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Mixed Problems and Layer Potentials for Harmonic and Biharmonic Functions

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ABSTRACT

The mixed problem is to find a harmonic or biharmonic function having prescribed Dirichlet data on one part of the boundary and prescribed Neumann data on the remainder. One must make a choice as to the required boundary regularity of solutions. When only weak regularity conditions are imposed, the harmonic mixed problem has been solved on smooth domains in the plane by Wendland, Stephan, and Hsiao. Significant advances were later made on Lipschitz domains by Ott and Brown. The strain of requiring a square-integrable gradient on the boundary, however, forces a strong geometric restriction on the domain. Well-known counterexamples by Brown show this restriction to be a necessary condition.

This thesis proves that these harmonic counterexamples are an anomaly, in that the mixed problem can be solved for all data modulo a finite dimensional subspace. The geometric restriction now required is significantly less stringent than the one referred to above. This result is proved by representing solutions in terms of single and double layer potentials, establishing a mixed Rellich inequality, and applying functional analytic arguments to solve a two-by-two system of equations. These results are then extended to allow Robin data in place of Neumann data.

This thesis also establishes counterexamples for the biharmonic mixed problem with Poisson ratio in the interval [-1,-.5]. These counterexamples are biharmonic analogues to the harmonic ones referred to above. Their exact form is obtained by solving a four-by-four system of equations.

MIXED PROBLEMS AND LAYER POTENTIALS FOR HARMONIC AND BIHARMONIC FUNCTIONS

By

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DISSERTATION

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Contents

List of Figures

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Introduction

In a 1994 paper [**Bro94**], R. Brown formulated and solved a *Mixed Problem* for certain domains in $\mathbb{R}^n$. To illustrate the required restriction on the class of domains used, we may consider the following special case of his result in $\mathbb{R}^2$:

THEOREM 0.0.1. *Let $\Omega = \{(x, y) : y > \phi(x)\}$, where ϕ is a real-valued Lipschitz function on $\mathbb{R}$ with $\phi(0) = 0$. Define $D = \partial\Omega \cap \{(x, y) : x < 0\}$ and $N = \partial\Omega \cap \{(x, y) : x \geq 0\}$. Furthermore, assume there is an $\epsilon > 0$ such that $\phi' < -\epsilon$ for a.e.$x < 0$ and $\phi \equiv 0$ for $x \geq 0$.*

Given $\lambda \geq 0$, $h_D \in W^{1,2}(D)$, and $h_N \in L^2(N)$ there exists a unique function u with nontangential maximal function bounds

$$\int_{\partial\Omega} |\nabla u^*|^2 + \lambda^2 |u^*|^2 \, d\sigma \leq \int_D |\partial_T h_D|^2 + \lambda^2 h_D^2 \; d\sigma + \int_N h_N^2 \; d\sigma$$

satisfying the conditions

$$\Delta u + \lambda u = 0 \text{ in } \Omega$$

$$u|_D = h_D$$

$$\partial_\nu u|_N = h_N$$

Here Δ is Laplace's operator, σ is surface measure, ∂_ν is the outer normal derivative, and boundary values are taken using nontangential limits. The superscript * indicates the nontangential maximal function.

When extending this result to bounded domains, Brown obtains a requirement that the sets N and D meet at interior angles strictly less than π. This thesis aims to loosen this restriction so that any interior angle $\neq \pi$ is allowed. There is a cost, however; we will now require the mixed data to be in a certain subspace of $W^{1,2}(D) \times L^2(N)$. Fortunately this new restriction presents only a finite dimensional problem. In **Chapter 3.1** we prove:

THEOREM 0.0.2. *Let Ω be a creased Lipschitz domain in the plane and $D \subset \partial\Omega$ be the finite union of m connected open sets with pairwise disjoint closures and $\overline{D} \neq \partial\Omega$. Then the quotient space $W^{1,2}(D)/W_0^{1,2}(D)$ has dimension $2m$.*

The conditions on ϕ in **Theorem 0.0.1** allow us to create a larger domain, $\tilde{\Omega}$ (with boundary $\partial\tilde{\Omega} = D \cup \{(x, -\phi(x)) : x \leq 0\}$), by reflecting Ω over N. We may then solve the Dirichlet Problem

$$\Delta u + \lambda u = 0 \text{ in } \tilde{\Omega}$$

$$u(x, \phi(x)) \;=\; u(x, -\phi(x)) \;=\; h_D(x).$$

Any such u is also a solution to the Mixed Problem on Ω with data

$$u|_D = h_D$$

$$\partial_\nu u|_N = 0.$$

Such a reflection is not possible for all domains in the plane. [**Bro94**] partially overcomes this by utilizing a Rellich inequality and the method of continuity. In **Chapter 3.2** we establish a Rellich Inequality better suited for our new approach to the method of continuity. With these tools, we are able to prove our main result in **Chapter 3.3**. Unlike the methods in [**Bro94**], our proof of the existence and regularity of solutions utilizes the theory of Layer Potentials.

Two additional mixed problems are addressed in this thesis. **Chapter 3.4** extends the above results to the Mixed Robin Problem, while **Chapter 4** establishes counterexamples for the Biharmonic Mixed Problem.

CHAPTER 1

Definitions

1.1. Function Spaces

Throughout this thesis Ω is a bounded and connected Lipschitz domain in the plane with connected boundary.[1] A *dissection* of Ω is a boundary decomposition $\partial\Omega = D \cup N$, where D is open, $N = \partial\Omega \setminus D$, and both sets have non-empty interior.

Our domains will be given a special type of dissection:

DEFINITION 1.1.1. A domain Ω with dissection $\partial\Omega = D \cup N$ is *strongly dissected* if there is a $C_0^\infty(\mathbb{R}^2)$ vector field α and a constant $\delta > 0$ such that $\alpha \cdot \nu > \delta$ on N and $\alpha \cdot \nu < -\delta$ on D.[2]

We will be working with a variety of function spaces throughout this thesis. If U is an open set in the plane, $C_0^\infty(U)$ denotes the space of infinitely differentiable functions with compact support in U. By $C_0^\infty(D)$ we mean the restriction of $C_0^\infty(\mathbb{R}^2 \backslash N)$ functions to $\partial\Omega$. Similarly, $C^\infty(\partial\Omega)$ denotes the restriction of $C^\infty(\mathbb{R}^2)$ functions to $\partial\Omega$.

[1] Lipschitz domains will be defined in **Chapter 1.2**.

[2] Strongly dissected domains are more general than the *creased Lipschitz domains* considered in [**Bro94**]. The differences between the two are examined in **Example 3.5.1**

$L^2(\partial\Omega)$ is the space of functions g defined almost everywhere on $\partial\Omega$ and bounded in the L^2 norm

$$||g||^2_{L^2(\partial\Omega)} := \int_{\partial\Omega} g^2\, d\sigma,$$

where $d\sigma$ denotes surface measure. If N is a closed subspace of $\partial\Omega$, we define $L^2_0(N)$ as the set of $g \in L^2(\partial\Omega)$ with support in N and $\int_N g\, d\sigma = 0$. Such g are said to have *mean value zero.*

Let D be an open subset of $\partial\Omega$. Since Ω is Lipschitz, every h in $C_0^\infty(D)$ has well-defined tangential derivatives. For such functions we may define the Sobolev norm

$$||h||^2_{W^{1,2}(D)} := \int_D h^2 + |\partial_T h|^2\, d\sigma.$$

We define the *Sobolev space* $W^{1,2}(D)$ as the set of functions $f \in L^2(D)$, having a companion function $f_t \in L^2(D)$ satisfying the *integration by parts* formula

$$\int_D f_t\, h\, d\sigma = -\int_D f\, \partial_T h\, d\sigma,$$

for every $h \in C_0^\infty(D)$. In this case we set $\partial_T f := f_t$ and call this a *tangential derivative.* We might also refer to f_t as being a *weak derivative.* Finally, we define $W_0^{1,2}(D)$ to be the closure of the set of $W^{1,2}(D)$ functions with support compactly contained in D.

Chapter 3 of this thesis is dedicated to mixed problems for harmonic functions in the plane.

DEFINITION 1.1.2. A twice-differentiable function u is *harmonic* on an open set $U \subset \mathbb{R}^n$ if $\sum_{i=1}^{n} \partial_i^2 u(x) = 0$ on U. The partial differential operator $\Delta := \sum_{i=1}^{n} \partial_i^2$ is called the *Laplacian* and $\Delta u = 0$ is *Laplace's equation*.

Definitions and function spaces for the Biharmonic Mixed Problem will be given in **Chapter 4**.

1.2. Boundary Values and Layer Potentials

Recall Ω is a bounded and connected Lipschitz domain in the plane with connected boundary. A *bounded Lipschitz domain* is defined by a finite collection of neighborhoods U_j and Lipschitz functions ϕ_j, where the U_j cover $\overline{\Omega}$ and

$$\Omega \cap U_j = \{(x_1, x_2) : \phi_j(x_1) < x_2\} \cap U_j \,,$$

after an appropriate translation and rotation of the coordinate system.

Given positive constants α and r, we define the *truncated cones*

$$\Gamma_{\alpha,r}(x) := \{z \in \Omega \cap B_r(x) : \ |x - z| < (1 + \alpha)\,\mathrm{dist}(z, \partial\Omega)\}$$

for each $x \in \partial\Omega$. We also define the *nontangential maximal function* of a harmonic function u on Ω by

$$u^*_{\alpha,r}(x) := \sup_{z \in \Gamma_{\alpha,r}(x)} |u(z)|\,.$$

When $u^*_{\alpha,r} \in L^2(\partial\Omega)$, u has well-defined *nontangential boundary values*

$$u(x) := \lim_{\substack{z\to x \\ z\in\Gamma_{\alpha,r}(x)}} u(z)$$

for almost every $x \in \partial\Omega$.[3]

Given an $L^2(\partial\Omega)$ function f, there is a unique harmonic function u and a parameter (α, r), depending only on Ω, such that $u^*_{\alpha,r} \in L^2(\partial\Omega)$ and $u(x) = f(x)$ on $\partial\Omega$. Since α and r are independent of f we omit them from the notation, instead writing $\Gamma(x)$ and $u^*(x)$.

We use ∇u to denote the *gradient* vector $(\partial_1 u, \partial_2 u)$. When $\nabla u^* \in L^2(\partial\Omega)$, the gradient of u also has nontangential boundary values, in the sense that

$$\vec{v}\cdot\nabla u(x) = \lim_{\substack{z\to x \\ z\in\Gamma(x)}} \vec{v}\cdot\nabla u(z),$$

for any constant vector $\vec{v}$ and almost every $x \in \partial\Omega$.

By considering the exterior domain $\Omega^e := \overline{\Omega}^c$, we may define *exterior* cones, maximal functions, and nontangetial limits. All exterior objects are written with the superscript 'e'. For example if x is on the boundary, $u^e(x)$ denotes the nontangential limit taken using exterior cones. On the other hand, both $u(x)$ and $u^i(x)$ denote the

[3] See Corollary 1.4.3 in [**Ken94**]

nontangential limit at x using *interior* cones.

On a Lipschitz domain the unit outer normal vector $\nu(x) = (\nu_1(x), \nu_2(x))$ and tangent vector $T(x) = (\nu_2(x), -\nu_1(x))$ are defined at almost every x on the boundary. $\langle \cdot, \cdot \rangle$ will denote the standard inner product in $\mathbb{R}^2$. When $\nabla u^* \in L^2(\partial\Omega)$ we may define the directional derivatives $\partial_\nu u(x) := \langle \nu(x), \nabla u(x) \rangle$ and $\partial_T u(x) := \langle T(x), \nabla u(x) \rangle$ for almost every $x \in \partial\Omega$.

K is the *double layer potential* operator

$$K(h)(x) := -\frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle x - y,\, \nu(y) \rangle}{|x - y|^2}\, h(y)\, d\sigma(y)$$

and S is the *single layer potential* operator

$$S(h)(x) := \frac{1}{2\pi} \int_{\partial\Omega} \log|x - y|\, h(y)\, d\sigma(y),$$

for any L^2 function h, and $x \in \mathbb{R}^2 \setminus \partial\Omega$. For $x \in \partial\Omega$ we also define the four following *principal value* integral operators:

$$\mathbf{K}(h)(x) := p.v.\ \frac{-1}{2\pi}\int_{\partial\Omega} \frac{\langle x-y,\, \nu(y)\,\rangle}{|x-y|^2}\ h(y)\ d\sigma(y)$$

$$\mathbf{S}(h)(x) := \frac{1}{2\pi}\int_{\partial\Omega} \log|x-y|\ h(y)\ d\sigma(y)$$

$$\partial_T\mathbf{S}(h)(x) := p.v.\ \frac{1}{2\pi}\int_{\partial\Omega} \frac{\langle x-y,\, T(x)\,\rangle}{|x-y|^2}\ h(y)\ d\sigma(y)$$

$$\begin{aligned}\partial_\nu\mathbf{S}(h)(x) &:= p.v.\ \frac{1}{2\pi}\int_{\partial\Omega} \frac{\langle x-y,\, \nu(x)\,\rangle}{|x-y|^2}\ h(y)\ d\sigma(y)\\ &= \mathbf{K}^*(h)(x)\end{aligned}$$

These are all bounded operators on $L^2(\partial\Omega)$.[4] When $f \in W^{1,2}(\Omega)$ and α is a $C^1(\mathbb{R}^2)$ vector field we may also define $\alpha \cdot \nabla\mathbf{K}$, via the identities of **Lemma 1.2.1**. This is a bounded operator from $W^{1,2}(\partial\Omega)$ to $L^2(\partial\Omega)$.

With this notation we can state the well-known *jump relations*.[5] For a.e. $x \in \partial\Omega$ we have

[4] See the paper [**CMM82**] by Coifman, McIntosh, and Meyer.
[5] See page 218 in [**McL00**].

$$\begin{aligned}\alpha\cdot\nabla K^i(f)(x) :&= \text{n.t.}\lim_{z\to x}\alpha(x)\cdot\nabla K(f)(z)\\ &= \frac{1}{2}\alpha(x)\cdot T(x)\;\partial_T f(x)\;+\;\alpha(x)\cdot\nabla\mathbf{K}(f)(x)\end{aligned}$$

$$\begin{aligned}\alpha\cdot\nabla S^i(g)(x) :&= \text{n.t.}\lim_{z\to x}\alpha(x)\cdot\nabla S(g)(z)\\ &= -\frac{1}{2}\alpha(x)\cdot\nu(x)\;g(x)\;+\;\alpha(x)\cdot\nabla\mathbf{S}(g)(x),\end{aligned}$$

where $f\in W^{1,2}(\partial\Omega)$, $g\in L^2(\partial\Omega)$, and the non-tangential limits are taken using an interior family of regular cones. For limits using *exterior* cones we instead have

$$\alpha\cdot\nabla K^e(f)(x) = -\frac{1}{2}\alpha(x)\cdot T(x)\;\partial_T f(x)\;+\;\alpha\cdot\nabla\mathbf{K}(f)(x)$$

$$\alpha\cdot\nabla S^e(g)(x) = \;\frac{1}{2}\alpha(x)\cdot\nu(x)\;g(x)\;+\;\alpha\cdot\nabla\mathbf{S}(g)(x).$$

The operators $\alpha\cdot\nabla\mathbf{K}$ and $\alpha\cdot\nabla\mathbf{S}$ are related to one another by the following lemma.

LEMMA 1.2.1. *Let $f\in W^{1,2}(\partial\Omega)$. For almost every $x\in\partial\Omega$ we have the pointwise equalities*

1) $\partial_T \mathbf{K}(f)(x) = -\partial_\nu \mathbf{S}(\partial_T f)(x)$

2) $\partial_\nu \mathbf{K}(f)(x) = \partial_T \mathbf{S}(\partial_T f)(x)$

PROOF.

1. First restrict x_0 to a compactly contained open subset $U \subset \Omega$ and restrict y to a compactly contained open subset of $\overline{U}^c$. With these restrictions $\log|x_0 - y|$ is harmonic in both the x_0 and y variables.

Fix $x, y \in \partial\Omega$, and $x_0 \in \Omega$. Expanding the dot product, we have

$$\begin{aligned} &T(x) \cdot \nabla_{x_0} \partial_{\nu,y} \log|x_0 - y| \\ &\qquad = \sum_{k=1,2} [\nu_2(x)\partial_{1,x_0}\nu_k(y)\partial_{k,y} - \nu_1(x)\partial_{2,x_0}\nu_k(y)\partial_{k,y}] \log|x_0 - y| \\ &\qquad = \sum_{k=1,2} [\nu_2(x)\partial_{\mathbf{k},x_0}\nu_k(y)\partial_{\mathbf{1},y} - \nu_1(x)\partial_{\mathbf{k},x_0}\nu_k(y)\partial_{\mathbf{2},y}] \log|x_0 - y|, \end{aligned}$$

where we have used $\partial_{k,x_0}\partial_{j,y} \log|x_0 - y| = \partial_{k,y}\partial_{j,x_0} \log|x_0 - y|$ for the second equality, $j = 1, 2$.

Since $\log|x_0 - y|$ is harmonic and $\partial_{k,y} \log|x_0 - y| = -\partial_{k,x_0} \log|x_0 - y|$ we may add

$$0 = \sum_{k=1,2} [-\nu_2(x)\partial_{k,x_0}\nu_1(y)\partial_{k,y} + \nu_1(x)\partial_{k,x_0}\nu_2(y)\partial_{k,y}] \log|x_0 - y|$$

to the right hand side of the equation. After collecting terms, we arrive at

$$T(x) \cdot \nabla_{x_0} \partial_{\nu,y} \log|x_0 - y| = \nu(x) \cdot \nabla_{x_0} \partial_{T,y} \log|x_0 - y|.$$

Integrating against f and using integration by parts to pass the tangential derivative onto f yields

$$T(x) \cdot \nabla_{x_0} \int_{\partial\Omega} \partial_{\nu,y} \log|x_0 - y|\, f(y)\, d\sigma(y) = -\nu(x) \cdot \nabla_{x_0} \int_{\partial\Omega} \log|x_0 - y|\, \partial_T f(y)\, d\sigma(y).$$

Letting $x_0 \to x$ nontangentialy, we produce singular integrals on both sides of the equation:

$$\partial_T \mathbf{K}(f)(x) + \frac{1}{2}\partial_T f(x) = -\partial_\nu \mathbf{S}(\partial_T f)(x) + \frac{1}{2}\partial_T f(x)$$

Canceling the $\frac{1}{2}\partial_T f(x)$ yields the desired equality.

2. We use a process similar to part **1**. Fix $x, y \in \partial\Omega$, and $x_0 \in \Omega$. Then "$\nu(x) \cdot \nabla_{x_0} \partial_{\nu,y} \log|x_0 - y|$" equals

$$\begin{aligned}
&\sum_{j,k} \nu_j(x) \partial_{\mathbf{k},\mathbf{x_0}} \nu_k(y) \partial_{\mathbf{j},\mathbf{y}} \log|x_0 - y| \\
&\qquad = \sum_{j,k} \nu_j(x) \partial_{k,x_0} \Big[\Big(\nu_k(y)\partial_{j,y} - \nu_j(y)\partial_{k,y} \Big) + \nu_j(y)\partial_{k,y} \Big] \log|x_0 - y| \\
&\qquad = \sum_{j \neq k} \nu_j(x) \partial_{k,x_0} \Big[\nu_k(y)\partial_{j,y} - \nu_j(y)\partial_{k,y} \Big] \log|x_0 - y|
\end{aligned}$$

where we have used the fact that $\log|x_0 - y|$ is harmonic for the second equality.

Summing over $1 \leq k, j \leq 2$ then gives

$$-\nu(x) \cdot \nabla_{x_0} \partial_{\nu,y} \log |x_0 - y| = T(x) \cdot \nabla_{x_0} \partial_{T,y} \log |x_0 - y|.$$

We may integrate against f and use integration by parts to pass the tangential derivative onto f. Taking limits then yields the desired equality.

$\square$

To conclude this section, we state the following well-known results [6]:

LEMMA 1.2.2. *Given any $f \in W^{1,2}(\partial\Omega)$ and $g \in L^2(\partial\Omega)$, the function $u = Kf - Sg$ is harmonic on $\mathbb{R}^2 \backslash \partial\Omega$, and satisfies $\int_{\partial\Omega} |u^*|^2 + |\nabla u^*|^2 d\sigma < \infty$.*

LEMMA 1.2.3. *Let u be a harmonic function on Ω with nontangential maximal function bound $\int_{\partial\Omega} |u^*|^2 + |\nabla u^*|^2 d\sigma < \infty$. Then the L^2 norms of ∇u^*, $\partial_\nu u$, and $\partial_T u$ are all comparable.*

1.3. The Harmonic Mixed Problem and Main Result

With the notation from previous sections at our disposal, we are now able to describe our main result.

DEFINITION 1.3.1. **The Harmonic Mixed Problem**

[6] See [**JK82**].

Let Ω be a strongly dissected Lipschitz domain, and fix $h_D \in W^{1,2}(D)$ and $h_N \in L^2(N)$. A harmonic function u is said to solve the *Mixed Problem with data h_D and h_N* if it satisfies the three boundary conditions:

$$\begin{aligned} u|_D &= h_D, \\ \partial_\nu u|_N &= h_N, \\ \text{and} \quad \int_{\partial\Omega} |u^*|^2 &+ |\nabla u^*|^2 \, d\sigma < \infty, \end{aligned}$$

where boundary values are taken via nontangential limits.

In **Chapter 3.3** we prove our main result:

THEOREM 1.3.2. *Let Ω be a strongly dissected Lipschitz domain in the plane such that D is the finite union of m connected open sets with pairwise disjoint closures.*

There exists a finite dimensional subspace $E \subset W^{1,2}(D) \times L^2(N)$ such that the Mixed Problem is uniquely solvable for all data $(h_D, h_N) \in E^\perp$. Furthermore, $\dim E \leq 2m+1$ and the solution to the Mixed Problem satisfies

$$\int_{\partial\Omega} |\nabla u^*|^2 \, d\sigma \leq C\left(\int_D u^2 + |\partial_T u|^2 \, d\sigma + \int_N |\partial_\nu u|^2 \, d\sigma\right)$$

Existence and regularity will be proved by representing solutions as $u = Kf - Sg$ and solving the system of equations

$$\begin{bmatrix} \frac{1}{2}I + \mathbf{K} & \mathbf{S} \\ \\ \partial_\nu \mathbf{K} & \frac{1}{2}I - \partial_\nu \mathbf{S} \end{bmatrix} \begin{bmatrix} f \\ \\ g \end{bmatrix} = \begin{bmatrix} u_D \\ \\ u_N \end{bmatrix}$$

for $f \in W_0^{1,2}(D)$ and $g \in L_0^2(N)$. The bound on $\dim E$ will be a consequence of **Theorem 0.0.2** and **Remark 3.3.8**.

CHAPTER 2

History

The formal definition of the mixed problem for harmonic functions can obscure just how important this problem is. Let us keep an example in mind, furnished by the paper *Harmonic solutions of a mixed boundary problem arising in the modeling of macromolecular transport into vessel walls* [**BNR10**]. This paper, written jointly by a mathematician, a biomedical engineer, and a chemical engineer, moves past the traditional areas of physics and engineering, and enters the realm of medicine.

As cholesterol builds in our veins, it constricts blood flow. This means blood pressure builds up by the cholesterol, but flows through the dangerously small opening remaining. Here, then, we can consider two different types of information: the value of blood density on one part of the vein surface, and a directional derivative of changing blood density on the remainder. If we can measure this data, can we discover what is happening within the vein? Is it time to consider Lipitor?

These questions might be beyond the reach of study– after all, it might be expensive or unsafe to measure these quantities in a healthy person. Instead we may wish to start with supposed boundary data, and ask “what if?” Does this data lead to a correct analysis of blood density in veins? This information could be used to determine criteria for unsafe cholesterol levels, and we need only test at-risk patients.

[**BNR10**] answers these questions affirmatively. If given reasonable mixed boundary data on a reasonably shaped vein, we can create a reasonably detailed analysis of interior blood density.

The main issue driving research in the mixed problem is to determine definitions for *reasonable* domains, data, and solutions.

2.1. Origins

The mixed problem has its origins in the experimental sciences, with Leopoldo Nobili observing colored rings forming on a charged silver plate. By 1824, Nobili had learned to create these rings by applying a negatively charged wire to the *positively* charged plate.[1] Bernard Riemann then translated this phenomenon into a mathematical framework in 1855 [**Rie55**], solving a mixed Dirichlet-Neumann problem for Laplace's Equation. This was Riemann's first paper in mathematical physics, and in fact was his first publication in a major journal.[2] It was not, however, his most famous. Herbert Weber is typically credited as the mathematical father of the field, having proposed a solution to Nobili's problem in the 1873 paper [**Web73**].[3]

The most influential early work on the mixed problem was conducted by Stanislaw Zaremba, whose 1910 paper *Sur un probleme mixte relatif a l'equation de Laplace*

[1] See page 41 in [**Duf08**].

[2] See [**Arc91**].

[3] It is interesting to note that Riemann was introduced to Nobili rings in a physics seminar at Göttingen– taught by *William* Weber. In turn, Riemann's writings greatly influenced *Herbert* Weber. Although this author has not been able to find a reference, it would seem quite a coincidence if the two Webers were unrelated. See [**Arc91**].

[**Zar10**] is possibly the most cited in the field. In fact, the mixed problem is often called "Zaremba's problem."

It was not long before further applications of the mixed problem were found, many of which involve Laplace's Equation.[4] Most developments have focused on second-order elliptic equations, although there has also been some progress on higher order and nonlinear equations. Despite these advances, the original mixed problem for *harmonic* functions holds secrets to be revealed.

There have been an overwhelming number of papers written on mixed problems over the years.[5] It is unfortunate to find that a large number of these are rather difficult to obtain, including Zaremba's influential paper.[6][7]

This thesis chapter presents selected results on the mixed problem for harmonic functions. It is organized thematically, rather than chronologically, and aims to give readers a sense of the different avenues that mathematicians have followed in taming the mixed problem.

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[4] The books of Duffy [**Duf08**] and Sneddon [**Sne66**] offer a good history of the developments through the 1960's, with emphasis on applications to physics and engineering.

[5] A recent MathSciNet search of articles with "mixed problem" in the title returned 2044 hits, a substantial proportion of which seem to be relevant. Combinations of "mixed, misti, problem, Zaremba, elliptic, Laplace, and harmonic" produce between 100 and 865 hits.

[6] Zaremba's paper is readily available... in Russian translation. See http://mi.mathnet.ru/eng/umn7059. The original journal containing the paper is also available online through the HathiTrust Digital Library, but Zaremba's paper has been omitted. An Inter-Library Loan request for the original work was filled only after several attempts, and the text was obtained from Purdue University.

[7] Although the papers of Riemann and Weber do not appear on a MathSciNet search, they are available for free online through Google Books.

In what follows Ω will always denote a simply-connected Lipschitz domain in $\mathbb{R}^n$. Its boundary is partitioned into disjoint sets N and D, which correspond to the location of prescribed Neumann and Dirichlet data.

2.2. Common Threads

We begin with the 1930 paper of Evans and Haskell, *The mixed problem for Laplace's equation in the plane discontinuous boundary values* [**EH30**].[8] It employs techniques that have been elaborated upon and rediscovered through the years.

Evans and Haskell first choose their domain Ω to be the upper-half unit disk, with N the linear part of the boundary. The authors then state necessary and sufficient conditions for the proposed solution u to have an integral representation of the form

$$u = \int_D \partial_\nu G \, dF - \int_N G \, dH,$$

where ν is the outward unit normal vector and G is the Green's function for the mixed problem, constructed as follows: Let $\tilde{G}$ be the Green's function for the unit disk with pole contained in the open upper half disk, and define $G(x, y) := \tilde{G}(x, y) + \tilde{G}(x, -y)$.

Since $\tilde{G}$ equals zero on D and G is even with respect to the x-axis, this *mixed* Green's function satisfies

[8] This does not appear in MathSciNet, but is instead found at http://www.jstor.org/stable/85356.

$$G|_D = 0$$

$$\text{and} \ \ \partial_\nu G|_N = 0.$$

The measures dF and dH are derived from functions of bounded variation such that

$$F(\theta_2) - F(\theta_1) = \int_{\theta_1}^{\theta_2} u(1, \theta)\, d\theta$$

and

$$H(b) - H(a) = \int_a^b \partial_\nu u(r, 0)\, dr,$$

where u has been written in polar coordinates.

This representation is then extended to more general domains in the plane. Two key observations are utilized: First, that a harmonic function on the upper-half disk taking on Neumann data identically equal to zero on N has a harmonic extension to the full disk, and is therefore smooth on $\Omega \cup N$; Second, that such a function can be conformally mapped to a harmonic function on a simply connected domain, with Neumann data identically equal to zero along a connected boundary portion of our choosing. By combining these facts the authors obtain necessary and sufficient conditions for a Green's representation to the Mixed Problem.

The main elements of this paper are *integral representations*, *reflections*, and *conformal mappings*. Along with the *Lax-Milgram theorem*, these tools are at the heart of nearly all advances to the mixed problem in the 20^{th} century. Of the four, integral representations– along with their natural counterparts, series representations– have received the most attention.

2.3. Integral Representations, Part 1

The successful use of a Green's function representation for solutions leads one to wonder if alternative integral representations may also be utilized. In fact, Weber's original paper [**Web73**] obtains a solution as an integral transform of the form

$$u(\rho, z) = \int_0^\infty \xi^{-1} A(\xi) e^{-\xi z} J_0(\xi\rho)\ d\xi,$$

where J_0 is a Bessel function. The function A is obtained by solving dual integral equations corresponding to the two types of data (Neumann and Dirichlet) in a mixed problem.

In his paper, Weber considers a 3-dimensional space (written in cylindrical coordinates)

$$\mathbb{R}^3 := \{(\rho, \theta, z) : 0 < \rho, \quad 0 < \theta < 2\pi, \quad -\infty < z < \infty\}$$

containing an electrically charged plate

$$D = \{0 < \rho < 1, \quad z = 0\}.^{9}$$

His goal is to find a function U describing the electric potential at any given point of $\mathbb{R}^3$. Such a function will be harmonic in both the upper-half and lower-half spaces. In fact, U must also be even and rotation invariant with respect to the plate, i.e.

$$U(\rho, \theta_1, z) = U(\rho, \theta_2, -z)$$

for any triples (ρ, θ_1, z) and (ρ, θ_2, z) in the domain.

Using this symmetry Weber reduces to a 2-dimensional mixed problem in the upper half plane $\mathbb{R}^2_+$ with $D = \{(\rho, z) : |\rho| < 1, \ z = 0\}$ and $N = \{(\rho, z) : |\rho| \geq 1, \ z = 0\}$. He then solves for a harmonic function u with constant mixed data of $u = 1$ on D and $\partial_\nu u = 0$ on N.

A great number of papers follow Weber's lead by representing solutions in terms Bessel functions, Hankel functions, sine expansions, and other familiar tools from applied mathematics. Such methods have the advantage of giving concrete formulas that can be computed to any desired degree of accuracy. Furthermore they typically do not require new theory, instead relying on long established techniques. For

[9] Weber's paper considers several boundary value problems. In some texts "Weber's problem" refers instead to a mixed problem on an infinite strip in the plane.

example, the paper [**BNR10**] mentioned earlier mirrors Weber's method, first taking advantage of the symmetries of a cylindrical blood vessel, and then representing solutions in terms of Bessel and hyperbolic sine functions

These benefits come at a price: a great deal of apriori knowledge is required, on both the type of data and the specific shape of the domain. Cylinders, sectors, rectangles and upper-half spaces are commonly used. The mixed data is often assumed to be constant on all or part of the boundary. Each type of problem must be addressed on a case-by-case basis.

Because of these limitations, the methods are mainly used by engineers and applied mathematicians with a specific problem in mind. As such, the typical complaint can be levied: experimental evidence is given priority over mathematical rigor. A quote from *Fourier Analysis and Boundary Value Problems* by E. A. González-Valesco describes the philosophy espoused by many authors:

> "What matters... is that we have found such a solution. What does not matter is that certain steps of our procedure were unjustified. In fact, this is not untypical of work in applied mathematics and will be a recurrent theme in this book. When looking for the solution of a particular problem we will not hesitate in making any number of reasonable assumptions, justified or unjustified, and if they lead to a solution that can be verified *a posteriori* the matter is settled, and that is that."[10]

[10] Page 10 in [**GV96**].

There are, of course, many papers conducted in a more aesthetically pleasing fashion. One such example of *rigorous* work conducted under Weber's influence is the 1954 proof by Green and Zerna [**GZ54**] described in [**Sne66**]. Their paper is set up as in Weber's problem, except the Dirichlet data f is no longer assumed constant. They obtain the representation

$$u(\rho, z) = \frac{1}{2}\int_0^1 \frac{g(t)}{\sqrt{\rho^2 + (z+it)^2}}\, dt + \frac{1}{2}\int_0^1 \frac{g(t)}{\sqrt{\rho^2 + (z-it)^2}}\, dt,$$

with

$$g(t) = \frac{2}{\pi} f(0) + \frac{2t}{\pi}\int_0^t \frac{f'(\rho)}{\sqrt{t^2-\rho^2}}\, d\rho.$$

As in previous examples, the solution is first constructed using Bessel functions, the computation of which require apriori knowledge of the domain, the upper-half plane.

The reader might notice, however, that u is given as a single layer potential. Rather than computing Bessel functions it would seem reasonable to simply begin with a single layer representation, freeing one to work on arbitrary Lipschitz domains.[11]

2.4. Integral Representations, Part 2

In the works referred to above, very little concern is given to the manner in which boundary values are computed. Typically, the mixed data is expected to be

[11] This approach relies on theory that was not developed until much later, such as [**CMM82**].

attained continuously, or sometimes via an approach perpendicular to the boundary. This causes an imbalance: solutions only have a derivative on the N portion of the boundary.

To address this issue we utilize the well-known trace theorems. Let Ω be a bounded Lipschitz domain with connected boundary, and define the trace operator

$$\gamma : u \to u|_{\partial\Omega}.$$

If $u \in C^\infty(\overline{\Omega})$ then γ is simply the restriction to the boundary and $\gamma(u) \in C^\infty(\partial\Omega)$.

The trace operator has a unique extension to a bounded linear operator

$$\gamma : W^{s,2}(\Omega) \to W^{s-\frac{1}{2},2}(\partial\Omega),$$

for $\frac{1}{2} < s < \frac{3}{2}$. A stronger results holds for smooth domains: if Ω is a $C^{k-1,1}$ domain with $k \geq 1$, then the extension is bounded for all $\frac{1}{2} < s \leq k$.[12]

In light of the Green's identity for smooth functions,

$$\int_\Omega u\,\Delta u\,dx = -\int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} u\,\partial_\nu u\,d\sigma(x),$$

a solution u is typically required to be in $W^{1,2}(\Omega)$, and its Dirichlet data will therefore be in $W^{\frac{1}{2},2}(D)$. Since the *regularity problem* can be solved in this space, u also has Neumann data in $W^{-\frac{1}{2},2}(N)$. Continuing this line of thought we see that, in general,

[12] See page 102 in [**McL00**].

our mixed data should reside in $W^{t,2}(D) \times W^{t-1,2}(N)$ for some $t > 0$. [13]

When Ω is smooth one may consider larger values of t, as is done in the 1979 paper *On the Integral Equation Method for the Plane Mixed Boundary Value Problem of the Laplacian*, by Wendland, Stephan, Darmstadt, and Hsiao [**WSH79**]. Their paper considers $1 < t < 2$, with Ω an appropriately smooth, bounded, and simply connected domain in the plane. The boundary sets D and N are connected and have positive measure. For simplicity we will describe $t = \frac{3}{2}$, the case where the mixed data is in $W^{\frac{3}{2},2}(D) \times W^{\frac{1}{2},2}(N)$.

The paper begins by establishing a variational solution u to the mixed problem for harmonic functions. While this method guarantees $u \in W^{1,2}(\Omega)$, a greater degree of regularity is required in order to apply the trace theorems above– an amount of regularity that the authors recognized as *unattainable* in general. This difficulty is an issue mathematicians have grappled with throughout the history of the mixed problem, and every investigator is forced to compromise in some way.

Wendland and his collaborators solve the mixed problem, modulo a 2-dimensional subspace. Specifically, their variational solution can be decomposed as $u = g + c_1 b_1 + c_2 b_2$, where $g \in W^{2,2}(\Omega)$, the c_i are constants, and each b_i is a rotation of $Im(z - v_i)^{\frac{1}{2}}$, where $\{v_1, v_2\} = \overline{D} \cap \overline{N}$. Since the b_i are smooth away from the interface points v_1 and v_2, they do not interfere with u attaining its mixed data.

These authors are not satisfied with such an abstract solution. They proceed to represent u in terms of single and double layer potentials, and then solve the system

[13] This thesis considers the endpoint case $t = 1$ on Lipschitz domains.

of equations they generate. Furthermore they establish superconvergence results for the Galerkin method, showing that their system can be obtained using a well-known approximation technique.

The variational approach used by [**WSH79**] is a significant departure from the methods of the previous section, and untethers the authors from the case-by-case analysis previously required. Similarly, their layer potential representations are able to handle a wide class of domains and mixed data with a single argument.

In 2005 Wendland and Hsiao, and Cakoni, successfully applied this method to a mixed problems for *biharmonic* functions on bounded and simply connected $C^{1,1}$ domains in the plane [**CHW05**]. In analogy to the harmonic case, the singularity they now encounter is comparable to that of $Im(z^{\frac{3}{2}})$ at the origin.

The layer potential operators used above for the *harmonic* mixed problem in the plane are the *single layer potential*

$$S(g)(x) = \frac{1}{2\pi} \int_{\partial\Omega} \log|x-y|\, g(y)\; d\sigma(y),$$

and the *double layer potential*

$$K(f)(x) = -\frac{1}{2\pi} \int_{\partial\Omega} \frac{\langle x-y, \nu(y) \rangle}{|x-y|^2}\, f(y)\; d\sigma(y).$$

Here f and g are continuous functions, $x \in \mathbb{R}^2 \setminus \partial\Omega$., and ν denotes the unit outer normal vector.

The functions $S(g)$ and $K(f)$ are harmonic in Ω, and, by letting x approach the boundary nontangentially, give rise to bounded linear operators:

$$\begin{aligned}
\frac{1}{2}I + \mathbf{K} &: W^{s+\frac{1}{2},2}(\partial\Omega) \to W^{s+\frac{1}{2},2}(\partial\Omega) \\
\partial_\nu \mathbf{K} &: W^{s+\frac{1}{2},2}(\partial\Omega) \to W^{s-\frac{1}{2},2}(\partial\Omega) \\
\mathbf{S} &: W^{s-\frac{1}{2},2}(\partial\Omega) \to W^{s+\frac{1}{2},2}(\partial\Omega) \\
-\frac{1}{2}I + \partial_\nu \mathbf{S} &: W^{s-\frac{1}{2},2}(\partial\Omega) \to W^{s-\frac{1}{2},2}(\partial\Omega),
\end{aligned}$$

where $-\frac{1}{2} \leq s \leq \frac{1}{2}$. If Ω is a $C^{r+1,1}$ domain then this holds on the wider interval $-r-1 \leq s \leq r+1$.[14]

If our solution can be represented in the form $u = K(f) - S(g)$, and if u_D and u_N denote the desired mixed data, we are led to the matrix equation

$$\begin{bmatrix} \frac{1}{2}I + \mathbf{K} & \mathbf{S} \\ \partial_\nu \mathbf{K} & \frac{1}{2}I - \partial_\nu \mathbf{S} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} u_D \\ u_N \end{bmatrix}.$$

Any solution (f, g) to this equation gives a solution to the the mixed problem with mixed data (u_D, u_N).

[14] See Theorem 7.1 and the note on page 209 in [**McL00**].

The 1979 and 2006 papers by Wendland, Hsiao, and their collaborators utilized this approach. The book *Strongly Elliptic Systems and Integral Equations* [**McL00**] by William McLean takes this method and runs with it.

McLean's domains are only assumed to be bounded and Lipschitz in $\mathbb{R}^n$ $(n \geq 2)$, and he works on a large class of second-order differential operators: those that are formally self-adjoint, have smooth coefficients, and are coercive on both $W^{1,2}(\Omega)$ and $W^{1,2}(\mathbb{R}^n \backslash \overline{\Omega})$. Under these assumptions, McLean proves the matrix equation above has finite dimensional cokernel, as an operator on $W^{\frac{1}{2},2}(\partial\Omega) \times W^{-\frac{1}{2},2}(\partial\Omega)$.[15] Thus he has solved a mixed problem for functions $u \in W^{1,2}(\Omega)$, modulo a finite dimensional subspace.[16]

McLean is able to prove his result with remarkably little work because he is only requiring the solution to have a gradient in $W^{-\frac{1}{2},2}(\partial\Omega)$. This is a rather weak condition, and it would be much more satisfying if the gradient was continuous, or at least in some L^p space with $p \geq 1$.

In [**AK82**] Azzam and Kreyszig did even better. Working on sectors with sufficiently small angles and mixed data equal to zero near the interface points, they produce solutions that are in $C^2(\overline{\Omega})$. Their method uses barrier functions, not integral representations, and suggests that the geometry of the domain is a key factor in the boundary regularity solutions might possess.

[15] It is not clear who this result is due to, although McLean does specifically mention Wendland at several points earlier in the book.

[16] Unlike [**WSH79**], the size of this subspace is not computed.

2.5. Reflections and Apriori Estimates

Sectors in the plane are natural domains for the harmonic mixed problem. For one thing, working in the complex plane gives one access to the Riemann mapping theorem and the theory of conjugate harmonic functions. For another, sectors can be used to separate out the level of regularity one can expect from a solution. Azzam and Kreyszig's result show that sectors with small interior angles allow for smooth solutions.

A 2008 paper by Lanzani, Capogna, and Brown [**LCB08**] gives a way to extend such a result. Conformal maps of the form $i(-iz)^q$ transform mixed problems on sectors with large angles into mixed problems on sectors with *smaller* angles, where a solution is known to exist. There is some distortion in the process, however, and rather than having a full derivative in $L^2(\partial\Omega)$, solutions will only be guaranteed a derivative in some Hardy or weighted L^p space.

The paper makes use of apriori estimates of solutions, and these are also at the heart of the 1994 paper [**Bro94**] by R. Brown. If Ω is a convex polyhedral domain, then Brown can guarantee solutions have a gradient in $L^2(\partial\Omega)$ when each edge is given entirely by either Dirichlet data or Neumann data. In fact his result holds for a much more general subclass of Lipschitz domains in $\mathbb{R}^n$, subject to a convexity-type requirement:

A domain in $\mathbb{R}^2$ is called a *Creased Lipschitz graph domain* if it is the area above a Lipschitz function ϕ such that $D = \{(x, \phi(x)) : x < 0\}$, $\phi'(x) < -\epsilon$ on D, and $\phi'(x) \geq 0$ on N, for some constant $\epsilon > 0$. A bounded Lipschitz domain with connected boundary is a *Creased Lipschitz domain* if it agrees locally with orthogonal motions of Lipschitz and Creased Lipschitz graph domains. (Here we assume the Lipschitz graph portions are given either pure Dirichlet or pure Neumann conditions.)

As in [**LCB08**], Brown extends results on "good" domains to a wider class. Here, the good domains are creased Lipschitz graphs with $\phi'(x) \equiv 0$ on N. For such domains a mixed problem with Neumann data identically zero is equivalent to a purely unmixed Dirichlet problem obtained on the domain obtained by reflecting across N.[17]

By using a *Rellich identity* of the form

$$\int_{\partial\Omega} |\nabla u|^2 \alpha \cdot \nu - 2\partial_\nu u \cdot \alpha \; d\sigma = -2 \int_\Omega k^2 u \, \alpha \cdot \nabla u \; dx,$$

for solutions to $\Delta u = k^2 u$, Brown obtains the estimate

$$\int_{\partial\Omega} (\nabla u^*)^2 + k^2 (u^*)^2 \, dP \leq C(\int_N u_N^2 \, dP + \int_{\partial\Omega} (\partial_T u_D)^2 + k^2 (f)^2 \, dP.$$

Here u_D and u_N are the Dirichlet and Neumann data, respectively, * indicates the nontangential maximal function, ν is the outer unit normal vector, and α is a smooth vector field with compact support. Using this inequality, a sequence of approximating domains, a corresponding sequence of operators, and a localization argument,

[17] This is exactly the idea used in [**EH30**] to construct a mixed Green's function.

Brown is able to extend his results to Creased Lipschitz domains.[18] Similar methods were subsequently employed to solve mixed problems for Lamé and Stokes systems [**BMMW10**], [**BM09**].

The 2008 paper [**VV08**] by G. Verchota and Venouziou extended Brown's result to compact polyhedra in $\mathbb{R}^3$, obtaining a similar requirement on the interior angles of the domain. This paper also uses a sequence of approximating domains, but in a somewhat different manner. While Brown's functional analytic arguments required a family of operators with uniform bound from below, this paper also needs a uniform bound from above to prove convergence of a sequence of harmonic functions.[19]

The idea of approximating solutions by a bounded sequence of harmonic functions had appeared earlier, in the work of Ohtsuka and Strebel.[20] There, a mixed problem for smooth bounded domains in $\mathbb{R}^2$ was converted to a purely unmixed Dirichlet problem by reflection onto a doubled Riemann surface. To solve the resulting Dirichlet problem, Ohtsuka and Strebel employed the Schwartz alternating method, generating a sequence of uniformly bounded harmonic functions converging to the desired solution. Conformal maps allowed them to transfer between the manifold and the original domain, and as in [**LCB08**] this causes a loss of boundary regularity.

Ohtsuka and Strebel's solution does not necessarily have a gradient in $L^2(\partial\Omega)$. Furthermore, their reliance on conformal maps ties this result to domains in the complex plane. In [**Lie86**] G. Lieberman is able to work in higher dimensions by

[18] The methods employed in this thesis are a blend of Brown's approach with McLean's.
[19] Polyhedra need not be Lipschitz. The approximating domains used are Lipschitz, however, and Brown's bound from below must be adapted to the situation.
[20] See p. 446 in [**GM08**], where the result is credited to conversations held in 1970. As pointed out in the book, finding a concrete reference to this has proved illusive.

instead using a modified Perron process, approximating the solution by a convergent sequence of *subharmonic* functions. This result, however, also suffers from a lack of boundary regularity.[21]

2.6. A Final Example

This thesis chapter is an attempt to describe a broad swath of techniques using a minimal collection of references. As such we conclude this history with some remarks and a final example.

The Regularity problem can be solved on all bounded Lipschitz domains, with solutions having a full gradient in $L^2(\partial\Omega)$. As seen in previous sections, however, the mixed problem requires one to make a compromise– either impose additional restrictions on the domain or weaken the required level of boundary regularity.

The results of Ohtsuka, Strebel, and Lieberman, along with all the results from **Integral Representations, Part 1**, drop the requirement of having a gradient on the entire boundary. Wendland, McLean, and others allow their gradients to exist in relatively weak function spaces. Azzam, Kreysig, and Brown obtain strong boundary regularity, but only on a restricted class of domains.

The paper *The mixed problem for the Laplacian in Lipschitz domains* [**OB**], by K. Ott and R. Brown finds a new compromise: gradients are expected to exist in L^p for some $p \geq 1$. Unlike the weak function spaces of McLean, this means the gradient is defined pointwise almost everywhere. In their paper Ott and Brown solved the

[21] There are benefits to these approaches. The proofs use locally conformal maps and the Perron process, tools that do not require the domain to be simply connected.

mixed problem on all bounded Lipschitz domains in $\mathbf{R^n}$ with connected boundary, with mixed data in $W^{1,p}(D) \times L^p(N)$ for small $p > 1$ or in the Hardy space $H^{1,1}(D) \times H^1(N)$. As should be expected for a Hardy space result, the proof relies in part on reverse-Holder estimates and modern real-variable techniques.

Many authors have examined the nature of admissible mixed data, and have obtained results on *weighted* L^p, Besov, and other function spaces. The Ott and Brown result, however, uses the most classical function spaces of measure theory, L^p. At the same time it solves a mixed problem on *all* Lipschitz domains in $\mathbb{R}^n$, domains sufficiently general for most applications.[22]

Out of all the L^p spaces, L^2 often plays the most critical role. For instance, it is the dividing line between the Neumann and Dirichlet Problems for harmonic functions on Lipschitz domains. This thesis examines mixed data in $W^{1,2}(D) \times L^2(N)$. As always, there is a required compromise. Here it resembles that of [**WSH79**], but allows for a more general class of Lipschitz domains than in [**Bro94**].

[22] It is worth noting, however, that many polyhedral domains are not Lipschitz.

CHAPTER 3

Harmonic Mixed Problems

This chapter is devoted to solving the following Mixed Problem:[1]

Definition 3.0.1. **The Harmonic Mixed Problem**

Let Ω be a strongly dissected Lipschitz domain, and fix $h_D \in W^{1,2}(D)$ and $h_N \in L^2(N)$. A harmonic function u is said to solve the *Mixed Problem with data h_D and h_N* if it satisfies the three boundary conditions:

$$u|_D = h_D,$$

$$\partial_\nu u|_N = h_N,$$

$$\text{and} \quad \int_{\partial\Omega} |u^*|^2 + |\nabla u^*|^2 \, d\sigma < \infty,$$

where boundary values are taken via nontangential limits.

In Section 3.3 we prove our main result:

Theorem 3.0.2. *Let Ω be a strongly dissected Lipschitz domain in the plane such that D is the finite union of m connected open sets with pairwise disjoint closures.*

[1] The mixed problem with Robin data in place of Neumann data is addressed in **Section 3.4**.

There exists a finite dimensional subspace $E \subset W^{1,2}(D) \times L^2(N)$ such that the Mixed Problem is uniquely solvable for all data $(h_D, h_N) \in E^{\perp}$. Furthermore, $\dim E \leq 2m+1$ and the solution to the Mixed Problem satisfies

$$\int_{\partial\Omega} |\nabla u^*|^2 \, d\sigma \leq C\Big(\int_D u^2 + |\partial_T u|^2 \, d\sigma + \int_N |\partial_\nu u|^2 \, d\sigma\Big).$$

Existence and regularity is proved by representing solutions in terms of layer potentials, $u = Kf - Sg$, and then solving the system of equations

$$\begin{bmatrix} \frac{1}{2}I + \mathbf{K} & \mathbf{S} \\ \partial_\nu \mathbf{K} & \frac{1}{2}I - \partial_\nu \mathbf{S} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} u_D \\ u_N \end{bmatrix}$$

for $f \in W_0^{1,2}(D)$ and $g \in L_0^2(N)$. The bound on $\dim E$ is obtained by calculating the dimension of the quotient space $W^{1,2}(D)/W_0^{1,2}(D) \times L^2(N)/L_0^2(N)$.

3.1. The Dimension of $W^{1,2}(D)/W_0^{1,2}(D)$

LEMMA 3.1.1. $W^{1,p}(0,1)$ *functions are absolutely continuous for all* $p \geq 1$.[2]

PROOF.

Let f be in $W^{1,p}(0,1)$ for some $p \geq 1$ and define $F_a(b) = \int_a^b f'(t)\, dt$, for $0 < a, b < 1$. F_a is continuous on $[0,1]$ since $f' \in L^p([0,1]) \subset L^1([0,1])$.

By the Lebesgue Dominated Convergence theorem we can write

[2] $W^{1,p}(0,1)$ is defined by replacing 2 with p in the definition of $W^{1,2}(D)$ given in **Chapter 1.1**.

$$F_a(b) = \lim_{r\to\infty} \int_0^1 f'(x)\chi_r(x)dx,$$

where χ_r is an approximate indicator function, constructed as follows:

Choose a $C_0^\infty(0,1)$ function δ with $\int_0^1 \delta(t)\,dt = 1$. We then set $\delta_r(t) = r\delta(rt)$ and define $\chi_r(x) = \int_0^x \delta_r(t-a) - \delta_r(t-b)\,dt$. In the sense of distributions, $\lim\limits_{r\to\infty} \delta_r(x)$ is the dirac-delta mass at $x = 0$.[3]

Using the Lebesgue dominated convergence theorem and the definition of a weak derivative, we then have

$$\begin{aligned} F_a(b) &= -\lim_{r\to\infty} \int_0^1 f(x)\big[\delta_r(x-a) - \delta_r(x-b)\big]\,dx \\ &= f(b) - f(a) \end{aligned}$$

for almost every $0 < a, b < 1$.

Fix an a where this limit holds. Then $f(x) = f(a) + F_a(x)$ for almost every $x \in [0,1]$, and f can be adjusted on a set of measure zero to make it agree with the continuous function $f(a) + F_a(x)$. $\square$

Recall **Theorem 0.0.2**: *Let Ω be a bounded Lipschitz domain and $D \subset \partial\Omega$ be the finite union of m connected open sets with pairwise disjoint closures and $\overline{D} \neq \partial\Omega$. Then the quotient space $W^{1,2}(D)/W_0^{1,2}(D)$ has dimension $2m$.*

Proof of Theorem 0.0.2.

[3] Theorem 2, Ch. 3 of [**Ste70**]

Case 1: $D = (0,1) \subset \partial\Omega \cap \mathbb{R}$

First we show $h(x) = 1$ and $h(x) = x$ are not in $W_0^{1,2}(D)$, so that the dimension of $W^{1,2}(D)/W_0^{1,2}(D)$ is at least 2. The following argument applies to either formula for h:

If h were in $W_0^{1,2}(D)$ we could approximate it by $C_0^\infty(D)$ functions. On the compact set $[0,1]$ these functions would also approximate h in the $W^{1,1}(D)$ norm. Assume we have found a g in $C_0^\infty(D)$ with $||g-h||_{W^{1,1}(D)} < \frac{1}{4}$. Since $h(x) \geq x$ there must exist an x_0 in the interval $(\frac{1}{2},1)$ where $g(x_0) > \frac{1}{4}$.

Using $g(1) = 0$, $h(x_0) \leq 1$, and $h(1) = 1$ we see:

$$\begin{aligned}
\int_{(0,1)} |g'(t) - h'(t)|\, dt &\geq \left| \int_{(x_0,1)} g'(t) - h'(t) dt \right| \\
&= |g(1) - g(x_0) - h(1) + h(x_0)| \\
&= |-g(x_0) + (h(x_0) - 1)| \\
&> \frac{1}{4} \,,
\end{aligned}$$

a contradiction. Now set $f(x) = 1$ and $g(x) = x$. Given any $h \in W_0^{1,2}(D)$ and $c \in \mathbb{R}$,

$$f(x) \neq cg(x) + h(x),$$

since $f(0) = 1$ and $g(0) = h(0) = 0$. It follows that $\dim[W^{1,2}(D)/W_0^{1,2}(D)] \geq 2$.

Next we show $\dim[W^{1,2}(D)/W_0^{1,2}(D)] \le 2$. Choose $f(x) \in W^{1,2}(D)$. By **Lemma 3.1.1** f is continuous and there is a linear function $p(x)$ such that $f(0) = p(0)$ and $f(1) = p(1)$. Define $g(x) = f(x) - p(x)$ so that $g(0) = g(1) = 0$. To prove the claim we show $g \in W_0^{1,2}(D)$.

For each $0 < r < \frac{1}{2}$, choose $\chi_r \in C_0^\infty(D)$ such that $\chi_r(x) = 1$ on $(r, 1-r)$, $0 \le \chi_r \le 1$, and $|\chi_r'| \le \frac{2}{r}$. Define the $W_0^{1,2}(D)$ functions $g_r = g \cdot \chi_r$. The g_r approximate g in the $W^{1,2}(D)$ norm. To see this we use the product rule, the bound on χ_r' and Hölder's Inequality to obtain the estimate:

$$\begin{aligned}
||g_r - g||^2_{W^{1,2}(D)} &= ||g\,(\chi_r - 1)||^2_{W^{1,2}(D_r)} \\
&\le \int_{D_r} |g|^2 + |g'(\chi_r - 1) + g\,\chi_r'|^2 \\
&\le ||g||^2_{W^{1,2}(D_r)} + \frac{4}{r}\int_{D_r} |g' \cdot g| + \frac{4}{r^2}\int_{D_r} |g|^2 \\
&\le ||g||^2_{W^{1,2}(D_r)} + \frac{4}{r}||g||_{L^2(D_r)}\Big(||g'||_{L^2(D_r)} + \frac{||g||_{L^2(D_r)}}{r}\Big),
\end{aligned}$$

where $D_r = D \setminus [r, 1-r]$.

The first term converges to zero as $r \to 0$. For the other term we use the absolute continuity and vanishing of g on the boundary, together with Hölder's inequality to obtain

$$\int_{D_r} |g(x)|^2 dx = \int_{[0,r]} \left| \int_{[0,x]} g'(t)dt \right|^2 dx + \int_{[1-r,1]} \left| \int_{[x,1]} g'(t)dt \right|^2 dx$$
$$\leq \frac{r^2}{2} \int_{D_r} |g'(t)|^2 \, dt,$$

which shows $\frac{1}{r}||g||_{L^2(D_r)} \to 0$.

Case 2: D connected

Since D is connected and $\overline{D} \neq \partial\Omega$ we may parametrize D as $\{(x(t), y(t)) : 0 \leq t \leq 1\}$, with two distinct boundary points. Given any $H \in W^{1,2}(D)$ we may define $h(t) = H(x(t), y(t))$, and since D is Lipschitz, the $W^{1,2}(D)$ norm is equivalent to the $W^{1,2}((0,1))$ norm under this relationship. Furthermore h vanishes at the boundary if and only if H does, and h has compact support if and only if H does. Therefore the result for connected D follows from that for $D = (0,1)$.

Case 3: D has multiple components

It is enough to prove the theorem when D has two components, D_1 and D_2, with disjoint closures. From the previous case we know each component contributes two functions (extended to be zero outside that component) in $W^{1,2}(D)/W_0^{1,2}(D)$.

Given any $f \in W^{1,2}(D)$ we can find $p(x)$, a linear combination of the four functions referred to above, such that $g = f - p$ is zero at the (four) boundary points of D. If

χ is the indicator function, **Case 2** implies the functions $g_i = g\chi_{D_i}$ are in $W_0^{1,2}(D_i)$, $i = 1, 2$. Therefore $g = g_1 + g_2$ is in $W_0^{1,2}(D)$. $\square$

3.2. The Mixed Rellich Inequality

Recall that Ω is a bounded Lipschitz domain in the plane with connected boundary. In addition, Ω is given the boundary dissection $\partial\Omega = D \cup N$.

In order to make use of layer potentials we will establish some important integral estimates. We start with a useful algebraic identity.

LEMMA 3.2.1. *Let $\alpha = (\alpha_1, \alpha_2)$ and $\tilde{\alpha} = (-\alpha_2, \alpha_1)$ be continuous vector fields on the plane, and let u be C^1 in a neighborhood of $\partial\Omega$. Then the following pointwise identity holds:*

$$\partial_\nu u(\alpha \cdot \nabla u) = (\alpha \cdot \nu) |\nabla u|^2 + (\tilde{\alpha} \cdot \nabla u)\partial_T u$$

PROOF.

$$\partial_\nu u(\alpha \cdot \nabla u) = \sum_{i,j} \nu_i \partial_i u \ \alpha_j \partial_j u$$

We reorder the α and ν terms, and add "$\alpha_i \partial_j u \ \nu_i \partial_j u - \alpha_i \partial_j u \ \nu_i \partial_j u$" to get

$$\partial_\nu u(\alpha \cdot \nabla u) = \sum_{i,j} (\alpha_j \partial_i - \alpha_i \partial_j) u \ \nu_i \partial_j u + \alpha_i \partial_j u \ \nu_i \partial_j u.$$

Summing over $1 \leq i, j \leq 2$ the right-hand side becomes

$$(\alpha_2\partial_1 - \alpha_1\partial_2)u\,\nu_1\partial_2 u + (\alpha_1\partial_2 - \alpha_2\partial_1)u\,\nu_2\partial_1 u + (\alpha\cdot\nu)\,|\nabla u|^2$$

$$= (\tilde{\alpha}\cdot\nabla u)\,\partial_T u + (\alpha\cdot\nu)\,|\nabla u|^2. \quad \square$$

This result can be extended by an application of the dominated convergence theorem.

COROLLARY 3.2.2. *Let u be the Poisson extension of a $W^{1,2}(\partial\Omega)$ function. Then $\int_D \partial_\nu u(\alpha\cdot\nabla u)\,d\sigma = \int_D (\alpha\cdot\nu)\,|\nabla u|^2 + (\tilde{\alpha}\cdot\nabla u)\partial_T u\,d\sigma$.*

The next theorem establishes a Rellich identity for the Mixed Problem.

THEOREM 3.2.3. *Let $u = K(f) - S(g)$ for some $f \in W_0^{1,2}(D)$ and $g \in L^2(\partial\Omega)$ with supp g in N, $\lambda \in \mathbb{R}$, and let α be a $C_0^\infty(\mathbb{R}^2)$ vector field.*

The following mixed Rellich Identity holds:

$$\int_N (\alpha\cdot\nu)\,|\nabla u|^2\,d\sigma + \int_D (-\alpha\cdot\nu)\,|\nabla u|^2\,d\sigma + \lambda\int_{\partial\Omega}(\alpha\cdot\nu)g^2 - (\alpha\cdot\nu)\,|\partial_T f|^2\,d\sigma$$
$$= 2\int_N \alpha\cdot\nabla u\,(\partial_\nu u+\lambda g)\,d\sigma + 2\int_D \tilde\alpha\cdot\nabla u\,(\partial_T u+\lambda\partial_T f)\,d\sigma$$
$$+ \lambda\int_{\partial\Omega} g\,C(g) - \partial_T f\;C(\partial_T f) + 2g\,\tilde C(\partial_T f)\,d\sigma$$
$$+ \int_\Omega div(\alpha)\,|\nabla u|^2 - 2\,\nabla\alpha(\nabla u)\cdot\nabla u\,dx,$$

where $\tilde\alpha = (-\alpha_2,\alpha_1)$, and C and $\tilde C$ are compact operators.

When $\lambda = 0$ this also holds for any harmonic function on Ω with $u^*+\nabla u^* \in L^2(\partial\Omega)$.

Proof. Since u^* and ∇u^* are in $L^2(\partial\Omega)$ by **Lemma 1.2.2**, a standard approximation argument allows us to apply the Divergence Theorem to the vector field "$|\nabla u|^2\alpha - 2(\alpha\cdot\nabla u)\nabla u$" and get

$$\int_{\partial\Omega}(\alpha\cdot\nu)\,|\nabla u|^2\,d\sigma = 2\int_{\partial\Omega}\partial_\nu u(\alpha\cdot\nabla u)\,d\sigma + \int_\Omega \mathrm{div}(\alpha)\,|\nabla u|^2 - 2\nabla\alpha(\nabla u)\cdot\nabla u\,dx,$$

where we have used $\Delta u = 0$, and $\nabla\alpha(\nabla u)\cdot\nabla u$ is used to denote $\sum\partial_i\alpha_j\,\partial_i u\,\partial_j u$, the sum being over $1\le i,j\le 2$.[4]

Applying **Corollary 3.2.2** , we replace $\int_{\partial\Omega}\partial_\nu u(\alpha\cdot\nabla u)\,d\sigma$ with $\int_N \partial_\nu u(\alpha\cdot\nabla u)\,d\sigma$ $+ \int_D (\alpha\cdot\nu)\,|\nabla u|^2 + (\tilde\alpha\cdot\nabla u)\;\partial_T u\;d\sigma$. After collecting terms we have

[4] The idea to use the Divergence Theorem in this way first appeared in [**Bro94**]. There, α is a constant vector and u satisfies $\Delta u = k\,u$ for some real number k.

$$\begin{aligned}\int_N (\alpha\cdot\nu)\,|\nabla u|^2\,d\sigma + \int_D (-\alpha\cdot\nu)\,|\nabla u|^2\,d\sigma\\ = 2\int_N \partial_\nu u(\alpha\cdot\nabla u)d\sigma + 2\int_D (\tilde{\alpha}\cdot\nabla u)\partial_T u\;d\sigma\\ + \int_\Omega \operatorname{div}(\alpha)\,|\nabla u|^2 - 2\nabla\alpha(\nabla u)\cdot\nabla u\;dx.\end{aligned}$$

This is the mixed Rellich identity with $\lambda = 0$, and we have only used the nontangential maximal function bounds on u and ∇u.

We now introduce the λ-terms by adding and subtracting

$$2\lambda\int_N g(\alpha\cdot\nabla u)\;d\sigma \;+\; 2\lambda\int_D (\tilde{\alpha}\cdot\nabla u)\partial_T f\;d\sigma$$

to the right side of the equation. This now gives us:

$$\begin{aligned}\int_N (\alpha\cdot\nu)\,|\nabla u|^2\,d\sigma + \int_D (-\alpha\cdot\nu)\,|\nabla u|^2\,d\sigma\\ = 2\int_N (\partial_\nu u + \lambda g)(\alpha\cdot\nabla u)\,d\sigma + 2\int_D (\tilde{\alpha}\cdot\nabla u)(\partial_T u + \lambda\partial_T f)\,d\sigma\\ - 2\lambda\int_N g\;(\alpha\cdot\nabla u)\;d\sigma - 2\lambda\int_D (\tilde{\alpha}\cdot\nabla u)\;\partial_T f\,d\sigma\\ + \int_\Omega \operatorname{div}(\alpha)\,|\nabla u|^2 - 2(\nabla\alpha(\nabla u)\cdot\nabla u)\,dx.\end{aligned}$$

We name the two middle terms from the right-hand side:

$$\mathbf{Error}_1 = -2\lambda \int_N g(\alpha \cdot \nabla u)\, d\sigma$$

$$\text{and} \quad \mathbf{Error}_2 = -2\lambda \int_D (\tilde{\alpha} \cdot \nabla u)\, \partial_T f\, d\sigma.$$

We deal with each in turn. Note that $\partial_\nu u = \partial_\nu \mathbf{K} f - \partial_\nu \mathbf{S} g$ a.e. on D since g is supported in N. Equivalently, $\partial_\nu u^i = \partial_\nu u^e$ on D. Similarly, we have $\partial_T u = \partial_T \mathbf{K} f - \partial_T \mathbf{S} g$ a.e. on N since f is supported in D, and hence $\partial_T u^i = \partial_T u^e$ on N. (Recall that in our notation $u|_{\partial\Omega}$ and $u^i|_{\partial\Omega}$ both refer to the boundary values obtained via nontangential limits in the *interior* domain, while $u^e|_{\partial\Omega}$ is obtained using limits from the *exterior* domain.)

1. Error$_1$

On N we have

$$\begin{aligned}\alpha \cdot \nabla u &= (\alpha \cdot \nu)\partial_\nu u + (\alpha \cdot T)\partial_T u \\ &= (\alpha \cdot \nu)[\,\frac{1}{2} g + \partial_\nu \mathbf{K} f - \partial_\nu \mathbf{S} g] + (\alpha \cdot T)[\partial_T \mathbf{K} f - \partial_T \mathbf{S} g]\end{aligned}$$

almost everywhere.

We regroup terms on the right hand side to get

$$\frac{1}{2}(\alpha \cdot \nu)g + [\,(\alpha \cdot \nu)\partial_\nu \mathbf{K} f + (\alpha \cdot T)\partial_T \mathbf{K} f\,] - [\,(\alpha \cdot \nu)\partial_\nu \mathbf{S} g + (\alpha \cdot T)\partial_T \mathbf{S} g\,].$$

We may then use the identities $\partial_\nu \mathbf{K} = \partial_T \mathbf{S} \partial_T$ and $\partial_T \mathbf{K} = -\partial_\nu \mathbf{S} \partial_T$ of **Lemma 1.2.1** along with the pointwise equality $(\alpha \cdot T)\nu - (\alpha \cdot \nu)T = \tilde{\alpha}$ to get

$$\alpha \cdot \nabla u = \frac{1}{2}(\alpha \cdot \nu)\ g - \ [\tilde{\alpha} \cdot \nabla]\mathbf{S}(\partial_T f) - [\alpha \cdot \nabla]\mathbf{S} g.$$

Integrating against g gives us the equality

$$\int_N g(\alpha \cdot \nabla u)\, d\sigma = \int_N \frac{1}{2}(\alpha \cdot \nu)\, g^2\, d\sigma - \int_N g\, [\tilde{\alpha} \cdot \nabla]\mathbf{S}(\partial_T f)\, d\sigma - \int_N g\, [\alpha \cdot \nabla]\mathbf{S} g\, d\sigma.$$

Since supp $g \subset N$, we may take advantage of the adjoint operator to rewrite the third integral on the right hand side as

$$\begin{aligned}\int_N g\, [\alpha \cdot \nabla]\mathbf{S}(g)\, d\sigma &= \frac{1}{2}\int_N g\, \big(\, [\alpha \cdot \nabla]\mathbf{S} + ([\alpha \cdot \nabla]\mathbf{S})^*\, \big)(g)\, d\sigma \\ &= \frac{1}{2}\int_N g\, C(g)\, d\sigma,\end{aligned}$$

where C is the integral operator with kernel

$$\frac{1}{2\pi}\frac{(x-y)\cdot(\alpha(x)-\alpha(y))}{|x-y|^2}.$$

Since this kernel is bounded, C is a compact operator.

We can now rewrite Error_1 as

$$\mathbf{Error_1} = \lambda\int_N -(\alpha\cdot\nu)g^2 \; + \; 2g\,[\tilde{\alpha}\cdot\nabla]\mathbf{S}(\partial_T f) + g\,C(g)\,d\sigma$$

2. $\mathbf{Error}_2$

We mimic the process used for Error_1. On D we have

$$\begin{aligned}\tilde{\alpha}\cdot\nabla u &= (\tilde{\alpha}\cdot\nu)\partial_\nu u + (\tilde{\alpha}\cdot T)\partial_T u\\ &= (\tilde{\alpha}\cdot\nu)[\,\partial_\nu\mathbf{K}f - \partial_\nu\mathbf{S}g\,] + (\tilde{\alpha}\cdot T)[\,\frac{1}{2}\partial_T f + \partial_T\mathbf{K}f - \partial_T\mathbf{S}g\,]\end{aligned}$$

almost everywhere.

After regrouping, applying **Lemma 1.2.1**, and noting that $\tilde{\alpha}\cdot T = -\alpha\cdot\nu$ and $\tilde{\alpha}\cdot\nu = \alpha\cdot T$, we obtain

$$\tilde{\alpha}\cdot\nabla u = -\frac{1}{2}(\alpha\cdot\nu)\partial_T f + [\alpha\cdot\nabla]\mathbf{S}\partial_T f - [\tilde{\alpha}\cdot\nabla]\mathbf{S}g.$$

We now integrate against $\partial_T f$ to get

$$\text{Error}_2 = \lambda \int_D (\alpha \cdot \nu)\, |\partial_T f|^2 - \partial_T f\, C(\partial_T f) + 2\ \partial_T f\, [\tilde{\alpha} \cdot \nabla]\mathbf{S}(g)\, d\sigma,$$

where C is the same compact operator as in $\mathbf{Error}_1$.

3. $\mathbf{Error}_1$ + $\mathbf{Error}_2$

Adding the two error terms and regrouping gives us

$$\begin{aligned}\text{Error}_1 + \text{Error}_2 \ = \ & -\lambda \int_{\partial\Omega} (\alpha \cdot \nu) g^2 + (-\alpha \cdot \nu)\, |\partial_T f|^2\, d\sigma \\ & + \lambda \int_{\partial\Omega} g\, C(g) - \partial_T f\, C(\partial_T f))\, d\sigma \\ & + 2\lambda \int_D \partial_T f\, [\tilde{\alpha} \cdot \nabla]\mathbf{S}(g) + 2\lambda \int_N g\, [\tilde{\alpha} \cdot \nabla]\mathbf{S}(\partial_T f)\, d\sigma.\end{aligned}$$

The last two terms can be combined using adjoints as

$$2\lambda \int_N g\, (\, [\tilde{\alpha} \cdot \nabla\mathbf{S}]^* + [\tilde{\alpha} \cdot \nabla\mathbf{S}]\,)\, (\partial_T f)\, d\sigma = 2\lambda \int_N g\, \tilde{C}(\partial_T f)\, d\sigma,$$

where $\tilde{C}$ is the compact integral operator with kernel

$$\frac{1}{2\pi} \frac{(x - y) \cdot (\tilde{\alpha}(x) - \tilde{\alpha}(y))}{|x - y|^2}.$$

Using these expressions for the error terms and then regrouping proves the theorem.

□

REMARK 3.2.4. Allowing f in $W^{1,2}(\partial\Omega)$ (and not just $W_0^{1,2}(D)$) would introduce the unwanted term $\lambda \int_N (\alpha \cdot T)\, g\, f\, d\sigma$ into $\mathbf{Error}_1$. To compensate we could instead require $\alpha \cdot T = 0$ on $N \cap \operatorname{supp}(f)$ and obtain the same equality. The restriction on the support of f, however, will be required in the proof of **Theorem 3.2.12**.

REMARK 3.2.5. Choosing α to be a vector field with components the real and imaginary parts of a holomorphic function would remove the last integral in the mixed Rellich Identity, and simplify our work on the exterior domain $\overline{\Omega}^c$. Although we will not use the result, it may be of interest to note the following lemma from [**LCB08**]:

LEMMA 3.2.6. *Let u be a harmonic function on a Lipschitz Graph domain Ω, with u^* and ∇u^* in $L^2(\partial\Omega)$. Also let α be a vector field whose components are the real and imaginary parts of a holomorphic function. Then $div\big(\,|\nabla u|^2\,\alpha \,-\, 2(\alpha\cdot\nabla u)\nabla u\big) = 0$.*

COROLLARY 3.2.7. *Let u and α be as in* **Lemma 3.2.6**. *The following mixed Rellich Identity then holds:*

$$\int_N (\alpha\cdot\nu)\,|\nabla u|^2\,d\sigma \;+ \int_D (-\alpha\cdot\nu)\,|\nabla u|^2\,d\sigma$$
$$= \;2\int_N \alpha\cdot\nabla u\;\partial_\nu u\,d\sigma \;+\; 2\int_D \tilde{\alpha}\cdot\nabla u\;\partial_T u\,d\sigma$$

REMARK 3.2.8. In **Theorem 3.2.3** $u = Kf - Sg$ approaches the boundary from the interior domain Ω. We can obtain a very similar equality for u^e on the *exterior* domain when $\int_{\partial\Omega} g\,d\sigma = 0$.[5] This is accomplished by replacing λ with $-\lambda$ and noting that in the jump relations we now have $\partial_\nu u^e|_N = -\frac{1}{2}g + \partial_\nu\mathbf{K}f - \partial_\nu\mathbf{S}g$ and $\partial_T u^e|_D = -\frac{1}{2}\partial_T f + \partial_T\mathbf{K}f - \partial_T\mathbf{S}g$. Otherwise the proof is identical, and we obtain

$$\int_N (\alpha\cdot\nu)\,|\nabla u^e|^2\,d\sigma + \int_D (-\alpha\cdot\nu)\,|\nabla u^e|^2\,d\sigma + \lambda\int_{\partial\Omega} (\alpha\cdot\nu)g^2 - (\alpha\cdot\nu)\,|\partial_T f|^2\,d\sigma$$
$$= 2\int_N \alpha\cdot\nabla u^e\,(\partial_\nu u^e - \lambda g)\,d\sigma \;+\; 2\int_D \tilde{\alpha}\cdot\nabla u^e\,(\partial_T u^e - \lambda\partial_T f)\,d\sigma$$
$$-\;\lambda\int_{\partial\Omega} g\,C(g) \;-\; \partial_T f\;C(\partial_T f) \;+\; 2g\,\tilde{C}(\partial_T f)\,d\sigma$$
$$+\int_{\overline{\Omega}^c} \operatorname{div}(\alpha)\,|\nabla u^e|^2 \;-\; 2\,\nabla\alpha(\nabla u^e)\cdot\nabla u^e\,dx.$$

[5] Without the assumption that g has mean value zero we would end up with an additional compact operator. Unlike the other compact operators involved, however, it does not vanish when $\lambda = 0$.

Notice that some of the terms on the right-hand side have changed sign, but those on the left have not. The condition that g has mean value zero guarantees that u has sufficient decay at infinity to justify our application of the divergence theorem.[6]

Of special use to us is the case $\lambda = 0$, where we have

$$\begin{aligned}\int_N (\alpha\cdot\nu)\,|\nabla u^e|^2\,d\sigma + \int_D (-\alpha\cdot\nu)\,|\nabla u^e|^2\,d\sigma \\ = 2\int_N (\alpha\cdot\nabla u^e)\,\partial_\nu u^e\,d\sigma + 2\int_D (\tilde{\alpha}\cdot\nabla u^e)\,\partial_T u^e\,d\sigma \\ + \int_{\overline{\Omega}^c} \operatorname{div}\,\alpha\,|\nabla u^e|^2 - 2\,\nabla\alpha(\nabla u^e)\cdot\nabla u^e\,dx.\end{aligned}$$

Just as in **Theorem 3.2.3**, this holds for *any* harmonic function for which the divergence theorem can be applied on the exterior domain. In particular, the specific choice of boundary dissection does not matter.

Our next result requires us to divide by $|\alpha\cdot\nu|$. We therefore require this term to be bounded from below. Recall that a bounded and connected Lipschitz domain $\Omega \subset \mathbb{R}^2$ with connected boundary is *strongly dissected* (with respect to the C_0^∞ vector field α and boundary dissection $\partial\Omega = D \cup N$) if there is a $\delta > 0$ such that $\alpha\cdot\nu > \delta$ on N and $\alpha\cdot\nu < -\delta$ on D.

COROLLARY 3.2.9. *Let Ω be a strongly dissected Lipschitz domain, $\lambda \geq 0$, $f \in W_0^{1,2}(D)$, $g \in L^2(\partial\Omega)$ with support in N, and $u = Kf - Sg$.*

The following inequality holds:

[6] See Theorem 3.40 and the proof of Proposition 3.4 in [**Fol95**].

$$\begin{aligned}||\nabla u||^2_{L^2(\partial\Omega)} + \lambda||g||^2_{L^2(N)} + \lambda||\partial_T f||^2_{L^2(D)} \\ \leq c\Big[||\partial_\nu u + \lambda g||^2_{L^2(N)} + ||\partial_T u + \lambda \partial_T f||^2_{L^2(D)} \\ + \lambda||R(f,g)||^2_{L^2(\partial\Omega)} + ||T(f,g)||^2_{L^2(\partial\Omega)}\Big],\end{aligned}$$

for some constant c and compact operators R and T. The constant $c \to \infty$ as $\inf|\alpha\cdot\nu| \to 0$, and T is the map $(f,g) \to \int_{\partial\Omega} |u|^2\, d\sigma$.

Proof. Applying Young's Inequality $\left(2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}\right)$, the Schwartz Inequality, the sign change of $\alpha\cdot\nu$ between N and D, and the bound from below on $|\alpha\cdot\nu|$ to **Theorem 3.2.3**, we may find a constant c_0 such that

$$\begin{aligned}\int_N |\nabla u|^2 + \lambda\, g^2 d\sigma + \int_D |\nabla u|^2 + \lambda\,|\partial_T f|^2\, d\sigma \\ \leq \mathrm{c}_0\Big[\int_N \epsilon\,|\nabla u|^2 + \frac{1}{\epsilon}\,|\partial_\nu u + \lambda\, g|^2\, d\sigma + \int_D \epsilon\,|\nabla u|^2 + \frac{1}{\epsilon}\,|\partial_T u + \lambda\,\partial_T f|^2\, d\sigma \\ + \ \lambda \int_N \frac{\epsilon}{2}\, g^2 + \frac{2}{\epsilon}|C(g)|^2 d\sigma \ + \lambda \int_D \epsilon\,|\partial_T f|^2 + \frac{1}{\epsilon}|C(\partial_T f)|^2 d\sigma \\ + \lambda\int_N \frac{\epsilon}{2}\, g^2 + \frac{8}{\epsilon}|\tilde{C}(\partial_T f)|^2 d\sigma + \int_\Omega |\nabla u|^2\, dx\Big].\end{aligned}$$

By moving the "ϵ" and "$\lambda\epsilon$" terms to the left hand side, we get

$$
\begin{aligned}
(1 - c_0\epsilon)\Big[\int_N |\nabla u|^2 + \lambda\, g^2 d\sigma + \int_D |\nabla u|^2 + \lambda\, |\partial_T f|^2\, d\sigma \Big] \\
\le \mathrm{c}_0 \Big[\int_N \frac{1}{\epsilon}\, |\partial_\nu u + \lambda\, g|^2\, d\sigma + \int_D \frac{1}{\epsilon}\, |\partial_T u + \lambda\, \partial_T f|^2\, d\sigma \\
+\ \lambda \int_N \frac{2}{\epsilon} |C(g)|^2 d\sigma\ +\ \lambda \int_D \frac{1}{\epsilon} |C(\partial_T f)|^2 d\sigma \\
+ \lambda \int_N \frac{8}{\epsilon} |\tilde{C}(\partial_T f)|^2 d\sigma + \int_\Omega |\nabla u|^2\, dx \Big].
\end{aligned}
$$

Taking ϵ small, and dividing by $1 - c_o\epsilon$ gives

$$
\begin{aligned}
\int_{\partial\Omega} |\nabla u|^2\, d\sigma + \lambda \int_{\partial\Omega} g^2 + |\partial_T f|^2\, d\sigma \\
\le \mathrm{c}_1 \Big[\int_N |\partial_\nu u + \lambda g|^2\, d\sigma + \int_D |\partial_T u + \lambda \partial_T f|^2\, d\sigma + \int_\Omega |\nabla u|^2\, dx \Big] \\
+ c_2 \lambda \Big[||C(g)||^2_{L^2(N)} + ||C(\partial_T f)||^2_{L^2(D)} + ||\tilde{C}(\partial_T f)||^2_{L^2(N)} \Big],
\end{aligned}
$$

where c_1 and c_1 are positive constants depending only on $\inf |\alpha \cdot \nu|$.

We can bound the solid term using Green's identity and then Young's inequality:

$$
\begin{aligned}
2 \int_\Omega |\nabla u|^2\, dx &= 2 \int_{\partial\Omega} u\ \partial_\nu u\ d\sigma \\
&\le \frac{1}{\epsilon} \int_{\partial\Omega} u^2\ d\sigma + \epsilon \int_{\partial\Omega} |\partial_\nu u|^2\, d\sigma.
\end{aligned}
$$

Taking ϵ small, we move $\epsilon \int_{\partial\Omega} |\partial_\nu u|^2 \, d\sigma$ to the left-hand side. Since (by Rellich's Theorem) the map $T : (f, g) \to \int_{\partial\Omega} |u|^2 \, d\sigma$ is compact, we may combine the above inequalities to prove the corollary. $\square$

REMARK 3.2.10. At the end of **Corollary 3.2.9** we could have instead used the bound

$$\begin{aligned} 2\int_\Omega |\nabla u|^2 \, dx &= 2\int_{\partial\Omega} u \; \partial_\nu u \; d\sigma \\ &\leq \frac{1}{\epsilon}\int_D u^2 \; d\sigma + \frac{1}{\epsilon}\int_N |\partial_\nu u|^2 \, d\sigma + \epsilon \int_N u^2 \; d\sigma + \epsilon \int_D |\partial_\nu u|^2 \, d\sigma. \end{aligned}$$

Since Ω is connected, $\int_N u^2 \; d\sigma \leq k(\int_D u^2 \; d\sigma + \int_\Omega |\nabla u|^2 \, dx)$, for some constant k depending on Ω. By rearranging terms we then have

$$(2 - k\epsilon)\int_\Omega |\nabla u|^2 \, dx - k\epsilon \int_D |\partial_\nu u|^2 \, d\sigma \leq (k\epsilon + \frac{1}{\epsilon})\int_D u^2 \; d\sigma + \frac{1}{\epsilon}\int_N |\partial_\nu u|^2 \, d\sigma.$$

By using these estimates in the proof of **Corollary 3.2.9** and then setting $\lambda = 0$, we obtain

$$||\nabla u||^2_{L^2(\partial\Omega)} \leq c\,(||\partial_\nu u||^2_{L^2(N)} + ||u||_{W^{1,2}(D)}),$$

for some constant c. Since $\lambda = 0$ this holds for any harmonic u with $u^* + \nabla u^* \in L^2(\partial\Omega)$.

Notice that the we now have the $W^{1,2}(D)$ norm of u on the right-hand side of the inequality. Because of the operator T, taking $\lambda = 0$ in **Corollary 3.2.9** would have instead introduced the unwanted term $||u||_{L^2(N)}$ to the right-hand side of the inequality. The inequality of **Remark 3.2.10**, however, only uses the mixed data of u on the right-hand side.

Recall that $L^2_0(N)$ is the subspace of $L^2(N)$ whose elements g have support in N and *mean value zero*, i.e. $\int_N g\, d\sigma = 0$.

THEOREM 3.2.11. *Let u^i be a harmonic function in Ω with $|u^i|^* + |\nabla u^i|^* \in L^2(\partial\Omega)$. Then there is a constant c such that*

$$||\nabla u^i||^2_{L^2(\partial\Omega)} \leq c\,(||\partial_\nu u^i||^2_{L^2(D)} + ||u^i||_{W^{1,2}(N)}).$$

Let u^e be a harmonic function in $\overline{\Omega}^c$ with $|u^e|^* + |\nabla u^e|^* \in L^2(\partial\Omega)$. Then there is constant c such that

$$||\nabla u^e||^2_{L^2(\partial\Omega)} \leq c\,(||\partial_\nu u^e||^2_{L^2(D)} + ||u^e||^2_{W^{1,2}(N)}).$$

Notice that we have swapped N and D from their usual locations. If $u = Kf - Sg$ with $f \in W^{1,2}_0(D)$ and $g \in L^2_0(N)$, then both inequalities hold.

PROOF. If Ω is strongly dissected by α, D, and N then it is also strongly dissected by $-\alpha$, $\tilde{D} = N$, and $\tilde{N} = D$. Using this new dissection and $\lambda = 0$ in **Remark 3.2.10**, we have

$$||\nabla u^i||^2_{L^2(\partial\Omega)} \leq c\,(||\partial_\nu u^i||^2_{L^2(D)} + ||u^i||_{W^{1,2}(N)}),$$

where we have swapped the locations of N and D. This proves the first part of the theorem.

For the second inequality we use **Remark 3.2.8** with $\lambda = 0$:

$$\begin{aligned}\int_N (\alpha\cdot\nu)\,|\nabla u^e|^2\,d\sigma\;+\int_D (-\alpha\cdot\nu)\,|\nabla u^e|^2\,d\sigma \\ =\;2\int_N (\alpha\cdot\nabla u^e)\,\partial_\nu u^e\,d\sigma\;+\;2\int_D (\tilde{\alpha}\cdot\nabla u^e)\,\partial_T u^e\,d\sigma \\ +\int_{\overline{\Omega}^c} \mathrm{div}\;\alpha\,|\nabla u^e|^2\;-\;2\,\nabla\alpha(\nabla u^e)\cdot\nabla u^e\,dx.\end{aligned}$$

As in the proof of **Corollary 3.2.9**, we may use Young's inequality to find a constant $C > 0$ such that

$$\begin{aligned}\int_N |\nabla u^e|^2\,d\sigma\;+\int_D |\nabla u^e|^2\,d\sigma \\ \leq C\Big[\int_N |\partial_\nu u^e|^2\,d\sigma\;+\int_D |\partial_T u^e|^2\,d\sigma+\int_{\overline{\Omega}^c} |\nabla u^e|^2\,dx\Big].\end{aligned}$$

Let B_r denote the ball of radius r centered at the origin, and choose r large enough so that $\Omega \subset B_r$. Applying the divergence theorem to the domain $B_r \setminus \Omega$ we have

$$\int_{B_r \setminus \Omega} |\nabla u^e|^2 \, dx = \int_{\partial\Omega} u^e \, \partial_\nu u^e \, d\sigma + \int_{\partial B_r} u^e \; \partial_r u^e \, d\sigma,$$

where r is the outer normal vector to the ball B_r. (ν is still the outer normal vector for the *interior* domain Ω.) Since g has mean value zero, u is harmonic at infinity. Therefore, $|u|$ and $r^2 \, |\partial_r u|$ are both uniformly bounded uniformly on B_r^c.[7] Taking limits, we obtain

$$\begin{aligned}\int_{\overline{\Omega}^c} |\nabla u^e|^2 \, dx &= \lim_{r\to\infty} \int_{B_r \setminus \overline{\Omega}} |\nabla u^e|^2 \, dx \\ &= \int_{\partial\Omega} u^e \; \partial_\nu u^e \; d\sigma \; + \; \lim_{r\to\infty} \int_{\partial B_r} u^e \; \partial_\nu u^e \; d\sigma \\ &= \int_{\partial\Omega} u^e \; \partial_\nu u^e \; d\sigma.\end{aligned}$$

We now proceed as in **Remark 3.2.10** to obtain

$$||\nabla u^e||^2_{L^2(\partial\Omega)} \leq c \, (||\partial_\nu u^e||^2_{L^2(N)} + ||u^e||^2_{W^{1,2}(D)}),$$

for some constant c.

Using this inequality with the boundary dissection $-\alpha$, $\tilde{D} = N$, and $\tilde{N} = D$, we obtain

[7] Proposition 2.75 in [**Fol95**].

$$||\nabla u^e||^2_{L^2(\partial\Omega)} \leq c\,(||\partial_\nu u^e||^2_{L^2(D)} + ||u^e||^2_{W^{1,2}(N)}),$$

the second part of the theorem. Finally we note that if $u = Kf - Sg$ with $f \in W_0^{1,2}(D)$ and $g \in L_0^2(N)$, then $|u^i|^*$, $|u^e|^*$, $|\nabla u^i|^*$, and $|\nabla u^e|^*$ are all in $L^2(\partial\Omega)$. $\square$

We will use **Theorem 3.2.11** to strengthened **Corollary 3.2.9**. The next theorem says that the left-hand side can be made independent of λ. This improvement is essential to our later application of the Method of Continuity.

Theorem 3.2.12 (Mixed Rellich Inequality).

Let Ω be a strongly dissected Lipschitz domain, $\lambda \geq 0$, $f \in W_0^{1,2}(D)$, $g \in L_0^2(\partial\Omega)$, and $u = Kf - Sg$. We then have the following inequality:

$$\begin{aligned} ||\nabla u^i||^2_{L^2(\partial\Omega)} &+ ||g||^2_{L^2(N)} + ||\partial_T f||^2_{L^2(D)} \\ &\leq c\Big(||\partial_\nu u^i + \lambda\, g||^2_{L^2(N)} + ||\partial_T u^i + \lambda\, \partial_T f||^2_{L^2(D)} \\ &\qquad + \lambda||R(f,g)||^2_{L^2(\partial\Omega)} + ||T(f,g)||^2_{L^2(\partial\Omega)}\Big), \end{aligned}$$

for some constant c and compact operator B. T is the compact operator $T : (f, g) \to \int_{\partial\Omega} |u|^2\, d\sigma$. Furthermore, when $\lambda = 0$ the inequality also holds for u^e.

Proof. Using the jump relations for layer potentials, the continuity of $S(g)$ across the boundary, and our definition of u, we have

$$
\begin{aligned}
||\partial_T f||^2_{L^2(D)} &= ||\partial_T (Kf)^i - \partial_T (Kf)^e||^2_{L^2(D)} \\
&= ||\partial_T u^i - \partial_T u^e||^2_{L^2(D)} \\
&\leq ||\partial_T u^i||^2_{L^2(D)} + \ ||\partial_T u^e||^2_{L^2(D)}.
\end{aligned}
$$

Applying the inequality of **Theorem 3.2.11** for the exterior domain, we may bound $||\partial_T u^e||^2_{L^2(D)}$ by a multiple of $||u^e||^2_{W^{1,2}(N)} + ||\partial_\nu u^e||^2_{L^2(D)}$.

We now take advantage of the supports of f and g, along with the jump relations. Since $f = 0$ on N, $\partial_T u^e = \partial_T u^i$ on N. Similarly $\partial_\nu u^e = \partial_\nu u^i$ on D because $g = 0$ on D. This means

$$
\begin{aligned}
||\partial_T u^e||^2_{L^2(D)} &\leq \mathrm{c}\left(||\, u^i||^2_{W^{1,2}(N)} + ||\partial_\nu u^i||^2_{L^2(D)}\right) \\
&\leq \mathrm{c}\, ||\, u^i||^2_{W^{1,2}(\partial\Omega)},
\end{aligned}
$$

where we have used **Lemma 1.2.3** for the second inequality.

Using these inequalities and combining terms, we have $||\partial_T f||^2_{L^2(D)} \leq \mathrm{c}||u^i||^2_{W^{1,2}(\partial\Omega)}$, and a similar process shows $||g||^2_{L^2(N)} \leq \mathrm{c}\, ||\, u^i||^2_{W^{1,2}(\partial\Omega)}$.

We can rephrase these bounds using the map $T : (f, g) \to \int_{\partial\Omega} u^2 \, d\sigma$, as

$$
||\partial_T f||^2_{L^2(D)} + ||g||^2_{L^2(N)} \leq \ \mathrm{c}(||\nabla u^i||^2_{L^2(\partial\Omega)} + ||T(f, g)||^2_{L^2(\partial\Omega)}).
$$

Combining these inequalities with **Corollary 3.2.9**, we arrive at the desired result for u^i. The proof for u^e is nearly identical. □

COROLLARY 3.2.13. *Let u be as in **Theorem 3.2.12**. Then the following inequality holds:*

$$||\partial_T f||^2_{L^2(D)} + ||g||^2_{L^2(N)} \leq c\,(||u||^2_{W^{1,2}(D)} + ||\partial_\nu u||^2_{L^2(N)}),$$

for some constant c.

PROOF. The proof of **Theorem 3.2.12** shows

$$\begin{aligned} ||\partial_T f||^2_{L^2(D)} + ||g||^2_{L^2(N)} &\leq \mathrm{c}||u||^2_{W^{1,2}(\partial\Omega)} \\ &= \mathrm{c}(||\nabla u||^2_{L^2(\partial\Omega)} + ||u||^2_{L^2(D)} + ||u||^2_{L^2(N)}). \end{aligned}$$

Since Ω is connected, $\int_N u^2\, d\sigma \leq c(\int_D u^2\, d\sigma + \int_{\partial\Omega} |\nabla u^*|^2\, d\sigma)$, for some constant c depending on Ω. By using **Lemma 1.2.3** and combining inequalities, we get

$$||\partial_T f||^2_{L^2(D)} + ||g||^2_{L^2(N)} \leq \mathrm{c}(||\nabla u||^2_{L^2(\partial\Omega)} + ||u||^2_{L^2(D)}).$$

Applying **Remark 3.2.10** with $\lambda = 0$ to bound $||\nabla u||^2_{L^2(\partial\Omega)}$ proves the corollary. □

3.3. Proof of the Main Result

Let us recall some definitions from Functional Analysis.

DEFINITION 3.3.1. On a Hilbert space, a bounded linear operator A is *left-semi Fredholm* if it has closed range and finite dimensional kernel. Equivalently, A is left-semi Fredholm if and only if there is no sequence $\{x_n\}$ of unit vectors such that $x_n \to 0$ weakly and $\lim \|A(x_n)\| = 0$.[8]

DEFINITION 3.3.2. The *index* of a left-semi Fredholm operator A is the value of $\dim(ker\, A) - \dim(ker A^*)$. A is *Fredholm* if it has finite index.

DEFINITION 3.3.3. Let $S \subset \mathbb{R}$ be connected. A family $\{A_\lambda : \lambda \in S\}$ of bounded linear operators on a normed space is called *continuous* if the map $h : \lambda \to A_\lambda$ is continuous in the norm topology.

The following lemma will allow us to apply the Mixed Rellich Inequality in the proof of our main result, **Theorem 3.0.2**.

LEMMA 3.3.4 (Method of Continuity). *Let $\{A_\lambda\}$ be a continuous family of bounded linear operators on a Hilbert space with the uniform bound from below*

$$||x||^2 \;\leq\; C||A_\lambda(x)||^2 + ||K_\lambda(x)||^2,$$

where the K_λ are a collection of compact operators.

[8] Chapter 11, Theorem 2.3 in [**Con90**]

Then the operators A_λ are all left semi-Fredholm with the same index. In particular if one has index zero, all of them are Fredholm with index zero.

REMARK 3.3.5. The lemma remains true when the bound from below involves several compact operators, for instance

$$||x||^2 \ \leq \ C||A_\lambda(x)||^2 + ||K_{\lambda_1}(x)||^2 + ||K_{\lambda_2}(x)||^2.$$

The proof of this is nearly identical to that of the lemma.

REMARK 3.3.6. Similar theorems appear in the literature. The case where $A_\lambda = A + \lambda I$ and $K_\lambda \equiv 0$ is commonly used, as are two versions proved in the book [**GT01**]. Although probably well-known, this author was unable to find a reference to the version presented above.

PROOF OF LEMMA 3.3.4.

We prove the lemma by contradiction. Fix λ and assume $\{x_n\}$ is a sequence of unit vectors converging weakly to 0 such that $||A_\lambda(x_n)|| \to 0$. Being a compact operator, K_λ is completely continuous, and so $||K_\lambda(x_n)|| \to 0$. But then

$$1 = ||x_n||^2 \ \leq \ C||A_\lambda(x_n)||^2 + ||K_\lambda(x_n)||^2 \ \to \ 0,$$

a contradiction. Therefore A_λ is left-semi Fredholm.

Let h be the continuous map $h : \lambda \to A_\lambda$. The map i from left semi-Fredholm operators to their index (under the discrete topology) is continuous,[9] so the map $i \circ h$ is also continuous.

Assume one of the A_λ has finite index z. By continuity and the definition of the discrete topology, $(i \circ h)^{-1}(z)$ is open, closed and nonempty; hence it is the entire connected set S. Since the A_λ are left-semi Fredholm, this implies they all have the same index $z \in [-\infty, \infty)$. $\square$

Consider the operator $M : (f, g) \to (u|_D, \partial_\nu u|_N)$, where $u = Kf - Sg$. If the support of f is in D, then $\nabla u^* \in L^2(\partial\Omega)$ is only guaranteed when $f \in W_0^{1,2}(D)$. It is therefore natural to view M as a map from $W_0^{1,2}(D) \times L^2(N)$ to $W^{1,2}(D) \times L^2(N)$, where the domain and range are different.

Let π_D be the orthogonal projection of $W^{1,2}(D)$ onto $W_0^{1,2}(D)$ and π_N be the orthogonal projection of $L^2(N)$ onto $L_0^2(N)$, and equip the product space $W^{1,2}(D) \times L^2(N)$ with the norm

$$||(a, b)||^2_{W^{1,2}(D)\times L^2(N)} = ||a||^2_{W^{1,2}(D)} + ||b||^2_{L^2(N)}.$$

Define the continuous family of bounded linear operators

$$\{U_\lambda : W^{1,2}(D) \times L^2(N) \to W^{1,2}(D) \times L^2(N),\ \lambda \geq 0\}$$

[9] Chapter 11, Theorem 3.13 in [**Con90**]

by

$$U_\lambda(f,g) = M(\pi_D[f],\, \pi_N[g]) + \lambda(f,g).$$

When λ is sufficiently large U_λ is invertible, and we may apply the method of continuity. The difference between $W^{1,2}(D)$ and $W_0^{1,2}(D)$ therefore plays an essential role. As we have proved in **Chapter 3.1**, these spaces differ by only a finite dimensional vector space.[10]

REMARK 3.3.7. When $(f,g) \in W_0^{1,2}(D) \times L_0^2(N)$ the equation $U(f,g) = (u|_D, u|_N)$ can be written in matrix form as

$$\begin{bmatrix} \frac{1}{2}I + \mathbf{K} & \mathbf{S} \\ \partial_\nu \mathbf{K} & \frac{1}{2}I - \partial_\nu \mathbf{S} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} u_D \\ u_N \end{bmatrix},$$

This is the same matrix equation used in [**McL00**]. There, however, (f,g) lies in the weaker function space $W^{\frac{1}{2},2}(D) \times W^{-\frac{1}{2},2}(N)$, and $u = Kf - Sg$ does not necessarily satisfy $\nabla u^* \in L^2(\partial\Omega)$..

REMARK 3.3.8. Given any $g \in L^2(N)$ with support in N, the function $g - \frac{1}{|N|}\int_N g\, d\sigma$ is in $L_0^2(N)$. Hence these spaces differ by a one-dimensional vector space.

We are now in a position to prove our main result.

[10] This is not true in higher dimensions, which explains why all the results proved in this chapter are restricted to $\mathbb{R}^2$.

THEOREM 3.3.9. *Let Ω be a strongly dissected Lipschitz domain in the plane such that D is the finite union of m open sets with pairwise disjoint closures. Also let $U := U_0$ be the operator defined above. Then U is Fredholm with index 0, and its kernel has dimension $2m+1$.*

PROOF. We can adapt the Mixed Rellich Inequality to provide the uniform bound from below required in the Method of Continuity. By adding $||(1-\pi_D)f||^2$ and $||(1-\pi_N)g||^2$ to both sides of **Theorem 3.2.12** and using the triangle inequality, we obtain

$$\begin{aligned}
||\nabla u||^2_{L^2(\partial\Omega)} + ||g||^2_{L^2(N)} + \ ||\partial_T f||^2_{L^2(D)} \\
\leq c\,||(U+\lambda I_{2\times 2})(f,g)||^2_{W^{1,2}(D)\times L^2(N)} \\
+ ||B(\pi[f],\pi_N[g])||^2_{L^2(\partial\Omega)} \\
+ (1+c\lambda)||(\,(1-\pi_D)f,(1-\pi_N)g\,)||^2_{W^{1,2}(D)\times L^2(N)},
\end{aligned}$$

for $f\in W^{1,2}(D)$ and $g\in L^2(N)$. Here c is a constant independent of λ and B is compact.

By **Theorem 0.0.2** and **Remark 3.3.8**, the maps $1-\pi_D$ and $1-\pi_N$ have finite dimensional range, and are also compact. Since $U+\lambda I$ is invertible for large λ, we may apply **Lemma 3.3.4** to show U is Fredholm with index 0.

By **Corollary 3.2.13**, if $U(f,g)=0$ with $f \in W_0^{1,2}(D)$ and $g \in L_0^2(N)$, then $g=0$ and f is constant on each component of D. In fact, since f vanishes at each boundary point of D, it is identically zero. Therefore U is injective on $W_0^{1,2}(D) \times L_0^2(N)$, and

$$\begin{aligned}\ker U &= \ker((1-\pi_D, 1-\pi_N)\\ &= W^{1,2}(D)/W_0^{1,2}(D) \times L^2(N)/L_0^2(N).\end{aligned}$$

By **Theorem 0.0.2** and **Remark 3.3.8** the dimension of $\ker U$ is $2m+1$. $\square$

Remark 3.3.10. By **Lemma 1.2.2**, any solution $u = Kf - Sg$ to the mixed problem with $f \in W^{1,2}(D)$ and $g \in L_0^2(N)$ satisfies the estimate

$$|| \, u^*||^2_{L^2(\partial\Omega)} + ||\nabla u^*||^2_{L^2(\partial\Omega)} < \infty.$$

By **Lemmas 1.2.3**, and **Remark 3.2.10** we also have

$$\begin{aligned}||\nabla u^*||^2_{L^2(\partial\Omega)} &\leq c_1 ||\nabla u||^2_{L^2(\partial\Omega)}\\ &\leq c_2(||u||^2_{W^{1,2}(D)} + ||\partial_\nu u||^2_{L^2(N)}),\end{aligned}$$

where c_1 and c_2 are constants independent of u.

Combining these apriori estimates with **Theorem 3.3.9** concludes the proof of **Theorem 3.0.2**.

3.4. Robin Boundary Conditions

The results of the previous section remain true when the Neumann data is replaced with Robin boundary conditions.

Definition 3.4.1. **The Mixed Robin Problem**

Let Ω be a strongly dissected Lipschitz domain, and fix $h_D \in W^{1,2}(D)$ and $h_N, b \in L^2(N)$. A harmonic function u is said to solve the *Mixed Robin Problem with data h_D and h_N* if

$$u|_D = h_D,$$

$$\big(\partial_\nu u + b\, u\big)|_N = h_N,$$

and

$$\int_{\partial\Omega} |u^*|^2 + |\nabla u^*|^2 \, d\sigma < \infty,$$

where boundary values are taken via nontangential limits.

Theorem 3.4.2. *Let Ω be a strongly dissected Lipschitz Domain in the plane such that D is the finite union of m connected open sets with pairwise disjoint closures.*

There exists a finite dimensional subspace $F \subset W^{1,2}(D) \times L^2(N)$ such that the Mixed Robin Problem is uniquely solvable for all data $(h_D, h_N) \in F^{\perp}$. Furthermore the solution to the Mixed Problem satisfies

$$\int_{\partial\Omega} |\nabla u^*|^2 \, d\sigma \leq C(\int_D u^2 + |\partial_T u|^2 \, d\sigma + \int_N |\partial_\nu u|^2 \, d\sigma).$$

If $b > 0$ a.e. on N then $\dim F \leq 2m + 1$.

Remark 3.4.3. The proof below is modeled after the paper [**LS04**].

Proof. Let U, π_D, and π_N be the operators from **Theorem 3.3.9**, and define a new operator on $W^{1,2}(D) \times L^2(N)$ by

$$R(f, g) = \big(0|_D \, , b\,(K\pi_D[f] - S\pi_N[g])\,|_N\big).$$

Thus $U + R$ maps (f, g) to the mixed Robin data of $u = K\pi_D[f] - S\pi_N[g]$.

If $b \in L^\infty(N)$ then

$$||b\,u||_{L^2(N)} \leq ||b||_{L^\infty(N)} \, ||u||_{L^2(N)},$$

and we see that R is the composition of the bounded map $(f, g) \to u$, the compact inclusion map $W^{1,2}(N) \hookrightarrow L^2(N)$, and the bounded map $u \to (0, \, b\,u)$. Thus R is a compact operator when $b \in L^\infty(N)$.

Next fix b in $L^2(N)$. We may now obtain the following inequality:

$$\begin{aligned} ||b\,u||_{L^2(N)} &\leq ||b||_{L^2(\partial\Omega)}\,||u||_{L^\infty(N)} \\ &\leq C||b||_{L^2(\partial\Omega)}\,||u||_{W^{1,2}(\partial\Omega)} \\ &\leq C_1||b||_{L^2(\partial\Omega)}\,(||f||_{W^{1,2}(D)} + ||g||_{L^2(N)}), \end{aligned}$$

where C_1 is a constant independent of u. This shows R is a bounded operator when b is an $L^2(N)$ function. Since $L^\infty(N)$ is dense in $L^2(N)$ and the set of compact operators is closed,[11] R is a compact operator whenever $b \in L^2(N)$. The operator $U + R$ is therefore Fredholm with index 0.

We now compute the dimension of $\ker(U + R)$ assuming $b > 0$ a.e. on N. By the definition of $U + R$, $(f, g) \in \ker(U + R)$ when $f \in W^{1,2}(D)/W_0^{1,2}(D)$ and $g \in L^2(N)/L_0^2(N)$, so that $|\ker(U + R)| \geq 2m + 1$. On the other hand, if $(U + R)(f, g) = 0$, then $u|_D = 0$ and $\partial_\nu u|_N = -b\,u|_N$.

Applying the divergence theorem, and then using $b > 0$, we see:

$$\begin{aligned} \int_\Omega |\nabla u|^2\,dx &= \int_{\partial\Omega} u\,\partial_\nu u\,d\sigma(x) \\ &= -\int_N b\,|u|^2 d\sigma \\ &\leq 0. \end{aligned}$$

[11] Theorem 3.5 in chapter 6 of [**Con90**]

Thus u is constant in Ω. In fact, since $u|_D = 0$ we must have $u \equiv 0$ and $U(f,g) = 0$.

By **Remark 3.2.13**, if $U(f,g) = 0$ with $f \in W_0^{1,2}(D)$ and $g \in L_0^2(N)$, then $g = 0$ and f is constant on each component of D. Since f vanishes at each boundary point of D, it is identically zero. This proves $U + R$ is injective on $W_0^{1,2}(D) \times L_0^2(N)$ and $|\ker(U + R)| = 2m + 1$.

The nontangential maximal function bounds on u are a consequence of its representation as layer potentials, in conjunction with **Lemmas 1.2.3**, and **Remark 3.2.10**. □

3.5. Remarks on Bad Mixed Data

The results above solve the mixed problem on strongly dissected Lipschitz domains, but only up to a finite dimensional subspace F of dimension $\leq 2m + 1$. In contrast, there is no exceptional subspace when applying the results in [**Bro94**] to creased Lipschitz domains. One benefit of the results contained above is that while every creased Lipschitz domain is strongly dissected, the reverse inclusion is not true.

Example 3.5.1. All polygonal domains P can be strongly dissected given *any* partition of the sides into N and D. On the other hand, the polygon is a creased Lipschitz domain only if N and D sides form interior angles less than π.

To see this, consider a vertex at the origin where N and D meet at an angle $\theta > \pi$. Near that vertex the domain agrees with a sector, say $\Omega_\theta = \{re^{i\phi} : 0 < r < 1,\, 0 < \phi < \theta\}$. Assume D is a subset of the x-axis. Then the harmonic function $u(z) = Im(z^{\frac{\pi}{2\alpha}})$

has mixed data identically zero near the vertex, and is also well-behaved on the entire boundary. However, the singularity of u is such that $\nabla u \notin L^2(\partial P)$. P cannot be a creased Lipschitz domain; otherwise this would contradict [**Bro94**].[12]

To see that P *is* strongly dissected we now construct the appropriate vector field α. First consider Ω_θ from above with $0 < \theta < 2\pi$, $\theta \neq \pi$. If both linear pieces are of type D we can take $\alpha(z)$ to be the vector with endpoints z and $e^{i\frac{\theta}{2}}$. If both linear pieces are of type N we use the vectors with endpoints z and $e^{i\frac{\theta-2\pi}{2}}$. These choices guarantee α points directly into D and out of N.

If N lies on the x-axis and D is the other linear part of the boundary, we must consider two cases. When $\theta < \pi$ we choose α with endpoints at z and $e^{i\frac{\theta-\pi}{2}}$. Finally, when $\theta > \pi$ we choose α with endpoints at z and $e^{i\frac{\theta+\pi}{2}}$.

This defines a strong dissection near each vertex, and we may extend this smoothly to the remainder of the boundary. We may therefore apply **Theorem 3.0.2** to solve the mixed problem for all data outside a finite dimensional subspace.

We may wish to categorize the bad mixed data, that is, data *not* perpendicular to F. In the case of *smooth* domains in the plane there is exactly $2m$ dimensions of bad data, generated by translations and rotations of the function $u(z) = Re(z^{\frac{1}{2}})$ [**WSH79**]. This thesis treats the case of strongly dissected Lipschitz domains, and requires them to be *nonsmooth* near the intersection of N and D. In this setting the mixed problem has *at most* a $2m + 1$ dimensional subspace of bad data. When an additional convexity-type condition is imposed, there is *no* bad data [**Bro94**].

[12] This example appears in [**Bro94**].

This opens some further questions to be address:

QUESTION 3.5.2. The results in [**WSH79**] suggest that we should have an upper bound of $2m$, not $2m+1$. Can we prove this without further restricting the class of domains allowed?

Replacing the vector field in the definition of a strong dissection with one with components the real and imaginary parts of a harmonic function would accomplish this, but at what cost? Can we describe the domains dissected by such vectors?

QUESTION 3.5.3. Can the results for strongly dissected domains and smooth domains be combined? Perhaps we may develop a theory where the domain is smooth at some meeting points of D and N, and strongly dissected at others?

QUESTION 3.5.4. For each $0 \leq n \leq 2m+1$, can we classify all domains for which the bad data has dimension n? Curvilinear polygons where some of the edges meet at convex angles seem like possible candidates.

CHAPTER 4

The Biharmonic Mixed Problem

The Neumann and Regularity problems for the biharmonic equation on Lipschitz domains were studied in [**Ver90**] and [**Ver05**]. The solutions to these problems lead us to consider a mixed problem for biharmonic functions. Just as with harmonic functions, the geometry of the domain appears to play a role in its solvability.

In this chapter we give examples of well-behaved mixed data which nonetheless yield biharmonic functions without the appropriate boundary regularity. These examples are constructed on sectors in the plane with interior angles greater than π. The specific range of bad angles will depend on the Poisson ratio, a parameter described in the definition of the biharmonic Neumann data.

4.1. Biharmonic Mixed Data

The function spaces involved in the biharmonic mixed problem are somewhat more complicated than those for the harmonic mixed problem. For one thing, biharmonic Dirichlet and Neumann problems each contribute two pieces of boundary data, for a total of four in the mixed problem. For another, some of this data only exists in the sense of distributions.

In this section we define mixed biharmonic data and the mixed biharmonic problem. These definitions are similar to those found in [**Ver05**]. As in previous sections, Ω is a connected Lipschitz domain in the plane with connected boundary, unit outernormal vector ν, and unit tangent vector T.

∇ denotes the gradient, and $\nabla \cdot \nabla = \Delta$ is Laplace's operator. When we write $\nabla\nabla u$, we are *not* using the dot product. Rather we are applying the gradient to each component of ∇u, producing the 2×2 Hessian matrix $(\partial_i \partial_j u)$ with Hilbert-Schmidt norm $|\nabla\nabla u|^2 = \sum_{i,j} |\partial_i \partial_j u|^2$.

$W^{2,2}(\Omega)$ is the closure of $C^\infty(\Omega)$ in the Sobolev norm

$$||u||^2_{W^{2,2}(\Omega)} = \int_\Omega u^2 + |\nabla u|^2 + |\nabla\nabla u|^2 \, dx.$$

A function $u \in W^{2,2}(\Omega)$ is *biharmonic* if it satisfies the *biharmonic equation* $\Delta^2 u = 0$ in Ω. If we also have $\nabla\nabla u^* \in L^p(\partial\Omega)$ for some $p > 1$, then the boundary values of $\partial_i \partial_j u$ are obtained by taking nontangential limits in the interior domain.

Given a relatively open subset $U \subseteq \partial\Omega$, we define $W^{-1,2}(U)$ to be the space dual to the Sobolev space $W^{1,2}_0(U)$. Since the Regularity problem for harmonic functions can be solved in $W^{1,2}(\partial\Omega)$, we will identify each $h \in W^{1,2}_0(U)$ with its Poisson extension on Ω, and note that $\nabla h^* \in L^2(\partial\Omega)$.

Definition 4.1.1. Let u be a biharmonic function satisfying $\nabla\nabla u^* \in L^2(\partial\Omega)$. We define the distribution $\partial_\nu \Delta u \in W^{-1,2}(\partial\Omega)$ by the formula

$$\int_{\partial\Omega} h\,\partial_\nu \Delta u\,d\sigma \;:=\; \int_{\partial\Omega} \Delta u\,\partial_\nu h\,d\sigma,$$

$h \in W^{1,2}(\partial\Omega)$.

Elements $\partial_\nu \Delta u \in W^{-1,2}(U)$ are defined by instead using $h \in W_0^{1,2}(U)$. In addition, we can sometimes define $\partial_\nu \Delta u$ when $\nabla\nabla u^* \notin L^2(\partial\Omega)$. Let $\Omega_j \subset \Omega$ be a sequence of smooth approximating domains.[1] *If* the following limits exist, we define

$$\int_{\partial\Omega} f\,\partial_\nu \Delta u\,d\sigma \;:=\; \lim_{j\to\infty} \int_{\partial\Omega_j} F\,\partial_\nu \Delta u\,d\sigma, \tag{4.1}$$

for $F \in C(\Omega)$ having nontangential limits f almost everywhere. When $\nabla\nabla u^* \in L^2(\partial\Omega)$ and $f \in L^2(\partial\Omega)$, this agrees with our original definition of $\partial_\nu \Delta u$.[2] One benefit of this definition is that it does not require F to be harmonic.

Definition 4.1.2. $W_0^{-1,2}(\partial\Omega)$ is the set of distributions $f \in W^{-1,2}(\partial\Omega)$ such that $\int_{\partial\Omega} f\,d\sigma = 0$.

Taking h to be constant in **Definition 4.1.1**, we see $\partial_\nu \Delta u \in W_0^{-1,2}(\partial\Omega)$.

Definition 4.1.3. Let w be a function satisfying $w^* \in L^2(\partial\Omega)$. We define the distribution $\partial_T w \in W_0^{-1,2}(\partial\Omega)$ by the integration-by-parts formula

[1] These are commonly used in the literature. See, for instance, definition 3.1 in [**Ver05**].
[2] Proposition 4.2 in [**Ver05**].

$$\int_{\partial\Omega} h\,\partial_T w\,d\sigma := -\int_{\partial\Omega} w\,\partial_T h\,d\sigma,$$

$h \in W^{1,2}(\partial\Omega)$.

DEFINITION 4.1.4. **Biharmonic Neumann Data**

Given $u \in W^{2,2}(\Omega)$ with $\nabla\nabla u^* \in L^2(\partial\Omega)$ we define two boundary operators, M_ρ, and K_ρ by:

$$M_\rho(u)(x) \;=\; \rho\Delta u(x) + (1-\rho)\sum_{i,j} \nu_i(x)\nu_j(x)\partial_i\partial_j u(x)$$

$$K_\rho(u)(x) = \partial_N \Delta u(x) + (1-\rho)\partial_T \sum_{i,j} \nu_i(x)T_j(x)\partial_i\partial_j u(x),$$

for $-1 \le \rho < 1$. Here Δu and $\partial_i\partial_j u$ are nontangential boundary values defined for almost every $x \in \partial\Omega$, while $\partial_\nu \Delta u$ and $\partial_T(\nu_i T_j \partial_i\partial_j u)$ should be understood as elements of $W_0^{-1,2}(\partial\Omega)$. Together $M_\rho(u)$ and $K_\rho(u)$ are known as the *Neumann data* for the biharmonic equation. The parameter ρ is called the *Poisson ratio.*

With this notation we can define the Biharmonic Neumann and Dirichlet Problems:

DEFINITION 4.1.5. **The Biharmonic Neumann Problem**

Let $h \in L^2(\partial\Omega)$ and $\Lambda \in W_0^{-1,2}(\partial\Omega)$. The *Biharmonic Neumann Problem* is to find a biharmonic function $u \in W^{2,2}(\Omega)$ satisfying the three boundary conditions

$$M_\rho(u) = h$$

$$K_\rho(u) = \Lambda$$

$$\nabla\nabla u^* \in L^2(\partial\Omega).$$

Definition 4.1.6. **The Biharmonic Dirichlet Problem**
Let $f \in W^{1,2}(\partial\Omega)$ and $g \in L^2(\partial\Omega)$. The *Biharmonic Dirichlet Problem* is to find a biharmonic function $u \in W^{1,2}(\Omega)$ satisfying the three boundary conditions

$$u = f$$

$$\partial_\nu u = g$$

$$\nabla u^* \in L^2(\partial\Omega),$$

in the sense of nontangential convergence.

While the biharmonic Neumann Problem requires integrability of $\nabla\nabla u^*$, the Dirichlet Problem only requires integrability of ∇u^*. As with the case for harmonic functions, we rectify this imbalance by establishing a regularity problem.

Definition 4.1.7. In $\mathbb{R}^2$, $WA^{2,2}(\partial\Omega)$ is the space of vectors $F = (f_0, f_1, f_2)$, where each f_i is in $W^{1.2}(\partial\Omega)$, and $\partial_T f_0 = N_2 f_1 - N_1 f_2$.

Definition 4.1.8. **The Biharmonic Regularity Problem**

Let $F = (f, f_1, f_2)$ be an element of $WA^{2,2}(\partial\Omega)$. The *Biharmonic Regularity Problem* is to find a biharmonic function $u \in W^{2,2}(\Omega)$ satisfying the three boundary conditions

$$\begin{aligned} u &= f \\ \partial_\nu u &= \nu_1 f_1 + \nu_2 f_2 \\ \nabla\nabla u^* &\in L^2(\partial\Omega), \end{aligned}$$

in the sense of nontangential convergence.

If u solves a regularity problem, then $(u, \nabla u) \in WA^{2,2}(\partial\Omega)$ and $\partial_\nu u \in L^2(\partial\Omega)$. For such u we set $g = \nu_1 f_1 + \nu_2 f_2$. In a slight abuse of notation, both F and (f, g) are called the *biharmonic regularity data*.

Individually, the Biharmonic Neumann and Regularity Problems can be solved. More precisely, we have the following theorem from [**Ver05**] and [**Ver90**]:

Theorem 4.1.9. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with connected boundary, and $-1 \le \rho < 1$.*

The Biharmonic Neumann problem is uniquely solvable for any Neumann data in $L^2(\partial\Omega) \times W_0^{-1,2}(\partial\Omega)$. The Biharmonic Regularity Problem is uniquely solvable for any biharmonic regularity data in $WA^{2,2}(\partial\Omega)$.

The range $-1 \leq \rho < 1$ for the Poisson ratio is important. When $-3 < \rho < -1$ there exist biharmonic functions with $L^2(\partial\Omega) \times W_0^{-1,2}(\partial\Omega)$ Neumann data, but for which $\nabla u^* \notin L^2(\partial\Omega)$. Examples of this include the functions $u(z) = Im(\beta z^q + \gamma \bar{z} z^{q-1})$, with the proper choice of q, β, and γ [**Ver05**].

As with the Laplacian, we will consider a mixture of the Neumann and Regularity problems. Recall the following definition:

Definition 4.1.10. Given a domain Ω, a *dissection* of $\partial\Omega$ is a decomposition $\partial\Omega = N \cup D$, where D is open, $N = \partial\Omega \backslash D$, and both sets have non-empty interior.

Definition 4.1.11. Let Ω be a Lipschitz domain and let $D \cup N$ be a dissection of $\partial\Omega$. Given $F = (f, g) \in WA^{2,2}(\partial\Omega)$, $h \in L^2(\partial\Omega)$, and $\Lambda \in W_0^{-1,2}(\partial\Omega)$, the *Biharmonic Mixed Problem* is to find a biharmonic function $u \in W^{2,2}(\Omega)$ satisfying the following five boundary conditions:

$$M_\rho(u) = h, \quad K_\rho(u) = \Lambda, \text{ on } N,$$

$$u = f, \quad \partial_\nu u = g, \text{ on } D,$$

$$\nabla\nabla u^* \in L^2(\partial\Omega).$$

This definition warrants further explanation. The boundary data f, g, and h should be attained pointwise almost everywhere in the sense of nontangential limits. Meanwhile, by $K_\rho(u) = \Lambda$ on N we mean $K_\rho(u)$ and Λ agree as linear functionals on $W_0^{1,2}(N)$.

Whereas the Neumann and Regularity problems are solvable for $-1 \leq \rho < 1$, we will construct counterexamples showing that the Biharmonic Mixed Problem is *not* solvable in general when $-1 \leq \rho \leq -\frac{1}{2}$.

Remark 4.1.12. In their 2005 paper [**CHW05**], Cakoni, Hsiao, and Wendland solved a similar mixed biharmonic problem for $0 < \rho < 1$. There, however, the mixed data is in the weaker function spaces $(f, g) \in W^{\frac{3}{2},2}(\partial\Omega) \times L^{\frac{1}{2},2}(\partial\Omega)$ and $(h, \Lambda) \in W^{-\frac{1}{2},2}(\partial\Omega) \times W^{-\frac{3}{2},2}(\partial\Omega)$, where Ω is a smooth domain. These conditions allow them to solve the mixed problem using compact operators. Unfortunately, these operators are not compact in our setting.

4.2. Counterexamples

In this section we establish counterexamples to the general solvability of the biharmonic mixed problem. To do this we will explicitly define biharmonic functions $u \in W^{2,2}(\Omega)$ that have mixed data meeting the requirements of definition 4.1.11, but do not satisfy the boundary regularity requirement $\nabla\nabla u^* \in L^2(\partial\Omega)$. We also show that there is no solution to the mixed problem having the same mixed data u.

For each $\theta \in (0, 2\pi)$, define the domain $\Omega_\theta := \{z = re^{i\phi} : \ 0 < r < 1, \ 0 < \phi < \theta\}$. When $\theta \geq \pi$, functions of the form $v(z) = Im(z^\alpha)$ serve as counterexamples for the *Harmonic* Mixed Problem [**Bro94**]. When $-3 < \rho < -1$, functions of the form

$w = Im(Ez^q + F\overline{z}z^{q-1})$ serve as counterexamples for the *Biharmonic* Neumann Problem [**Ver05**]. We will show that when $-1 \le \rho \le -\frac{1}{2}$ and θ is near $\frac{3\pi}{2}$, functions of the form

$$u = Im(Ez^q + F\overline{z}z^{q-1}) + Re(Gz^q + H\overline{z}z^{q-1})$$

serve as counterexamples for the Biharmonic Mixed Problem. $E, F, G, H,$ and q are constants we will need to determine.

We begin by dissecting the boundary of Ω_θ, setting $N = \{re^{i\phi} : 0 \le r \le 1,\ \phi = 0\}$ and $D = \partial\Omega_\theta \backslash N$. The function u above has well-behaved mixed data on this domain. In fact, $\nabla\nabla u \in L^\infty(\overline{\Omega_\theta \backslash B})$ for any ball B centered at the origin.

Let $\tilde{D} = \{re^{i\theta} : 0 < r < 1\}$. The mixed data of u on N and $\tilde{D}$ is:[3]

$$\begin{aligned}
u &= r^q\big[E\sin(q\theta) + F\sin((q-2)\theta) + G\cos(q\theta) + H\cos((q-2)\theta)\big] \text{ on } \tilde{D},\\
\partial_\nu u &= -r^{q-1}\big[Eq\cos(q\theta) + F(q-2)\cos((q-2)\theta)\\
&\qquad - Gq\sin(q\theta) - H(q-2)\sin((q-2)\theta)\big] \text{ on } \tilde{D},\\
M_\rho(u) &= r^{q-2}(1-q)\big[G(1-\rho)q + H((1-\rho)q-4)\big] \text{ on } N,\\
K_\rho(u) &= -r^{q-3}(1-q)(2-q)\big[E(1-\rho)q + F((1-\rho)(q-2)+4)\big] \text{ on } N,
\end{aligned}$$

[3] See the calculation in Section 21 of [**Ver05**]

Setting this boundary data equal to zero produces the matrix equation $A(\rho, \theta, q)x = 0$, where

$$A(\rho,\theta,q) = \begin{bmatrix} \sin(q\theta) & \sin((q-2)\theta) & \cos(q\theta) & \cos((q-2)\theta) \\ q\cos(q\theta) & (q-2)\cos((q-2)\theta) & -q\sin(q\theta) & -(q-2)\sin((q-2)\theta) \\ 0 & 0 & (1-\rho)q & (1-\rho)q-4 \\ (1-\rho)q & (1-\rho)(q-2)+4 & 0 & 0 \end{bmatrix}$$

and

$$x = \begin{bmatrix} E \\ F \\ G \\ H \end{bmatrix}.$$

Solutions to this equation give nontrivial coefficients to u, so that u has smooth mixed data on all of $\partial\Omega_\theta$. In particular, u has zero Dirichlet data on $\tilde{D}$, is in $WA^{2,2}(D)$, and has zero Neumann data pointwise on N. Furthermore, when $1 < q \leq \frac{3}{2}$, $\nabla\nabla u \in L^2(\Omega)$. On the other hand, $|\nabla\nabla u|^* \notin L^2(\partial\Omega)$ for this range of q, and u is not a solution to the corresponding mixed problem.

Triples $(\rho,\, \theta,\, q)$ such that $\det(A(\rho,\, \theta,\, q)) = 0,\ -1 \leq \rho < 1,\ 0 < \theta < 2\pi$ and $1 < q \leq \frac{3}{2}$ therefore correspond to *potential* counterexamples for the mixed problem.[4]

[4] This is the same strategy employed in [**Ver05**] to establish counterexamples for the biharmonic Neumann problem.

Before identifying such triples, we prove a uniqueness result showing these to be true counterexamples.

PROPOSITION 4.2.1. *Let* $u = Im(Ez^q + F\overline{z}z^{q-1}) + Re(Gz^q + H\overline{z}z^{q-1})$, $\det(A(\rho,\, \theta,\, q)) = 0,\ -1 \le \rho < 1,\ 0 < \theta < 2\pi$ *and* $1 < q \le \frac{3}{2}$. *Then there is no solution to the biharmonic mixed problem with the same data as* u *on* $\partial\Omega_\theta$.

REMARK 4.2.2. We must define what we mean by saying u and w to have the same mixed data.

By inspection $|\nabla u|$ is continuous on $\partial\Omega$. Since $\nabla\nabla w^* \in L^2(\partial\Omega)$, an application of **Lemma 3.1.1** shows $|\nabla w|$ is absolutely continuous on $\partial\Omega$. u and w having the same Dirichlet on D therefore means $|u - w| + |\nabla(u - w)| = 0$ pointwise on D.

By inspection $\nabla\nabla u^* \in L^p(\partial\Omega)$ for $1 < p < \frac{1}{2-q}$. Therefore $M_\rho(u - w)$ is a well-defined L^p function, and we are assuming it equals zero pointwise a.e. on N.

Some care must be taken when considering the third-order Neumann data on N. While $K_\rho(u) = 0$ *pointwise* on N, it is not a bounded distribution in $W^{-1,2}(N)$. On the other hand, since w is a solution to the mixed problem, $K_\rho(w)$ must equal zero as a *distribution* acting on $W_0^{1,2}(\overline{D}^c)$. It need not necessarily be defined pointwise a.e. on N, however. We will therefore need to analyze $K_\rho(u)$ and $K_\rho(w)$ separately.

PROOF. Set $\Omega = \Omega_\theta$. Assume that w is a biharmonic function with the same mixed data as u, but satisfying $\nabla\nabla w^* \in L^2(\partial\Omega)$. We would like to apply Green's identity [5]

[5] See **Equation 4.2** below.

$$(1-\rho)\int_{\Omega} L_{il}(u-w)L_{il}(u-w)dx$$
$$= \int_{\partial\Omega} \partial_\nu(u-w)M_\rho(u-w)\,d\sigma - \int_{\partial\Omega}(u-w)K_\rho(u-w)d\sigma,$$

but must do so carefully. The left-hand side integral is finite because both $|\nabla\nabla u|$ and $|\nabla\nabla w|$ are in $L^2(\Omega)$. The first integral on the right-hand side is also finite because $\nabla\nabla(u-w)^* \in L^p(\partial\Omega)$ for $1 < p < \frac{1}{2-q}$, and $|\nabla(u-w)|$ is in $L^{p'}(\partial\Omega)$ for the Hölder conjugate exponent $p' = \frac{p}{p-1}$. As noted in **Remark 4.2.2**, however, the second integral on the right-hand side is not well-defined.

To overcome this, we will work on a sequence of smooth approximating domains $\Omega_j \subset \Omega$, where $\frac{1}{j} < \mathrm{dist}(\partial\Omega_j, \partial\Omega) < \frac{2}{j}$. On these domains we have $\nabla\nabla(u-w)^* \in L^2(\partial\Omega_j)$. An application of Green's Identity to the biharmonic function $u-w$ gives

$$(1-\rho)\int_{\Omega_j} L_{il}(u-w)L_{il}(u-w)dx$$
$$= \int_{\partial\Omega_j} \partial_\nu(u-w)M_\rho(u-w)\,d\sigma - \int_{\partial\Omega_j}(u-w)K_\rho(u-w)d\sigma, \tag{4.2}$$

where $L_{il} = \partial_i\partial_l + t\delta_{il}\Delta,\ t = -1 + \frac{1}{2}\frac{\sqrt{1+\rho}}{\sqrt{1-\rho}},$ and the summation convention is used.[6]

Since $|\nabla\nabla(u-w)| \in L^2(\Omega)$, the left-hand side can be bounded uniformly in j and converges to $(1-\rho)\int_\Omega L_{il}(u-w)L_{il}(u-w)\,dx$. Our goal is to show that this integral

[6] See Section 6 and Equation 10.2 in [**Ver05**].

equals zero, for this would imply $u(x) - w(x) = a + \vec{b} \cdot x + c\,|x|^2$ for some constants a, $\vec{b}$, and c.[7] This would mean $|\nabla\nabla u| \leq |\nabla\nabla w| + 4c$, contradicting $\nabla\nabla u^* \notin L^2(\partial\Omega)$.

To accomplish this goal we examine each term on the right-hand side of **Equation 4.2**. By inspection $\left|\partial^k u(z)\right| \leq |z|^{q-|k|}$, where $k = (k_1, k_2)$ is a multi-index with $|k| = k_1 + k_2$ and $0 \leq |k| \leq 3$. Since $\partial^m w = \partial^m u = 0$ on D for $0 \leq |m| \leq 1$, we may obtain a similar estimate on $|w(z)|$ as follows:

Fix $z \in \Omega$ and let γ be the line segment connecting z to the origin. Applying the Fundamental Theorem of Calculus, $w(0) = 0$, and then Hölder's Inequality, we have

$$
\begin{aligned}
|w(z)|^2 &= \left| \int_\gamma \partial_T w(y)\, d\sigma(y) + w(0) \right|^2 \\
&= \left| \int_\gamma \partial_T w(y)\, d\sigma(y) \right|^2 \\
&\leq |\gamma| \int_\gamma |\nabla w(y)|^2 \, d\sigma(y).
\end{aligned}
$$

Repeating this argument, but now using $|\nabla w(0)| = 0$, we obtain

$$
|\nabla w(z)|^2 \leq |\gamma| \int_\gamma |\nabla\nabla w(y)|^2 \, d\sigma(y).
$$

Together with the measurement $|\gamma| = |z|$ and the nontangential maximal bound on $\nabla\nabla w$, these inequalities show $|\partial^m w(z)| \leq C\,|z|^{\frac{3}{2}-|m|}$ for $0 \leq |m| \leq 1$ and some constant C. Since $q \leq \frac{3}{2}$ and $|z| \leq 1$ for $z \in \overline{\Omega}$, we also have $|w(z)| \leq C\,|z|^q$.

[7] See the proof of Lemma 10.5 in [**Ver05**].

Using the above inequalities, nontangential bounds, and Hölder's Inequality, we may obtain the following estimates for the right-hand side of **Equation 4.2**.

$$\int_{\partial\Omega_j} \partial_\nu(u-w)M_\rho(u-w)\,d\sigma \;\le\; \|\nabla(u-w)^*\|_{L^{p'}(\partial\Omega)}\,\|\nabla\nabla(u-w)^*\|_{L^p(\partial\Omega)}$$

$$\int_{\partial\Omega_j} (u-w)K_\rho(u)\,d\sigma \;\le\; C\int_{\partial\Omega_j} |z|^q\,|z|^{q-3}\,d\sigma$$

Let us consider each estimate individually.

The L^1 function $\nabla(u-w)^*\nabla\nabla(u-w)^*$ dominates the first integrand pointwise almost everywhere. $\int_{\partial\Omega_j}\partial_\nu(u-w)M_\rho(u-w)\,d\sigma$ therefore converges to $\int_{\partial\Omega}\partial_\nu(u-w)M_\rho(u-w)\,d\sigma$ by the Dominated Convergence Theorem. Since u and w have the same mixed data by assumption, this integral equals zero.

For the second inequality, we should notice that $2q-3>-1$. The integrand on the right-hand side therefore provides the L^1 function needed to justify another application of the Dominated Convergence Theorem. This shows $\int_{\partial\Omega_j}(u-w)K_\rho(u)\,d\sigma \to \int_{\partial\Omega}(u-w)K_\rho(u)\,d\sigma$. By assumption $K_\rho(u) = 0$ pointwise a.e. on N. Meanwhile, $u-w=0$ on D since they share the same Dirichlet data there. Therefore, $\int_{\partial\Omega}(u-w)K_\rho(u)\,d\sigma = 0$.

This now leaves us with only

$$\int_{\partial\Omega_j}(u-w)K_\rho(w)\,d\sigma$$

on the right-hand side of **Equation 4.2**. By Proposition 4.2 and Lemma 4.9 in [**Ver05**] this integral equals $\langle u - w, K_\rho(w)\rangle$, the action of the distribution $K_\rho(w)$ on the $W_0^{1,2}(\overline{D}^c)$ function $u - w$. This equals zero since, by assumption, $K_\rho(w) = 0$ as a distribution acting on $W_0^{1,2}(\overline{D}^c)$.

Taking limits in **Equation 4.2**, we conclude

$$(1-\rho)\int_\Omega L_{il}(u-w)L_{il}(u-w)dx = 0.$$

As seen before, this implies $|\nabla\nabla u| \leq |\nabla\nabla w| + 4c$, contradicting $\nabla\nabla u^* \notin L^2(\partial\Omega)$.

□

We now turn our attention to finding triples $(\rho,\, \theta,\, q)$ such that $\det(A(\rho,\, \theta,\, q)) = 0$, $-1 \leq \rho < 1,\ 0 < \theta < 2\pi$, and $1 < q \leq \frac{3}{2}$.

LEMMA 4.2.3.

1. $\det(A(\rho,\, \theta,\, 1)) =\ 16$

2. $\det(A(\rho,\, \frac{3\pi}{2},\, \frac{4}{3}\ )) =\ \frac{64}{27}(2+\rho)(1+2\rho)$

PROOF.

1. Setting $q = 1$ we have

$$A(\rho,\theta,1) = \begin{bmatrix} \sin(\theta) & -\sin(\theta) & \cos(\theta) & \cos(\theta) \\ \cos(\theta) & -\cos(\theta) & -\sin(\theta) & -\sin(\theta) \\ 0 & 0 & (1-\rho) & -(3+\rho) \\ 1-\rho & 3+\rho & 0 & 0 \end{bmatrix},$$

Expanding by minors along the fourth and then third rows gives

$$\det(A(\rho,\theta,1) = -(1-\rho) \begin{vmatrix} -\sin(\theta) & \cos(\theta) & \cos(\theta) \\ -\cos(\theta) & -\sin(\theta) & -\sin(\theta) \\ 0 & (1-\rho) & -(3+\rho) \end{vmatrix}$$

$$+ (3+\rho) \begin{vmatrix} \sin(\theta) & \cos(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) & -\sin(\theta) \\ 0 & (1-\rho) & -(3+\rho) \end{vmatrix}$$

$$= (1-\rho)^2 \begin{vmatrix} -\sin(\theta) & \cos(\theta) \\ -\cos(\theta) & -\sin(\theta) \end{vmatrix}$$

$$+ (1-\rho)(3+\rho) \begin{vmatrix} -\sin(\theta) & \cos(\theta) \\ -\cos(\theta) & -\sin(\theta) \end{vmatrix}$$

$$- (3+\rho)(1-\rho) \begin{vmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{vmatrix}$$

$$- (3+\rho)^2 \begin{vmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{vmatrix}.$$

Simplifying, we obtain

$$\begin{aligned} \det(A(\rho,\theta,1)) &= (1-\rho)^2 + 2(1-\rho)(3+\rho) + (3+\rho)^2 \\ &= [(1-\rho)+(3+\rho)]^2 \\ &= 16 \end{aligned}$$

2. With $\theta = \frac{3\pi}{2}$ and $q = \frac{4}{3}$, we have $q\theta = 2\pi$ and $(q-2)\theta = -\pi$. The determinant of $A(\rho, 2\pi, \frac{4}{3})$ therefore equals

$$\begin{bmatrix} 0 & 0 & 1 & -1 \\ \frac{4}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3}(1-\rho) & \frac{4}{3}(1-\rho) - 4 \\ \frac{4}{3}(1-\rho) & -\frac{2}{3}(1-\rho) + 4 & 0 & 0 \end{bmatrix}$$

Expanding by minors along the first row and then the third column, we compute

$$\det(A(\rho, 2\pi, \frac{4}{3})) = \begin{vmatrix} \frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3}(1-\rho) - 4 \\ \frac{4}{3}(1-\rho) & -\frac{2}{3}(1-\rho) + 4 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} \frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3}(1-\rho) \\ \frac{4}{3}(1-\rho) & -\frac{2}{3}(1-\rho) + 4 & 0 \end{vmatrix}$$

$$= -\frac{4}{3}(1-\rho) + 4 \begin{vmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{4}{3}(1-\rho) & -\frac{2}{3}(1-\rho) + 4 \end{vmatrix}$$

$$- \frac{4}{3}(1-\rho) \begin{vmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{4}{3}(1-\rho) & -\frac{2}{3}(1-\rho) + 4 \end{vmatrix}$$

$$= \frac{4}{3}(1+2\rho) \begin{vmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{4}{3}(1-\rho) & -\frac{2}{3}(1-\rho) + 4 \end{vmatrix}.$$

The determinant of the matrix on the last line equals $\frac{16}{9}(2+\rho)$. We therefore conclude $\det(A(\rho, \frac{3\pi}{2}, \frac{4}{3})) = \frac{64}{27}(2+\rho)(1+2\rho)$.

□

THEOREM 4.2.4. *For each* $-2 \leq \rho \leq -\frac{1}{2}$ *there exists a number* $q \in (1, \frac{4}{3}\,]$ *such that* $\det(A(\rho, \frac{3\pi}{2}, q)) = 0$.

PROOF. The determinant of A is continuous on the box

$$B = \Big\{(\rho, \theta, q) :\ -2 \le \rho \le 1,\ \ 0 \le \theta \le 2\pi,\ \ 1 \le q \le \tfrac{3}{2}\Big\}.$$

By the previous lemma, the determinant is positive on the set

$$P = \Big\{(\rho, \theta, 1) :\ -2 \le \rho \le 1,\ \ 0 \le \theta < 2\pi\Big\}.$$

On the other hand, the the lemma also states

$$\det\big(A(\ \rho,\ \tfrac{3\pi}{2},\ \tfrac{4}{3})\big) = \tfrac{64}{27}(2+\rho)(1+2\rho).$$

When $-2 < \rho < -\frac{1}{2}$, this determinant is negative.

Applying the intermediate value theorem we find that there is at least one zero for every $\rho \in [-2, -\frac{1}{2}\,]$. □

By continuity, the determinant of A must be negative in a neighborhood of $\{(\rho,\ \frac{3\pi}{2},\ \frac{4}{3}) : -2 < \rho < -\frac{1}{2}\} \subset \mathbb{R}^3$. Another application of the intermediate value theorem proves the following theorem.

THEOREM 4.2.5. *For each* $\delta > 0$ *and* $-2 < \rho < -\frac{1}{2} - \delta$ *there exist constants* $\epsilon > 0$ *and* $q \in (1, \frac{4}{3}]$ *such that the determinant of* $A(\rho,\ \theta,\ q)$ *equals zero on the set*

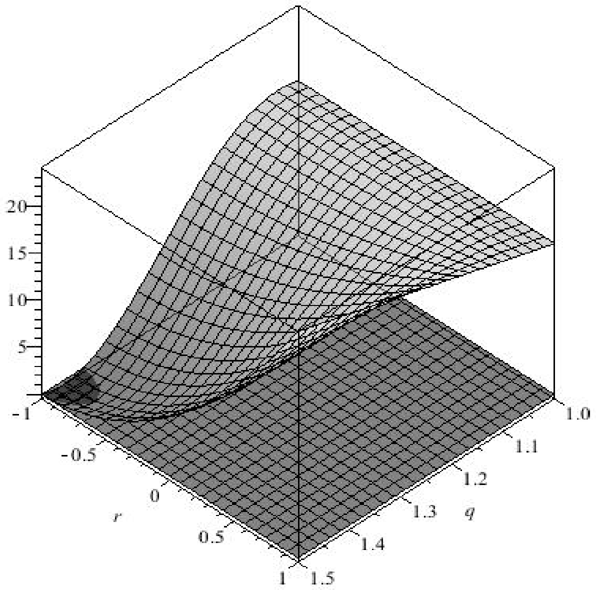
$$\{\,(\rho, \theta, q) : \tfrac{3\pi}{2} - \epsilon < \theta < \tfrac{3\pi}{2} + \epsilon\,\}.$$

When $\rho \geq -1$ each of these points corresponds to a counterexample for the Biharmonic Mixed Problem.

3-dimensional plots created using the computer program *Maple* suggest that the interval $\rho \in (-1, -\frac{1}{2}]$ is optimal. They also indicate that $\theta \geq \pi$ is required for all our counterexamples. We should keep in mind, however, that these counterexamples are all of the form $u = Im(Ez^q + F\overline{z}z^{q-1}) + Re(Gz^q + H\overline{z}z^{q-1})$. There may very well be counterexamples for other values of ρ and θ.

To illustrate the relationship between ρ and θ we present 3-D plots created using *Maple*. In all these examples, the curved surface shows the value of the determinant. Intersections with the plane represent determinants equal to zero. The axis labels indicate the Poisson ration r, the exponent q, and the angle t in radians.

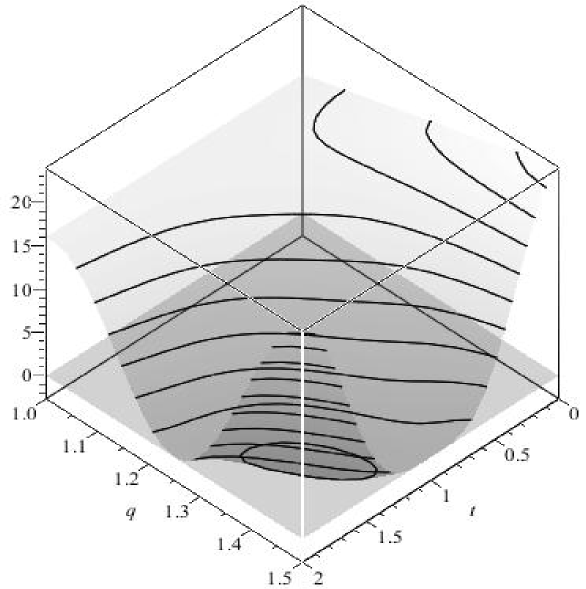
FIGURE 1. Biharmonic Counterexample with $\theta = \pi$



The angle $\theta = \pi$ is attained, as a straightforward computation shows $\det(A(-1, \pi, \frac{3}{2}) = 0$. The following 3-D plot indicates that this is the *only* such example. We have fixed $\theta = \pi$, and are letting ρ and q vary.

Notice that there is only a single intersection point, at $\rho = -1$, $\theta = \pi$, and $q = \frac{3}{2}$.

FIGURE 2. Biharmonic Counterexamples with $\rho = -1$

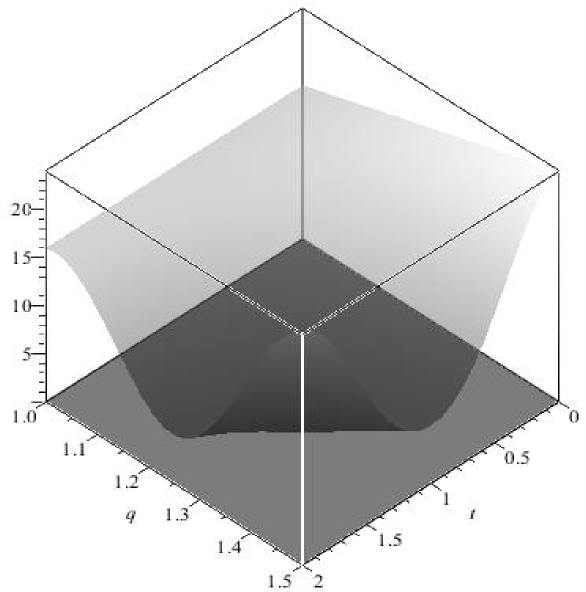


To better illustrate the relationship between ρ and θ we now set $\rho = -1$ and let q and θ vary:

The intersection of the plane and surface forms a curve, roughly in the shape of an ellipse. Notice that $\theta \geq \pi$ at all these points. Similar graphs can be obtained for $-1 < \rho < -\frac{1}{2}$.

On $-1 < \rho < -\frac{1}{2}$, the area enclosed by the intersection curve decreases as ρ increases. When $\rho = -\frac{1}{2}$ there appears to be a single point of intersection.

FIGURE 3. Lack of Biharmonic Counterexamples for $\rho = 0$



Plots show no intersections when $\rho > -\frac{1}{2}$. To illustrate this we also show the graph for $\rho = 0$:

Here we see that there are no intersection points to indicate counterexamples for the Biharmonic Mixed Problem.

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Index

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