11-3-2009

Area Contraction for Harmonic Automorphisms of the Disk

Ngin-Tee Koh
Syracuse University

Leonid V. Kovalev
Syracuse University

AREA CONTRACTION FOR HARMONIC AUTOMORPHISMS OF THE DISK

NGIN-TEE KOH AND LEONID V. KOVALEV

Abstract. A harmonic self-homeomorphism of a disk does not increase the area of any concentric disk.

1. Introduction

The unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ can be endowed with the hyperbolic metric

$$d\sigma = \frac{|dz|}{1-|z|^2}.$$

The Schwarz-Pick lemma (e.g., [1]) implies that any holomorphic map $f \colon \mathbb{D} \to \mathbb{D}$ does not increase distances in the hyperbolic metric. This is no longer true for harmonic maps, which verify the Laplace equation $\partial\bar{\partial} f = 0$ but not necessarily the Cauchy-Riemann equation $\bar{\partial} f = 0$. The harmonic version of the Schwarz lemma ([5], see also [2]) states that any harmonic map $f \colon \mathbb{D} \to \mathbb{D}$ with normalization $f(0) = 0$ satisfies

$$|f(z)| \leqslant \frac{4}{\pi} \arctan|z|, \quad z \in \mathbb{D}.$$

This inequality is sharp [4, p. 77]. More precisely, for any $r \in (0,1)$ and any small $\epsilon > 0$ there is a bijective harmonic map $f \colon \mathbb{D} \to \mathbb{D}$ such that $f(0) = 0$ and

$$f(r) = -f(-r) = \frac{4}{\pi} \arctan r - \epsilon.$$

This map is not a contraction in either Euclidean or hyperbolic metric. With respect to either metric, the diameter of the disk $\mathbb{D}_r = \{z \in \mathbb{C} \colon |z| < r\}$ is strictly less than the diameter of $f(\mathbb{D}_r)$.

In this note we prove that a bijective harmonic map $f \colon \mathbb{D} \to \mathbb{D}$ does not increase the area of $\mathbb{D}_r$ for any $0 < r < 1$. We write $|E|$ for the area (i.e., planar Lebesgue measure) of a set E.

Theorem 1.1. *Let $f \colon \mathbb{D} \to \mathbb{D}$ be a bijective harmonic map. Then*

$$|f(\mathbb{D}_r)| \leqslant |\mathbb{D}_r|, \qquad 0 < r < 1. \tag{1.1}$$

If (1.1) turns into an equality for some $r \in (0,1)$, then f is an isometry.

Date: November 2, 2009.
2000 *Mathematics Subject Classification.* Primary 31A05; Secondary 30C62, 58E20.
Key words and phrases. Harmonic map, homeomorphism, area.
Kovalev was supported by the NSF grant DMS-0913474.

It should be noted that the class of harmonic automorphisms of $\mathbb{D}$ is much wider than the class of holomorphic automorphisms, which consists of Möbius maps only. Harmonic homeomorphisms of $\mathbb{D}$ form an interesting and much-studied class of planar maps, see [3, 7, 8] or the monograph [4]. Theorem 1.1 is different from most known estimates for harmonic maps in that it remains sharp when specialized to the holomorphic case.

An immediate consequence of (1.1) is $|f(\mathbb{D} \setminus \mathbb{D}_r)| \geqslant |\mathbb{D} \setminus \mathbb{D}_r|$. If f is sufficiently smooth, we can divide by $1-r$ and let $r \to 1$ to obtain the following.

Corollary 1.2. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a bijective harmonic map that is continuously differentiable in the closed disk $\overline{\mathbb{D}}$. Then*

$$\int_{|z|=1} |\det Df|\,|dz| \geqslant 2\pi,$$

where $\det Df = |\partial f|^2 - |\bar\partial f|^2$ *is the Jacobian determinant of f.*

Corollary 1.2 was proved in a different way in [6] where it serves as an important part of the proof of Nitsche's conjecture on the existence of harmonic homeomorphisms between doubly-connected domains. In fact, Corollary 1.2 is what led us to think that (1.1) might be true.

If $f\colon \mathbb{D} \to \mathbb{D}$ is holomorphic, then (1.1) holds without the assumption of f being bijective. Indeed, in this case $f(\mathbb{D}_r)$ is contained in a hyperbolic disk D of the same hyperbolic radius as $\mathbb{D}_r$. Since the density of the hyperbolic metric increases toward the boundary, it follows that the Euclidean radius of D is at most r, which implies (1.1).

Question 1.3. Does the area comparison (1.1) hold for general harmonic maps $f\colon \mathbb{D} \to \mathbb{D}$? Does it hold in higher dimensions?

We conclude the introduction by comparing the behavior of $|f(\mathbb{D}_r)|$ for holomorphic and harmonic maps. If $f\colon \mathbb{D} \to \mathbb{C}$ is holomorphic and injective, one can use the power series $f(z) = \sum c_n z^n$ to compute

$$|f(\mathbb{D}_r)| = \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

Since the right-hand side is a convex function of r^2, it follows that

$$|f(\mathbb{D}_r)| \leqslant r^2 |f(\mathbb{D})|, \tag{1.2}$$

which includes (1.1) as a special case. However, (1.2) fails for harmonic maps. Indeed, let $f(z) = z + c\bar z^2$ where $0 < |c| < 1/2$. It is easy to see that $f\colon \mathbb{D} \to \mathbb{C}$ is harmonic and one-to-one, but

$$|f(\mathbb{D}_r)| = r^2 - 2|c|^2 r^4$$

is a strictly concave function of r^2. Therefore, $|f(\mathbb{D}_r)| > r^2|f(\mathbb{D})|$ for $0 < r < 1$. This example does not contradict Theorem 1.1 since $f(\mathbb{D})$ is not a disk.

2. Preliminaries

Let f be as in Theorem 1.1. We may assume that f is orientation-preserving; otherwise consider $f(\bar{z})$ instead. In this section we derive an identity that relates the area of $f(\mathbb{D}_r)$ with the boundary values of f, which exist a.e. in the sense of nontangential limits.

The Poisson kernel for $\mathbb{D}$ will be denoted $P_r(t)$,

$$P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}, \quad 0 \leqslant r < 1,\ t \in \mathbb{R}.$$

We represent f by the Poisson integral

$$f(re^{i\theta}) = \frac{\omega}{2\pi}\int_0^{2\pi} e^{i\xi(t)} P_r(\theta - t)\, dt, \tag{2.1}$$

where $\xi\colon [0,2\pi) \to [0,2\pi)$ is a nondecreasing function and ω is a unimodular constant. By Green's formula we have

$$|f(\mathbb{D}_r)| = \frac{1}{2}\int_0^{2\pi} \operatorname{Im}\left(\overline{f(re^{i\theta})} f_\theta(re^{i\theta})\right) d\theta,$$

where f_θ indicates the derivative with respect to θ. Since

$$f_\theta(re^{i\theta}) = \frac{\omega}{2\pi}\int_0^{2\pi} e^{i\xi(t)} P_r'(\theta - t)\, dt,$$

it follows that

$$\overline{f(re^{i\theta})} f_\theta(re^{i\theta}) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} e^{-i\xi(t)} e^{i\xi(s)} P_r(\theta - t) P_r'(\theta - s)\, dt\, ds. \tag{2.2}$$

Integrating (2.2) with respect to θ and reversing the order of integration, we find

$$|f(\mathbb{D}_r)| = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \mathcal{K}_r\, \sin(\xi(s) - \xi(t))\, dt\, ds \tag{2.3}$$

where $\mathcal{K}_r$ is a function of r, s, and t,

$$\mathcal{K}_r = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t) P_r'(\theta - s)$$

Recall that the Poisson kernel has the semigroup property [9, p.62],

$$P_{r\sigma}(t) = \frac{1}{2\pi}\int_0^{2\pi} P_r(s) P_\sigma(t-s)\, ds, \quad 0 \leqslant r, \sigma < 1. \tag{2.4}$$

We will only use (2.4) with $\sigma = r$. Differentiation with respect to t yields

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(s) P_r'(t-s)\, ds = P_{r^2}'(t) = -\frac{2r^2(1-r^4)\sin t}{(1-2r^2\cos t + r^4)^2}. \tag{2.5}$$

Identity (2.5) provides an explicit formula for $\mathcal{K}_r$,

$$\mathcal{K}_r = \mathcal{K}_r(s-t) = \frac{2\rho^2(1-\rho^4)\sin(s-t)}{(1-2\rho^2\cos(s-t) + \rho^4)^2}. \tag{2.6}$$

Now we can rewrite (2.3) as

$$|f(\mathbb{D}_r)| = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \mathcal{K}_r(s-t)\,\sin(\xi(s)-\xi(t))\,dt\,ds. \tag{2.7}$$

In the next section we will estimate (2.7) from above.

3. Proof of Theorem 1.1

We continue to use the Poisson representation (2.1). The function ξ, originally defined on $[0,2\pi)$, can be extended to $\mathbb{R}$ so that $\xi(t+2\pi)=\xi(t)+2\pi$ for all $t\in\mathbb{R}$. By (2.7) we have

$$|f(\mathbb{D}_r)| = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \mathcal{K}_r(s-t)\,\sin(\xi(s)-\xi(t))\,dt\,ds. \tag{3.1}$$

When f is the identity map, (3.1) tells us that

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \mathcal{K}_r(s-t)\,\sin(s-t)\,dt\,ds = |\mathbb{D}_r|.$$

The desired inequality $|f(\mathbb{D}_r)|\leqslant|\mathbb{D}_r|$ now takes the form

$$\int_0^{2\pi}\int_0^{2\pi} \mathcal{K}_r(s-t)\,\big\{\sin(s-t)-\sin(\xi(s)-\xi(t))\big\}\,dt\,ds \geqslant 0. \tag{3.2}$$

Neither the kernel $\mathcal{K}_r$, which is defined by (2.6), nor the other factor in the integrand are nonnegative. We will have to transform the integral in (3.2) before effective pointwise estimates can be made. It will be convenient to use the notation

$$\alpha = s-t, \quad \text{and} \quad \gamma=\gamma(\alpha,t)=\xi(\alpha+t)-\xi(t), \tag{3.3}$$

so that the integral in (3.2) becomes

$$\int_0^{2\pi}\int_{-\pi-t}^{\pi-t} \mathcal{K}_r(\alpha)\,(\sin\alpha-\sin\gamma)\,d\alpha\,dt.$$

Since the integrand is 2π-periodic with respect to α, our goal can be equivalently stated as

$$\int_0^{2\pi}\int_0^{2\pi} \mathcal{K}_r(\alpha)\,(\sin\alpha-\sin\gamma)\,d\alpha\,dt \geqslant 0. \tag{3.4}$$

Note that $\gamma\in[0,2\pi]$ for all $\alpha,t\in[0,2\pi]$.

Step 1. We claim that

$$\int_0^{2\pi}\int_0^{2\pi} \mathcal{K}_r(\alpha)(\gamma-\alpha)\cos\alpha\,d\alpha\,dt = 0. \tag{3.5}$$

Indeed, the function $\zeta(t):=\xi(t)-t$ is 2π-periodic, which implies

$$\int_0^{2\pi}\{\zeta(\alpha+t)-\zeta(t)\}\,dt = 0 \tag{3.6}$$

for every $\alpha \in \mathbb{R}$. Multiplying (3.6) by $\mathcal{K}_r(\alpha)\cos\alpha$ and integrating over $\alpha \in [0, 2\pi]$, we obtain

$$\int_0^{2\pi}\int_0^{2\pi} \mathcal{K}_r(\alpha)\{\zeta(\alpha+t)-\zeta(t)\}\cos\alpha\, d\alpha\, dt = 0$$

It remains to note that $\zeta(\alpha+t)-\zeta(t) = \gamma-\alpha$, completing the proof of (3.5).

We take advantage of (3.5) by adding it to (3.4), which reduces our task to proving that

$$\int_0^{2\pi}\int_0^{2\pi} \mathcal{K}_r(\alpha)\left\{\sin\alpha + (\gamma-\alpha)\cos\alpha - \sin\gamma\right\} d\alpha\, dt \geqslant 0. \tag{3.7}$$

Step 2. Let us now consider the function

$$H(\alpha,\beta) := \sin\alpha + (\beta-\alpha)\cos\alpha - \sin\beta, \quad (\alpha,\beta)\in[0,2\pi]\times[0,2\pi] \tag{3.8}$$

which appears in (3.7). It has a simple geometric interpretation in terms of the graph of the sine function $y = \sin x$. Indeed, the tangent line to this graph at $x = \alpha$ has equation $y = \sin\alpha + (x-\alpha)\cos\alpha$. The quantity $H(\alpha,\beta)$ represents the difference in the y-values of the tangent line and the graph at $x = \beta$. Since the sine curve is strictly concave on $[0, \pi]$, it follows that

$$H(\alpha,\beta) \geqslant 0, \qquad 0 \leqslant \alpha,\beta \leqslant \pi, \tag{3.9}$$

with equality only when $\alpha = \beta$. The upper bound on β in (3.9) can be weakened to $\beta \leqslant 2\pi - \alpha$ thanks to the monotonicity with respect to β,

$$\frac{\partial H}{\partial \beta} = \cos\alpha - \cos\beta \geqslant 0, \quad 0 \leqslant \alpha \leqslant \pi,\ \alpha \leqslant \beta \leqslant 2\pi-\alpha.$$

Note that the product $\mathcal{K}_r(\alpha)H(\alpha,\beta)$ is invariant under the central symmetry of the square $[0,2\pi]\times[0,2\pi]$, i.e., the transformation $(\alpha,\beta)\mapsto(2\pi-\alpha, 2\pi-\beta)$. Hence

$$\mathcal{K}_r(\alpha)H(\alpha,\beta) \geqslant 0, \qquad (\alpha,\beta)\in\big([0,2\pi]\times[0,2\pi]\big)\setminus(T_1\cup T_2) \tag{3.10}$$

where

$$T_1 = \{(\alpha,\beta)\colon 0<\alpha<\pi,\ 2\pi-\alpha<\beta\leqslant 2\pi\};$$
$$T_2 = \{(\alpha,\beta)\colon \pi<\alpha<2\pi,\ 0\leqslant\beta<2\pi-\alpha\}.$$

Within the triangles T_1 and T_2 the product $\mathcal{K}_r(\alpha)H(\alpha,\beta)$ may be negative. However, for all $(\alpha,\beta)\in[0,2\pi]\times[0,2\pi]$ the following holds.

$$\mathcal{K}_r(\alpha)H(\alpha,\beta) + \mathcal{K}_r(2\pi-\alpha)H(2\pi-\alpha,\beta) = 2\mathcal{K}_r(\alpha)H(\alpha,\pi) \geqslant 0, \tag{3.11}$$

where the last inequality follows from (3.10). We will use (3.11) to control the contribution of triangles T_1 and T_2 to the integral (3.7).

Step 3. For each fixed t the function $\alpha \mapsto \gamma(\alpha,t)$ defined by (3.3) is nondecreasing and it maps the interval $[0,2\pi]$ onto itself. Thus, inequality (3.7) will follow once we show that for any nondecreasing function

$\Gamma\colon [0,2\pi]\to[0,2\pi]$

$$\int_0^{2\pi} \mathcal{K}_r(\alpha)H(\alpha,\Gamma(\alpha))\,d\alpha \geqslant 0. \tag{3.12}$$

The integral in (3.12) remains unchanged if we replace $\Gamma(\alpha)$ with $\widetilde{\Gamma}(\alpha) = 2\pi-\Gamma(2\pi-\alpha)$. Thus we lose no generality in assuming that $\Gamma(\pi)\leqslant\pi$. By virtue of (3.10) the integrand in (3.12) is nonnegative outside of the interval $[\pi,\alpha_0]$, where

$$\alpha_0 = \sup\{\alpha\in[\pi,2\pi]\colon \alpha+\Gamma(\alpha)\leqslant 2\pi\}$$

We claim that

$$\mathcal{K}_r(\alpha)H(\alpha,\Gamma(\alpha)) \geqslant \mathcal{K}_r(\alpha)H(\alpha,\Gamma(\pi)), \quad 2\pi-\alpha_0<\alpha<\alpha_0. \tag{3.13}$$

Indeed, the inequality

$$\frac{\partial H}{\partial\beta} = \cos\alpha-\cos\beta\leqslant 0, \quad |\alpha-\pi|\leqslant|\beta-\pi|\leqslant\pi,$$

implies

$$H(\alpha,\beta_1)\geqslant H(\alpha,\beta_2), \quad 0\leqslant\beta_1\leqslant\beta_2\leqslant\min(\alpha,2\pi-\alpha). \tag{3.14}$$

To see that (3.14) applies in our situation, note that $\Gamma(\alpha)\leqslant 2\pi-\alpha_0$ for $\alpha<\alpha_0$. Inequality (3.14) yields

$$\begin{aligned} &H(\alpha,\Gamma(\alpha))\leqslant H(\alpha,\Gamma(\pi)), \quad \pi\leqslant\alpha<\alpha_0;\\ &H(\alpha,\Gamma(\alpha))\geqslant H(\alpha,\Gamma(\pi)), \quad 2\pi-\alpha_0<\alpha\leqslant\pi. \end{aligned} \tag{3.15}$$

Multiplying (3.15) by $\mathcal{K}_r(\alpha)$, we arrive at (3.13).

Finally, we combine (3.10), (3.13), and (3.11) to obtain

$$\begin{aligned} \int_0^{2\pi} \mathcal{K}_r(\alpha)H(\alpha,\Gamma(\alpha))\,d\alpha &\geqslant \int_{2\pi-\alpha_0}^{\alpha_0} \mathcal{K}_r(\alpha)H(\alpha,\Gamma(\alpha))\,d\alpha\\ &\geqslant \int_{2\pi-\alpha_0}^{\alpha_0} \mathcal{K}_r(\alpha)H(\alpha,\Gamma(\pi))\,d\alpha\\ &= 2\int_{\pi}^{\alpha_0} \mathcal{K}_r(\alpha)H(\alpha,\pi)\,d\alpha\geqslant 0, \end{aligned} \tag{3.16}$$

completing the proof of (3.7).

Step 4. It remains to prove the equality statement in Theorem 1.1. Suppose that $\Gamma\colon[0,2\pi]\to[0,2\pi]$ is a nondecreasing function such that $\Gamma(\pi)\leqslant\pi$, and equality holds everywhere in (3.16). Returning to the geometric interpretation of $H(\alpha,\gamma)$ in (3.8), we note that

$$\mathcal{K}_r(\alpha)H(\alpha,\pi)>0, \quad 0<|\alpha-\pi|<\pi.$$

This forces $\alpha_0=\pi$, which by definition of α_0 implies

$$\mathcal{K}_r(\alpha)H(\alpha,\Gamma(\alpha))\geqslant 0, \quad 0\leqslant\alpha\leqslant 2\pi. \tag{3.17}$$

Hence, (3.17) must turn into an equality for almost all $\alpha\in[0,2\pi]$. In view of (3.9) and of the monotonicity of Γ this is only possible if $\Gamma(\alpha)=\alpha$ for all $\alpha\in[0,2\pi]$.

If $|f(\mathbb{D}_r)| = |\mathbb{D}_r|$, then equality holds in (3.7). Then for almost all $t \in [0, 2\pi]$ the function $\Gamma(\alpha) = \xi(\alpha + t) - \xi(t)$, or its reflection $\widetilde{\Gamma}(\alpha) = 2\pi - \Gamma(2\pi - \alpha)$, turns (3.16) into an equality. Hence $\xi(\alpha + t) - \xi(t) = \alpha$ for almost all $t \in [0, 2\pi]$ and all $\alpha \in [0, 2\pi]$. Thus ξ is the identity function and $f\colon \mathbb{D} \to \mathbb{D}$ is an isometry. Theorem 1.1 is proved.

Acknowledgements

We thank Tadeusz Iwaniec and Jani Onninen for valuable discussions on the subject of this paper.

References

1. L. V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.
2. H. Chen, P. M. Gauthier, and W. Hengartner, *Bloch constants for planar harmonic mappings*, Proc. Amer. Math. Soc. **128** (2000), no. 11, 3231–3240.
3. J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 3–25.
4. P. Duren, *Harmonic mappings in the plane*, Cambridge University Press, Cambridge, 2004.
5. E. Heinz, *On one-to-one harmonic mappings*, Pacific J. Math. **9** (1959), 101–105.
6. T. Iwaniec, L. V. Kovalev, and J. Onninen, *The Nitsche conjecture*, preprint, arXiv:0908.1253.
7. D. Kalaj, *On harmonic diffeomorphisms of the unit disk onto a convex domain*, Complex Var. Theory Appl. **48** (2003), no. 2, 175–187.
8. M. Pavlović, *Boundary correspondence under harmonic quasiconformal homeomorphisms of the unit disk*, Ann. Acad. Sci. Fenn. Math. **27** (2002), no. 2, 365–372.
9. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.

Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA
E-mail address: `nkoh@syr.edu`

Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA
E-mail address: `lvkovale@syr.edu`