Potential Theory on Compact Sets

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ABSTRACT

The primary goal of this work is to extend the notions of potential theory to compact sets. There are several equivalent ways to define continuous harmonic functions $H(K)$ on a compact set K in $\mathbb{R}^n$. One may let $H(K)$ be the uniform closure of all functions in $C(K)$ which are restrictions of harmonic functions on a neighborhood of K, or take $H(K)$ as the subspace of $C(K)$ consisting of functions which are finely harmonic on the fine interior of K. In [9] it was shown that these definitions are equivalent.

We study the Dirichlet problem on a compact set $K \subset \mathbb{R}^n$ in Chapter 4. As in the classical theory, our Theorem 4.1 shows $C(\partial_f K) \cong H(K)$ for compact sets with $\partial_f K$ closed, where $\partial_f K$ is the fine boundary of K. However, in general a continuous solution cannot be expected even for continuous data on $\partial_f K$ as illustrated by Theorem 4.1. Consequently, we show that the solution can be found in a class of finely harmonic functions. Moreover by Theorem 4.3, in complete analogy with the classical situation, this class is isometrically isomorphic to $C_b(\partial_f K)$ for all compact sets K.

To study these spaces, two notions of Green functions have previously been introduced. One by [22] as the limit of Green functions on domains D_j where the domains D_j are decreasing to K. Alternatively, following [12, 13] one has the fine Green function on the fine interior of K. Our Theorem 3.14 shows that these are equivalent notions.

Using a localization result of [3] one sees that a function $h \in H(K)$ if and only if it is continuous and finely harmonic on every fine connected component of the fine interior of K. Such collection of sets is usually called a *restoring covering*. Another equivalent definition of $H(K)$ was introduced in [22] using the notion of Jensen measures which leads to another restoring collection of sets.

In Section 5.1 a careful study of the set of Jensen measures on K, leads to an interesting extension result (Corollary 5.8) for subharmonic functions. This has a number of applications. In particular we show that the restoring coverings of [9] and [22] are the same. We are also able to extend some results of [18] and [22] to higher dimensions.

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DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

in the Graduate School of Syracuse University

May 2011

Contents

Acknowledgments

This thesis would not have been possible without the help of many people. First and foremost, my deepest gratitude goes to Eugene Poletsky for his excellent guidance, support and encouragement. He has taught me what it means to be a mathematician and professor of the highest caliber.

The Department of Mathematics at Syracuse University has generously supplied all the support and facilities I needed to complete my studies. In particular the wonderful faculty, staff and graduate students have provided me with invaluable knowledge, support and friendship for which I am sincerely thankful. Furthermore I am exceptionally grateful to all of the members of my defense committee, Mark Bowick, Dan Coman, Jani Onninen, Eugene Poletsky, Gregory Verchota and Andrew Vogel, for helping me through this final stage.

Last, but not the least, I wish to thank my family and friends for their patience and understanding.

Chapter 1

Introduction

There are several ways to define the spaces $(S(K))$-$H(K)$ of continuous (sub)-harmonic functions on a compact set K in $\mathbb{R}^n$. Let $C(K)$ denote the space of all continuous real functions on K. The natural definition is to let $H(K)$ or $S(K)$ be the uniform closure of all functions in $C(K)$ which are restrictions of harmonic (resp. subharmonic) functions on a neighborhood of K. More fashionably, we can define $H(K)$ and $S(K)$ as the subspaces of $C(K)$ consisting of functions which are finely harmonic (resp. finely subharmonic) on the fine interior of K. The equivalence of these definitions was shown in [2] and [3].

Another definition was introduced in [22] using the notion of Jensen measures. A measure μ supported by K is Jensen with barycenter $x \in K$ if for every open set V containing K and every subharmonic function u on V we have $u(x) \leq \mu(u)$. The set of such measures will be denoted by $\mathcal{J}_x(K)$. Then $H(K)$ is the subspace of $C(K)$

consisting of functions h such that $h(x) = \mu(h)$ for all $\mu \in \mathcal{J}_x(K)$ and $x \in K$. It was shown in [22] that this definition is equivalent to the definitions above.

The main goal of this work is to extend the classic potential theory to compact sets $K \subset \mathbb{R}^n$. We consider two main problems in this arena. The first is a Dirichlet problem on compact sets and the second is to prove a natural restoring property of harmonic functions on compact sets with respect to the fine topology.

1.1 A Dirichlet problem on compact sets

The Dirichlet problem for harmonic functions on domains in $\mathbb{R}^n$ is not only important by itself but also by its influence on potential theory. Many now standard notions, e.g. regular points, fine topology, etc., first appeared in the study of this problem.

One possible extension can be found in the abstract theory of balayage spaces, see [4, 19]. However we feel that the gain in transparency following from a direct geometric approach more than justifies the use of new techniques.

The Dirichlet problem can be thought of as having two components; the data set and the data itself. One uses an initial function defined on the data set to construct a solution (a harmonic function) on the rest of the domain which must have a prescribed regularity as it approaches the data set. Classically, the data set is taken to be the topological boundary of the domain. One of our main goals here is to establish that the natural choice for the data set on a compact set K is the fine boundary of K, $\partial_f K$, which is shown by Lemma 3.3 to be the Choquet boundary of K with respect to subharmonic functions on K. We limit ourselves to initial functions that are continuous and bounded on $\partial_f K$ as in the classical case.

In Section 3.1, we introduce Jensen measures as our main tool and begin extending potential theory to compact sets $K \subset \mathbb{R}^n$ by defining harmonic functions and subharmonic functions on K. We devote Section 3.3 to the construction and study of harmonic measure on compact sets. The harmonic measure on K is shown to be a maximal Jensen measure. This is used to see the important fact (Corollary 3.12) that harmonic measures are concentrated on the fine boundary. In Chapter 4 we study the Dirichlet problem for compact sets. As in the classical theory, our Theorem 4.1 shows $C(\partial_f K) \cong H(K)$ for a class of compact sets. However, in general a continuous solution cannot be expected even for continuous data on $\partial_f K$ as illustrated by Example 4.1. Consequently, we show that the solution can be found in the class of finely harmonic functions introduced in this section. Moreover by Theorem 4.3, in complete analogy with the classical situation, this class is isometrically isomorphic to $C_b(\partial_f K)$ for all compact sets K.

1.2 Restoring properties of harmonic functions

Despite the existence of so many equivalent definitions of harmonic functions on compact sets it is still difficult to verify whether a function on a compact set is harmonic or subharmonic. In [9] it was shown that $h \in H(K)$ if and only if h is continuous and finely harmonic on the fine interior of K. A localization result from

[3] implies that $h \in H(K)$ if and only if h is continuous and finely harmonic on every fine connected component of the fine interior of K. Such collection of sets is usually called a *restoring covering*.

In its turn another restoring collection of sets was introduced in [22]. For $x \in K$ let $I(x)$ be the set of all points $y \in K$ such that $\mu(V) > 0$ for every $\mu \in \mathcal{J}_x(K)$ and every open set V containing y. It was shown that the sets $I(x)$ form the restoring covering.

The main goal of Chapter 5 is to reconcile the results in [9] and [22]. It required the understanding of a connection between fine topology and Jensen measures. For this we use the fact from [22] that $I(x)$ is the closure of the set $Q(x)$ of all $y \in K$ such that $G_K(x, y) > 0$, where the Green function G_K on K is defined as the limit of Green functions on domains D_j decreasing to K.

Fuglede [12, 13, 14] defined a Green function on K as the fine Green function on the fine interior $int_f(K)$ of K. We denote the fine Green function on a finely open set U by $G^f_U(x, y)$ (see [13, 14, 15] for the definition, and Section 3.4 for some basic properties).

As the first step we show (Theorem 3.14) that these two notions of Green functions are constant multiples of each other. This leads to Proposition 3.15 which shows that the set $Q(x)$ is a fine connected component of $int_f(K)$.

To finish the reconciliation process in Section 5.1 we study closely the set $\mathcal{J}_x(K)$. The main result (Theorem 5.6) provides Corollary 5.7 showing that $\mu \in \mathcal{J}_x(K)$ if and only if $\mu \in \mathcal{J}_x(I(x))$. This corollary proves to be quite useful. From it we are able to derive a number of applications in Section 5.2. In particular Corollary 5.8 an extension result for subharmonic functions shows that for every $f \in S(I(x))$ there is a $\hat{f} \in S(K)$ such that $\hat{f}|_{I(x)} = f$. Also following from Corollary 5.7 is the desired reconciliation of the restoring theorem of Poletsky [22] and the [9] result, proved here as Theorem 5.9.

In 1983, Gamelin and Lyons have shown [18, Theorem 3.1] that for $K \subset \mathbb{R}^2$

$$H(K)^{\perp} = \bigoplus H(\overline{A}_j)^{\perp}$$

where A_j are the fine components (fine open, fine connected) of the fine interior of K. However their work follows from an estimate for harmonic measure of the radial projection of a set, proved by Beurling in his thesis, which has no analog in $\mathbb{R}^n$ for $n > 2$. By using Theorem 5.9 we are now able to extend this result to higher dimensions in Corollary 5.10. As an application of this we are able to show, Proposition 5.11, that every Jensen set is Wermer, which was first proved by Poletsky in [22] for $n = 2$.

Chapter 2

Fundamental Ideas

We begin by developing some standard concepts which are basic to the theory developed below.

2.1 Classical Potential Theory

Potential theory is generally defined as the study of harmonic and subharmonic functions. Subharmonic functions are a generalization of convex functions. Convex functions are characterized by a subaveraging property with respect to lines. Indeed consider a convex open set D in $\mathbb{R}^n$, $n \geq 2$. One says that a continuous function $f\colon D \to \mathbb{R}$ is *convex* on D if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in D$ and $0 \leq \lambda \leq 1$. In reality, it is an easy exercise [24, Chp 4, Ex 24] to see that it is sufficient to take $\lambda = 1/2$ above, that is, a continuous function f is convex if and only if the property

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

holds for all $x, y \in D$. In other words if f is continuous and subaveraging over all one-dimensional spheres, i.e. end-points of line segments. Actually continuity is somewhat stronger than is needed in this case. However one cannot drop the condition entirely for the second definition to remain equivalent to the first.

Subharmonic functions are those that satisfy the same subaveraging inequality with n-dimensional spheres replacing their one-dimensional counterparts. Specifically, an upper semicontinuous function $f\colon D \to [-\infty, \infty)$ is *subharmonic* if

$$f(x) \leq \frac{1}{\mathrm{SA}(\partial B(x,r))} \int\limits_{\partial B(x,r)} f(\zeta)\, d\sigma(\zeta)$$

for every $x \in D$ and every ball $B(x, r)$, centered at x of radius r, compactly contained in D, i.e. $\overline{B(x,r)} \subset D$, where $SA(\partial B(x, r))$ is the surface area of the n-sphere and σ is the standard surface measure. This is easily seen ([20, Sections 2.3-2.4]) to be equivalent to subaveraging over balls, that is, an upper semicontinuous function $f\colon D \to [-\infty, \infty)$ is *subharmonic* if

$$f(x) \leq \frac{1}{\mathrm{vol}(B(x,r))} \int\limits_{B(x,r)} f(\zeta)\, dm(\zeta)$$

for every $x \in D$ and every ball $B(x, r)$ compactly contained in D.

Perhaps the most remarkable characteristic of subharmonicity is that it is an entirely local property, see [20, Thm 2.3.8]. A function is subharmonic if either of the above properties holds for only arbitrarily small radii. This property is not at all obvious from the definitions given above.

To remove pathologies we do not allow the function f to be identically equal to $-\infty$ on any connected component of D. The set of subharmonic functions and harmonic functions on D are denoted $S(D)$ and $H(D)$, respectively. A function g is *superharmonic* if $-g$ is subharmonic. A function h is *harmonic* if it is both subharmonic and superharmonic.

The central question of study in potential theory is the Dirichlet problem. For any $f \in C(\partial D)$, the *Dirichlet problem* on D is to find a unique function h which is harmonic on D and continuous on $\overline{D}$ such that $h|_{\partial D} = f$. The function f is commonly referred to as the *boundary data*, and the corresponding h is said to be the *solution* of the Dirichlet problem on D with boundary data f. The punctured disk in $\mathbb{R}^2$ is a fundamental example which shows that the Dirichlet problem can not be solved for any continuous boundary data.

However for a bounded open set D the method of Perron allows one to assign a function which is harmonic on D to any continuous (or simply measurable) boundary data. Given $f \in C(\partial D)$ Perron considered the function

$$h(x) = \sup\{u(x) \colon u \in S(D) \text{ and } \limsup_{\zeta \to p} u(\zeta) \leq f(p) \text{ for all } p \in \partial D\}$$

called the *Perron solution* which he then showed to be harmonic in D.

Later the concept of a regular domain was developed to establish the continuity of the Perron solution to the boundary. A bounded open set $D \subset \mathbb{R}^n$ is a *regular* domain if the Dirichlet problem is solvable on D for any continuous boundary data. Therefore on a regular domain, the space of boundary data functions $C(\partial D)$ is isometrically isomorphic to $H(D)$, the space of continuous functions on $\overline{D}$ which are harmonic on D.

For any $f \in C(\partial D)$ let $h_f \in H(D)$ denote the solution of the Dirichlet problem on D with boundary data f. Let $z \in D$. The point evaluation $H_z \colon f \mapsto h_f(z)$ is a positive bounded linear functional on $C(\partial D)$. By the Riesz Representation Theorem, there is a Radon measure $\omega_D(z, \cdot)$ on ∂D which represents H_z, that is

$$h_f(z) = \int_{\partial D} f(\zeta)\, d\omega_D(z, \zeta),$$

for all $f \in C(\partial D)$. The measure $\omega_D(z, \cdot)$ is called the *harmonic measure* of D with barycenter at z. See [1, 20] for more details on potential theory.

2.2 The Fine Topology

In solving the Dirichlet problem people wanted to characterize regular boundary points. It turns out that this is a local problem and leads directly to the development of the fine topology.

The *fine topology* on $\mathbb{R}^n$ is the coarsest topology on $\mathbb{R}^n$ such that all subharmonic functions are continuous in the extended sense of functions taking values in $[-\infty, \infty]$.

One easily sees that the metric topology is coarser than the fine topology. Hence all usual open sets are finely open. Furthermore since there exist finite valued discontinuous subharmonic functions the fine topology is strictly finer than the metric topology. For example the function

$$u(z) = \sum_{n=1}^{\infty} 2^{-n} \log |z - 2^{-n}|$$

is subharmonic on the complex plane and discontinuous at the origin, see [23, pg. 41-42].

When referring to a topological concept in the fine topology we will follow the standard policy of either using the words "fine" or "finely" prior to the topological concept or attaching the letter f to the associated symbol. For example, the fine boundary of K, $\partial_f K$, is the boundary of K in the fine topology. The fine topology is strictly finer than the Euclidean topology.

A set E is said to be *thin* at a point x_0 if x_0 is not a fine limit point of E, i.e. if there is a fine neighborhood U of x_0 such that $E \setminus \{x_0\}$ does not intersect U. For an open set D a boundary point $p \in \partial D$ is regular for the Dirichlet problem if and only if the complement of D is not thin at p.

An example of a set which is thin at the origin is given by the Lebesgue spine in $\mathbb{R}^3$ defined by

$$L = \{(x, y, z) \colon x > 0 \text{ and } y^2 + z^2 < \exp(-c/x)\},$$

where $c > 0$.

Fuglede's [12, p. 147] observation that a fine open set U in $\mathbb{R}^n$ has at most countably many fine open connected components will be useful later.

Many of the key concepts of classical potential theory have analogous definitions in relation to the fine topology. Presently we will recall a few of them. Relative to a finely open set V in $\mathbb{R}^n$ the *harmonic measure* $\delta_x^{V^c}$ is defined as the swept-out of the Dirac measure δ_x on the complement of V. A function u is said to be *finely hyperharmonic* on a finely open set U if it is lower finite, finely lower semicontinuous, and

$$-\infty < \delta_x^{V^c}(u) \leq u(x),$$

for all $x \in V$ and all relatively compact finely open sets V with fine closure contained in U. We say that u is *finely superharmonic* if u is finely hyperharmonic and not identically equally to ∞ in any fine component of U. Then u is called *finely subharmonic* $-u$ is finely superharmonic. A function h is said to be *finely harmonic* if h and $-h$ are finely hyperharmonic, or equivalently finely superharmonic. Furthermore, the *fine Dirichlet problem on U* for a finely continuous function f defined on the fine boundary of a bounded finely open set U consists of finding a finely harmonic extension of f to U. The development of the fine Dirichlet problem is quite similar to that of the classical. In the seventies Fuglede [12] establishes a Perron solution for the fine Dirichlet problem. His [12, Theorem 14.6] shows that there exists a Perron solution H_f^U which is finely harmonic on U for any numerical function f on $\partial_f U$ which is $\delta_x^{\partial_f U}$ integrable for every $x \in U$. Furthermore [12, Theorem 14.6] provides us with the desired continuity at the boundary, i.e. that the fine limit of $H_f^U(x)$ tends to $f(y)$ as $x \in U$ goes to y for every finely "regular" boundary point $y \in \partial_f U$ at which f is finely continuous.

The two books [5, 12] are classical references on the fine topology and many books on potential theory contain chapters on the topic, e.g. [1, Chapter 7].

2.3 Functional Analysis

We will often use $\mu(f)$ to denote $\int f \, d\mu$ where the integral is taken over the entire support of μ.

We will be primarily concerned with continuous real functions defined on either a domain or a compact subset of $\mathbb{R}^n$. Therefore our prerequisites from this beautiful subject are rather limited. The aim of this section is to present a rather focused account of the theory.

Let $\mathcal{M}(\mathbb{R}^n)$ denote the space of finite signed Radon measures on $\mathbb{R}^n$ and let $C_0(\mathbb{R}^n)$ denote the space of continuous functions on $\mathbb{R}^n$ which vanish at infinity. Observe that $C_0(\mathbb{R}^n)$ is a separable Banach space with the supremum norm, that is $||f|| = \sup_{z \in \mathbb{R}^n} |f(z)|$. Furthermore by the Riesz Representation Theorem the space $C_0^*(\mathbb{R}^n)$ of bounded linear functionals on $C_0(\mathbb{R}^n)$ is isometrically isomorphic to $\mathcal{M}(\mathbb{R}^n)$.

A useful concept in analysis is the notion of weak* convergence . Let $\{\mu_j\}$ be a sequence in $\mathcal{M}(\mathbb{R}^n)$. We say that μ_j converges to μ in the weak* topology, if $\mu_j(f)$ converges to $\mu(f)$ for every $f \in C_0(\mathbb{R}^n)$. This topology is particularly useful because of the theorem of Alaoglu, which states that for any normed space X, the unit ball in X^* is compact in the weak* topology. To check the weak* convergence of a sequence $\{\mu_j\}$ whose supports lie in a closed ball B, it suffices to check the weak* convergence in $C^*(B)$.

These standard definitions and results from functional analysis may be found in most functional analysis books, for example Conway [8].

2.4 Jensen Measures

If D is an open set in $\mathbb{R}^n$, we say that μ is a *Jensen measure* on D with barycenter $z \in D$ if μ is a probability measure (a positive Radon measure of unit mass) whose support is compactly contained in D and for every subharmonic function f on D the *sub-averaging inequality* $f(z) \leq \mu(f)$ holds.

The set of Jensen measures on D with barycenter $z \in D$ will be denoted $\mathcal{J}_z(D)$. Examples of Jensen measures with barycenter at $z \in D$ include the Dirac measure at z, i.e. δ_z, the harmonic measure with barycenter at z for any regular domain which is compactly contained in D, and the average over any ball (or sphere) centered at z which is contained in D.

It is important to note that the Jensen measures and in particular the harmonic measures are in the unit ball of $\mathcal{M}(\mathbb{R}^n) \cong C_0^*(\mathbb{R}^n)$ which is a compact set in the weak* topology.

One could define the set of Jensen measures $\mathcal{J}_z^c(D)$ with respect to the continuous subharmonic functions on D. However the following theorem shows that the set of Jensen measures would not be changed.

Theorem 2.1. *Let D be a bounded open subset of $\mathbb{R}^n$. For every $z \in D$, the sets $\mathcal{J}_z(D)$ and $\mathcal{J}_z^c(D)$ are equal.*

Proof. Since it is clear that $\mathcal{J}_z(D) \subseteq \mathcal{J}_z^c(D)$ for all $z \in D$, we will now show the reverse inclusion.

Pick some $z_0 \in D$ and let $\mu \in \mathcal{J}_{z_0}^c(D)$. Then we must show $f(z_0) \leq \mu(f)$ for every function f which is subharmonic on D. The support of μ is compactly contained in D.

Since f is subharmonic on D we can find ([20, Lemma 2.5.1]) a decreasing sequence $\{f_n\}$ of continuous subharmonic functions which converge to f. As $\mu \in \mathcal{J}_{z_0}^c(D)$ we have $f_n(z_0) \leq \mu(f_n)$ for every f_n. By the Lebesgue Monotone Convergence Theorem it follows that $f(z_0) \leq \mu(f)$. Thus $\mu \in \mathcal{J}_{z_0}(D)$. □

Since $\mathcal{J}_z(D) = \mathcal{J}_z^c(D)$ for all $z \in D$, to check that $\mu \in \mathcal{J}_z(D)$, it suffices to check that μ has the sub-averaging property for every continuous subharmonic function.

The following proposition of Cole and Ransford [7, Proposition 2.1] will demonstrate some basic properties of sets of Jensen measures.

Proposition 2.2. *Let D_1 and D_2 be open subsets of $\mathbb{R}^n$ with $D_1 \subset D_2$. Let $z \in D_1$.*

i. *If $\mu \in \mathcal{J}_z(D_1)$ then also $\mu \in \mathcal{J}_z(D_2)$.*

ii. *If* $\mu \in \mathcal{J}_z(D_2)$ *and supp*$(\mu) \subset D_1$, *and if each bounded component of* $\mathbb{R}^n \setminus D_1$ *meets* $\mathbb{R}^n \setminus D_2$, *then* $\mu \in \mathcal{J}_z(D_2)$.

Jensen measures and subharmonic functions are, in a sense, dual to each other. This duality is illustrated by the following theorem of Cole and Ransford [6, Corollary 1.7].

Theorem 2.3. *Let* D *be an open subset of* $\mathbb{R}^n$ *which possesses a Green's function. Let* $\phi\colon D \to [-\infty, \infty)$ *be a Borel measurable function which is locally bounded above. Then, for each* $z \in D$,

$$\sup \{v(z)\colon v \in S(D), v \leq \phi\} = \inf \{\mu(\phi)\colon \mu \in \mathcal{J}_z(D)\},$$

where $S(D)$ *denotes the set of subharmonic functions on* D.

Chapter 3

Potential theory on compact sets

We now begin our study of potential theory on compact sets. For compact sets which are not connected, the Hausdorff property will allow us to reduce Dirichlet type problems on the compact set to solving separate problems on each connected component. Therefore in what follows we will work on compact sets K in $\mathbb{R}^n$ which need not be connected, with the understanding that we can always separate the problem by working on the connected components of K individually.

3.1 Harmonic and Subharmonic Functions on Compact Sets

There are currently three equivalent ways to define harmonic and subharmonic functions on compact sets.

Definition 3.1 (Exterior)**.** Let $H(K)$ (or $S(K)$) be the uniform closure of all functions in $C(K)$ which are restrictions of harmonic (resp. subharmonic) functions on a neighborhood of K.

Definition 3.2 (Interior)**.** One can define $H(K)$ (or $S(K)$) as the subspaces of $C(K)$ consisting of functions which are finely harmonic (resp. finely subharmonic) on the fine interior of K.

The equivalence of these definitions of $H(K)$ was shown in [9] and of $S(K)$ in [2, 3].

For the third definition of $H(K)$ we must to extend the notion of Jensen measures to compact sets.

Definition 3.3. We define the set of Jensen measures on K with barycenter at $z \in K$ as the intersection of all the sets $\mathcal{J}_z(U)$, that is

$$\mathcal{J}_z(K) = \bigcap_{K \subset U} \mathcal{J}_z(U),$$

where U is any open set containing K.

Another definition of $H(K)$ was introduced in [22] using the notion of Jensen measures.

Definition 3.4 (Via Jensen measures)**.** The set $H(K)$ is the subspace of $C(K)$ consisting of functions h such that $h(x) = \mu(h)$ for all $\mu \in \mathcal{J}_x(K)$ and $x \in K$.

It was shown in [22] that this definition is equivalent to the exterior definition above.

Our first lemma shows that this last construction of Poletsky extends to subharmonic functions in the ideal way.

Lemma 3.1. *A function is in $S(K)$ if and only if it is continuous and satisfies the subaveraging property with respect to every Jensen measure on K, that is*

$$S(K) = \{f \in C(K)\colon f(z) \leq \mu(f),\ \textit{for all}\ \mu \in \mathcal{J}_z(K)\ \textit{and every}\ z \in K\}\,.$$

Proof. We use the exterior definition of $S(K)$ to show "$\subseteq$". Take $f \in C(K)$ and let $\{f_j\}$ be a sequence of subharmonic functions defined in a neighborhood of K such that $\{f_j\}$ is converging uniformly to f. Then $f_j(z) \leq \mu(f_j)$ for any $\mu \in \mathcal{J}_z(K)$. Since the convergence is uniform we have $f(z) \leq \mu(f)$.

Now suppose that f is in the set on the right. The subaveraging condition implies that f is finely subharmonic on the fine interior of K, and by assumption f is continuous. Therefore f satisfies the interior definition of $S(K)$. □

Recall the (exterior) definition of $S(K)$ as the uniform limits of continuous functions subharmonic in neighborhoods of K. The following proposition shows that the defining sequence for any function in $S(K)$ may be taken to be increasing. This result is a simple consequence of a duality theorem of Edwards.

Proposition 3.2. *Every function in $S(K)$ is the limit of an increasing sequence of continuous subharmonic functions defined on neighborhoods of K.*

Proof. Recall (see [16, Theorem 1.2]and [6]) Edwards Theorem states: If p is a continuous function on K, then for all $z \in K$ we have

$$Ep(z) := \sup\{f(z) \colon f \in S(K),\ f \leq p\} = \inf\{\mu(p) \colon \mu \in \mathcal{J}_z(K)\}.$$

From the proof of this theorem it follows that Ep is lower semicontinuous and is the limit of an increasing sequence of continuous subharmonic functions on neighborhoods of K. The result follows by observing that $p = Ep$ whenever $p \in S(K)$. □

3.2 The Choquet Boundary

In the book [16], Gamelin introduces a version of Choquet theory for cones of functions on compact sets. (Actually it applies to sets of functions which are slightly weaker than the cones we define.)

Following his guidance we consider a set $\mathcal{R}$ of functions mapping a compact set $K \subset \mathbb{R}^n$ to $[-\infty, \infty)$ with the following properties:

i. $\mathcal{R}$ includes the constant functions,

ii. if $c \in \mathbb{R}^+$ and $f \in \mathcal{R}$ then $cf \in \mathcal{R}$,

iii. if $f, g \in \mathcal{R}$ then $f + g \in \mathcal{R}$, and

iv. $\mathcal{R}$ separates the points of K.

One then considers a set of $\mathcal{R}$-measures for $z \in K$ defined as the set of probability measures μ on K such that

$$f(z) \leq \mu(f)$$

for all $f \in \mathcal{R}$.

Naturally our model for $\mathcal{R}$ will be $S(K)$. It then follows that when $\mathcal{R} = S(K)$ the $\mathcal{R}$-measures for $z \in K$ are precisely $\mathcal{J}_z(K)$. We now state some classic results from [16] which we will need in the following sections.

One can define the Choquet boundary of K with respect to $S(K)$ as

$$Ch_{S(K)}K = \{z \in K \colon \mathcal{J}_z(K) = \{\delta_z\}\}.$$

Many nice properties of the Choquet boundary are known. In particular, we will need the following characterization, see also, for example, [4, VI.4.1] and [19].

Lemma 3.3. *The Choquet boundary of K with respect to $S(K)$ is the fine boundary of K, i.e.*

$$Ch_{S(K)}K = \partial_f K.$$

Proof. Since the fine topology is strictly finer than the Euclidean topology, any point in the interior of K will also be in the fine interior of K, and any point of $\mathbb{R}^n \setminus K$ can be separated from K by an Euclidean (therefore fine) open set. Therefore the fine boundary of K is contained in ∂K. The result follows immediately from [22, Theorem 3.3] or [4, Proposition 3.1] which states that $\mathcal{J}_z(K) = \{\delta_z\}$ if and only if the complement of K is non-thin at z, that is z is a fine boundary point of K. $\square$

In particular,

Corollary 3.4. *If $\mathcal{J}_x(K) \neq \{\delta_x\}$, then $x \in int_f K$.*

The set $\partial_f K$ is also called the stable boundary of K. In fact the lemma shows that $Ch_{S(K)}K$ is the finely regular boundary of the fine interior of K. For more details on finely regular boundary points and other related concepts, see [4, VII.5-7] and [19].

With this association, the result in [5, p. 89] of Brelot about the stable boundary points of K shows that $Ch_{S(K)}K$ is dense in ∂K. We present a more geometric proof here.

Theorem 3.5. *The fine boundary of K (and therefore the Choquet boundary of K with respect to $S(K)$) is dense in the topological boundary of K.*

For the proof we will need the following notation. Recall that $B(x, r)$ is the open ball of radius r centered at x in $\mathbb{R}^n$. The sphere of radius r centered at x in $\mathbb{R}^n$ is then denoted $S(x, r) = \partial B(x, r)$. The surface measure on $S(x, r)$ will be denoted σ, and take s_{n-1} to be the surface area of the unit $(n-1)$-sphere.

Proof. Consider $x_0 \in \partial K$. Suppose we have an arbitrary ball centered at x_0. Then it contains a point y_0 which does not belong to K. Take $r_0 = ||y_0 - x_0||$.

From now on we will call $B = B(x_0, r_0)$ and let $\overline{B}$ denote the closure of B. Let H be the hyperplane tangent to B at y_0. It is given by the equation $H = \{x \in \mathbb{R}^n : \langle x, y_0 - x_0 \rangle = r_0^2\}$, where $\langle , \rangle$ is the standard inner product on Euclidean space. Let us find the maximal $t < r_0$ such that the hyperplane $H_t = \{x \in \mathbb{R}^n : \langle x, y_0 - x_0 \rangle = tr_0\}$ contains some point $x_1 \in K \cap \overline{B}$. Since $x_0 \in K$ the number $t \geq 0$.

There are two possibilities: firstly, $x_1 \in B$ or, secondly, $x_1 \in \partial B$. In the first case for every sufficiently small $r > 0$ all points y of the sphere $S(x_1, r)$ for which $\langle y, y_0 -$

$x_0\rangle > tr_0$ lie in the complement K^c of K. Hence $\sigma(S(x_1,r)\cap K^c) > \sigma(S(x_1,r))/2$ and

$$\liminf_{r\to 0} \frac{\sigma(S(x_1,r)\cap K^c)}{\sigma(S(x_1,r))} \geq \frac{1}{2}.$$

In the second case, we take a small neighborhood V of y_1 in ∂B, lying in the set $\{y \in \partial B : \langle y, y_0 - x_0\rangle > tr_0\}$ and note that due to convexity all points of the intervals connecting x_1 with $y \in V$, except x_1, lie in the set $\{y \in \overline{B} : \langle y, y_0 - x_0\rangle > tr_0\}$ and, consequently, in K^c. Since the rays $x_1 + sy$, $s > 0$, $y \in V$, form a cone of positive aperture with vertex at x_1 we see that there is a constant $c > 0$ such that $\sigma(S(x_1,r)\cap K^c) > cs_{n-1}r^n$ when $r > 0$ is sufficiently small. Hence

$$\liminf_{r\to 0} \frac{\sigma(S(x_1,r)\cap K^c)}{\sigma(S(x_1,r))} \geq c > 0.$$

There is a standard criteria for thinness [20, Corollary 5.6.5, p. 227] which states that if E is thin at a point x then

$$\liminf_{r\to 0} \frac{\sigma(S(x,r)\cap E)}{\sigma(S(x,r))} = 0.$$

Thus K^c is non-thin at x_1, which means that x_1 is in the fine boundary of K. $\square$

3.3 Harmonic Measure on a Compact Set

To use the exterior definition of $H(K)$ we will commonly want to approximate K by a decreasing sequence of regular domains. A decreasing sequence of regular domains $\{U_j\}$ is said to be *converging* to K if for every $\epsilon > 0$ there is a j_0 such that U_j lies in the ϵ-neighborhood K_ϵ of K when $j \geq j_0$ and contains K. Furthermore, we require that U_{j+1} is compactly contained in U_j, i.e. $\overline{U}_{j+1} \subset U_j$, for all j. The existence of such a sequence is provided by [21, Prop 7.1].

The next theorem will allow us to define a harmonic measure on K. For a decreasing sequence of regular domains $\{U_j\}$, we will let $\omega_{U_j}(z, \cdot)$ denote the harmonic measure on U_j with barycenter at $z \in U_j$.

Theorem 3.6. *If $\{U_j\}$ is a sequence of regular domains converging to a compact set $K \subset \mathbb{R}^n$, then for every $z \in K$ the harmonic measures $\omega_{U_j}(z, \cdot)$ converge weak*. Furthermore, this limit does not depend on the choice of the sequence of domains $\{U_j\}$.*

Proof. Since ω_{U_j} are measures of unit mass supported on a compact set in $\mathbb{R}^n$, by Alaoglu's Theorem they must have a limit point. To show that this point is unique it suffices to show that for every $z \in K$ the limit

$$\lim_{j\to\infty} \int\limits_{\partial U_j} u(\zeta)\, d\omega_{U_j}(z, \zeta) \tag{3.1}$$

exists for every $u \in C(\overline{U}_1)$.

First, we show the limit in (3.1) exists when u is continuous and subharmonic in a neighborhood of K. The solution u_j of the Dirichlet problem on U_j with boundary value u is equal to

$$u_j(z) = \int\limits_{\partial U_j} u(\zeta)\, d\omega_{U_j}(z, \zeta).$$

Since u is subharmonic, we have $u_j \geq u$ on U_j. Then as $u_{j+1} = u$ on ∂U_{j+1} and $u_j \geq u = u_{j+1}$ on ∂U_{j+1}, the maximum principle for harmonic functions implies that $u_j \geq u_{j+1}$ on U_{j+1}. Thus $\{u_j\}$ is a decreasing sequence on K and we see that for every $z \in K$ the limit in (3.1) exists.

If $u \in C^2(\overline{U}_1)$, then we may represent u as a difference of two $C^2(\overline{U}_1)$ functions which are subharmonic on U_1. By the argument above the limit in (3.1) exists.

Since $C^2(\overline{U}_1)$ is dense in $C(\overline{U}_1)$ we see that the limit in (3.1) always exists. □

Definition 3.5. We define the harmonic measure $\omega_K(z, \cdot)$ on a compact set K with $z \in K$ as the weak* limit of $\omega_{U_j}(z, \cdot)$ chosen as above.

To use this definition for the Dirichlet problem we must check that the support of $\omega_K(z, \cdot)$ lies on the boundary of K. Actually in Section 3.2 we will be able to give more specific information about $\omega_K(z, \cdot)$, see Corollary 3.12.

Lemma 3.7. *The support of $\omega_K(z, \cdot)$ is contained in ∂K.*

Proof. Let W be a neighborhood of ∂K. Let $\{U_j\}$ be a sequence of domains converging to K and take a sequence $z_j \in \partial U_j$. Then there exists a subsequence $\{z_{j_k}\}$ which must be converging to some $z_0 \in K$. As $z_j \in \partial U_j$, then z_j is not in K. Therefore the limit of z_{j_k} cannot be in the interior of K. Thus z_0 is in $\partial K \subset W$. Consequently, there is j_0 such that $\partial U_j \subset W$ for each $j \geq j_0$,

Let $x \in \mathbb{R}^n \setminus \partial K$ and take W to be a neighborhood of ∂K so that $x \notin \overline{W}$. There is an $r > 0$ so that $\overline{B(x,r)} \cap \overline{W} = \emptyset$. Since $\omega_{U_j}(z, \cdot)$ has support on ∂U_j, which is contained is $\overline{W}$ for large j, we have $\omega_{U_j}(z, B(x,r)) = 0$. Since $B(x,r)$ is open, the Portmanteau Theorem shows

$$\liminf_{j\to\infty} \omega_{U_j}(z, B(x,r)) \geq \omega_K(z, B(x,r)).$$

Hence $\omega_K(z, B(x,r)) = 0$ and x is not in the support of $\omega_K(z,\cdot)$. $\square$

The following theorem brings our study back to the topic of Jensen measures.

Theorem 3.8. *The harmonic measure on K is a Jensen measure on K.*

Proof. Since $\omega_K(z,\cdot)$ is defined as the weak* limit of probability measures, $\omega_K(z,\cdot)$ is a probability measure.

Recall that for $z \in K$ we have defined $\mathcal{J}_z(K) = \cap \mathcal{J}_z(U)$, where $K \subset U$. However it is sufficient to see that $\mathcal{J}_z(K) = \cap \mathcal{J}_z(U_j)$ where $\{U_j\}$ is any sequence of domains converging to K. We will show $\omega_K(z,\cdot) \in \mathcal{J}_z(U_j)$ for all j.

Pick some j. Then let f be a continuous subharmonic function on U_j. Then

$$f(z) \leq \int_{\partial U_l} f(\zeta)\, d\omega_{U_l}(z,\zeta),$$

for all $l > j$. Then by taking the weak* limit, we have that

$$f(z) \leq \int_{\partial K} f(\zeta)\, d\omega_K(z,\zeta).$$

Then $\omega_K(z,\cdot)$ satisfies the sub-averaging inequality for every continuous subharmonic function on U_j and $\omega_K(z,\cdot)$ is a probability measure with support contained in U_j. Thus $\omega_K(z,\cdot)$ must be in $\mathcal{J}^c_z(U_j)$, which is equal to $\mathcal{J}_z(U_j)$ by Theorem 2.1. Therefore $\omega_K(z,\cdot) \in \mathcal{J}_z(K)$. $\square$

Following [16, p. 16] a partial ordering on the set of Jensen measures is defined below. The notation $\mathcal{J}(K)$ is used to stand for the union of all Jensen measures on K, that is

$$\mathcal{J}(K) = \bigcup_{z \in K} \mathcal{J}_z(K).$$

Definition 3.6. For $\mu, \nu \in \mathcal{J}(K)$ we say that $\mu \succeq \nu$ if for every $\phi \in S(K)$ we have $\mu(\phi) \geq \nu(\phi)$. Furthermore, a Jensen measure μ is maximal if there is no $\nu \succeq \mu$ with $\nu \neq \mu$ where $\nu \in \mathcal{J}(K)$.

We start with a simple observation.

Lemma 3.9. *If $\mu \in \mathcal{J}_{z_1}(K)$ and $\nu \in \mathcal{J}_{z_2}(K)$ with $z_1 \neq z_2$ then μ and ν are not comparable.*

Proof. To see this simply recall that the coordinate functions π_i are harmonic. As $z_1 \neq z_2$ they must differ in at least one coordinate, say the i^{th}. Assume with out loss of generality that $\pi_i(z_1) > \pi_i(z_2)$. Then $\mu(\pi_i) > \nu(\pi_i)$. However $-\pi_i$ is also harmonic and so $\nu(-\pi_i) > \mu(-\pi_i)$. Therefore μ and ν are not comparable and if $\mu \succeq \nu$ then they have the common barycenter. □

We will now show that the harmonic measure is maximal with respect to this ordering. The maximality of harmonic measure proved below is the Littlewood Subordination Principle (see [11, Theorem 1.7]) when K is the closed unit ball in the plane.

Theorem 3.10. *For all $z \in K$, the measure $\omega_K(z, \cdot)$ is maximal in $\mathcal{J}(K)$.*

Proof. By Lemma 3.9 it suffices to show that for any $z \in K$, $\omega_K(z,\cdot)$ is maximal in $\mathcal{J}_z(K)$.

Pick any $z_0 \in K$. Now we will show that $\omega_K(z_0,\cdot)$ majorizes every measure $\mu \in \mathcal{J}_{z_0}(K)$. Consider a decreasing sequence of regular domains $\{U_j\}$ converging to K. Take any $\phi \in S(K)$. By Proposition 3.2 we may find a sequence $\phi_j \in S(U_j) \cap C(\overline{U_j})$ increasing to ϕ. Furthermore we extend ϕ as $\tilde{\phi} \in C_0(\mathbb{R}^n)$ while keeping $\tilde{\phi} \geq \phi_j$ for all j. Define harmonic functions Φ_j on U_j by

$$\Phi_j(x) = \int\limits_{\partial U_{j+1}} \phi_j(\zeta)\, d\omega_{U_{j+1}}(x,\zeta).$$

Therefore as ϕ_j is subharmonic, $\Phi_j \geq \phi_j$ on U_{j+1}, so

$$\int\limits_{\partial U_{j+1}} \phi_j(\zeta)\, d\omega_{U_{j+1}}(z_0,\zeta) = \Phi_j(z_0) = \mu(\Phi_j) \geq \mu(\phi_j).$$

As $\tilde{\phi} \geq \phi_j$, we have

$$\int\limits_{\partial U_{j+1}} \tilde{\phi}(\zeta)\, d\omega_{U_{j+1}}(z_0,\zeta) \geq \mu(\phi_j), \tag{3.2}$$

for all j. By taking weak* limits, we have that

$$\lim_{j\to\infty} \int\limits_{\partial U_{j+1}} \tilde{\phi}(\zeta)\, d\omega_{U_{j+1}}(z_0,\zeta) = \int\limits_{\partial K} \phi(\zeta)\, d\omega_K(z_0,\zeta).$$

The Lebesgue Monotone Convergence Theorem provides

$$\lim_{j\to\infty} \mu(\phi_j) = \mu(\phi).$$

Therefore by taking the limit by j of 3.2 we see

$$\int\limits_{\partial K} \phi(\zeta)\, d\omega_K(z_0,\zeta) \geq \mu(\phi).$$

We now have $\omega_K(z_0, \cdot) \succeq \mu$. If any $\nu \in \mathcal{J}_{z_0}(K)$ has the property $\nu \succeq \omega_K(z_0, \cdot)$, by the antisymmetry property of partial orderings $\nu = \omega_K(z_0, \cdot)$. Thus the measure $\omega_K(z_0, \cdot)$ is maximal in $\mathcal{J}_{z_0}(K)$. □

The maximality of harmonic measures implies that they are trivial at the points $z \in K$ such that $\mathcal{J}_z(K) = \{\delta_z\}$, which by Lemma 3.3 are precisely the fine boundary points.

Corollary 3.11. *The harmonic measure* $\omega_K(z_0, \cdot) = \delta_{z_0}$ *if and only if* $\mathcal{J}_{z_0}(K) = \{\delta_{z_0}\}$.

Proof. Suppose $\omega_K(z_0, \cdot) = \delta_{z_0}$. Consider the function $\rho(z) = ||z - z_0||^2 \in S^c(K)$. Then for any $\mu \in \mathcal{J}_{z_0}$, by the maximality of $\omega_K(z_0, \cdot)$ we have

$$0 = \rho(z_0) \leq \mu(\rho) \leq \int_{\partial K} \rho(\zeta)\, d\omega_K(z_0, \zeta) = \rho(z_0) = 0.$$

As $\rho(z) > 0$ for all $z \neq 0$ and as μ is a probability measure, we see that $\mu = \delta_{z_0}$. Thus $\mathcal{J}_{z_0}(K) = \{\delta_{z_0}\}$.

For the reverse implication we have already proved Theorem 3.8 that $\omega_K(z_0, \cdot) \in \mathcal{J}_{z_0}(K)$. □

In general the fine boundary is not closed, as Example 4.1 will show. So we cannot claim that it is the support of measures. Moreover, as Theorem 3.5 just showed the closure of $\partial_f K$ is the boundary of K. In particular, it may coincide with K for porous Swiss cheeses, see [17, pg. 25-26].

Recall that a measure $\mu \in \mathcal{M}(K)$ is concentrated on a set E, if for every set $F \subset K \setminus E$, $\mu(F) = 0$. A probability measure μ is concentrated on a set E if and only if $\mu(E) = 1$. From [16, p. 19] we know that all maximal measures are concentrated on $Ch_{S(K)}K = \partial_f K$. With this observation, the next corollary immediately follows from Theorem 3.10 which stated that the harmonic measure is maximal.

Corollary 3.12. *For every z in K, the harmonic measure with barycenter at z is concentrated on $\partial_f K$.*

3.4 On the Green function associated to a compact set

We now proceed to study the Green function on K. Recall [10, Theorem (b) 1.VII.6, p. 94] that if D is an open Greenian set in $\mathbb{R}^n$ so that $\{D_j\}$ is a decreasing sequence of open sets converging to D, then the sequence $\{G_{D_j}(\cdot, y)\}$ of Green functions associated to $\{D_j\}$ is decreasing to $G_D(\cdot, y)$ for every $y \in D$. By analogy one can define a Green function on a compact set K as the limit of the sequence $\{G_{D_j}(\cdot, y)\}$ where $y \in K$ and $\{D_j\}$ is any decreasing sequence of open sets converging to K. In the article [22] Poletsky defines a Green function on a compact set in this way.

Recall, [10, p. 90], that for a regular open set D the associated Green function $G_D(\cdot, y)$ extends continuously as $\hat{G}_D(\cdot, y)$ to $\mathbb{R}^n$ for any $y \in D$ where $\hat{G}_D(\cdot, y) = 0$ on $\complement D$, the complement of D, and this extension $\hat{G}_D(\cdot, y)$ is subharmonic on $\mathbb{R}^n \setminus \{y\}$.

In the following proposition we outline some of the basic properties of $\hat{G}_K$.

Proposition 3.13. *For all $y \in K$, the function $\hat{G}_K(\cdot, y) \colon \mathbb{R}^n \to [0, \infty]$ defined as $\hat{G}_K(\cdot, y) = \liminf_j \hat{G}_{D_j}(\cdot, y)$ has the following properties:*

i. *$\hat{G}_K(x, y) = 0$ when $x \in \complement K := \mathbb{R}^n \setminus K$ and $y \in K$,*

ii. *$\hat{G}_K$ does not depend on the sequence $\{D_j\}$ chosen,*

iii. *$\hat{G}_K \geq 0$ and $\hat{G}_K(y, y) = +\infty$ for all $y \in K$,*

iv. *$\hat{G}_K$ is symmetric, i.e. $\hat{G}_K(x, y) = \hat{G}_K(y, x)$, for all $x, y \in K$,*

v. *$\hat{G}_K(\cdot, y)$ is super-averaging on K, i.e. $\hat{G}_K(x, y) \geq \int \hat{G}_K(\zeta, y)\, d\mu(\zeta)$ for all $\mu \in \mathcal{J}_x(K)$ with $x \in K$, and*

vi. *$\hat{G}_K(\cdot, y)$ is subharmonic on $\mathbb{R}^n \setminus \{y\}$.*

proof of i. This follows from the fact that $\hat{G}_{D_j}(x, y) = 0$ whenever $x \notin \overline{D}_j$. □

proof of ii. If $D_1 \supset D_2$ then $\hat{G}_{D_1}(\cdot, y) \geq \hat{G}_{D_2}(\cdot, y)$ for any Greenian sets D_1 and D_2. Alternatively we could have defined $\hat{G}_K$ by

$$\hat{G}_K(\cdot, y) = \inf\{\hat{G}_D(\cdot, y) \colon D \supset K, D \text{ Greenian}\}, \qquad y \in K.$$ □

proof of iii. As $\hat{G}_D \geq 0$ and $\hat{G}_D(y, y) = +\infty$ for all $x, y \in D$ for any Greenian D. □

proof of iv. Since $\hat{G}_D(x, y) = \hat{G}_D(y, x)$ for all $x, y \in D$ for any Greenian D. □

proof of v. For any Greenian set D the function $\hat{G}_D(\cdot, y)$ is superharmonic on D. Then $\hat{G}_D(x, y) \geq \int \hat{G}_D(\zeta, y)\, d\mu(\zeta)$ for all $\mu \in \mathcal{J}_x(D)$ with $\zeta \in D$. If D_j is a decreasing sequence of domains converging to K, then $\hat{G}_{D_j}(\cdot, y)$ is decreasing to $\hat{G}_K(\cdot, y)$. Therefore by the Lebsegue Monotone Convergence Theorem $\hat{G}_K(x, y) \geq \int \hat{G}_K(\zeta, y)\, d\mu(\zeta)$ for all $\mu \in \cap_j \mathcal{J}_x(D_j) := \mathcal{J}_x(K)$ with $x \in K$. $\square$

proof of vi. Let $\{D_j\}$ be a decreasing sequence of regular domains converging to K. Then $\hat{G}_{D_j}(\cdot, y)$ is continuous, and so $\hat{G}_K(\cdot, y)$ must be upper semicontinuous. For any j and any $y \in D_j$, by [10, p. 90] the extension $\hat{G}_{D_j}(\cdot, y)$ of Green function $\hat{G}_{D_j}(\cdot, y)$ by 0 is subharmonic on $\mathbb{R}^n \setminus \{y\}$. Therefore by the Lebesgue Monotone Convergence Theorem $\hat{G}_K(\cdot, y)$ is subaveraging on $\mathbb{R}^n \setminus \{y\}$ as it is the decreasing limit of a sequence of subharmonic functions. Since $\hat{G}_K(\cdot, y)$ is upper semicontinuous and subaveraging, $\hat{G}_K(\cdot, y)$ is subharmonic on $\mathbb{R}^n \setminus \{y\}$. $\square$

It was shown in [13] that every bounded fine open set U admits a fine Green function which we shall denote by $G^f_U(x, y)$. The following result shows that for a compact set K the functions $\hat{G}_K(x, y)$ and $G^f_{int_f K}(x, y)$ are scalar multiples of each other.

Theorem 3.14. *For any compact set $K \subset \mathbb{R}^n$ there is $c > 0$ such that $\hat{G}_K(x, y) = cG^f_{int_f K}(x, y)$ for any $y \in int_f K$.*

Proof. Fuglede has given a simple characterization of the fine Green function up to multiplication by a positive constant. Indeed, if a function $g\colon U \times U \to \mathbb{R}$ has the following properties

1. $g(\cdot, y)$ is a nonnegative finely superharmonic function on U,

2. if v is finely subharmonic on U and $v \leq g(\cdot, y)$, then $v \leq 0$,

3. $g(\cdot, y)$ is finely harmonic on $U \setminus \{y\}$ for any $y \in U$, and

4. $g(y, y) = +\infty$

then $g(x, y) = cG_U^f(x, y)$ for some $c > 0$ for all $x, y \in U$.

Hence to prove the theorem we need only to check these properties. Firstly, we note that by Lemma 3.13 $\hat{G}_K(\cdot, y)$ is subharmonic (and thereby finely subharmonic) on $\mathbb{R}^n \setminus \{y\}$, which implies ([12, Theorem 9.10]) fine continuity on $\mathbb{R}^n \setminus \{y\}$.

In fact, we will shall now see that $\hat{G}_K(\cdot, y)$ is finely continuous at y when $y \in int_f K$. Every bounded fine open set admits a fine Green function, cf. [13, 15]. Let $G^f_{int_f K}$ denote the fine Green function corresponding to the bounded fine open set $int_f K$. Since $int_f K \subset D_j$ we have $G^f_{int_f K}(\cdot, y) \leq \hat{G}_{D_j}(\cdot, y)$. As $\hat{G}_K(\cdot, y)$ is the decreasing limit of $\hat{G}_{D_j}(\cdot, y)$ we have the inequalities

$$G^f_{int_f K}(\cdot, y) \leq \hat{G}_K(\cdot, y) \leq \hat{G}_{D_j}(\cdot, y),$$

for all $y \in int_f K$. Since $G^f_{int_f K}(\cdot, y)$ and $\hat{G}_{D_j}(\cdot, y)$ are finely continuous, $\hat{G}_K(\cdot, y)$ must be finely continuous at y as

$$\infty = \text{f}\,\text{-}\lim_{x \to y} G^f_{int_f K}(x, y) \leq \text{f}\,\text{-}\lim_{x \to y} \hat{G}_K(x, y) \leq \text{f}\,\text{-}\lim_{x \to y} \hat{G}_{D_j}(x, y) = \infty.$$

Therefore $\hat{G}_K(\cdot, y)$ is finely continuous on $\mathbb{R}^n$ when $y \in int_f K$.

Thus $\hat{G}_K(\cdot, y)$ is finely superharmonic on $int_f K$ as it is finely continuous and the decreasing limit of $\{\hat{G}_{D_j}(\cdot, y)\}$, a sequence of finely superharmonic functions on $int_f K$ and this implies that 1. holds.

Suppose that $\hat{G}(x_0, y) > 0$ for $x_0 \in \partial_f K$. Then there is a fine neighborhood V of x_0 such that $\hat{G}_K(x, y) > 0$ for all $x \in V$. By definition $x \in \partial_f K$ if and only if $\complement K$ is non-thin at x. As $x_0 \in \partial_f K$, this means that $V \cap \complement K \neq \emptyset$. However by Lemma 3.13, $\hat{G}_K(x, y) = 0$ for $x \in \complement K$ and $y \in K$, a contradiction. Therefore $\hat{G}_K(x, y) = 0$ for all $x \in \partial_f K$ and $y \in int_f K$. So $\hat{G}_K$ is a fine potential on $int_f K$ by the minimum principle [4, III.4.1] (see also [12, Theorem 9.1]) and this implies 2.

We have seen above that $\hat{G}_K(\cdot, y)$ is finely superharmonic on $int_f K$. By Proposition 3.13.vi $\hat{G}_K(\cdot, y)$ is finely subharmonic on $int_f K \setminus \{y\}$. Therefore $\hat{G}_K(\cdot, y)$ is finely harmonic on $int_f K \setminus \{y\}$ and we checked 3.

The property 4. follows immediately from Proposition 3.13.iii and the theorem is proved. □

Proposition 3.15. *The Green function $\hat{G}_K(x, y) > 0$ for $x, y \in K$ if and only if x and y are in the same fine connected component of $int_f K$.*

Proof. By the previous proposition $\hat{G}_K(\cdot, y)$ is finely superharmonic on $int_f K$. If $\hat{G}_K(x, y) = 0$, then by [12, Theorem 12.6] for all ζ in the fine component of y we have $\hat{G}_K(\zeta, y) = 0$. Therefore $\hat{G}_K(\cdot, y) > 0$ on the fine component containing y.

Suppose that $int_f K$ has multiple components. Each component is fine open and therefore has its own Green function. We can define a function $g(x, y)$ on $int_f K$ by

$$g(x, y) = \begin{cases} G^f_{Q_x}(x, y), & y \in Q_x \\ \\ 0, & y \in (int_f K) \setminus Q_x \end{cases}$$

where Q_x is the fine component containing x. Since fine subharmonicity and fine harmonicity are local properties, g satisfies the requirements mentioned in the proof of Theorem 3.14 to be a positive multiple of the fine Green function on $int_f K$. Therefore $G^f_{int_f K}(x, y)$ is positive if and only if x and y are in the same fine component of $int_f K$. So $\hat{G}_K(x, y) = 0$ when x and y are in different fine connected components.

In the proof of the previous proposition we proved that $\hat{G}_K(x, y) = 0$ for $x \in \partial_f K$ and $y \in K \setminus \{x\}$. $\square$

In [22] Poletsky introduced the sets

$$Q(x) = \{y \in K \colon \hat{G}_K(x, y) > 0\},$$

for every $x \in K$. The following corollary directly follows from Proposition 3.15 and characterizes these sets in terms of the fine topology.

Corollary 3.16. *For all $x \in int_f K$, the set $Q(x)$ is the fine connected component of $int_f K$ which contains x. Additionally the point $x \in K$ is in $\partial_f K$ if and only if $Q(x) = \{x\}$.*

Chapter 4

A Dirichlet problem on compact sets

In the classical setting we know that any continuous function in the boundary of a domain $D \subset \mathbb{R}^n$ extends harmonically to D and continuously to $\overline{D}$ if and only if every point of the boundary is regular. For general compact sets in $\mathbb{R}^n$ we have the following result.

From this result it also follows that the swept-out point mass at z onto the complement of K is just $\omega_K(z, \cdot)$.

Theorem 4.1. *If K is a compact set in $\mathbb{R}^n$ then any function $\phi \in C(\partial_f K)$ extends to a function in $H(K)$ if and only if the set $\partial_f K$ is closed. Moreover, the solution is given by*

$$\Phi(z) = \int_{\partial_f K} \phi(\zeta)\, d\omega_K(z, \zeta) \qquad z \in K$$

and $H(K)$ is isometrically isomorphic to $C(\partial_f K)$.

Proof. Suppose that the set $\partial_f K$ is closed. Consider a continuous function ϕ on $\partial_f K$. Let

$$\Phi(z) = \int\limits_{\partial_f K} \phi(\zeta)\, d\omega_K(z,\zeta) \qquad z \in K.$$

As $\partial_f K$ is closed, by Theorem 3.5, we have $\partial_f K = \partial K$. Also as $\omega_K(z,\cdot) = \delta_z$ for every $z \in \partial_f K$, we see that $\Phi = \phi$ on $\partial_f K$.

Let z_j be a sequence in K converging to $z_0 \in \partial_f K$. As z_0 is in $\partial_f K = Ch_{S(K)}K$, so $\mathcal{J}_{z_0}(K) = \{\delta_{z_0}\}$. Since (see [16, p. 3]) $\mathcal{J}(K)$ is weak* compact, any sequence of measures $\mu_j \in \mathcal{J}_{z_j}(K)$ must converge weak* to δ_{z_0}. In particular, $\omega_{U_j}(z_j,\cdot)$ is weak* converging to δ_{z_0}. Hence $\Phi(z_j)$ is converging to $\Phi(z_0) = \phi(z_0)$, and Φ is continuous at the boundary of K.

As $\partial_f K$ is closed, we have $\phi \in C(\partial_f K) = C(\partial K)$. We extend ϕ continuously as $\tilde{\phi} \in C_0(\mathbb{R}^n)$, and then define the harmonic functions

$$h_j(z) = \int\limits_{\partial U_j} \tilde{\phi}(\zeta)\, d\omega_{U_j}(z,\zeta).$$

As $\tilde{\phi}$ is continuous and $\omega_{U_j}(z,\cdot)$ converges weak* to $\omega_K(z,\cdot)$,

$$\lim_{j\to\infty} h_j(z) = \lim_{j\to\infty} \int\limits_{\partial U_j} \tilde{\phi}(\zeta)\, d\omega_{U_j}(z,\zeta) = \int\limits_{\partial K} \phi(\zeta)\, d\omega_K(z,\zeta) = \Phi(z).$$

Therefore Φ is the pointwise limit of a sequence $\{h_j\}$ of functions harmonic in a neighborhood of K. Furthermore we can take the extension $\tilde{\phi}$ of ϕ in such a way that the sequence $\{h_j\}$ is uniformly bounded. It now easily follows that Φ is continuous on the interior of K. Indeed, consider a point z in the interior of K. Then there exists a ball B centered at z contained in the interior of K. The h_j are harmonic functions on B and converging pointwise to Φ. Thus Φ is continuous on B by the Harnack principle, and so Φ is continuous on K. Therefore we have a continuous function Φ with representation

$$\Phi(z) = \int_{\partial K} \phi(\zeta)\, d\omega_K(z,\zeta) \qquad z \in K.$$

Since Φ is continuous on K by [22] to check that $\Phi \in H(K)$ all that remains is to show that Φ is averaging with respect to Jensen measures, i.e. the equivalence of the external definition of $H(K)$ and the definition by Jensen measures. So we need to see that $\Phi(z) = \mu_z(\Phi)$ for every $\mu_z \in \mathcal{J}_z(K)$ and for every $z \in K$. As h_j is harmonic on U_j, $h_j(z) = \mu_z(h_j)$. However by the Lebesgue Dominated Convergence Theorem

$$\mu_z(\Phi) = \lim_{j\to\infty} \mu_z(h_j) = \lim_{j\to\infty} h_j(z) = \Phi(z).$$

Thus $\Phi \in H(K)$.

For the converse, suppose $\partial_f K$ is not closed. Then there is a point $z_0 \in \partial K \setminus \partial_f K$. Since z_0 is not in $\partial_f K$, by Corollary 3.11, $\omega_K(z_0,\cdot)$ is not trivial. Therefore we can find a set $E \subset \partial K$ such that $\omega_K(z_0, E) > 0$ with E in the complement of $B(z_0, r)$ for some $r > 0$. Consider a continuous function f on ∂K such that $f = 1$ on ∂K outside $B(z_0, r)$ is 1 and $f = 0$ on $B(z_0, r/2) \cap \partial K$. Then

$$\int_{\partial K} f(\zeta)\, d\omega_K(z_0,\zeta) > \omega_K(z_0, E) \qquad z \in K.$$

However $f(z_0) = 0$. Thus there can be no function in $H(K)$ which agrees with f on the boundary of K. $\square$

Example 4.1. The following set provides a simple example of a compact set $K \subset \mathbb{R}^n$, $n \geq 3$, in which the fine boundary is not closed. The set K is obtained from the closed unit ball $\overline{B} \subset \mathbb{R}^n$ by deleting a sequence $\{B(z_n, r_n)\}_{n=1}^{\infty}$ of open balls whose centers and radii tend to zero. We take the centers to be $z_n = (2^{-n}, 0, \ldots, 0) \in \mathbb{R}^n$ and the radii $0 < r_n < 2^{-n-2}$. This example is analogous to the "road runner" example of Gamelin [17, Figure 2, pg 52] and the Lebesgue spine [1, pg 187].

By Theorem 4.1 one can not expect a continuous solution for the Dirichlet problem on an arbitrary compact set even with continuous boundary data. Therefore at this point we consider the following broader class of solutions with weaker continuity requirement.

Definition 4.1. Let $fH^c(K)$ denote the class of finely continuous functions on K which are finely harmonic on the fine interior of K and continuous and bounded on $\partial_f K$.

We have seen (the definition via Jensen measures) that $H(K)$ consists of the functions in $C(K)$ satisfying the averaging property with respect to $\mathcal{J}(K)$ and by the interior definition of $H(K)$ can also be seen as the $C(K)$ functions which are finely harmonic on the fine interior of K. Therefore in the definition of $fH^c(K)$ we have maintained the finely harmonic requirement while requiring continuity only on the boundary $\partial_f K$ (to match the boundary data). In fact Theorem 4.3 below shows that the functions in $fH^c(K)$ also satisfy the averaging property with respect to $\mathcal{J}(K)$.

Theorem 4.3 will show that the Dirichlet problem on compact sets $K \subset \mathbb{R}^n$ is solvable in the class of functions $fH^c(K)$ for boundary data that is continuous and bounded on $\partial_f K$. The functions which are continuous and bounded on $\partial_f K$ will be denoted $C_b(\partial_f K)$. For this we will need the following [12, Theorem 11.9] of Fuglede.

Theorem 4.2. *The pointwise limit of a pointwise convergent sequence of finely harmonic functions u_m in U, a finely open subset of $\mathbb{R}^n$, is finely harmonic provided that $\sup_m |u_m|$ is finely locally bounded in U.*

Theorem 4.3. *For every $\phi \in C_b(\partial_f K)$, i.e. continuous and bounded on $\partial_f K$, there is a unique $h_\phi \in fH^c(K)$ equal to ϕ on $\partial_f K$. Moreover, h_ϕ satisfies the averaging property for $\mathcal{J}(K)$ and in particular*

$$h_\phi(x) = \int\limits_{\partial_f K} \phi(\zeta)\, d\omega_K(x, \zeta), \qquad x \in K.$$

Proof. Let $\phi \in C_b(\partial_f K)$ and for $x \in \overline{\partial_f K}$ define

$$\tilde{\phi}(x) = \limsup_{y \to x,\, y \in \partial_f K} \phi(y).$$

Since ϕ is continuous on $\partial_f K$, if $x \in \partial_f K$ then $\tilde{\phi}(x) = \phi(x)$. Furthermore, $\tilde{\phi}$ is upper semicontinuous, and as such we may find a decreasing sequence of functions $\{\phi_k\}$ which are continuous on $\overline{\partial_f K}$ and converge pointwise to $\tilde{\phi}$. Then we extend the ϕ_k to $C_0(\mathbb{R}^n)$ as $\hat{\phi}_k$. By taking $\tilde{\phi}_k = \min\{\hat{\phi}_1, \hat{\phi}_2, \cdots, \hat{\phi}_k\}$ we can make the extensions be decreasing. Consider a decreasing sequence of regular domains U_j converging to K. Let $u_{j,k}$ be the solution of the Dirichlet problem on U_j for $\tilde{\phi}_k$. As the measures $\omega_{U_j}(x,\cdot)$ weak* converge to $\omega_K(x,\cdot)$, we have that $\lim_j u_{j,k} = \int \tilde{\phi}_k \, d\omega_K := u_k$. As the $\tilde{\phi}_k$ are decreasing, u_k must also be decreasing. Indeed, we will let $h_\phi = \lim u_k$.

Take any $\mu \in \mathcal{J}(K)$. Then $\mu \in \mathcal{J}_{z_0}(U_j)$ for all j and some $z_0 \in K$. As $u_{j,k}$ is harmonic, we have $\mu(u_{j,k}) = u_{j,k}(z_0)$. However by the Lebesgue Dominated Convergence Theorem we have $\lim_j \mu(u_{j,k}) = \mu(u_k)$, and so $\mu(u_k) = u_k(z_0)$. Since the sequence $\{u_k\}$ is decreasing pointwise to h_ϕ we have that $\mu(h_\phi) = h_\phi(z_0)$ by the Lebesgue Monotone Convergence Theorem. Thus h_ϕ satisfies the averaging property on $\mathcal{J}(K)$. As $\omega_K(z,\cdot) \in \mathcal{J}(K)$ for all $z \in K$ we see that

$$h_\phi(z) = \int_{\partial_f K} h_\phi(\zeta) \, \omega_K(z,\zeta).$$

We will now show that $h_\phi = \phi$ on $\partial_f K$. For any $x \in \mathcal{O}_k$, we know $\omega_K(x,\cdot) = \delta_x$, and

$$u_k(x) = \lim_{j\to\infty} u_{j,k}(x) = \int \tilde{\phi}_k(\zeta) \, d\omega_K(x,\zeta) = \tilde{\phi}_k(x).$$

Thus $u_k(x) = \tilde{\phi}_k(x)$ for all $x \in \partial_f K$, and so

$$h_\phi(x) = \lim_{k\to\infty} u_k(x) = \lim_{k\to\infty} \tilde{\phi}_k(x) = \phi(x),$$

for all $x \in \partial_f K$.

To see that h_ϕ is finely harmonic we use Theorem 4.2. Observe that u_k is the pointwise limits of the harmonic (and therefore finely harmonic) functions $u_{j,k}$, and the solution h_ϕ is the pointwise limit of u_k. From the construction of these functions it is clear that they are bounded. $\square$

Corollary 4.4. *The set $C_b(\partial_f K)$ is isometrically isomorphic to $fH^c(K)$.*

Proof. The previous theorem establishes the homomorphism taking $C_b(\partial_f K)$ to $fH^c(K)$. Observe that $h|_{\partial_f K} \in C_b(\partial_f K)$ for every $h \in fH^c(K)$. The uniqueness of the solution shows that $h|_{\partial_f K}$ extends as h. Furthermore, the isometry follows directly from the integral representation in the previous theorem. $\square$

Chapter 5

Restoring properties of harmonic functions on compact sets

5.1 A return to Jensen measures

Some results from [22] now follow from standard properties of the fine potential theory and the fine topology. For example [22, Theorem 3.6 (2)] is the partitioning the set K into the fine connected components of $int_f K$ and singleton sets for peak points (i.e. the set $\partial_f K$) forms an equivalence relation, [22, Theorem 3.6 (3)] is the fine minimum principle, and [22, Theorem 3.6 (4)] is that fine connected components have positive measure. We can now extend/rephrase some results of [22] and use them to obtain some new results.

Theorem 5.1. *For $x \in K$ and any $\varepsilon > 0$ there exists a $\mu \in \mathcal{J}_x(K)$ with $\mu(B(y,\varepsilon)) > 0$*

if and only if the point y is in the (Euclidean) closure $\overline{Q(x)}$ of the fine component of x.

Proof. In [22] Poletsky defines $I(x)$ as the set of points $y \in K$ with the property that for any $\varepsilon > 0$ there exists a $\mu \in \mathcal{J}_x(K)$ with $\mu(B(y,\varepsilon)) > 0$ and in [22, Theorem 3.6 (1)] proves that $I(x) = \overline{Q(x)}$. The result follows from Corollary 3.16. □

The following corollary is an immediate consequence of the previous theorem.

Corollary 5.2. *Let K be a compact set in $\mathbb{R}^n$. Then supp$(\mu) \subset \overline{Q(x)}$ for all $\mu \in \mathcal{J}_x(K)$.*

For use in the following proposition we recall the notion of a reduced function, see [1, Definition 5.3.1]. Fix a Greenian open set $\Omega \subset \mathbb{R}^n$. Let $U_+(\Omega)$ be the set of non-negative superharmonic functions on Ω. For $u \in U_+(\Omega)$ and $E \subset \Omega$, the reduced function of u relative to E in Ω is defined by

$$R_u^E(x) = \inf\{v(x) \colon v \in U_+(\Omega) \text{ and } v \geq u \text{ on } E\}, \qquad x \in \Omega.$$

Also note that $\hat{R}_u^E$ is the lower semicontinuous regularization of R_u^E.

Proposition 5.3. *Let U and V be disjoint fine open sets. Then $V \cap \overline{U}$ is a polar set.*

Proof. It suffices to prove this statement when U and V are bounded. Otherwise, we may consider intersections of these sets with increasing sequence of open balls.

Let Ω be any open Greenian set containing U and V. Since U is disjoint from V, U is thin at y for every $y \in V$. Then by [1, Theorem 7.3.5] there is a bounded continuous potential $u^{\#}$ on Ω with the property that $\hat{R}^U_{u^{\#}}(y) < u^{\#}(y)$ for all $y \in V \cap \overline{U}$. By construction $R^U_{u^{\#}} \geq u^{\#}$ and $R^U_{u^{\#}}(x) = \hat{R}^U_{u^{\#}}(x) = u^{\#}(x)$ for all $x \in U$. Therefore $V \cap \overline{U} \subset \{R^U_{u^{\#}} \neq \hat{R}^U_{u^{\#}}\}$, and by [1, Theorem 5.7.1] the set $\{R^U_{u^{\#}} \neq \hat{R}^U_{u^{\#}}\}$ is polar. $\square$

Corollary 5.4. *For a compact set $K \subset \mathbb{R}^n$, let $\{A_i\}$ be the collection of disjoint fine connected components of the fine interior of K. Then $int_f\, \overline{A_i} = A_i$ for all i.*

Proof. We will show that $int_f\, \overline{A_i}$ has only one fine component and so it must be A_i. Suppose that $int_f\, \overline{A_i} = A \cup V$ where A is the fine component containing A_i and V is fine open and disjoint from A. First we note that $A_i = A$ as $A_i \subset A \subset int_f K$ and A_i is a fine component of $int_f K$. Secondly, V is disjoint from A_i and contained in $int_f\, \overline{A_i}$, hence $V \subset \overline{A_i} \setminus A_i$. Therefore by Proposition 5.3, we have that V must be polar and cannot be fine open. $\square$

The following corollary tells us that the only trivial Jensen measures can have support in the closure of two fine components. We use the notation $\mathcal{J}(K) := \cup_{x \in K} \mathcal{J}_x(K)$ to denote the collection of all Jensen measures on K.

Corollary 5.5. *Let $\{A_j\}$ be the fine connected components of the fine interior of K. Then*

$$\mathcal{J}(\overline{A}_i) \bigcap \mathcal{J}(\overline{A}_j) = \bigcup_{x \in \overline{A}_i \cap \overline{A}_j} \{\delta_x\},$$

where $i \neq j$.

Proof. Let $\mu \in \mathcal{J}(\overline{A}_i) \bigcap \mathcal{J}(\overline{A}_j)$ with $i \neq j$. Then there is an $x_i \in \overline{A}_i$ and $x_j \in \overline{A}_j$ so that $\mu \in \mathcal{J}_{x_i}(\overline{A}_i) \bigcap \mathcal{J}_{x_j}(\overline{A}_j)$. As the coordinate functions are harmonic, this implies that $x_i = x_j$. Let us call $x_0 := x_i = x_j \in \overline{A}_i \bigcap \overline{A}_j$. As A_i and A_j are disjoint, we have by Corollary 5.4 that x_0 must be in the fine boundary of either $\overline{A}_i$ or $\overline{A}_j$. However the only way that x_0 can be in the fine boundary (see Lemma 3.3) is if $\mu = \delta_{x_0}$. $\square$

The following theorem gives sufficient condition on a subset E of K so that the Jensen measures on K with barycenter $x \in E$ belong to the Jensen measures on E.

Theorem 5.6. *Let $A \subset K \subset \mathbb{R}^n$ with K compact with A and $int_f K \setminus A$ fine open, that is A is a union of some of the fine connected components of $int_f K$. Suppose that supp(μ) $\subset \overline{A}$ for all $\mu \in \mathcal{J}_x(K)$ and all $x \in A$ then $\mathcal{J}_x(K) = \mathcal{J}_x(\overline{A})$ for all $x \in A$.*

Proof. The inclusion $\mathcal{J}_x(\overline{A}) \subseteq \mathcal{J}_x(K)$ is trivial.

We will now check that $\mathcal{J}_x(K) \subseteq \mathcal{J}_x(\overline{A})$. Pick $\mu \in \mathcal{J}_{x_0}(K)$. To see that $\mu \in \mathcal{J}_{x_0}(\overline{A})$ we must show that $f(x_0) \leq \mu(f)$ for all $f \in S(\overline{A})$. Hence we will assume there exists $f \in S(\overline{A})$ so that $\mu(f) < f(x_0)$ and construct $h \in S(K)$ with h close to f at x_0 and on a large ($d\mu$) subset of supp(μ). This h will then have the property $\mu(h) < h(x_0)$ contradicting that $\mu \in \mathcal{J}_{x_0}(K)$.

Suppose there exists $f \in S(\overline{A})$ such that $\mu(f) < f(x_0)$. As $cf + c'$ is also in $S(\overline{A})$ for $c > 0$ and since the functions in $S(\overline{A})$ are uniform limits of continuous subharmonic functions defined in neighborhoods of $\overline{A}$, we may assume that $f \in C(G) \cap S(G)$ for some open set $G \supset \overline{A}$ with the properties $0 < \mu(f) < f(x_0) < 1$ and $0 < f < 1$. Let $a := f(x_0) - \mu(f) > 0$ and take G' open with $\overline{A} \subset G'$ and $\overline{G'} \subset G$.

Pick $\phi \in C(\mathbb{R}^n)$ with $\phi = 0$ on $\overline{A}$, $\phi = -1$ on $\mathbb{R}^n \setminus G'$ and $-1 < \phi < 0$ on $G' \setminus \overline{A}$.

By Edwards Theorem (see [6])

$$E\phi(y) = \sup\{f(y)\colon f \in S(K),\ f \leq \phi\} = \inf\{\nu(\phi)\colon \nu \in \mathcal{J}_y(K)\}.$$

By assumption $\mathrm{supp}(\nu) \subset \overline{A}$ for all $\nu \in \mathcal{J}_y(K)$ and every $y \in A$. So $E\phi(y) = 0$ for every $y \in A$. Thus for any $0 < \varepsilon < a/3$ there exists a $g \in S(K)$ with $-1 \leq g \leq \phi \leq 0$ and $g(x_0) > -\varepsilon > -a/3$.

Actually we can say a little more. By Corollary 3.4, we know that $\mathcal{J}_y(K) \neq \{\delta_y\}$ if and only if $y \in int_f K$. This allows us to decompose $\overline{A}$ into three sets; A, $\partial_1 A \subset \partial A$ where $\mathcal{J}_y(K) = \{\delta_y\}$ for $y \in \partial_1 A$, and $\partial_2 A = \overline{A} \setminus (A \cup \partial_1 A)$. Each point in $\partial_2 A$ belongs to $int_f K \setminus A$. Recall that by hypothesis $int_f K \setminus A$ is fine open. Therefore $\partial_2 A \subset \overline{A} \cap (int_f K \setminus A)$, which means that $\partial_2 A$ is polar by Proposition 5.3. Since $\partial_2 A$ is a polar set, we see that $\mu(\partial_2 A) = 0$.

Thus there exists C a compact neighborhood of x with $C \subset A \cup \partial_1 A$ so that $\mu(C) > 1 - \varepsilon$. As $E\phi|_{A \cup \partial_1 A} = 0$, trivially $E\phi|_C = 0$. For every $y \in C$ there are a continuous and subharmonic function $g_y \leq \phi$ in a neighborhood of K and an open neighborhood U_y of y with $g_y > -\varepsilon$ on U_y. The sets U_y cover C, so by compactness we can pick up $y_1, \ldots, y_N$ so that $C \subset U_{y_1} \cup \cdots \cup U_{y_N}$. Then $g = \max\{g_{y_1}, \ldots, g_{y_n}\}$ has the property $g|_C > -\varepsilon$ and $\mu(\{g < -\varepsilon\}) < \varepsilon$.

Consider the function $f + g$. As $g \leq 0$ we have

$$\mu(f + g) = \mu(f) + \mu(g) \leq f(x_0) - a + g(x_0) - g(x_0) < (f(x_0) + g(x_0)) - a + \varepsilon.$$

As $g \leq \phi$, we have that $f + g \geq 0$ on $K \setminus \overline{G'}$. Note also that

$$f(x_0) + g(x_0) = \mu(f) + a + g(x_0) > a + g(x_0) > a - \varepsilon > 0.$$

So

$$h(y) = \begin{cases} 0, & K \setminus \overline{G} \\ \max\{f + g, 0\}, & G \cap K \end{cases}$$

is in $C(K)$, $h \equiv 0$ on $K \setminus G'$ and $h(x_0) = f(x_0) + g(x_0)$.

To see that h is in $S(K)$ we use a localization argument. Let $\mathcal{V}$ be a covering of the fine interior of K by fine open sets such that $V \in \mathcal{V}$ has the property: if $V \cap G' \neq \emptyset$ then $V \subset G$. If $V \subset G$, then $h = \max\{f + g, 0\} \subset S(V)$. If $V \cap G' = \emptyset$ then $h \equiv 0 \in S(V)$. Thus $h \in S(K, int_f K, \mathcal{V}) = S(K)$, by [3, Proposition 3.5].

Thus

$$\mu(h) = \int\limits_{\{f+g>0\}} (f + g)\, d\mu = \mu(f + g) - \int\limits_{\{f+g\leq 0\}} (f + g)\, d\mu.$$

Now $\mu(f + g) < f(x_0) + g(x_0) - a + \varepsilon$ and

$$\int\limits_{\{f+g\leq 0\}} (f + g)\, d\mu = \int\limits_{\{f+g\leq 0\}} f\, d\mu + \int\limits_{\{f+g\leq 0\}} g\, d\mu.$$

The first integral on the right is positive (as $0 < f < 1$) and because $-1 < g \leq 0$

$$\int\limits_{\{f+g\leq 0\}} g\, d\mu \geq \int g\, d\mu.$$

But the last integral is equal to

$$\int\limits_{\{g\geq -\varepsilon\}} g\, d\mu + \int\limits_{\{g<-\varepsilon\}} g\, d\mu \geq -\varepsilon - \mu(\{g < -\varepsilon\}).$$

Recall that $\mu(\{g < -\varepsilon\}) < \varepsilon$. Thus

$$\mu(h) \leq f(x_0) + g(x_0) + a - 3\varepsilon < h(x_0).$$

However this contradicts that $\mu \in J_{x_0}(K)$ and $h \in S(K)$. Hence $\mathcal{J}_x(K) \subseteq \mathcal{J}_x(\overline{A})$ for all $x \in A$. $\square$

We also get the following useful restriction property of Jensen measures.

Corollary 5.7. *Let $K \subset \mathbb{R}^n$ be a compact set. For all $x \in K$, we have*

$$\mathcal{J}_x(\overline{Q(x)}) = \mathcal{J}_x(K).$$

Proof. We will show $\mathcal{J}_x(\overline{Q(x)}) \subset \mathcal{J}_x(K)$ first. Consider $\mu \in \mathcal{J}_x(\overline{Q(x)})$ and $u \in S(K)$. Then $u|_{\overline{Q(x)}} \in S(\overline{Q(x)})$, so that $u(x) \leq \mu(u)$. Thus $\mu \in \mathcal{J}_x(K)$.

By Corollary 5.2, $\operatorname{supp}(\mu) \subset \overline{Q(x)}$ for all $\mu \in \mathcal{J}_x(K)$, which by Theorem 5.6 means that $\mathcal{J}_x(K) \subset \mathcal{J}_x(\overline{Q(x)})$. $\square$

5.2 Applications

An interesting corollary follows immediately from the proof of Theorem 5.6. For any cone of functions $\mathcal{R}$, we define the closure $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ as all continuous functions which can be represented as the supremum of functions from $\mathcal{R}$.

Corollary 5.8. *Let $K \subset \mathbb{R}^n$ be a compact set with $\{A_j\}$ the fine connected components of the fine interior of K. Then*

$$S(\overline{A}_j) = \widetilde{S(K)|_{\overline{A}_j}}$$

for every component A_j.

Proof. It is clear that $S(K)|_{\overline{A}_j} \subset S(\overline{A}_j)$. Consider any function $f \in S(\overline{A}_j)$. By Edwards Theorem (see [6])

$$f(x) = \sup\{\phi(x) \colon \phi \in S(\overline{A}_j) \text{ and } \phi \leq f \text{ on } \overline{A}_j\} = \inf\{\mu(f) \colon \mu \in \mathcal{J}_x(\overline{A}_j)\},$$

for all $x \in \overline{A}_j$. From Corollary 5.7 we have $\mathcal{J}_x(\overline{A}_j) = \mathcal{J}_x(K)$ when $x \in A_j$. Therefore we may apply Edwards Theorem again to see that

$$f(x) = \inf\{\mu(f) \colon \mu \in \mathcal{J}_x(K)\} = \sup\{\phi(x) \colon \phi \in S(K) \text{ and } \phi \leq f \text{ on } K\},$$

for $x \in A_j$. Thus $f \in \widetilde{S(K)|}_{\overline{A}_j}$. ☐

The following theorem shows that the restoring covering of [22] is given by the fine connected components of $int_f K$.

Theorem 5.9. *Let $K \subset \mathbb{R}^n$ be a compact set with $\{A_j\}$ denoting the fine components (fine open, fine connected) of the fine interior of K. For any $f \in C(K)$, $f \in H(K)$ if and only if $f \in H(\overline{A}_j)$ for all j.*

Proof. Recall that

$$H(K) = \{f \in C(K) \colon f(x) = \mu(f) \text{ for all } \mu \in \mathcal{J}_x(K) \text{ and every } x \in K\}$$

However if x is in A_j by Corollary 5.7 we have that $\mathcal{J}_x(\overline{A}_j) = \mathcal{J}_x(K)$ which implies the result. ☐

As a corollary we may extend the [18] result to higher dimensions. Recall that for any compact set E, the set $H(E)^\perp$ is the set of Radon measures μ with $supp(\mu) \subset E$ such that $\mu(h) = 0$ for all $h \in H(E)$, and if $m(E)$ is any set of Radon measures with support in E the set ${}^\perp m(E)$ consists of all $f \in C(E)$ such that $\mu(f) = 0$ for all $\mu \in m(E)$.

Corollary 5.10. *For any $K \subset \mathbb{R}^n$ compact*

$$H(K)^\perp = \bigoplus H(\overline{A}_j)^\perp$$

where A_j are the fine components (fine open, fine connected) of the fine interior of K.

Proof. Consider any $\mu \in \oplus H(\overline{A}_j)^\perp$ and $h \in H(K)$. Then $h|_{\overline{A}_j} \in H(\overline{A}_j)$, so $\mu(h) = 0$. Thus $\oplus H(\overline{A}_j)^\perp \subset H(K)^\perp$.

Conversely, suppose that $h \in C(K)$ and $h \in {}^\perp\big(\oplus H(\overline{A}_j)^\perp\big)$. Then $h|_{\overline{A}_j} \in {}^\perp\big(H(\overline{A}_j)^\perp\big) = H(\overline{A}_j)$. The restoring property (Theorem 5.9) then implies that $h \in H(K)$. Therefore ${}^\perp\big(\oplus H(\overline{A}_j)^\perp\big) \subset H(K)$ and so $H(K)^\perp \subset \oplus H(\overline{A}_j)^\perp$. □

Recall the following definitions of Poletsky [22, Def 3.9, 3.15].

Definition 5.1. A compact set $K \subset \mathbb{R}^n$, $n \geq 2$, is called Jensen if $K = \overline{Q(x)}$ for some $x \in K$, and Wermer if for all $x \in K$, either $\overline{Q(x)} = K$ or $\overline{Q(x)} = \{x\}$.

It has been shown in [22, Corollary 3.16] that every Jensen set is a Wermer set in the plane. We can now provide a proof of this in $\mathbb{R}^n$.

Proposition 5.11. *A Jensen set is Wermer.*

Proof. Suppose K is Jensen. Then $K = \overline{Q(x_0)}$ for some $x_0 \in K$. Every $y \in K$ is either a fine boundary point or in the fine interior. If y is in the fine boundary of K, then $Q(y) = \{y\}$ by Corollary 3.16.

We will show that $int_f K$ has only one fine component which must be $Q(x_0)$. Suppose that $int_f K = Q(x_0) \cup V$ were V is fine open and disjoint from $Q(x_0)$. Since $\overline{Q(x_0)} = K$, by Proposition 5.3, we have that V must be polar and cannot be fine open.

Thus for any $y \in int_f K$, we have $Q(y) = Q(x_0)$ and so $\overline{Q(y)} = K$. $\square$

The set $K = [0,1] \subset \mathbb{R}^2$ provides a simple example of a Wermer set that is not Jensen. Every point is a fine boundary point, so $Q(x) = \{x\}$ for all $x \in K$. However there is no point $x_0 \in K$ such that $K = \overline{Q(x_0)}$. Proposition 5.11 can be interpreted as saying that if K is Wermer then either $H(K) = C(K)$ or K is Jensen.

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