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An N-Dimensional Version of the Beurling-Ahlfors Extension

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AN N-DIMENSIONAL VERSION OF THE BEURLING-AHLFORS EXTENSION

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Abstract. We extend monotone quasiconformal mappings from dimension n to $n+1$ while preserving both monotonicity and quasiconformality. The extension is given explicitly by an integral operator. In the case $n=1$ it yields a refinement of the Beurling-Ahlfors extension.

1. Introduction

Extension Problem. Given a mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathscr{A}$, find $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of class $\mathscr{A}$ such that the restriction of F to $\mathbb{R}^n$ agrees with f.

Let us introduce coordinate notation $x=(x^1,\dots,x^n)$ and $f=(f^1,\dots,f^n)$. By setting $F^i=f^i$ for $i=1,\dots,n$ and $F^{n+1}=x^{n+1}$ one immediately obtains a solution to the extension problem for many classes $\mathscr{A}$ such as continuous ($\mathscr{A}=C^0$), smooth ($\mathscr{A}=C^k$), homeomorphic, diffeomorphic, and (bi-)Lipschitz mappings.

When $\mathscr{A}=\mathcal{QC}$, the class of quasiconformal mappings, the extension problem is much more difficult. It was solved

- for $n=1$ by Beurling and Ahlfors [4] in 1956,
- for $n=2$ by Ahlfors [1] in 1964,
- for $n\le 3$ by Carleson [8] in 1974, and
- for all $n\ge 1$ by Tukia and Väisälä [16] in 1982.

The Tukia-Väisälä extension uses, among other things, Sullivan's theory [15] of deformations of Lipschitz embeddings. Our goal is to give an explicit extension for a subclass of $\mathcal{QC}$. Quasiconformal mappings can be defined as orientation-preserving quasisymmetric mappings [11, 17].

Definition 1.1. A homeomorphism $f\colon \mathbb{R}^n\to\mathbb{R}^n$ is quasisymmetric if there is a homeomorphism $\eta\colon [0,\infty)\to[0,\infty)$ such that

$$\frac{|f(x)-f(z)|}{|f(y)-f(z)|}\le \eta\left(\frac{|x-z|}{|y-z|}\right). \tag{1.1}$$

for $x,y,z\in\mathbb{R}^n$, $z\ne y$.

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One can say that quasisymmetry is a three-point condition. But there are two subclasses of $\mathcal{QC}$ that are defined by *two-point* conditions, namely bi-Lipschitz class $\mathcal{BL}$ and the class of nonconstant delta-monotone mappings [2, Chapter 3]. Recall that a mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is *monotone* if

$$\langle f(x) - f(y), x - y\rangle \geq 0 \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{1.2}$$

We called f *delta-monotone* if there exists $\delta > 0$ such that

$$\langle f(x) - f(y), x - y\rangle \geq \delta |f(x) - f(y)||x - y| \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{1.3}$$

The class of nonconstant delta-monotone mappings is denoted by $\mathcal{DM}$. When we want to specify the value of δ we write that f is δ-monotone.

In contrast to the bi-Lipschitz case, the extension problem for the class $\mathcal{DM}$ cannot be solved by means of the trivial extension. For example, the mapping $f(x) = |x|^p x$, $p > -1$, belongs to $\mathcal{DM}$ but its trivial extension does not (unless $p = 0$).

Main Result. *Let $n \geq 2$. For any mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{DM}$ there exists $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of class $\mathcal{DM}$ such that the restriction of F to $\mathbb{R}^n$ agrees with f.*

Our proof is by an explicit construction that can be viewed as an n-dimensional version of the Beurling-Ahlfors extension. Suppose $f \in \mathcal{DM}$. Let $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times [0, \infty)$ and

$$\phi(x) = (2\pi)^{-\frac{n}{2}} e^{-|x|^2/2}, \qquad x \in \mathbb{R}^n. \tag{1.4}$$

We define $F\colon \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}_+$ by

$$F^i(x,t) = \int_{\mathbb{R}^n} f^i(x + ty)\, \phi(y)\, dy \qquad i = 1, \dots, n \tag{1.5}$$

$$F^{n+1}(x,t) = \int_{\mathbb{R}^n} \langle f(x + ty), y\rangle\, \phi(y)\, dy \tag{1.6}$$

where $x \in \mathbb{R}^n$, $t \geq 0$ (see §4 for the convergence of these integrals). Observe that $F(x, 0) = (f(x), 0)$. Furthermore, $F^{n+1}(x,t) \geq 0$ because

$$\int_{\mathbb{R}^n} \langle f(x + ty), y\rangle\, \phi(y)\, dy = \int_{\mathbb{R}^n} \langle f(x + ty) - f(x), y\rangle\, \phi(y)\, dy \geq 0$$

due to the monotonicity of f. Finally, we extend F to $\mathbb{R}^{n+1}$ by reflection

$$F^i(x,t) = F^i(x,-t) \quad i = 1, \dots, n \qquad \text{and} \qquad F^{n+1}(x,t) = -F^{n+1}(x,-t).$$

Theorem 1.2. *Let $n \geq 2$. If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is δ-monotone, then $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is δ_1-monotone where δ_1 depends only on δ and n. In addition, $F\colon \mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ is bi-Lipschitz in the hyperbolic metric.*

Here $\mathbb{H}^{n+1} = \mathbb{R}^n \times (0, \infty)$ and the hyperbolic metric on $\mathbb{H}^{n+1}$ is $|dx|/x^{n+1}$. Theorem 1.2 can be also formulated for $n = 1$, in which case it becomes a refinement of the Beurling-Ahlfors extension theorem.

Proposition 1.3. *If $f\colon \mathbb{R} \to \mathbb{R}$ is increasing and quasisymmetric, then $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is δ_1-monotone where δ_1 depends only on η in Definition 1.1. Furthermore, $F\colon \mathbb{H}^2 \to \mathbb{H}^2$ is bi-Lipschitz in the hyperbolic metric.*

Fefferman, Kenig and Pipher [9, Lemma 4.4] proved that F in Proposition 1.3 is quasiconformal. Proposition 1.3 was originally proved in [12] using their result. In this paper we give a direct proof.

Theorem 1.2 has an application to mappings with a convex potential [7], i.e., those of the form $f = \nabla u$ with u convex. The basic properties and examples of quasiconformal mappings with a convex potential are given in [13].

Corollary 1.4. Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, is a K-quasiconformal mapping with a convex potential. Then f can be extended to a K_1-quasiconformal mapping $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with a convex potential, where K_1 depends only on K and n.

2. Preliminaries

Let $e_1, \dots, e_{n+1}$ be the standard basis of $\mathbb{R}^{n+1}$. All vectors are treated as column vectors. The transpose of a vector v is denoted by v^T. We use the operator norm $\|\cdot\|$ for matrices. A Borel measure μ on $\mathbb{R}^n$ is *doubling* if there exists $\mathscr{D}_\mu$, called the doubling constant of μ, such that

$$\mu(2B) \leq \mathscr{D}_\mu \, \mu(B)$$

for all balls $B = B(x, r)$. Here $2B = B(x, 2r)$.

The geometric definition of class $\mathcal{QC}$ given in the introduction is equivalent to the following analytic definition [11, 17].

Definition 2.1. A homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ ($n \geq 2$) is quasiconformal if $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and there exists a constant K such that the differential matrix $Df(x)$ satisfies the distortion inequality

$$\|Df(x)\|^n \leq K \det Df(x) \quad \text{a.e. in } \mathbb{R}^n.$$

Delta-monotone mappings also have an analytic definition.

Lemma 2.2. *Let Ω be a convex domain in $\mathbb{R}^n$, $n \geq 2$. Suppose $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is continuous. The following are equivalent:*

(i) f is δ-monotone in Ω for some $\delta > 0$; that is, (1.3) *holds for all $x, y \in \Omega$;*

(ii) there exists $\delta > 0$ such that for a.e. $x \in \Omega$ the matrix $Df(x)$ satisfies

$$v^T Df(x) v \geq \delta |Df(x)v||v| \qquad \text{for every vector } v \in \mathbb{R}^n;$$

(iii) there exists $\gamma > 0$ such that for a.e. $x \in \Omega$ the matrix $Df(x)$ satisfies

$$v^T Df(x) v \geq \gamma \|Df(x)\| |v|^2 \qquad \text{for every vector } v \in \mathbb{R}^n.$$

The constants δ and γ depend only on each other.

Proof. The equivalence of (i) and (ii), with the same constant δ, was proved in [12, p. 397]. It is obvious that (iii) implies (ii) with $\delta = \gamma$. It remains to establish the converse implication (ii) $\Longrightarrow$ (iii). To this end we need the following

Claim: if a real square matrix A satisfies

$$v^T A v \geq \delta |Av||v| \qquad \text{for every } v \in \mathbb{R}^n$$

then

$$|Av| \geq c\|A\||v| \qquad c = c(\delta) > 0. \tag{2.1}$$

Although this claim is known, even with a sharp constant [3], we give a proof for the sake of completeness. It suffices to estimate $|Av|$ from below under the assumptions that $Av \neq 0$ and $\|A\| = 1 = |v|$. Let u be a unit vector in $\mathbb{R}^n$ such that $|Au| = 1$. Replacing u by $-u$ if necessary we may assume that $u^T A v + v^T A u \leq 0$. Let $\lambda = \sqrt{|Av|}$. On one hand we have

$$(\lambda u + v)^T A(\lambda u + v) \leq \lambda^2 u^T A u + v^T A v \leq \lambda^2 + \lambda^2 = 2\lambda^2. \tag{2.2}$$

On the other hand

$$(\lambda u + v)^T A(\lambda u + v) \geq \delta |\lambda A u + Av||\lambda u + v| \geq \delta(\lambda - \lambda^2)(1 - \lambda). \tag{2.3}$$

Combining (2.2) and (2.3) we obtain $2\lambda \geq \delta(1-\lambda)^2$, hence

$$\lambda \geq \delta^{-1} + 1 - \sqrt{(\delta^{-1}+1)^2 - 1} > 0.$$

This proves the claim. $\square$

3. Delta-monotone mappings and doubling measures

The following result shows that $\mathcal{DM} \subset \mathcal{QC}$. In particular, $f \in \mathcal{DM}$ implies that f is a continuous Sobolev mapping, and therefore (ii)–(iii) of Lemma 2.2 hold.

Proposition 3.1. [12, Theorem 6] *Every nonconstant δ-monotone mapping is η-quasisymmetric where η depends only on δ.*

It is well-known that quasisymmetric mappings are closely related to doubling measures [11]. The following lemma is another instance of this relation.

Lemma 3.2. *For any nonconstant δ-monotone mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ ($n \geq 2$) the measure $\mu = \|Df(x)\|\,dx$ is doubling. The doubling constant $\mathscr{D}_\mu$ depends only on δ and n.*

Proof. Recall that f is quasisymmetric. Lemma 3.2 in [14] implies the existence of a constant $C = C(\delta, n)$ such that

$$C^{-1}\frac{\operatorname{diam} f(B)}{\operatorname{diam} B} \leq \frac{1}{|B|}\int_B \|Df\|\,dx \leq C\frac{\operatorname{diam} f(B)}{\operatorname{diam} B} \tag{3.1}$$

for all balls $B \subset \mathbb{R}^n$. Since $\operatorname{diam} f(2B) \leq C \operatorname{diam} f(B)$ with $C = C(\eta)$, the lemma follows. $\square$

Recall that $\phi\colon \mathbb{R}^n \to (0,\infty)$ is the Gaussian kernel (1.4). Let $\mathbb{B} = B(0,1)$ be the open unit ball in $\mathbb{R}^n$.

Lemma 3.3. *Let μ be a doubling measure in $\mathbb{R}^n$ and $p \geq 0$. Let Ω be either $\mathbb{R}^n$ or the half space $\{y\colon \langle y,\xi\rangle \geq 0\}$ for some $\xi \in \mathbb{R}^n$. Then*

$$C^{-1}\mu(\mathbb{B}) \leq \int_\Omega |y|^p \phi(y)\, d\mu(y) \leq C\mu(\mathbb{B}) \tag{3.2}$$

where the constant C depends only on $\mathscr{D}_\mu$, p and n.

Proof. We begin by estimating the integral in (3.2) from above as follows

$$\int_{\mathbb{R}^n} |y|^p\phi(y)\, d\mu(y) = \int_{\mathbb{B}} |y|^p\phi(y)\, d\mu(y) + \sum_{k=0}^{\infty}\int_{2^k<|y|\leq 2^{k+1}} |y|^p\phi(y)\, d\mu(y),$$

where

$$\int_{\mathbb{B}} |y|^p\phi(y)\, d\mu(y) \leq \phi(0)\mu(\mathbb{B}) = (2\pi)^{-\frac{n}{2}}\mu(\mathbb{B})$$

and

$$\begin{aligned}\int_{2^k<|y|\leq 2^{k+1}} |y|^p\phi(y)\, d\mu(y) &\leq 2^{p(k+1)}(2\pi)^{-\frac{n}{2}} e^{-2^{2k-1}}\mu(B(0,2^{k+1}))\\ &\leq 2^{p(k+1)}(2\pi)^{-\frac{n}{2}} e^{-2^{2k-1}}\mathscr{D}_\mu^{k+1}\mu(\mathbb{B}).\end{aligned}$$

Summing over $k = 0,1,2\dots$ we obtain

$$\int_{\mathbb{R}^n} \phi(y)\, d\mu(y) \leq C\,\mu(\mathbb{B})$$

where $C = C(\mathscr{D}_\mu, p, n) > 0$.

We turn to the left side of (3.2). The inequality

$$|y|^p\phi(y) \geq \frac{e^{-1/2}}{2^p(2\pi)^{n/2}} \qquad \text{for} \quad \frac{1}{2} \leq |y| \leq 1$$

implies

$$\int_\Omega |y|^p\phi(y)\, d\mu(y) \geq \frac{e^{-1/2}}{2^p(2\pi)^{n/2}}\mu(\Omega\cap\{1/2\leq |y|\leq 1\}).$$

Since $\mu(\Omega\cap\{1/2\leq|y|\leq 1\}) \geq \mathscr{D}_\mu^{-3}\mu(\mathbb{B})$, the left side of (3.2) follows. □

4. Proof of main results

Proof of Theorem 1.2. Since f is quasisymmetric by Proposition 3.1, it satisfies the growth condition $|f(x)| \leq \alpha|x|^p + \beta$ for some constants α, β, p, see [11, Theorem 11.3]. Therefore, the integrals (1.5) and (1.6) converge and F is C^∞-smooth in $\mathbb{H}^{n+1}$. Let $\gamma = \gamma(\delta) > 0$ be as in part (iii) of Lemma 2.2.

Our first step is to prove that for $(x,t) \in \mathbb{H}^{n+1}$ the matrix $\mathscr{B} := DF(x,t)$ satisfies the condition

$$w^T\mathscr{B}w \geq \gamma_1\|\mathscr{B}\||w|^2 \qquad \text{for every vector } w \in \mathbb{R}^{n+1} \tag{4.1}$$

where $\gamma_1 = \gamma_1(\delta, n) > 0$. Fix $x \in \mathbb{R}^n$ and $t > 0$. We compute the partial derivatives of F at $(x,t) \in \mathbb{H}^{n+1}$ as follows.

$$\begin{aligned}
\frac{\partial F^i}{\partial x_j} &= \int_{\mathbb{R}^n} f^i_j(x+ty)\phi(y)dy, \quad 1 \le i,j \le n; \\
\frac{\partial F^i}{\partial t} &= \int_{\mathbb{R}^n} \sum_{j=1}^n f^i_j(x+ty)y^i\phi(y)dy, \quad 1 \le i \le n; \\
\frac{\partial F^{n+1}}{\partial x_j} &= \int_{\mathbb{R}^n} \sum_{i=1}^n f^i_j(x+ty)y^j\phi(y)dy, \quad 1 \le j \le n; \\
\frac{\partial F^{n+1}}{\partial t} &= \int_{\mathbb{R}^n} \sum_{i=1}^n \sum_{j=1}^n f^i_j(x+ty)y^iy^j\phi(y)dy.
\end{aligned}$$

To simplify formulas we write $A(y) = Df(x+ty)$ and let $B(y)$ be the $(n+1)\times(n+1)$ matrix written in block form below.

$$B(y) = \left(\begin{array}{c:c} A(y) & A(y)y \\ \hdashline y^T A(y) & y^T A(y)y \end{array}\right). \tag{4.2}$$

With this notation we have

$$DF(x,t) = \int_{\mathbb{R}^n} B(y)\phi(y)\,dy. \tag{4.3}$$

First we show that the norm of $\mathscr{B}$ is dominated by the quantity

$$\alpha := \int_{B(0,1)} \|A(y)\|\,dy.$$

Indeed,

$$\|\mathscr{B}\| \le \int_{\mathbb{R}^n} \|B(y)\|\phi(y)\,dy \le \int_{\mathbb{R}^n} \|A(y)\|(1+|y|)^2\phi(y)\,dy.$$

By Lemma 3.2 the measure $\mu = \|A(y)\|\,dy$ is doubling. Applying Lemma 3.3 we obtain

$$\|\mathscr{B}\| \le C\alpha, \qquad C = C(\delta, n). \tag{4.4}$$

Next we estimate the quadratic form $w \mapsto w^T\mathscr{B}w$ generated by $\mathscr{B}$ from below. For this we fix a vector $w \in \mathbb{R}^{n+1}$, written as $w = v + se_{n+1}$ with $v \in \mathbb{R}^n$ and $s \in \mathbb{R}$. It is easy to see that

$$w^T B(y) w = (v+sy)^T A(y)(v+sy).$$

Let $\Omega = \{y \in \mathbb{R}^n \colon \langle v, sy\rangle \geq 0\}$. Then

$$\begin{aligned} w^T \mathscr{B} w &= \int_{\mathbb{R}^n} \left\{(v+sy)^T A(y)(v+sy)\right\} \phi(y)\, dy \\ &\geq \gamma \int_{\mathbb{R}^n} \|A(y)\| |v+sy|^2 \phi(y)\, dy \\ &\geq \gamma \int_{\Omega} \|A(y)\| |v+sy|^2 \phi(y)\, dy \\ &\geq \gamma |v|^2 \int_{\Omega} \|A(y)\| \phi(y)\, dy + \gamma s^2 \int_{\Omega} \|A(y)\| |y|^2 \phi(y)\, dy. \end{aligned}$$

Applying Lemma 3.3 with $\mu = \|A(y)\|\, dy$ we obtain

$$w^T \mathscr{B} w \geq c\,\alpha\gamma(|v|^2 + s^2) = c\,\alpha\gamma |w|^2, \qquad c = c(\delta, n). \tag{4.5}$$

Combining (4.4) and (4.5) we obtain (4.1) with $\gamma_1 = (c/C)\gamma$. By virtue of Lemma 2.2 F is δ_1-monotone in the upper half-space $\mathbb{H}^{n+1}$ where $\delta_1 = \delta_1(\delta, n)$. By symmetry, F is also δ_1-monotone in the lower half-space.

To prove that F is δ_1-monotone in the entire space $\mathbb{R}^{n+1}$, we consider two points $a, b \in \mathbb{R}^{n+1}$ such that the line segment $[a, b]$ crosses the hyperplane $\mathbb{R}^n$ at some point c. We have

$$\begin{aligned} \langle F(a) - F(b), a-b\rangle &= \langle f(a) - f(c), a-b\rangle + \langle F(c) - F(b), a-b\rangle \\ &\geq \delta_1 |F(a) - F(c)||a-b| + \delta_1 |F(c) - F(b)||a-b| \\ &\geq \delta_1 |F(a) - F(b)||a-b| \end{aligned}$$

Therefore, $F \in \mathcal{DM}$.

It remains to show that $F \colon \mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ is bi-Lipschitz in the hyperbolic metric. Since $F \in \mathcal{QC}$ and $\mathbb{H}^{n+1}$ is a geodesic space, it suffices to prove that

$$\|DF(x,t)\| \approx \frac{F^{n+1}(x,t)}{t}. \tag{4.6}$$

Here $X \approx Y$ means that X and Y are comparable, i.e., $C^{-1} Y \leq X \leq CY$ where $C = C(\delta, n)$. It follows from (4.4) and (4.5) that $\|DF(x,t)\|$ is comparable to the integral average of $\|Df\|$ over the ball $B(x,t)$. By (3.1) this average is comparable to $t^{-1} \operatorname{diam} f(B(x,t))$. The quasisymmetry of F implies (cf. [11, 11.18])

$$\operatorname{diam} f(B(x,t)) \approx |F(x,t) - F(x, t/2)| \approx F^{n+1}(x,t).$$

This proves (4.6). □

Proof of Proposition 1.3. The proof of Theorem 1.2 also works in the case $n = 1$ with the following interpretation. Since quasisymmetric mappings on the line need not be absolutely continuous [4], the derivative f' must be understood in the sense of distributions. In fact, $\mu := f'$ is a positive doubling measure with $\mathscr{D}_\mu = \mathscr{D}_\mu(\eta)$ [11, 13.20]. Lemma 3.2 is not needed in this case. The rest of the proof carries over with $\gamma = 1$ and $\gamma_1 = \gamma_1(\mathscr{D}_\mu)$. □

Proof of Corollary 1.4. According to [12, Lemma 18], a K-quasiconformal mapping with a convex potential is also δ-monotone with $\delta = \delta(K, n)$. Let F be the δ_1-monotone extension of f provided by Theorem 1.2. Since the differential matrix Df is symmetric, the formulas (4.2) and (4.3) show that DF is symmetric as well. In addition, DF is positive semidefinite by Lemma 2.2. Thus, $F = \nabla U$ for some convex function $U\colon \mathbb{R}^{n+1} \to \mathbb{R}$. □

5. Concluding remarks

Both classes $\mathcal{QC}$ (quasiconformal) and $\mathcal{BL}$ (bi-Lipschitz) are groups under composition. However, the class of delta-monotone mappings $\mathcal{DM}$ is not closed under composition (consider the rotation of the complex plane given by $z \mapsto e^{i\theta}z$ where $|\theta| < \pi/2$). Let $\mathcal{QC}_d \subset \mathcal{QC}$ be the group generated by $\mathcal{BL}$ and $\mathcal{DM}$. In other words, f belongs to $\mathcal{QC}_d$ if it can be decomposed into bi-Lipschitz and delta-monotone mappings. This should be compared with the notion of polar factorization of mappings introduced by Brenier [6].

Theorem 1.2 together with the trivial extension of bi-Lipschitz mappings yield a solution to the extension problem for $\mathcal{QC}_d$.

Corollary 5.1. Let $n \geq 2$. For any mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{QC}_d$ there exists $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of class $\mathcal{QC}_d$ such that the restriction of F to $\mathbb{R}^n$ agrees with f.

It seems likely that $\mathcal{QC}_d$ is a proper subset of $\mathcal{QC}$. This motivates the following question:

Question 5.2. Which quasiconformal mappings are decomposable?

Both bi-Lipschitz and delta-monotone mappings take smooth curves into rectifiable curves [2, Theorem 3.11.7]. This is no longer true for their composition. More precisely, for any $1 < \alpha < 2$ one can construct a mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f \in \mathcal{QC}_d$ and $f(\mathbb{R})$ has Hausdorff dimension at least α. To this end, one first finds a bi-Lipschitz mapping $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(\mathbb{R})$ contains a planar Cantor set E of dimension $0 < \beta < 1$ (see Lemma 3.1[5] and the comment after its proof). Second, there is a delta-monotone mapping $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the Hausdorff dimension of $h(E)$ is equal to α (see the construction in [10, Theorem 5]). Finally, let $f = h \circ g$.

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