iteration for comparison. This randomization introduces an additional source of variability into the exercise, which may cause some instability in the convergence results, but the instability is the price we pay for computational ease. Fortunately, the additional variability is common to all estimators considered, so any instability will be globally manifest.

3.1 Simulation Procedure

The experiment is designed to assess $\widehat{\theta}_j$, $\widehat{P}^{j}_{\Omega\max}$, and $\widehat{P}^{j}_{\Omega\min}$ over a range of common panel sizes (n and T) and variance ratios (γ). We use eight panel configurations: $T = 5$ and $n = 25, 100, 500$; $T = 10$ and $n = 25, 100, 500$; and $T = 20$ and $n = 25, 100$.[6] In all cases we are concerned with the usual panel setting of large n and fixed T, so asymptotic arguments are along the dimension n. For each panel configuration we conduct simulation exercises for five variance ratios $\gamma = 0.1,\ 0.5,\ 1,\ 5$, and 10, so there are forty simulations in total. For reasons discussed above, we fix the number of firms for calculation of $\widehat{\theta}_j$, $\widehat{P}^{j}_{\Omega\max}$, and $\widehat{P}^{j}_{\Omega\min}$ to five (randomly selected from $N = 1, ..., n$) .[7]

Each iteration within a simulation exercise (indexed by $m = 1, \ldots, M$), goes through the following sampling and estimation procedure, which is repeated $M = 5,000$ times. First, the errors u_{jm} and v_{jtm} are drawn from the appropriate half-normal and normal distributions (with respective variances σ_u^2 and σ_v^2), and the regressors x_{jtm} are drawn from an indepen-

[6] We omitted N = 500, T = 20 to save computing time.

[7] This also allowed us to indirectly examine the validity of the subset efficiency probabilities introduced in equations 7 and 8.

dent uniform [0,1] distribution.[8] Then y_{jtm} is generated for $\beta_0 = 0$ and $\beta_1 = 1$ (the only parameterization of the conditional mean function considered). Since each ϵ_{jtm} is observed, we can calculate the true values of μ_{*jm} and σ_{*m} for *each* draw, m. These map into the true values for θ_{jm}, $P^{jm}_{\Omega\max}$, and $P^{jm}_{\Omega\min}$ for each m, so the "parameters" of interest are *not* constant across m. Estimation of β_{0m}, β_{1m}, and e_{jtm} proceeds with corrected GLS (the "random effects" estimator).[9] After estimating $\widehat{\mu}_{*jm}$ and $\widehat{\sigma}_{*m}$, using e_{jtm} for ϵ_{jtm} and $\widehat{\sigma}^2_u, \widehat{\sigma}^2_v$ for σ^2_u, σ^2_v in equations 2 and 3, five firms are randomly selected to produce the subset $J_{\Omega m} \subset N$. From these results we calculate estimates $\widehat{\theta}_{jm}$, $\widehat{P}^{jm}_{\Omega\max}$, and $\widehat{P}^{jm}_{\Omega\min}$ for the five firms $j \in J_{\Omega m}$, using equations 4, 7, and 8.

In what follows it is *very* important to remember that the *true values* θ_{jm}, $P^{jm}_{\Omega\max}$, and $P^{jm}_{\Omega\min}$, $j \in J_{\Omega m}$ are not fixed across iterations, m. (This should be clear, since all three of these measures are indexed by m.) This produces nonstandard formulae for the MSE, bias, and MAPE, although their interpretations are, indeed, standard. It also underscores the difficulties in estimating efficiency in these models: we are trying to make inferences about the distribution of efficiency for each firm from what amounts to a single draw from the distribution, and that single draw u_j is not even observed; it is merely "estimated" from the convolution, e_{jt}.

With the results from the $5,000$ iterations for each simulation exercise, we calculate the mean square error of $\widehat{\theta}_j$, $\widehat{P}^{j}_{\Omega\max}$, and $\widehat{P}^{j}_{\Omega\min}$. Our nonstandard formula is (typically):

$$\mathrm{MSE}(\hat{\theta}) = \frac{1}{5M} \sum_{m=1}^{M} \sum_{j \in J_{\Omega m}} (\hat{\theta}_{jm} - \theta_{jm})^2,$$

[8] We could have allowed the x_{jtm} to be correlated within firms, but did not.
[9] When CGLS fails due to $\hat{\sigma}^2_u < 0$, we set $\hat{\sigma}^2_u = 0$, per Waldman (1982)

and similarly for $MSE(\widehat{P}_{\Omega \max})$, and $MSE(\widehat{P}_{\Omega \min})$.[10] Even though the MSE is nonstandard because it includes sampling variability across the true parameters (even asymptotically), it still seems theoretically sensible. As we shall see, it also produces results that are sensible. Again, this is an unavoidable feature of efficiency estimation from these models (in general).

For the bias and MAPE, we separately use only the best or worst firms within each five-firm subsample. This is necessary as the probability statements within a comparison group automatically sum to one (e.g., $\sum_{j \in J_\Omega} \widehat{P}^{jm}_{\Omega \max} = 1$), so there is no average bias for the whole group for these estimators. This is another artifact of their "relative nature" and perhaps a nice feature. More specifically, using the population ranking of u_{jm} among the five randomly selected firms, $u_{[1]m} < u_{[2]m} < ... < u_{[5]m}$, we calculate the bias and MAPE of $\widehat{P}^{[1]m}_{\Omega \max}$, $\widehat{P}^{[5]m}_{\Omega \min}$, $\hat{\theta}_{[1]m}$, and $\hat{\theta}_{[5]m}$ for each iteration. Hence, the biases for each extremum measure are (typically):

$$\text{Bias}(\hat{\theta}_{[1]}) = \frac{1}{M} \sum_{m=1}^{M} (\hat{\theta}_{[1]m} - \theta_{[1]m})$$

and

$$\text{Bias}(\hat{\theta}_{[5]}) = \frac{1}{M} \sum_{m=1}^{M} (\hat{\theta}_{[5]m} - \theta_{[5]m}),$$

and similarly for $\text{Bias}(\widehat{P}^{[1]}_{\Omega \max})$, and $\text{Bias}(\widehat{P}^{[5]}_{\Omega \min})$. We could have selected any firms in the ranking for this purpose (i.e., [2], [3] or [4]), but the best and the worst seemed appropriate for evaluating the performance of ranked estimators. Also, the extreme firms map into efficiency probabilities from the population that tend to be large, precluding a "divide-by-zero" problem in the MAPE calculation, as we shall see. $\text{Bias}(\hat{\theta}_{[1]})$ quantifies the extent to

[10] We also calculated mean absolute error for each measure, but the results were similar to those for MSE and are not reported.

which the estimate of technical efficiency for the *most* efficient firm in the randomly selected subsample is mis-measured on average. Similarly, the Bias($\widehat{P}^{[1]}_{\Omega\,\max}$) quantifies the extent to which the estimate of the probability of being most efficient for the most efficient firm in the randomly selected subsample is mis-measured on average. Finally, since the units of θ_j and $P^j_{\Omega\,\max}$ are different, the MSE and Bias measures are only relevant for making comparisons for a single measure (in isolation).

To make comparisons *across* measures we employ the unitless MAPE (typically):

$$\text{MAPE}(\hat{\theta}_{[w]}) = \frac{1}{M}\sum_{m=1}^{M}\left|\frac{\hat{\theta}_{[w]m} - \theta_{[w]m}}{\theta_{[w]m}}\right| \quad w = 1, 5.$$

With the MAPE, we wish to avoid division by numbers close to zero, so we calculate it only for $\widehat{P}^{[1]}_{\Omega\,\max}$ and $\widehat{P}^{[5]}_{\Omega\,\min}$, the efficiency probability of the most efficient firm and the inefficiency probability of the least efficient firm, respectively, in the population. That is, efficiency probabilities, like $\widehat{P}^{[5]}_{\Omega\,\max}$ and $\widehat{P}^{[1]}_{\Omega\,\min}$ may be very close to zero in the denominator of the MAPE formula, so it is only calculated for $\widehat{P}^{[1]}_{\Omega\,\max}$ and $\widehat{P}^{[5]}_{\Omega\,\min}$, which should both be fairly large in each draw. The results of the simulations and their discussion follow.

3.2 Results

First, the experiment shows that failure of the CGLS procedure ($\sigma_u^2 < 0$) is a problem only for extremely "noisy" variance ratios (small γ) and for small n in Tables 1-3. There are no failures with $\gamma > 1$, and with $\gamma = 1$ only a small number of failures (less that 1%) occur using the smallest sample $n = 25, T = 5$.

As expected, the MSE of all measures decreases with increasing n and fixed T. Of course, Tables 1-3 do not allow us to make comparisons across measures, since the units are different across measures. Also, it is not surprising that as the signal-to-noise ratio (σ_u^2/σ_v^2) increases, the MSE of the estimates is usually non-increasing, but not always. The MSE (Table 1-3) of $\widehat{P}_{\Omega\max}$ (the probability that j is most efficient) is always non-increasing in γ. However, this is not true for the MSE of $\widehat{\theta}$, and $\widehat{P}_{\Omega\min}$. For example, in Table 3 for $n = 25$ and moving from γ equal 1 to 5 to 10, the MSE of $\widehat{\theta}$ is *increasing* from 0.0032 to 0.0048 to 0.0055. Similarly the MSE of $\widehat{P}_{\Omega\min}$ is increasing across these $\gamma's$ in the same simulations. (The non-monotonicities are highlighted with asterisks in Table 1-3.) Why might these non-monotonicities in γ arise? It is well-known that the random effects estimator of β_1 is a weighted sum of the between estimator and the within (or fixed effects) estimator (e.g., see Hsiao, 1986 p36). The between estimator ignores the within firm variation, σ_u^2, so when σ_u^2 is large the random effects estimator places more weight on the within variation and the random effect estimator is close to the fixed effect estimator. It is also well-known that the random effects estimator is asymptotically efficient relative to the fixed effects estimator (e.g., see Baltagi, 2005 p17), so when σ_u^2 is very large, the random effects estimator may have a larger variance than when σ_u^2 is small. This imprecision feeds into the estimates $\widehat{\theta}_j$, $\widehat{P}^j_{\Omega\max}$, and $\widehat{P}^j_{\Omega\min}$, so non-monotonicities in Tables 1-3 may reflect this lack of precision. Notice that they (highlighted with asterisks) occur primarily for the largest γ (and hence for largest σ_u^2).[11] Another factor that may induce the non-monotonicities is the size of σ_*, which appears as $-\mu_{*j}/\sigma_*$ in the formulae for the conditional mean and efficiency probabilities.

[11]The imprecision may be worsen by the fact that the fixed effects estimator cannot exploit correlations between x and u, as they have not been built into the DGP.

For our simulations, the true value of σ_* reaches a maximum between $\gamma = 0.4$ and $\gamma = 0.7$ depending on the value of T. Obviously, smaller values of $\hat{\sigma}_*$ *ceteris paribus* inflate any error in the ratio μ_{*j}/σ_*, so the estimators may be less precise for large γ. (Of course there is no way to disentangle this phenomenon from the effect of the random effects estimator approaching the fixed effects estimator, but it is interesting to note.)

Why is the probability $\widehat{P}^{[1]}_{\Omega \max}$ non-increasing in γ? More accurately, why is the maximal efficiency probability immune to the variability of the random effects estimator when γ is large? When γ (and hence σ_u^2) is large, the probability of $u_{[1]} << u_j$, $j \neq [1]$ is large, so that differences in $\hat{\mu}_{*[1]}$ and $\hat{\mu}_{*j}$, $j \neq [1]$ tend to be large. The efficiency probabilities are based on differences of these means $(\hat{\mu}_{*[1]} - \hat{\mu}_{*j})$ and their relative variability. When the differences are large, the ability of the probabilities to distinguish the efficiency distributions is improved. It must be the case that this ability to distinguish outweighs the increased variability in the random effects estimator. Of course this phenomenon does not occur for $\widehat{P}^{[5]}_{\Omega \min}$. Why? It may be related to the shape of the half normal distribution from whence the realizations of u come. The distribution has most of its mass in the left tail ($u = 0$). As σ_u^2 gets large the right tail of the distribution becomes more uniform while the left tail maintains some of its shape. Realizations from the left tail of the distribution are more "informative" (to borrow a word form the Bayesians) than from the right tail. Therefore differences relative to the in the right tail, $u_{[1]} - u_j$, may be smaller in magnitude than differences relative to the left tail, $u_{[5]} - u_j$. Hence, it may be "easier" for $\widehat{P}^{[1]}_{\Omega \max}$ to distinguishing $(\hat{\mu}_{*[1]} - \hat{\mu}_{*j})$ than $\widehat{P}^{[5]}_{\Omega \min}$ to distinguish $(\hat{\mu}_{*[5]} - \hat{\mu}_{*j})$. Another (perhaps more plausible reason) is approximation error in $\Phi(-\mu_{*j}/\sigma_*)$ caused by very large (in absolute value) $\mu_{[5]}/\sigma_*$. Since $\widehat{P}^{[1]}_{\Omega\,\max}$ follows from relatively small $\mu_{[1]}/\sigma_*$, it is immune to approximation error. In fact, absent approximation error, we believe that $\widehat{P}^{[5]}_{\Omega\,\min}$ would exhibit the same monotonicities as $\widehat{P}^{[1]}_{\Omega\,\max}$.

The results for the MSE in Tables 1-3 are similar (for the most part) to the Bias results in Tables 4-6, which are tabulated for extreme-efficiency firms ([1] and [5]) from the ranked subsample of five. As expected, the biases of all measures are non-increasing in n (in absolute value), and they are generally decreasing in γ with a few exceptions that are similar in nature to those of Tables 1-3. While the imprecision of the random effects estimator for large σ_u^2 manifests itself in the variance of the efficiency estimates and, hence, the MSE of each estimator (Tables 1-3), it may also effect the bias of the estimates in this exercise. To see this, remember that that the nonstandard bias formula is not based on a fixed parameter across all 5,000 draws. Our formulation does not "average out" deviations around a fixed parameter, so the possibility for large deviations persists. These persistent deviations may appear as bias in our results. Notice also that the probability measures are always negatively biased, while the conditional mean measures are always positively biased. We suspect that this reversal comes from the fact that the probabilities are based on the distribution of u while the conditional means are based on the distribution of $\exp\{-u\}$. Across Tables 4-6, only $\widehat{\theta}_{[5]}$ is uniformly improving in both n and γ (in the sense that the absolute value of the bias is non-increasing). However, comparisons of the bias across different measures is not possible due to inconsistency of the units of measure.

To make comparisons across different measures, mean absolute percentage errors (MAPE) for the extreme ends of the population order statistic are presented in Tables 7-9. Across all three tables the results are clear: MAPE($\widehat{P}^{[1]}_{\Omega\,\max}$) is less than MAPE($\widehat{\theta}_{[1]}$) for values of $\gamma > 1$, and MAPE($\widehat{P}^{[5]}_{\Omega\,\min}$) is less than MAPE($\widehat{\theta}_{[5]}$) for values of $\gamma > 0.1$. In other words, the probabilities are out-performing the conditional mean measures, when the variance of inefficiency, σ_u^2, is large. For example in Table 7, $n = 25$, $\gamma = 5.0$, the MAPE for $\widehat{\theta}_{[1]}$, $\widehat{\theta}_{[5]}$, $\widehat{P}^{[1]}_{\Omega\,\max}$, $\widehat{P}^{[5]}_{\Omega\,\min}$ are 0.0890, 0.1633, 0.0688 and 0.0347, respectively. Our results are complicated by the fact that MAPE($\widehat{P}^{[5]}_{\Omega\,\min}$) had extremely large values in some simulations with large γ. These instances are indicated in the tables with double asterisks (**) and were due to a few draws where the true values of $u_{[5]m}$ were so large, that they generated approximation errors in the computer calculations of the probabilities. (This is the same approximation error discussed for the MSE, but made worse since we are now selecting $u_{[5]}$.) This is an unfortunate feature of the probabilities, but it is purely computational in nature (i.e., it could be corrected with a more accurate algorithm for calculating Φ). As for monotonicities in the MAPE, all measures improve with n as expected. Both $\widehat{P}^{[1]}_{\Omega\,\max}$ and $\widehat{P}^{[5]}_{\Omega\,\min}$ appear to have MAPE non-increasing in γ as well, except in one case for $\widehat{P}^{[5]}_{\Omega\,\min}$ (and this may be due to approximation error in Φ). The MAPE of $\widehat{\theta}_{[1]}$ and $\widehat{\theta}_{[5]}$ reaches a minimum MAPE at or below $\gamma = 1$ in all panel configurations.

4 Conclusions

This study provides evidence on the sampling performance of two very different technical efficiency estimators that are used to assess absolute and relative firm-level efficiency, based on parametric stochastic frontier models. We find that both the traditional conditional mean estimates and the efficiency probabilities appear to be monotonically more precise as n increase. However, the effect of the variance ratio ($\gamma = \sigma_u^2/\sigma_v^2$) is more complicated. The efficiency probabilities out-perform the conditional mean when γ is greater than one. This is the empirically (and theoretically) important case for the frontier model. Our precision assessments are based on the unitless mean absolute percentage error, the only measure that could be used for comparison of these different estimators.

We are aware that we have introduced two other source of variability in our study. One follows from the quantities of interest varying over m, and the other follows from our random sample of five firms for each m to calculate the measures of interest. The first source of variability could not be avoided and underscores the fact that efficiency "estimates" are not estimates of *traditional population parameters*. They are, in fact, proxies for an unobserved realization from inefficiency distributions. This is precisely the challenge that the frontier literature presents, and it is manifest in our study. The second source of variability was included by choice to relieve some computational burden. However, this variability is purely random and effects the all efficiency estimators in similar ways. Finally, approximation error in calculating Φ may have invalidated (or precluded) simulation results for the largest values of γ, but the results for moderate values of γ are to be believed.

In conclusion, we argue for use of efficiency probabilities rather than the conditional means of firm-specific inefficiency distributions to assess firm-level efficiency and its rank. Beyond the philosophical justifications, we find evidence that the probabilities out-perform the conditional means in terms of mean absolute percentage error when the signal-to-noise ratio of inefficiency is high. We encourage the continued use and study of the probabilities in future applied and theoretical work.

5 References

Aigner, D.J., C.A.K. Lovell And P. Schmidt, 1977, Formulation and estimation of stochastic frontier production functions, Journal of Econometrics, 6, 21-37.

Baltagi, B.H., 2005, Econometric Analysis Panel Data, Wiley, New York.

Battese, G.E. and T.J. Coelli, 1988, Prediction of firm-level technical efficiencies with a generalized frontier production function and panel data, Journal of Econometrics, 38, 387-399.

Battese, G.E. and T.J. Coelli, 1992, Frontier production functions, technical efficiency and panel data: with application to paddy farmers in India, Journal of Productivity Analysis, 3, 153-170.

Bera, A.K. and S.C. Sharma, 1999, Estimating production uncertainty in stochastic frontier models, Journal of Productivity Analysis, 12, 187-210.

Cuesta, R.A., 2000, A production model with firm-specific temporal variation in technical efficiency: with application to Spanish dairy farms, Journal of Productivity Analysis, 13, 139-58.

Fernandez, C., G. Koop and M.F.J. Steel, 2002, Multiple output production with undesirable outputs: an application to nitrogen surplus in agriculture, Journal of the American

Statistical Association, 97, 432-442.

Greene, W.H,. Forthcoming, Reconsidering heterogeneity in panel data estimators of the stochastic frontier model, Journal of Econometrics (this issue).

Horrace, W.C., 2005, On ranking and selection from independent truncated normal distributions Journal of Econometrics, 126 (2005): 335-354.

Horrace, W.C. and P. Schmidt, 1996, Confidence statements for efficiency estimates from stochastic frontier models, Journal of Productivity Analysis, 7, 257-82.

Horrace, W.C. and P. Schmidt, 2000, Multiple comparisons with the best, with economic applications, Journal of Applied Econometrics, 15, 1-26.

Jondrow, J., C.A.K. Lovell, I.S. Materov and P. Schmidt 1982, On the estimation of technical efficiency in the stochastic production function model, Journal of Econometrics, 19, 233-238.

Hsiao, C., 1986, The Analysis of Panel Data. Cambridge University Press, Cambridge.

Kim Y. and P. Schmidt, 2000, A review and empirical comparison of Bayesian and classical approaches to inference on efficiency levels in stochastic frontier models with panel data, Journal of Productivity Analysis, 14, 91-118.

Koop G., J. Osiewalski and M.F.J. Steel, 1997, Bayesian efficiency analysis through individual effects: hospital cost frontiers, Journal of Econometrics, 77-105.

Kumbhakar, S.C., 1990, Production frontiers, panel data, and time-varying technical inefficiency, Journal of Econometrics, 46, 201-11.

Kumbhakar, S.C., 1996, Estimation of cost efficiency with heteroscedasticity: an application to electric utilities in Texas, 1966-1985, Journal of The Royal Statistical Society, Ser. D (The Statistician), 45, 319-335.

Olson JA, Schmidt P, Waldman DM (1980). A Monte Carlo study of estimators of stochastic frontier production functions. Journal of Econometrics, 13, 67-82 .

Tsionas, E.G., 2002, Stochastic frontier models with random coefficients, Journal of Applied Econometrics, 17, 127- 47.

Table 1. Mean squared error, T = 5.

		Variance Ratio ($\gamma = \frac{\sigma_u^2}{\sigma_v^2}$)				
n	Statistic	0.1	0.5	1.0	5.0	10
25	*COLS failure rate*	0.3332	0.0454	0.0024	0	0
	$\widehat{\theta}_j$	0.0223	0.0145	0.0065	0.0046	0.0051*
	$\widehat{P}^j_{\Omega\,\max}$	0.0031	0.0021	0.0011	0.0007	0.0007
	$\widehat{P}^j_{\Omega\,\min}$	0.0053	0.0030	0.0010	0.0003	0.0002
100	*COLS failure rate*	0.1624	0	0	0	0
	$\widehat{\theta}_j$	0.0116	0.0020	0.0012	0.0010	0.0012*
	$\widehat{P}^j_{\Omega\,\max}$	0.0016	0.0003	0.0002	0.0002	0.0002
	$\widehat{P}^j_{\Omega\,\min}$	0.0030	0.0004	0.0002	0.0001	0.0001
500	*COLS failure rate*	0.0110	0	0	0	0
	$\widehat{\theta}_j$	0.0025	0.0004	0.0002	0.0002	0.0002
	$\widehat{P}^j_{\Omega\,\max}$	0.0004	0.0001	0.0000	0.0000	0.0000
	$\widehat{P}^j_{\Omega\,\min}$	0.0007	0.0001	0.0000	0.0000	0.0000

* - indicates a non-montonicity in γ.

Table 2. Mean squared error, T = 10.

		Variance Ratio ($\gamma = \frac{\sigma_u^2}{\sigma_v^2}$)				
n	Statistic	0.1	0.5	1.0	5.0	10
25	*COLS failure rate*	0.1942	0.0026	0	0	0
	$\widehat{\theta}_j$	0.0134	0.0052	0.0038	0.0045*	0.0051*
	$\widehat{P}^j_{\Omega\max}$	0.0028	0.0009	0.0006	0.0005	0.0004
	$\widehat{P}^j_{\Omega\min}$	0.0047	0.0007	0.0002	0.0001	0.0007*
100	*COLS failure rate*	0.0312	0	0	0	0
	$\widehat{\theta}_j$	0.0041	0.0010	0.0009	0.0011	0.0013*
	$\widehat{P}^j_{\Omega\max}$	0.0009	0.0002	0.0001	0.0001	0.0001
	$\widehat{P}^j_{\Omega\min}$	0.0015	0.0001	0.0001	0.0000	0.0004*
500	*COLS failure rate*	0	0	0	0	0
	$\widehat{\theta}_j$	0.0006	0.0002	0.0002	0.0002	0.0003*
	$\widehat{P}^j_{\Omega\max}$	0.0001	0.0000	0.0000	0.0000	0.0000
	$\widehat{P}^j_{\Omega\min}$	0.0002	0.0000	0.0000	0.0000	0.0002*

* - indicates a non-monotonicity in γ.

Table 3. Mean squared error, T = 20.

		Variance Ratio ($\gamma = \frac{\sigma_u^2}{\sigma_v^2}$)				
n	Statistic	0.1	0.5	1.0	5.0	10
25	*COLS failure rate*	0.0572	0.0004	0	0	0
	$\widehat{\theta}_j$	0.0063	0.0030	0.0032*	0.0048*	0.0055*
	$\widehat{P}^j_{\Omega\max}$	0.0019	0.0005	0.0004	0.0003	0.0003
	$\widehat{P}^j_{\Omega\min}$	0.0028	0.0002	0.0001	0.0006*	0.0051*
100	*COLS failure rate*	0	0	0	0	0
	$\widehat{\theta}_j$	0.0011	0.0007	0.0008*	0.0012*	0.0014*
	$\widehat{P}^j_{\Omega\max}$	0.0003	0.0001	0.0001	0.0001	0.0001
	$\widehat{P}^j_{\Omega\min}$	0.0003	0.0000	0.0000	0.0002*	0.0031*

* - indicates a non-monotonicity in γ.

Table 4. Bias at maximal or minimal efficiency, T = 5.

		Variance Ratio ($\gamma = \frac{\sigma_u^2}{\sigma_v^2}$)				
n	Statistic	0.1	0.5	1.0	5.0	10
25	$\widehat{\theta}_{[1]}$	0.0396	0.0251	0.0097	0.0068	0.0081
	$\widehat{P}^{[1]}_{\Omega\,\max}$	-0.0043	-0.0092	-0.0050	-0.0017	-0.0029
	$\widehat{\theta}_{[5]}$	0.0346	0.0436	0.0225	0.0077	0.0062
	$\widehat{P}^{[5]}_{\Omega\,\min}$	-0.0082	-0.0161	-0.0084	-0.0009	-0.0009
100	$\widehat{\theta}_{[1]}$	0.0281	0.0040	0.0026	0.0015	0.0003
	$\widehat{P}^{[1]}_{\Omega\,\max}$	-0.0033	-0.0012	-0.0014	-0.0009	-0.0004
	$\widehat{\theta}_{[5]}$	0.0297	0.0077	0.0058	0.0020	0.0010
	$\widehat{P}^{[5]}_{\Omega\,\min}$	-0.0086	-0.0026	-0.0017	-0.0002	-0.0003
500	$\widehat{\theta}_{[1]}$	0.0069	0.0004	0.0003	0.0004	0.0003
	$\widehat{P}^{[1]}_{\Omega\,\max}$	-0.0011	-0.0003	-0.0002	-0.0002	0.0000
	$\widehat{\theta}_{[5]}$	0.0079	0.0011	0.0008	0.0004	0.0003
	$\widehat{P}^{[5]}_{\Omega\,\min}$	-0.0024	-0.0004	-0.0003	0.0000	0.0000

Table 5. Bias at maximal or minimal efficiency, T = 10.

		Variance Ratio ($\gamma = \frac{\sigma_u^2}{\sigma_v^2}$)				
n	Statistic	0.1	0.5	1.0	5.0	10
25	$\widehat{\theta}_{[1]}$	0.0321	0.0102	0.0074	0.0041	0.0084
	$\widehat{P}^{[1]}_{\Omega\max}$	-0.0071	-0.0050	-0.0041	-0.0021	-0.0025
	$\widehat{\theta}_{[5]}$	0.0351	0.0213	0.0144	0.0055	0.0050
	$\widehat{P}^{[5]}_{\Omega\min}$	-0.0131	-0.0071	-0.0025	-0.0004	-0.0026
100	$\widehat{\theta}_{[1]}$	0.0105	0.0019	0.0016	0.0017	0.0015
	$\widehat{P}^{[1]}_{\Omega\max}$	-0.0031	-0.0011	-0.0012	-0.0004	-0.0007
	$\widehat{\theta}_{[5]}$	0.0137	0.0047	0.0033	0.0017	0.0011
	$\widehat{P}^{[5]}_{\Omega\min}$	-0.0070	-0.0014	-0.0006	-0.0001	-0.0004
500	$\widehat{\theta}_{[1]}$	0.0017	0.0003	0.0003	0.0006	-0.0003
	$\widehat{P}^{[1]}_{\Omega\max}$	-0.0003	-0.0002	-0.0001	0.0001	-0.0001
	$\widehat{\theta}_{[5]}$	0.0023	0.0008	0.0006	0.0002	0.0000
	$\widehat{P}^{[5]}_{\Omega\min}$	-0.0010	-0.0001	-0.0001	-0.0001	-0.0009

Table 6. Bias at maximal or minimal efficiency, T = 20.

		Variance Ratio ($\gamma = \frac{\sigma_u^2}{\sigma_v^2}$)				
n	Statistic	0.1	0.5	1.0	5.0	10
25	$\widehat{\theta}_{[1]}$	0.0153	0.0053	0.0049	0.0034	0.0052
	$\widehat{P}^{[1]}_{\Omega\,\max}$	-0.0078	-0.0031	-0.0026	-0.0020	-0.0016
	$\widehat{\theta}_{[5]}$	0.0233	0.0117	0.0087	0.0043	0.0043
	$\widehat{P}^{[5]}_{\Omega\,\min}$	-0.0157	-0.0027	-0.0010	-0.0013	-0.0019
100	$\widehat{\theta}_{[1]}$	0.0032	0.0017	0.0015	0.0002	0.0002
	$\widehat{P}^{[1]}_{\Omega\,\max}$	-0.0013	-0.0010	-0.0007	-0.0006	0.0002
	$\widehat{\theta}_{[5]}$	0.0055	0.0031	0.0021	0.0010	0.0010
	$\widehat{P}^{[5]}_{\Omega\,\min}$	-0.0031	-0.0006	-0.0001	-0.0009	-0.0017

Table 7. MAPE at maximal or minimal efficiency, T = 5.

		Variance Ratio ($\gamma = \frac{\sigma_u^2}{\sigma_v^2}$)				
n	Statistic	0.1	0.5	1.0	5.0	10
25	$\widehat{\theta}_{[1]}$	0.1344	0.0911	0.0731	0.0890	0.1044
	$\widehat{P}^{[1]}_{\Omega\,\max}$	0.1960	0.1339	0.1119	0.0688	0.0616
	$\widehat{\theta}_{[5]}$	0.1943	0.2170	0.1875	0.1633	0.1651
	$\widehat{P}^{[5]}_{\Omega\,\min}$	0.2455	0.1461	0.0912	0.0347	0.0266
100	$\widehat{\theta}_{[1]}$	0.0865	0.0363	0.0336	0.0422	0.0494
	$\widehat{P}^{[1]}_{\Omega\,\max}$	0.1258	0.0582	0.0471	0.0355	0.0315
	$\widehat{\theta}_{[5]}$	0.1255	0.0934	0.0862	0.0769	0.0780
	$\widehat{P}^{[5]}_{\Omega\,\min}$	0.1673	0.0565	0.0368	0.0176	0.0135
500	$\widehat{\theta}_{[1]}$	0.0360	0.0157	0.0145	0.0192	0.0219
	$\widehat{P}^{[1]}_{\Omega\,\max}$	0.0579	0.0244	0.0209	0.0166	0.0137
	$\widehat{\theta}_{[5]}$	0.0551	0.0405	0.0376	0.0347	0.0343
	$\widehat{P}^{[5]}_{\Omega\,\min}$	0.0727	0.0239	0.0165	0.0077	0.0053

Table 8. MAPE at maximal or minimal efficiency, T = 10.

		Variance Ratio ($\gamma = \frac{\sigma_u^2}{\sigma_v^2}$)				
n	Statistic	0.1	0.5	1.0	5.0	10
25	$\widehat{\theta}_{[1]}$	0.0872	0.0561	0.0588	0.0925	0.1067
	$\widehat{P}^{[1]}_{\Omega\max}$	0.1650	0.0908	0.0718	0.0494	0.0389
	$\widehat{\theta}_{[5]}$	0.1478	0.1398	0.1342	0.1447	0.1498
	$\widehat{P}^{[5]}_{\Omega\min}$	0.2065	0.0679	0.0409	0.0174	**
100	$\widehat{\theta}_{[1]}$	0.0408	0.0253	0.0286	0.0458	0.0523
	$\widehat{P}^{[1]}_{\Omega\max}$	0.0822	0.0403	0.0345	0.0259	0.0216
	$\widehat{\theta}_{[5]}$	0.0781	0.0641	0.0634	0.0714	0.0735
	$\widehat{P}^{[5]}_{\Omega\min}$	0.0989	0.0280	0.0184	0.0088	**
500	$\widehat{\theta}_{[1]}$	0.0162	0.0114	0.0124	0.0201	0.0234
	$\widehat{P}^{[1]}_{\Omega\max}$	0.0335	0.0176	0.0151	0.0109	0.0092
	$\widehat{\theta}_{[5]}$	0.0319	0.0279	0.0277	0.0313	0.0324
	$\widehat{P}^{[5]}_{\Omega\min}$	0.0403	0.0119	0.0081	0.0037	**

Table 9. MAPE at maximal or minimal efficiency, T = 20.

		Variance Ratio ($\gamma = \frac{\sigma_u^2}{\sigma_v^2}$)				
n	Statistic	0.1	0.5	1.0	5.0	10
25	$\widehat{\theta}_{[1]}$	0.0493	0.0448	0.0561	0.0973	0.1130
	$\widehat{P}^{[1]}_{\Omega\,\max}$	0.1233	0.0638	0.0568	0.0339	0.0278
	$\widehat{\theta}_{[5]}$	0.1073	0.1005	0.1070	0.1346	0.1464
	$\widehat{P}^{[5]}_{\Omega\,\min}$	0.1326	0.0301	0.0194	**	**
100	$\widehat{\theta}_{[1]}$	0.0201	0.0220	0.0276	0.0474	0.0558
	$\widehat{P}^{[1]}_{\Omega\,\max}$	0.0507	0.0313	0.0268	0.0185	0.0142
	$\widehat{\theta}_{[5]}$	0.0473	0.0476	0.0520	0.0662	0.0708
	$\widehat{P}^{[5]}_{\Omega\,\min}$	0.0496	0.0140	0.0093	0.0225	**