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Mathematical Knowledge for Teaching Teachers: The Case of Multiplication and Division of Fractions

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Abstract

This study attempts to answer the question, *What is the mathematical knowledge required by teachers of elementary mathematics content courses in the area of multiplication and division of fractions?* Beginning in the mid-1980s, when Shulman (1986) introduced the idea of pedagogical content knowledge, researchers have been looking at the knowledge needed to teach in a variety of different content areas. One area that has garnered much of the research is that of mathematics. Researchers have developed frameworks for what they call mathematical knowledge for teaching, but there has been little work done looking at the knowledge requirements for teachers of teachers. This study attempts to fill this gap by determining some aspects of a framework for the mathematical knowledge required to teach prospective elementary teachers multiplication and division of fractions.

In order to determine aspects of a framework for mathematics teacher educator knowledge in relation to multiplication and division of fractions, I interviewed, observed, and audiotaped three experienced teacher educators in different educational settings to determine the mathematical work of teaching prospective teachers fraction multiplication and division. My analysis focused on three of major tasks that came out of the work: introducing fraction multiplication, helping students make sense of fraction division, and assessing student understanding. Each of these tasks played a major role in the work of the teacher educators, and the knowledge required to perform these tasks was evident in varying degrees in each teacher educator.

After analyzing the three mathematical tasks and the knowledge required by them, I was able to determine some components of a framework for the mathematical knowledge needed for teaching teachers multiplication and division of fractions. These

aspects include: understanding multiple representations of fraction multiplication and division and how these representations relate to each other, to whole number ideas, and to the algorithms, deciding which aspects of the topics will help prospective teachers make the connections that they will need in order to teach these topics, especially since time often plays a factor in what gets taught in mathematics content classes for prospective teachers, setting specific goals of exactly what one wants one's students to know, rather than having a general goal of wanting prospective teachers to develop conceptual understanding of a topic, and being able to design and use assessments effectively to help decide if one is achieving one's goals.

While each of the aspects described above are components of a framework for the mathematical knowledge needed by teacher educators, the three teacher educators in my study all lacked or were unable to demonstrate some of the knowledge components that would have helped them to meet their goals, despite having a wealth of experience teaching and designing mathematics content courses for prospective elementary teachers. One possible reason for this is that each of the teacher educators in my study were basically alone in their departments, without opportunities to collaborate or discuss these ideas with anyone else. These results suggest a need for better professional development for teacher educators in the field of mathematics education.

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List of Relevant Abbreviations

CBMS: Conference Board of Mathematical Sciences

CCK: Common Content Knowledge

CGI: Cognitively Guided Instruction

KCS: Knowledge of Content and Students

KCT: Knowledge of Content and Teaching

LMT: Learning Mathematics for Teaching Project

MKT: Mathematical Knowledge for Teaching

MKTT: Mathematical Knowledge for Teaching Teachers

MT: Mathematics Teacher

MTE: Mathematics Teacher Educators

MTEE: Mathematics Teacher Educator Educator

PCK: Pedagogical Content Knowledge

PST: Pre-Service Teacher

PUFM: Profound Understanding of Fundamental Mathematics

SCK: Specialized Content Knowledge

SMK: Subject Matter Knowledge

Chapter 1—Introduction

In his 1985 Presidential address at the American Educational Research Association (AERA) annual meeting, Lee Shulman identified what he called the "missing paradigm" in the research on teaching (Shulman, 1986). This missing paradigm referred to the lack of a research focus on the content knowledge needed for teaching, and consequently in exams for teacher certification and evaluation. Specifically, Shulman introduced a type of knowledge which he referred to as *pedagogical content knowledge*, which linked knowledge of teaching pedagogy with knowledge of the specific content that was taught. In response to this idea, large numbers of studies (e.g., Ball, Hill, & Bass, 2005; Brophy, 1991; Grossman, 1990) have been undertaken in a variety of subjects to identify the kinds of knowledge that are required to teach well; specifically what types of knowledge do teachers have that set them apart from non-teachers?

One of the areas where the study of the knowledge needed for teaching has flourished has been in mathematics. Recent studies (e.g., Ball, Lubienski, & Mewborn, 2001; Ma, 1999) indicated that the mathematics teachers need to know, even at the elementary level, is much more complex than was originally thought. Furthermore, research has shown that elementary teachers, both preservice and inservice, may not be well equipped with this deeper knowledge (e.g., Ball, 1990; Borko et al., 1992; Ma, 1999; Simon, 1993; Thanheiser, 2009). The growing ideas about the knowledge needed to teach elementary mathematics increases demands on teacher educators to help teachers acquire this knowledge. While a large amount of research has been and continues to be done on what mathematical knowledge is necessary for elementary teachers in mathematics, there has been little research on the knowledge demands placed on mathematics teacher educators to support teacher learning. This project attempts to address this gap by looking at the mathematical knowledge needed for teaching teachers.

Rationale

It has generally been assumed that more knowledgeable teachers lead to better student performance. However, it is not clear what types of knowledge produce better student outcomes. Mixed results on studies looking at factors such as the number of mathematics classes taken, the number of mathematics methods classes taken, and teachers' results on various exams have led researchers to question what it is that teachers really need to know in order to teach mathematics (Begle, 1979). Recent studies (e.g., Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004), have worked to identify what specific types of teacher knowledge produce better student outcomes. The researchers in these studies have developed measures to test aspects of what they call "mathematical knowledge for teaching," and their studies have shown evidence that first and third grade teachers who performed better on these measures had higher achieving students. The researchers determined that "students got an extra one-third to one-half of a month's worth of learning growth for every standard-deviation rise on their teacher's test scores" (Viadero, 2004, p. 8). Thus, we are making progress as a field in determining the nature of mathematical knowledge of teachers that produce better student outcomes.

In a similar manner, we can assume that more knowledgeable mathematics teacher educators will produce better student outcomes where the students are prospective teachers. However, at this point, we are unaware of the nature of the knowledge required by teacher educators in order to improve teacher learning. It is commonly understood that teachers need to know more than their students. Thus the knowledge demands placed on teacher educators must be more complex than those placed on their students (pre- and inservice teachers). This research study attempted to delve into this complexity and determine some of the aspects of mathematical knowledge needed by teacher educators.

The benefits of clarifying the knowledge needed by mathematics teacher educators are many. Scholars in the field of mathematics education have been calling for the better preparation of mathematics teachers (e.g., Askey, 1999; Howe, 1999; Ma, 1999). Understanding the mathematical knowledge needed for teaching teachers will help the field of mathematics education in preparing doctoral students to become teacher educators, in writing textbooks and teachers' guides for content courses for prospective teachers, and in giving teacher educators the opportunity to reflect on the knowledge and skills necessary to help produce high quality teachers. In addition, the development of a knowledge base for teacher educators will help professionalize the field of education. Murray (1996) states, "Without a sure sense of what constitutes educational malpractice, teaching and teacher education are behind other professions that have fairly well-articulated codes of good practice, which by extension define malpractice as the failure to follow good practice." Thus, defining a knowledge base for teacher educators will give the field a framework for "good practice," which may help quiet the critics of the professionalism of teaching and teacher education.

Historical Perspective

In his presidential address, Shulman (1986) identified three different types of knowledge needed by teachers: content knowledge, pedagogical content knowledge, and curricular knowledge. By its name, *content knowledge* refers to knowledge of content.

However, Shulman says that it goes beyond this. "The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied" (p. 9). Thus, teachers must understand the organizing structure of their discipline, how the concepts are related, how truth is established, and so on.

Shulman (1986) defined *pedagogical content knowledge* as "subject matter knowledge for *teaching*" (p. 9). This entails making the subject accessible to students. It includes knowing the best representations of material, what makes the subject easy or difficult for students, and places where students commonly make mistakes. Pedagogical content knowledge provides a link between knowledge of teaching and knowledge of a subject, to give us a type of knowledge unique to the profession of teaching a specific content area.

The third form of knowledge, *curricular knowledge*, involves understanding the curriculum one is teaching. However, Shulman also included in this type of knowledge, knowing what students are studying in subjects other than the teacher's own content, as well as understanding what has come before and after the particular piece of the curriculum that one is teaching. In other words, teachers must know where their students have been and where they are going.

Building on Shulman's three categories of knowledge, Ball and her colleagues (e.g., Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004) have expanded and defined "the mathematical knowledge for teaching." By researching the work that teachers do, they have developed a framework that expanded Shulman's knowledge categories and applied them to elementary mathematics. This framework, illustrated below, breaks Shulman's categories of subject matter knowledge and pedagogical content knowledge into pieces. Included in subject matter knowledge are two main categories called *Common Content Knowledge (CCK)* and *Specialized Content Knowledge (SCK)*, as well as *Knowledge at the Mathematical Horizon*. Pedagogical content knowledge is broken down into *Knowledge of Content and Teaching (KCT), Knowledge of Content and Students (KCS)*, and *Knowledge of the Curriculum*. While each of the categories of subject matter knowledge and pedagogical content knowledge contain three sub-categories, these researchers have thus far only developed two of the three sub-categories in each domain.

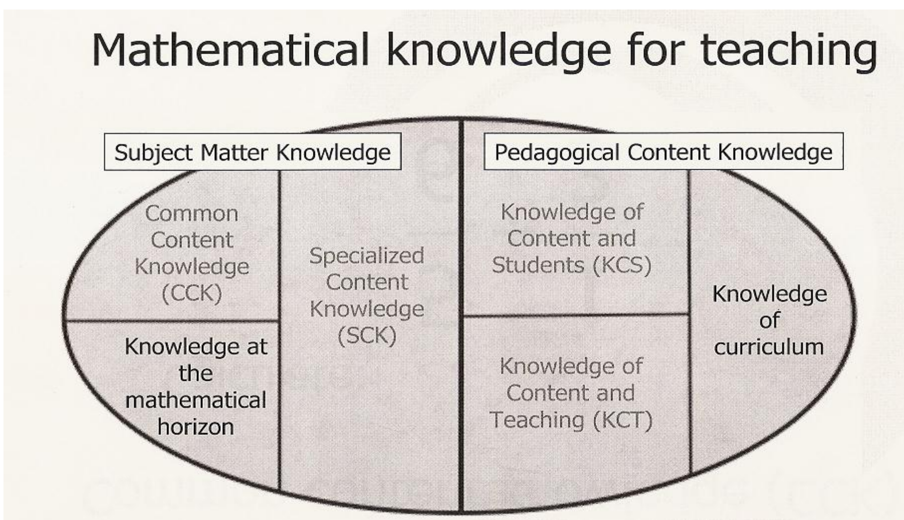

Figure 1.1. Mathematical knowledge for teaching (Ball et al., 2008).

Ball and her colleagues define *CCK* as "the mathematical knowledge and skill used in settings other than teaching" (Ball, Thames, & Phelps, 2008, p. 399). Thus it is the mathematical knowledge that anyone might know. Examples of common content knowledge include knowledge of algorithms and procedures for adding and subtracting, finding the area and perimeter of a given shape, or ordering a set of decimals.

Specialized content knowledge is defined as “the mathematical knowledge and skill uniquely needed by teachers in the conduct of their work” (Ball, Thames, & Phelps, 2008, p. 400). In studying this knowledge, the researchers determined that this type of knowledge is unique to teaching, in that others would not need it in the course of their work. Examples of specialized content knowledge include being able to evaluate student algorithms to determine their validity, explaining *why* we invert and multiply when we divide fractions, and understanding and being able to correctly use mathematical vocabulary. While these types of knowledge may be found in people other than teachers, the researchers argue that this knowledge is a necessity for teachers, but is generally not needed by the typical learner of mathematics.

“The third domain, *knowledge of content and students (KCS)*, is knowledge that combines knowing about students and knowing about mathematics” (Ball, Thames, & Phelps, 2008, p. 401). This type of knowledge includes anticipating student difficulties, understanding students’ reasoning, and knowing common errors and misconceptions that students will have with specific material. Ball, Thames, and Phelps (2008) explain how the first three types of knowledge work together in the classroom:

> Recognizing a wrong answer is common content knowledge (CCK), while sizing up the nature of the error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge (SCK). In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students. (p. 401)

The final domain that these researchers have expanded on in detail is *knowledge of content and teaching (KCT)*. This type of knowledge combines knowing about teaching with knowing about mathematics. It involves knowing how to sequence a particular set of topics and understanding the power and value of different mathematical representations. Ball and her colleagues (2008) describe one of the aspects of KCT in some of the roles that the teacher has to play in helping students: “During a classroom discussion, they have to decide when to pause for more clarification, when to use a student’s remark to make a mathematical point, and when to ask a new question or pose a new task to further students’ learning” (Ball, Thames, & Phelps, 2008, p. 401). These types of teaching tasks require that the teacher have both a deep understanding of the subject of mathematics, as well as understanding how their actions and decisions will affect how and what the students learn.

Much of the current research on teacher knowledge uses Ball and her colleagues’ framework as a starting point (Thanheiser et al., 2009). This project attempted to develop components of a framework for the mathematical knowledge required for teaching teachers. Since little is known on the learning trajectories of teacher educators, Stein, Smith, and Silver (1999) suggest that “we might turn to what is known about the learning of teachers in the context of the current reforms” (p. 243) in order to better understand how teacher educators might learn. Thus, we can use current frameworks for teacher knowledge as a basis for teacher educator knowledge.

While the Ball and colleagues’ “egg” (Figure 1.1) framework provides a basis for looking at mathematics educator knowledge, it does seem incomplete, as teacher educators need to know more than what is known by their students; future teachers.

Questions that we might ask about teacher educator knowledge include: Are there other aspects of teacher educator knowledge not covered by this framework? Is there knowledge for teaching that is unique to teacher educators? What does this knowledge entail? Is there some sort of *specialized*, specialized content knowledge that is deeper than what Ball and colleagues have identified as specialized content knowledge? In this study I attempted to answer these questions by using a grounded theory study of experienced teacher educators in practice, to determine how teacher educator knowledge is qualitatively different than what others have identified as teacher knowledge.

Content Area

Since the subject of mathematics is very broad, it would be difficult to look at elementary mathematics as a whole to determine aspects of a framework for teacher educator knowledge. I decided to narrow my content area by focusing on a domain that has been historically challenging for students and both pre and inservice teachers: multiplication and division of fractions (Ball, 1990a; Fischbein et al., 1985; Graeber & Tirosh, 1988; Ma, 1999).

Much of the current research dealing with teachers' knowledge of fractions has focused on division of fractions. Researchers give justification for this focus such as the fact that "division of fractions lies at the intersection of two mathematical concepts that many teachers never have had the opportunity to learn conceptually—division and fractions" (Sowder, Phillip, Armstrong, & Schappelle, 1998, p. 51), and "since division with fractions is most often taught algorithmically, it is a strategic site for examining the extent to which prospective teachers understand the meaning of division itself" (Ball, 1988, p. 61). As expected from these statements, researchers have found that both students and teachers struggle with teaching and learning this topic. While the majority of teachers are able to perform the “invert and multiply” division algorithm, researchers have found that teachers are unable to explain why the algorithm works (e.g., Borko et al., 1992; Eisenhart et al., 1993), or develop story problems that model division of fractions (e.g. Ball, 1988, 1990a; Ma, 1996, 1999). While there is less research on teachers’ understanding of multiplication of fractions, many of the difficulties teachers have with division result from not having a deep understanding of multiplicative ideas in general.

There are many reasons why multiplication and division of fractions are difficult for both students and teachers. First, unlike addition and subtraction, multiplication and division, both of fractions and whole numbers, are not unit-preserving operations. That is, when a person adds or subtracts, we can think of combining “like terms,” and the result is also the same like thing. For example, 2 *apples* added to 3 *apples* results in 5 *apples*. However, when one multiplies or divides, the units sometimes change. We do not multiply one number of apples by another number of apples. Instead, we would multiply 2 apples by 10 children, giving us 20 apples total. If we divide, we can divide 20 apples by 10 children, and we get 2 apples per child. Alternatively, we can divide 20 apples among children so that each child receives 2 apples, and our quotient would give us the number of children who would receive apples. While this idea is not difficult to understand, it can become more complicated when talking about fractional pieces of a number; knowing what one’s answer should even look like can be complicated.

Another reason why multiplication and division of fractions are difficult concepts for students and teachers is that many people have the misconception that multiplication always makes bigger, division always makes smaller, and when we divide, we must divide a larger number by a smaller number (Greer, 1994). When students first learn multiplication and division (with whole numbers) these ideas are true, however, with the introduction of fractions, this is not always the case. Multiplying a value by a number between 0 and 1 will decrease the original value, while dividing by a number between 0 and 1 will result in an increase. These misconceptions cause problems when students and teachers are unable to identify the correct operation to use to solve a word problem. For example, when students are asked a question such as: *Cheese costs $3.75 per pound. How much for 6 pounds of cheese?*, the inclination is to multiply the two quantities together to result in a larger value. However, if the question read: *Cheese costs $3.75 per pound. How much for* $\frac{3}{4}$ *pound of cheese?*, many students will choose to divide $3.75 by $\frac{3}{4}$, because they believe correctly that their answer should be less than $3.75, but incorrectly that division always makes the result smaller. This "nonconservation of operations" has been found both in elementary students and prospective and practicing teachers (Fischbein et al., 1985; Graeber, Tirosh, & Glover, 1989; Greer, 1994; Harel & Behr, 1995).

An added problem that researchers have identified regarding teachers' knowledge of rational numbers in general, is "that one critical aspect of teachers' knowledge of rational numbers is that they do not realize that they lack the understanding of rational numbers necessary to teach this topic in a meaningful way" (Sowder, Armstrong, et al., 1998, p. 145). Because many teachers know the procedures for multiplying and dividing fractions, they believe that they understand what they need to know in order to teach the topic. However, this procedural knowledge does not allow them to respond to student questions about why the algorithms work, examine alternative student algorithms, or pose meaningful problems for their students. Thus the job of the teacher educator becomes more complicated around the ideas of multiplication and division of fractions. Not only do teacher educators need to help prospective teachers deepen their knowledge of these topics, they may also need to convince prospective teachers of the need for this in the first place.

Research Questions

In this research study I attempted to answer the question: *What is the mathematical knowledge required by teachers of elementary mathematics content courses in the area of multiplication and division of fractions?* Specifically, using a qualitative study of teacher educators teaching mathematics content courses for prospective teachers, I attempted to determine some components of a framework for teacher educator knowledge as it relates to multiplication and division of fractions. Smith (2003) states that "one of the aims of doctoral theses for teacher educators should therefore be to build up a knowledge base for teacher education" (p. 205). This thesis attempts to answer this charge.

Chapter 2—Literature Review

My research question lies at the intersection of three areas of research: research on teacher knowledge, research on teacher educators, and research on the teaching and learning of multiplication and division of fractions. The first and third of these areas have a richly developed research base, while the second area is not as fully developed.

Research on Teacher Knowledge

The question of what teachers need to know in order to be effective has been around since the beginning of the study of teacher education. It has generally been assumed that more knowledgeable teachers lead to better student performance. However, it has not been clear exactly what types of knowledge will produce better student outcomes. Throughout the course of educational studies, researchers have worked to describe the knowledge that teachers need in order to teach, and also how teachers might develop this type of knowledge. This section presents a review of significant frameworks for looking at teacher knowledge, as well as some of the efforts that researchers have made to look at how teachers can develop this knowledge.

Frameworks for Teacher Knowledge

Prior to the 1980's, there were two major types of research studies done in education, both of which were fueled by the behaviorist movement. The first type of research was "process-product" oriented. This type of research can be described as "the large set of studies describing the relationship between teacher behaviors and student achievement" (Hill, Rowan, & Ball, 2005, p. 373). Some of the typical teacher behaviors examined in these studies included things such as having students work in groups and using classroom organizers. Although there were large numbers of these studies done, they were not particularly useful in producing data that would be useful in improving teaching. “Process-product studies of teaching provided precious little insight that was not already available through common sense; even the statistically significant findings that were produced were only true at the aggregate level and could in no way be applied to particular individuals or particular groups of students or teachers” (Donmoyer, 1996, p. 96). Thus, these types of studies, while plentiful, were not particularly useful for the education community at large.

The other type of behaviorist oriented studies had to do with teachers’ “knowledge” characteristics. Begel’s 1979 review of a number of these types of studies examined characteristics thought to determine teachers’ knowledge. He found that, although one might assume that teachers who take more mathematics courses or even courses in mathematics methods would be more knowledgeable, the studies did not show positive main effects for these characteristics in many cases. In fact, Begel found 21 studies with positive main effects for mathematics credits taken beyond calculus and 16 studies that showed negative main effects for more mathematics courses taken by teachers. He concluded that “the effects of a teacher’s subject matter knowledge. . . [were] far less powerful than most of us had realized” (p. 54), and suggested that since there were no promising lines for further research, that studying teacher characteristics was not a fruitful way of improving mathematics education.

In the early to mid-1980’s, this idea changed. With the rise of the cognitive science perspective in educational psychology, new ideas about teachers’ knowledge came into play. Rather than looking at characteristics such as the number of mathematics classes taken, which are not exclusive to teachers, educational researchers began to look at the idea that there was a “knowledge base” unique to the profession of teaching. In the area of cognitive science, knowledge base “refers to the set of rules, definitions, and strategies needed by a computer to perform as an expert would in a given task environment. . . In teaching, the knowledge base is the body of understanding, knowledge, skills, and dispositions that a teacher needs to perform effectively in a given teaching situation” (Wilson, Shulman, & Richert, 1987, pp. 105-106). The idea that there was a knowledge base unique to teaching led to a number of studies to determine what this knowledge base would look like and what kind of knowledge it would include.

Another impetus for studies looking at teacher knowledge was Lee Shulman’s 1985 Presidential address at the American Educational Research Association (AERA) annual meeting. In it, Shulman identified what he called the “missing paradigm” in the research on the study of teaching (Shulman, 1986). This missing paradigm referred to the lack of focus on the knowledge of the content in the process-product research on teaching. Shulman contended that too much emphasis was being given to characteristics having to do with classroom management and not enough emphasis was given to the specific knowledge that teachers had of the content they were teaching. In his address, Shulman described three types of knowledge necessary for teaching, including what he referred to as *pedagogical content knowledge*, which linked knowledge of teaching pedagogy with knowledge of the specific content that was taught.

The three types of knowledge for teaching that Shulman (1986) identified in his address were content knowledge, pedagogical content knowledge, and curricular knowledge. By its name, *content knowledge* refers to knowledge of content. However, Shulman says that it goes beyond this: “The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened and even denied" (p. 9). Thus, teachers must understand the organizing structure of their discipline, how the concepts are related, how truth is established, and so on.

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The third form of knowledge, *curricular knowledge*, involves understanding the curriculum one is teaching. However, Shulman (1986) also included in this type of knowledge, having knowledge of what students are studying in subjects other than the teacher's own content, as well as understanding what has come before and after the particular piece of the curriculum that one is teaching. In other words, one must know what has built up to where the students are at currently, and where the subject is going.

Shulman's ideas on pedagogical content knowledge sparked a huge interest in knowledge for teaching. "Since these ideas were first presented, Shulman's presidential address (1986) and the related *Harvard Education Review* article (1987) have been cited in more than 1,200 refereed journal articles" (Ball, Thames, & Phelps, 2008, p. 392) in a large variety of educational topics. What follows are some of the major studies that grew out of Shulman's work and those that are closely related to mathematics.

Knowledge growth in teaching. In the mid-1980s through the early 1990s, a number of groups of researchers attempted to study the knowledge base for teaching and to develop frameworks for what this knowledge base might look like. One of these groups was the Knowledge Growth in a Profession Project at Stanford University, of which Shulman was a member. Shulman and his colleagues (e.g., Grossman, 1990, 1991; Grossman, Wilson, & Shulman, 1989; Shulman, 1987; Wilson et al., 1987) focused on teachers' knowledge of subjects at the secondary level. Their focus was on fixing the "missing paradigm" (Shulman, 1986) and bringing subject matter knowledge to the forefront of research on teachers' knowledge.

Two of the major areas of focus for the Stanford research group were pedagogical content knowledge and subject matter knowledge for teaching. Shulman (1986, 1987) had a particular interest in pedagogical content knowledge because "it identifies the distinctive bodies of knowledge for teaching" (1987, p. 8). Thus, it was a key component in the professional knowledge base for teachers. It was specialized knowledge that only teachers had or needed, and it was something that gave teaching professional status.

Using Shulman's (1986) work in defining pedagogical content knowledge, Grossman (1990) defined four components of pedagogical content knowledge:

> the knowledge and beliefs about teaching a subject at different grade levels, . . . knowledge of students' understandings, conceptions, and misconceptions of particular topics in a subject matter, . . . knowledge of curriculum materials available for teaching particular subject matter, . . .

> and knowledge of instructional strategies and representations for teaching particular topics. (pp. 8-9)

Understanding these particular areas was important to the researchers because they believed that pedagogical content knowledge was key in teachers' development of *pedagogical reasoning* (Wilson et al., 1987). The deeper a teacher's pedagogical content knowledge, the more adept she would be at making pedagogical decisions that would facilitate learning.

In addition to pedagogical content knowledge, the Knowledge Growth in a Profession group was interested in subject matter knowledge as it was used in teaching. Grossman and her colleagues (Grossman et al., 1989) focused on four areas of subject matter knowledge for teaching: content knowledge for teaching, substantive knowledge, syntactic knowledge, and beliefs about subject matter. The researchers stressed the importance of content knowledge because of its importance in how teachers convey information to students. "Teachers' lack of content knowledge can also affect the style of instruction. In teaching material they are uncertain of, teachers may choose to lecture rather than soliciting student questions, which could lead them into unknown territory" (p. 28). Thus without a thorough understanding of the content, teachers may be unable or unwilling to engage their students in meaningful learning activities.

Substantive and syntactic knowledge play an important role in how and what teachers choose to teach and also in their learning of the subject matter. Substantive structures of a discipline deal with "the frameworks or paradigms that are used both to guide inquiry and to make sense of data" (Grossman et al., 1989, p. 29). Thus substantive knowledge of a discipline determines how teachers are able to facilitate their students' inquiry as well as their own learning about the subject. Syntactic knowledge, on the other hand, deals with knowing how new ideas and knowledge are brought into the field. Thus, both of these types of knowledge play an important role in teachers' development of knowledge and in their teaching it to students.

Another important part of subject matter knowledge according to Grossman and her colleagues (Grossman et al., 1989) includes teachers' beliefs about subject matter. These researchers state that while they did not intend from the onset to look at teachers' beliefs, many of the teachers that they studied treated their beliefs as knowledge. Thus, what they "knew" about a subject was determined by what they believed about it. For example, teachers who believe that mathematics is about getting the right answer and following step-by-step procedures "know" mathematics differently than those who believe mathematics is about the process of discovery and solving new problems.

Going along with beliefs, Grossman (1991) discussed what she calls "orientations" toward a discipline. In relation to English teaching, Grossman equates orientations with interpretive stances. She found that the way that teachers were oriented toward literature, in her case looking at teachers who had a reader-orientation, a text-orientation, and a context-orientation toward texts, greatly affected the ways that the teachers presented the texts to students, their goals for the lesson, and their methods of and content involved in assessment.

Conceptual and procedural knowledge. In terms of mathematics, two of the major "orientations" can be described as conceptual and procedural. While conceptual and procedural knowledge or orientations had been studied for a number of years prior to the 1980s, James Hiebert's edited book entitled *Conceptual and Procedural Knowledge:*

The Case of Mathematics (Hiebert, 1986) continues to be one of the major works describing these types of knowledge and how they relate to mathematics. In describing these two types of knowledge, the authors of the first chapter of the book say that *conceptual knowledge* can be thought of as "a connected web of knowledge" (Hiebert & Lefevre, 1986, p. 3). Conceptual knowledge is achieved when a person develops links between individual pieces of information. These links can be formed by tying together two pieces of information that are already known, or by connecting something already known with a new piece of knowledge. An example of this type of link would be relating rules of logarithms with laws of exponents.

Procedural knowledge is broken into two parts, recognition of proper "forms" and knowledge of rules, algorithms, and procedures (Hiebert & Lefevre, 1986). The recognition of proper forms involves knowing what certain mathematical elements are supposed to look like. For example, if we want to square a variable, x, we write x^2, rather than ${}^2x$. The former makes sense to us; the latter does not.

Once students have learned to recognize the proper form, they now incorporate rules, algorithms, or procedures. These are the steps that are used to solve the problems, or perform exercises. Some examples are the order of operations, using the quadratic formula, and finding a least common denominator before adding or subtracting fractions.

In discussing conceptual and procedural understandings, Hiebert and Leferve (1986) discuss the difference between rote and meaningful learning. They define meaningful learning in terms similar to conceptual knowledge. It comes from finding interrelationships between pieces of information. On the other hand, rote learning "produces knowledge that is notably absent in relationships and is tied closely to the context in which it is learned" (p. 8). The authors contend that when students learn by rote, (e.g., strict memorization), it is very difficult to develop conceptual knowledge and understanding. In rote learning, the emphasis is on memorization of procedures without an attempt to link new knowledge and procedures with information that is already known. The authors point out that while procedural knowledge can be developed from either rote or meaningful learning, it is impossible to generate conceptual knowledge directly from rote learning. That is, meaningful learning can occur when procedures are used, as long as the procedures are performed with conceptual understanding as their foundation.

A major objective of many of the contributors to Hiebert's (1986) book is to discuss the importance of having both conceptual and procedural knowledge. With only one or the other, it is difficult to have meaningful learning. When knowledge is based only on procedures, there can be no real understanding. They claim that mathematics has no meaning or importance for strict procedural knowers.

On the other hand, a strict conceptual knower would not be able to solve problems either. If asked to solve a quadratic equation, a student who did not know the quadratic formula would have difficulty, even if he or she knew exactly what the question was asking. In general, in order to have meaningful learning "procedural knowledge must rest on a conceptual knowledge base; in other words, one of the purposes of conceptual knowledge is to form a support system for procedural knowledge" (Silver, 1986, p. 184). Thus the two types of knowledge work together to provide a meaningful understanding of mathematics and a way in which to solve problems.

Recently the ideas of conceptual and procedural knowledge have again become an important topic of discussion in mathematics education. Jon Star (2005, 2007) has suggested that procedural knowledge has not gotten a fair amount of attention from mathematics education researchers. He claims, "the term *conceptual knowledge* has come to encompass not only what is known (knowledge of concepts) but also one way that concepts can be known (e.g., deeply and with rich connections). Similarly, the term *procedural knowledge* indicates not only what is known (knowledge of procedures) but also one way that procedures (algorithms) can be known (e.g., superficially and without rich connections)" (2005, p. 408). Star suggests that there is such a thing as "deep procedural knowledge," which can be separate from conceptual knowledge. This type of knowledge "is associated with comprehension, flexibility, and critical judgment" (2005, p. 408), and is exhibited in being flexible when choosing a procedure that is efficient for a given mathematical situation. In describing what he means by deep procedural knowledge, Star also calls for research to look deeper into this type of knowledge and how it develops.

In a response to Star's article, Baroody, Feil, and Johnson (2007) agree that there are such things as deep procedural knowledge and superficial conceptual knowledge. However, these authors, like many of those in Hiebert (1986), contend that deep procedural knowledge cannot exist without also having conceptual knowledge. They state that "although (relatively) superficial procedural and conceptual knowledge may exist (relatively) independently, (relatively) deep procedural knowledge cannot exist without (relatively) deep conceptual knowledge or vice versa" (Baroody, Feil, & Johnson, 2007, p. 123). They suggest that these types of deep knowledge come together in a form of adaptive expertise, which allows a person to respond to a variety of situations, either familiar or unfamiliar.

Leinhardt and colleagues. Around the same time that the Knowledge Growth in a Profession project was underway, a group of researchers from the Research and Development Center at the University of Pittsburgh were also looking at a knowledge base for teaching. While not specifically mathematics education researchers, this research did focus on teachers' knowledge of mathematics, and one of the lead researchers was Gaea Leinhardt. Looking at teaching as a "cognitive skill," Leinhardt and her colleagues (Leinhardt, 1989; Leinhardt & Greeno, 1986; Leinhardt, Putnam, Stein, & Baxter, 1991; Leinhardt & Smith, 1985) separated knowledge for teaching into two main areas: *lesson structure knowledge*, and *subject matter knowledge*, both of which build on the other. Briefly, "*lesson structure knowledge* includes the skills needed to plan and run a lesson smoothly, to pass easily from one segment to another, and to explain material clearly" (Leinhardt & Smith, 1985, p. 247), while "*subject matter knowledge* includes concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number system being drawn upon, the understanding of classes of student errors, and curriculum presentation" (Leinhardt & Smith, 1985, p. 247). It should be noted that the last two areas of Leinhardt and Smith's subject matter knowledge fall under what Shulman (1986) called pedagogical content knowledge.

In studying these two types of knowledge for teaching, Leinhardt and her colleagues compared the knowledge bases of "expert" and novice teachers with the idea that the expert teachers had developed the necessary knowledge for teaching, while the novices would be lacking in some areas. (Note that "experts" were determined by the growth in performance scores of their students.) In terms of lesson structure knowledge, the experts demonstrated well organized scripts, agendas, and routines, which they had developed over time. The novice teachers, on the other had, did not organize their time well, spent large amounts of time on tasks that the expert teachers did in minutes, and were unable to meet their goals (Leinhardt et al., 1991).

In terms of subject matter knowledge, the researchers investigated teachers' knowledge of fractions, specifically equivalent fractions (Leinhardt & Smith, 1985), and later subtraction with regrouping and multiplication of whole numbers, (Leinhardt et al., 1991). In general, like with the lesson structure knowledge, the "expert" teachers performed better than the novices at tests of subject matter knowledge, however the researchers were surprised that even some of the experts did not demonstrate a deep understanding of equivalent fractions (Leinhardt & Smith, 1985).

Fennema and Franke (1992) provide some critiques of Leinhardt and her colleagues' model. They claim that Leinhardt focused too much on knowledge of the procedures of developing lesson structure or of performing an algorithm for one particular topic, rather than getting in depth on the topic of subject matter knowledge. They write:

> As is often the case with researchers whose main interest is not mathematics education, the mathematics studied has been limited. The emphasis of the teaching studied has been on the learning of procedures [such as reducing a fraction to lowest terms (Leinhardt & Smith, 1985)], or the structure placed on the observed lessons by the researchers has reflected procedures rather than understanding. (p. 158)

Thus, they call for a more detailed look at frameworks that deal specifically with mathematical knowledge for teaching with a more conceptual basis.

Profound understanding of fundamental mathematics. By the mid-1990s, researchers had worked on and developed a number of frameworks of subject matter knowledge for teaching. However, Liping Ma (1996, 1999) identified a gap in the literature. In describing the impetus for her dissertation research, she writes:

> In spite of inspiring insights on what teachers' subject matter knowledge of mathematics should be, current research fails to provide a concrete vision of such knowledge. For example, what would an elementary teacher's deep understanding **of** mathematics look like? What would a teacher's sufficient knowledge **about** mathematics look like? How would these aspects coexist in teachers' knowledge of specific topics of elementary mathematics? These important questions that lead us to a better understanding of teachers' knowledge are not approached by current research. (1996, p. 12, emphasis in original)

The fact was that many of the studies up to this point (e.g., Ball, 1990a, 1990b; Borko et al., 1992; Graeber, Tirosh, & Glover, 1989) had established teachers' lack of subject matter knowledge of mathematics.

Having worked as an elementary teacher in China and as a researcher with elementary teachers there, Ma had seen evidence of a deep understanding of elementary mathematics in many of these teachers. She decided to investigate the subject matter knowledge of elementary teachers in both the United States and in China. Using questions on subtraction with regrouping, multidigit multiplication, division of fractions, and area and perimeter of a rectangle, Ma interviewed both experienced and novice teachers in the United States and in China. Based on these interviews, she developed a four-tiered hierarchical framework of the teachers' knowledge of elementary, or what she calls fundamental, mathematics.

The first level of Ma's (1996) framework is procedural understanding. She found that the majority of teachers that she interviewed were able to display "algorithmic competency" in doing the mathematics problems. She contends that being fluent with procedures is what many people consider to be "doing mathematics" and states that "at this level, there seems to be no difference between teachers' understandings of mathematics and that of laymen" (p. 218). Thus, while this type of knowledge is important to the elementary teacher, there is more to the knowledge necessary for teaching mathematics well at this level.

The second level of Ma's (1996) framework is conceptual understanding. Like many of the other researchers who discuss conceptual and procedural knowledge, Ma suggests that conceptual understandings are built on a foundation of procedural knowledge, but extend these ideas to be able to give a rationale for what one is doing. This deeper understanding of the procedures was something that Ma found to be a feature unique to teachers' knowledge. She writes, "Laymen usually do not tend to make an explicit explanation about the procedure of solving a math problem even when they are teaching you" (p. 92). However, Ma found that the majority of the teachers tended to provide at least a brief conceptual explanation for what they were doing.

The third level of Ma's (1996) framework deals with what she calls "knowledge packages." She writes, "A 'knowledge package,' in fact, represents a relationship between and among a group of mathematical ideas that specifically connect to the present topic which the teachers were addressing" (p. 226). For example, the knowledge package for subtraction with regrouping contains topics such as adding and subtracting "within ten" (numbers between one and ten), adding and subtracting "within 20," subtracting without regrouping, place value, and composition and decomposition of units within a place value. It is with these knowledge packages that a teacher's conceptual understanding becomes deeper and more rich. They provide a foundational basis for the mathematics that is taught and help the teachers see how the specific topic that they are teaching fits into the general curriculum of elementary mathematics.

The final piece of Ma's (1996) framework is what she calls the "structure of math," and it basically describes what Ma calls a "profound understanding of fundamental mathematics" or PUFM. In the book that she wrote from her dissertation, Ma defines profound understanding as "an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough" (1999, p. 120). Using the four topics she studied, subtraction with regrouping, multidigit multiplication, division of fractions, and area and perimeter in geometry, Ma constructed a "knowledge package" of fundamental mathematics, which had as its foundation the three principles of associativity, commutativity, and distributivity, and built up to the ideas using topics such as place value, the meaning of multiplication, the meaning of addition, and so on. Using this large knowledge package, she described *breadth* of a teacher's subject matter knowledge as being "comprised of his or her knowledge of links between and among those mathematical concepts that have 'similar-status'" (1996, p. 239). The *depth* of one's knowledge was the ability to link ideas to those that were more powerful, such as the three foundational ideas mentioned above. Most important to Ma, was the idea of *thoroughness*, which she describes as "one's capability of 'passing' through all parts of the field" (p. 240). This thoroughness is the "glue" that holds one's mathematical knowledge together and makes it complete.

While Ma's main focus of her investigation was looking at purely subject matter knowledge, she was also able to make some conclusions about pedagogical content knowledge. She writes, "our research reveals that teachers' subject matter knowledge of a topic is one of the major cornerstones on which their pedagogical content knowledge is built—before one can make anything teachable, one first has to understand the topic" (1996, pp. 113-114). Thus, one's ability to teach a topic for understanding depends on pedagogical content knowledge which rests on a foundation of deep, broad, and thorough subject matter knowledge.

As stated earlier, Ma published a book based on her dissertation work in 1999, which created a stir in the world of mathematics education. Prior to this point (as well as after), students in the United States had not been achieving as well as those in some other countries on international tests of mathematical achievement. Ma's book seemed to provide a reason why. First, Ma determined that elementary mathematics was much more complicated than many people had previously thought. Second, the problem seemed to be partially attributed to their teachers' knowledge. Mathematics educators (e.g., Askey, 1999; Howe, 1999) called for people to take action and change the way that teachers, teacher education programs, and the mathematics education community look at how elementary mathematics was taught and learned. Later in this literature review, I will discuss more about Ma's study and her suggestions for how to develop a profound understanding of fundamental mathematics.

The work of Deborah Ball. Unlike many of the other researchers studying knowledge for teaching, Deborah Ball's interest in mathematics knowledge for teaching came from her own need for this knowledge. Working as a fifth grade teacher, Ball came to realize that she did not know enough mathematics to feel comfortable teaching in the way that she wanted to. She writes, "I realized that I needed to learn more mathematics myself, for unlike the other areas in which I was developing my teaching, I was inadequately educated in mathematics" (2000, p. 369). So, Ball decided to go back to school to learn more mathematics and pursue her PhD. in the process. Through her dissertation (1988) and work on the Teacher Education and Learning to Teach Study (TELT) at Michigan State University, Ball explored many facets of mathematical knowledge for teaching.

In Ball's early work, (1988, 1991a, 1991b), she identified two areas, knowledge *of* mathematics and knowledge *about* mathematics, which she determined were important aspects for teachers to know. Knowledge of mathematics, which Ball also called *substantive knowledge*, entailed knowledge of the domain of mathematics, both conceptual and procedural, including the connections among and between them. Ball gave three criteria for determining knowledge of mathematics: correctness, meaning, and connectedness. She writes, "correctness is not the only criterion. Teachers should not just be able to 'do' mathematics: if they are to teach for understanding, they must also have a sense for the *mathematical meanings* underlying the concepts and procedures" (1991b, p. 74). Thus, the knowledge of mathematics entailed similar ideas to Ma's (1996) framework for PUFM and Hiebert's (1986) ideas of conceptual knowledge based on a procedural foundation.

Ball defined knowledge *about* mathematics as “knowledge of the nature and discourse of mathematics . . . [which] entails ideas about what is involved in doing mathematics and how truth or validity is established in the domain” (1991, p. 44). Ball contended that because of the way that many prospective and practicing teachers learned mathematics, that their assumptions about mathematics were that it involved following step-by-step procedures to arrive at an answer, that it was mostly an arbitrary collection of rules, and that little of mathematics related to real, everyday life. While Ball’s knowledge *about* mathematics seems similar to what Grossman (1991) called “orientations,” Ball contends that she sees orientations as broader than her idea of knowledge *about* mathematics.

After leaving Michigan State University and the TELT study, Ball moved on to the University of Michigan where she worked on two large-scale studies, the Mathematics Teaching and Learning to Teach Project (MTLT), and the Learning Mathematics for Teaching Project (LMT). Both of these projects had as a goal to investigate the mathematical knowledge entailed and needed in the work of teaching, with the current (LMT) project beginning to look at how this knowledge is developed through teacher training programs. In these projects, Ball worked with colleagues to look at what was involved in the work of teaching. She videotaped her own teaching of third grade mathematics for an entire year, and along with mathematicians and mathematics educators, analyzed the data to determine what she defined as “the work of teaching mathematics” and the mathematical knowledge that was required in order to do this work (Ball, 1999, 2000; Ball & Bass, 2002).

From this work as well as previous frameworks dealing with teacher knowledge, Ball and her colleagues were able to develop measures to test for the existence of different types of what they called "mathematical knowledge for teaching" (Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004) and also develop a framework to describe this type of knowledge (Ball et al., 2008; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008) .

In developing their framework for mathematical knowledge for teaching, Ball and her colleagues (Ball et al., 2005; Ball et al., 2008; Ball, Thames, & Phelps, 2008; Hill et al., 2008) looked to expand Shulman's (1996) knowledge categories. This framework, (see Figure 2.1) breaks Shulman's categories of subject matter knowledge and pedagogical content knowledge into pieces. Included in subject matter knowledge are two main categories called *Common Content Knowledge (CCK)* and *Specialized Content Knowledge (SCK)*. Pedagogical content knowledge is broken down into *Knowledge of Content and Teaching (KCT)* and *Knowledge of Content and Students (KCS)*. Also included in this framework are "knowledge at the mathematical horizon" and "knowledge of curriculum." Since this framework is so new, neither of these categories have been fully developed by the researchers, but the research group is continuing to work on developing each of their knowledge categories.

Ball and her colleagues define *common content knowledge* as "the mathematical knowledge and skill used in settings other than teaching" (Ball, Thames, & Phelps, 2008, p. 399). Thus it is the mathematical knowledge that everyone must know. Examples of common content knowledge include knowledge of algorithms and procedures for adding and subtracting, finding the area and perimeter of a given shape, or ordering a set of decimals.

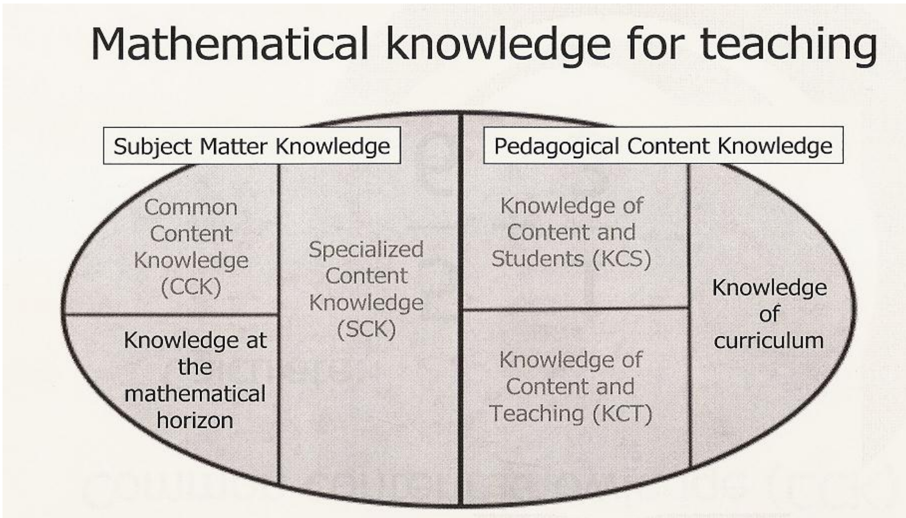


Figure 2.1. Mathematical knowledge for teaching (Ball et al., 2008).

Specialized content knowledge is defined as "the mathematical knowledge and skill uniquely needed by teachers in the conduct of their work" (Ball, Thames, & Phelps, 2008, p. 400). In studying this knowledge, the researchers determined that this type of knowledge is unique to teaching, in that others would not need it in the course of their work. Examples of specialized content knowledge include being able to evaluate student algorithms to determine their validity, explaining why we invert and multiply when we divide fractions, and understanding and being able to correctly use mathematical vocabulary. While these types of knowledge may be found in people other than teachers, the researchers argue that this knowledge is necessary for teachers, but is generally not needed by the typical learner of mathematics.

"The third domain, *knowledge of content and students (KCS)*, is knowledge that combines knowing about students and knowing about mathematics" (Ball, Thames, & Phelps, 2008, p. 401). This type of knowledge includes anticipating student difficulties, understanding students' reasoning, and knowing common errors and misconceptions that students will have with specific material. Ball, Thames, and Phelps (2008) explain how the first three types of knowledge work together in the classroom:

> Recognizing a wrong answer is common content knowledge (CCK), while sizing up the nature of the error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge (SCK). In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students. (p. 401)

The final domain that these researchers have expanded on in detail is *knowledge of content and teaching (KCT)*. This type of knowledge combines knowing about teaching with knowing about mathematics. It involves knowing how to sequence a particular set of topics and understanding the power and value of different mathematical representations. Ball and her colleagues (2008) describe one of the aspects of KCT in some of the roles that the teacher has to play in helping students: "During a classroom discussion, they have to decide when to pause for more clarification, when to use a student's remark to make a mathematical point, and when to ask a new question or pose a new task to further students' learning" (Ball, Thames, & Phelps, 2008, p. 401). These types of teaching tasks require that the teacher have both a deep understanding of the subject of mathematics, as well as understanding how their actions and decisions will affect how and what the students learn.

Ball, Thames, and Phelps (2008) define horizon knowledge as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). The researchers state that they provisionally place this knowledge as part of subject matter knowledge, however they are still looking into where and how it will fit in their framework. Similarly, they provisionally place "knowledge of content and curriculum," which they say corresponds to Shulman's curricular knowledge, as part of pedagogical content knowledge, but state that they also see it as potentially a subset of knowledge of content and teaching, or running across some other categories.

Other knowledge frameworks. While much of the current research on teacher knowledge uses Ball and her colleagues' framework as a starting point, there has been discussion that the "egg" framework (Figure 1.1) is sometimes difficult to use, with researchers having trouble determining how some pieces of knowledge fit into the framework (Thanheiser et al., 2009). Philipp (2008) suggests "scrambling the egg," implying that it does not matter exactly what we call the different types of knowledge, but that we recognize the importance of mathematical knowledge for teaching.

In addition to the "egg" framework, other recent groups have put forth frameworks for looking at mathematical knowledge or proficiency. The National Research Council (NRC) (2001) suggests five strands of mathematical proficiency, which they say captures "what we think it means for anyone to learn mathematics successfully" (p. 5). These five strands are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive dispositions. While much emphasis in the past has been placed on the first two strands, conceptual understanding and procedural fluency (e.g., Hiebert, 1986), the last three strands were newer ideas based on research in mathematics education, cognitive psychology, and the researchers' personal experiences in teaching and learning mathematics (NRC, 2001). *Strategic competence* is defined as the "ability to formulate, represent, and solve mathematical problems" (NRC, 2001, p. 5) while *adaptive reasoning* is the "capacity for logical thought, reflection, explanation, and justification" (NRC, 2001, p. 5). *Productive disposition* fits in with other research that looks at beliefs or orientations towards mathematics (Fennema & Franke, 1992; Grossman, 1990, 1991), and is defined as the "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (NRC, 2001, p. 5). Thus, part of being knowledgeable in mathematics means finding the topic worthwhile and also believing that one is capable of learning and doing mathematics. Since negative beliefs towards mathematics and mathematics anxiety have been found to be common traits in prospective elementary teachers (Barrantes & Blanco, 2006; Uusimaki & Nason, 2004), helping preservice teachers develop a productive disposition toward mathematics may be one of the goals of teacher educators. An important note about the NRC's five strands of mathematical proficiency, is that the researchers look at these strands as "interwoven and interdependent (NRC, 2001, p. 5). They depict their strands as intertwined, and emphasize that each of the strands works together to support one another. Similar to Hiebert's (1986) ideas of conceptual and procedural knowledge, these five strands must all combine for true mathematical proficiency to exist.

Another framework for looking at mathematical knowledge was developed by Davis and Simmt (2006) using complexity theory as a basis for their framework. This framework is composed of four branches: mathematical objects, curriculum structures, collective dynamics, subjective understanding. Their framework shows these four pieces as nested circles with subjective understanding being on the inside and working out to mathematical objects. They describe mathematical objects and curriculum structures as "stable" categories of knowledge, since they are not likely to change often, while collective dynamics (or classroom connectivity, as they also refer to it) and subjective understanding are seen as "dynamic," since these features vary with the students and teachers. While the researchers state that these four categories are "some aspects of teachers' mathematics-for-teaching" (p. 298), they claim that their categories are not exhaustive. Their main idea with their framework seems to be that knowledge for teaching mathematics is different than knowledge that students need, and that the place to see this knowledge in action is in the complex work of teachers.

Synthesizing the Frameworks

While many researchers have determined different frameworks for looking at the knowledge needed for teaching, both mathematical and otherwise, there are indeed many commonalities in all of the frameworks, which makes us able to look at the idea of knowledge for teaching as a whole. There are four areas touched on in the research on teachers' knowledge: the two main categories are subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Peripheral, but also important areas, are teachers' general pedagogical knowledge and also their beliefs and orientations toward the subject. Table 2.1 gives a summary of these knowledge frameworks.

Subject Matter Knowledge
1. General Knowledge of Content (Leinhardt et al., 1991; Leinhardt & Smith, 1985); Common Content Knowledge (Ball et al., 2005; Ball et al., 2007); Knowledge of Procedures (Ma, 1996).
2. Knowledge of Mathematics (Ball, 1991a); Conceptual Knowledge, Knowledge Packages, Profound Understanding of Fundamental Mathematics (Ma, 1996); Specialized Content Knowledge (Ball et al., 2005; Ball et al., 2007).
3. Syntactic and Substantive Knowledge (Grossman et al., 1989).
Pedagogical Content Knowledge
1. Knowledge of students' understandings and misconceptions (Grossman, 1990; Shulman, 1986); Knowledge of Content and Students (Ball et al., 2007; Hill et al., 2008)
2. Knowledge of curriculum materials (Grossman, 1990; Shulman, 1986)
3. Knowledge of instructional strategies (Grossman, 1990; Shulman, 1986); Knowledge of Content and Teachers (Ball et al., 2007)
General Pedagogical Knowledge
1. Lesson Structure Knowledge (Leinhardt, 1989; Leinhardt & Greeno, 1986; Leinhardt et al., 1991; Leinhardt & Smith, 1985)
Beliefs/Orientations Toward the Subject
1. Orientations (Grossman, 1990, 1991)
2. Knowledge about Mathematics (Ball, 1991a; Ball, 1991b)
3. Productive Dispositions Toward Mathematics (NRC, 2001)

Table 2.1. Summary of Knowledge Frameworks

Development of Teacher Knowledge

Once researchers had developed frameworks for organizing teachers' knowledge, a next step was looking at how this knowledge develops in teachers. While there are multiple frameworks for looking at teacher knowledge, there is less research that looks at the development of this knowledge. However, three of the major areas that researchers have looked for this development are: looking at experienced teachers, looking at teacher development for inservice teachers, and looking at training programs for prospective teachers.

Learning From Experienced Teachers

One of Ma's (1996, 1999) goals in interviewing teachers with Profound Understandings of Fundamental Mathematics (PUFM) was to determine how they were able to develop such a deep, broad, and thorough understanding of mathematics. Since evidence of PUFM only showed up in teachers who had a number of years of teaching experience, Ma concluded that this type of deep understanding developed through learning from teaching. An interesting, although not surprising, outcome of Ma's research was that none of the teachers, novice or experienced, in the United States displayed PUFM. Ma's rationale for this is the difference between the two cultures in the way that they conceptualize elementary mathematics: "In America, it seems to be taken for granted that teachers, especially elementary school teachers, **already know** about what they are supposed to teach. But in China, it is taken for granted that teachers, even those who teach in elementary schools, **should continually study** about what they are teaching" (1996, p. 6, emphasis in original). This process of studying mathematics and continually learning was evident in the interviews of teachers who displayed PUFM. Ma found four ways that these teachers with PUFM learned mathematics: studying teaching materials intensively, learning mathematics from colleagues, learning mathematics from students, and learning mathematics by doing it. The Chinese culture for learning encouraged teachers to study their curriculum materials in order to develop knowledge packages for the topic that they were covering, to meet with each other to discuss the curriculum, how to teach certain topics, and different ways of looking at mathematical ideas, to investigate student ideas and ways of looking at problems, and to approach mathematics with an inquisitive mind to search for new and perhaps better ways of solving a problem.

Unfortunately for teachers in the United States, their culture does not encourage this same type of inquiry. Much of this may have to do with teachers' orientations toward the subject. Borko, Eisenhardt and their colleagues (Borko et al., 1992; Eisenhart et al., 1993) found that their preservice student-teachers were not developing the necessary conceptual understandings in order to teach "hard" mathematics to their students, and the primary reason behind this was that they did not see these things as important to learn. The student-teachers' focus on their development was to learn new lessons that they could immediately bring into the classroom, rather than to develop deeper understandings of mathematical ideas. Even after being unable to answer a question posed by a student about why one inverts and multiplies when dividing fractions, the student teacher in Borko and her colleagues' study did not find it necessary to find an explanation of this rule either for herself or for the student (Borko et al., 1992).

As a result of these types of studies, researchers have called for a reconceptualization of the way that mathematics educators look at the mathematics education of teachers. As Ma (1996) writes, "I would argue that teacher education is a critical period for us to break the present 'vicious circle' whereby low quality math education and low quality teacher knowledge of school mathematics are both the cause of each other" (p. 278). Some attempts of various mathematics educators and mathematics education researchers are illustrated in the following discussion.

Inservice Teacher Development

A number of research projects have attempted to work with practicing teachers in teacher development projects in order to deepen their knowledge of mathematics. Duckworth (1987) was one of the early researchers to determine that although elementary mathematics was called "elementary," it was by no means easy. A rationale behind her teacher development project was that "adults in general, and teachers in particular, are rarely given the occasion to explore mathematical ideas, with no particular end-point in view" (p. 43). In her teacher development project, which took place over 15 weeks, elementary teachers were given the opportunity to look at, explore, discuss, and reflect on ideas of place value and division. In looking at and sharing their own work as well as discussing some of their students' ideas, these teachers were able to appreciate some of the complexities involved in these topics. The teachers concluded that by looking at mathematics problems in ways that they have developed themselves, they were better able to understand multiple ways of looking at a mathematical topic, and that if they let their students explore their own ways of doing mathematics, that they would perhaps be better equipped to understand the standard algorithm.

Among the most well known research that has looked at children's thinking about mathematics is Cognitively Guided Instruction (CGI) (Carpenter et al., 1989; Carpenter, Fennema, & Franke, 1996; Fennema, et al., 1993). This program used research on the ways that students solved addition and subtraction problems to design professional development programs for teachers. The idea was that if teachers understood how students came to learn and understand addition and subtraction ideas, they would increase their pedagogical content knowledge (specifically KCS, knowledge of content and students), and thus their students would learn more mathematics. In order to accomplish this goal, during the summers of 1986 and 1987, the researchers held seminars for first grade teachers in order to share with them the cognitive research that had been done on students' thinking about word problems. As stated by the designers of CGI, "the theme that tied together our analysis of students' mathematical thinking is that children intuitively solve word problems by modeling the action and relations described in them" (Carpenter et al., 2000, p. 2). The researchers did not want to give teachers a set curriculum to learn, but rather, they wanted teachers to look cognitively at their students' thinking and understanding of mathematics and use those ideas in guiding their own instruction.

Following the summer sessions, the researchers studied the teachers' classrooms to see if and how they were incorporating the cognitively guided ideas into their teaching. In a study of 40 teachers, 20 of whom attended the CGI seminars and 20 of whom did not, the researchers found that "CGI teachers spent significantly more time on word problems than did control teachers. In contrast, control teachers spent significantly more time on number facts than did CGI teachers" (Carpenter et al., 1989, p. 520). Thus the knowledge of the CGI principles aided teachers in focusing their instruction on problem solving. Another interesting finding in this study was that although the CGI teachers did not focus explicitly on number facts as much as non-CGI teachers, the students in the CGI classes tended to perform better not only on tests of problem solving, but also on recall of basic number facts. The researchers argue that because of the teachers' enhanced knowledge, they were able to accurately assess their students' knowledge using their cognitive skills, and they were aware of how the students were developing an understanding of number facts along with problem solving skills. Thus, one way of developing an aspect of the mathematical knowledge for teaching is to expose teachers to ways in which students solve problems, thus enhancing their knowledge of content and students.

One last set of professional development programs for inservice teachers is being developed at the University of Michigan as part of the Mathematics Teaching and Learning to Teach Project (MTLT). These researchers are working to develop tasks to challenge teachers' mathematical misunderstandings and increase their specialized content of mathematics. The key features of their task design include tasks that:

- Unpack, make explicit, and develop a flexible understanding of mathematical ideas that are central to the school curriculum
- Open opportunities to build connections among mathematical ideas
- Provoke a stumble due to a superficial "understanding" of an idea
- Lend themselves to alternative/multiple representations and solution methods
- Provide opportunities to engage in mathematical practices central to teaching (explaining, representing, using mathematical language, analyzing equivalences, proving, proof analysis, posing questions, writing on the board. (Ball et al., 2008, slide 18)

While these researchers are currently designing and piloting tasks to help develop teachers' mathematical knowledge, in the coming years, this research can provide us with a deeper understanding of how to help teachers develop the knowledge necessary for teaching.

Prospective Teachers' Knowledge Development

A major area of focus on the development of the mathematical knowledge for teaching looks at prospective teachers. The idea behind these studies is to help teachers develop the necessary knowledge for teaching while they are still in school. A number of studies (e.g., Ambrose, 2004; Gill, Ashton, & Algina, 2004; Szydlik, Szydlik, & Benson, 2003; Wilkins & Brand, 2004) have looked at ways of changing teachers' beliefs or conceptions about mathematics. These researchers are interested in increasing teachers' knowledge about the nature of mathematics. The idea is that if teachers have more conceptual orientations towards the teaching and learning of mathematics, they will be better equipped to teach mathematics in meaningful ways.

Other studies have looked at ways of increasing teachers' knowledge of mathematical topics. These studies have predominantly focused on one particular content area, and used a preservice mathematics (usually methods) course in order to do so. Chapman (2007) used arithmetic word problems to help prospective teachers understand multiple ways of looking at addition, subtraction, multiplication, and division. By having the prospective teachers focus on the different types of word problems, they were able to develop deeper understandings of arithmetic operations and their different meanings.

Huinker, Hedges, and Steinmeyer (2005) use the "core task" of asking teachers to generate strategies different from the traditional algorithm for solving division problems in order to "unpack" prospective teachers' knowledge of division. They contend that "prospective teachers enter [their] courses with compressed knowledge of division" (p.

478). That is, they are able to perform the procedures associated with division but have little conceptual understanding. By having teachers generate different ways of solving division problems and then discussing the various methods proposed by the class, the authors help the teachers to "unpack" their knowledge of division, building from what they already know. "The purpose of a core task is to reveal teachers' current knowledge and understandings and to provide a context for grounding discussion over several class sessions" (p. 478). Through these discussions, the researchers expose prospective teachers to multiple ways of looking at division and help them develop a knowledge package for looking at division which includes division as the inverse of multiplication, relationships among addition, subtraction, multiplication, and division, interpretations of division, flexible decomposition of numbers, and so on.

Lo, Grant, and Flowers (2008) discuss their efforts to help preservice teachers deepen their understandings of multiplication through the use of justifying strategies for alternative ways of multiplying. Similar to the previous study with division, prospective teachers are asked to generate strategies for multiplication that are different from the traditional algorithm. They are also provided with strategies used by other students and asked to justify why the strategy is correct or incorrect. They write,

> Based on our experience working with these students in and outside of classrooms, the inability to explain one's thinking is frequently tied to an incomplete understanding of the problem at hand. Thus, we hypothesize that prospective elementary teachers' understanding of multiplication enhances their ability to justify their own thinking, and that their

> understanding of how to justify their thinking also enhances their understanding of multiplication. (p. 19)

Thus, the authors contend that having teachers justify why they used a particular multiplication algorithm will help them in their ability to explain to children how and why the algorithms work.

A final study that also looked at helping teachers develop deeper understandings of multiplication and area was conducted by Simon and Blume (1994). The authors say that in their study, they "studied teacher candidates' development from very traditional views of learning and teaching and narrow views of mathematics toward views of mathematics, learning, and teaching embodied in recent reform documents" (p. 473). During the course of several class meetings, Simon, as the teacher of a class of prospective teachers, posed problems for students dealing with different ways of thinking about multiplication ideas. By looking at and analyzing the students' thinking and their solutions strategies, Simon was able to design other instructional tasks to help the prospective teachers to develop deeper understandings of multiplication and its relationship to the area of a rectangle.

Summary

As evidenced above, there has been much research done on looking at the knowledge necessary for teaching, specifically mathematics. However, much research still remains to be done. While researchers have begun to look at ways of enhancing preservice and practicing teachers' knowledge, little work has been done to look at how teachers develop this knowledge. Similar to work that the CGI researchers did in looking at how children develop understandings of addition and subtraction, research should be done on the ways that adults develop deeper understandings of these topics. Since they already are familiar with the arithmetic algorithms and formal rules for solving these types of problems, how do adults develop deep, connected, conceptual understandings of elementary mathematics?

Another area in need of study involves the knowledge demands of teacher educators. In looking at the different professional development programs for prospective and practicing teachers, the question arises of what knowledge demands do these types of activities put on teacher educators in terms of helping prospective and practicing teachers to develop this mathematical knowledge for teaching. What is the knowledge base for teacher educators? It can be understood that the knowledge base for the mathematical knowledge for teaching can be a starting point, but research has shown that teachers need to know more than their students. What does this "more" entail? These questions and many others arise by looking at research on the mathematical knowledge for teaching.

Research on Teacher Educators

There is not a large body of research on teacher educators, and this fact is echoed in the research literature, especially the lack of research on the process of becoming a teacher educator (Abell et al., 2009; Chapman, 2008; Cochran-Smith, 2003; Ducharme & Ducharme, 1996; Smith, 2003; Sztajn, Ball, & McMahon, 2006, Tzur, 2001; Van Zoest, Moore, & Stockero, 2006; Zaslavsky & Leikin, 2004). The research that is available is varied in its scope, with much of the focus on characteristics of teacher educators, such as what they teach or their demographics, rather than their knowledge or what they need to know (Cochran-Smith, 2003). However, there is a growing consensus that educating (particularly novice) teacher educators is an important step in improving teacher education and thus improving teaching (Abell et al., 2009; Cochran-Smith, 2003; Murray & Male, 2005; Rider & Lynch-Davis, 2006; Smith, 2003, 2005; Van Zoest, Moore, & Stockero, 2006, Wilson & Ball, 1996).

Characteristics of Teacher Educators

While my study deals mainly with teacher educator *knowledge*, it helps frame the question to look at who teacher educators are and what their jobs entail. In her 2005 study of teacher educators, Smith asked both experienced and novice teacher educators what were the characteristics of good teacher educators. Her respondents gave eight general functions that teacher educators fulfill: facilitator of the learning process of the student teacher, encourager of reflective skills, developer of new curricula, gatekeeper, researcher, stimulator of professional development for school teachers, team-member, and collaborator (with external contacts) (p. 178). Smith also adds that an additional function of teacher educators "is to create new knowledge in and about teaching. . . Teacher educators create new knowledge of two types: practical, in the form of new curricula for teacher education and for schools; and theoretical knowledge generated from research" (p. 178). Thus the job of the teacher educator is to enhance the field of education by designing and implementing curricula for teacher education, working with teachers and others in the field of education to develop new knowledge, and contributing to the educational knowledge base.

Oesterle and Liljedhal (2009) conducted a qualitative study of two different instructors of mathematics content courses for prospective teachers. The purpose of the study was to get a better understanding of the people who teach these types of courses. They found that their two teacher educators, while teaching similar courses on the surface, had very different beliefs about their students and the teaching of the course. Both teacher educators had no formal training in education courses, but each had taught the course for prospective teachers multiple times (six for Harriet and nine for Bob.) However, while Harriet seemed to put a lot of time into thinking about what it meant for her to teach prospective teachers, Bob did not appear to do so. "Bob needed to be pressed by the interviewer to consider what aspects of the course content might be particularly relevant to prospective teachers as opposed to general learners of mathematics" (p. 1257). Bob seemed to think that it was most important that his students master the subject content of mathematics. Bob believed that content knowledge, along with general pedagogical skills, which his students would acquire outside of his course, were sufficient for his students to become teachers of mathematics. The researchers claim that Bob's views on this subject are common to other mathematics teacher educators.

Harriet, on the other hand, had a different view on teaching prospective teachers. "For Harriet, access to a variety of representations and approaches is mathematics content that is particularly relevant for her students as prospective teachers" (Oesterle & Liljedhal, 2009, p. 1256). She also spent time in class trying to help her students develop positive beliefs about mathematics and teaching mathematics.

While both Bob and Harriet present contrasting ideologies about teaching mathematics to prospective teachers, they both could be considered typical instructors of mathematics content courses for prospective teachers. The researchers conclude their paper by suggesting further research be done on teacher educators, specifically in the actual practice of teaching (their study was done only through interviews), in order to gain a better idea into who is teaching mathematics content courses for prospective teachers.

Mathematics Content Courses for Elementary Teachers

Much of the literature on teacher educators in general focuses on teacher educators located in departments or schools of education (e.g., Ducharme & Ducharme, 1996; Smith, 2003, 2005). However, with the case of instructors of mathematics content courses for prospective elementary teachers, this is not always the case. McCrory and Cannata (2007), as part of the Mathematical Education of Elementary Teachers (ME.ET) project, investigated 70 institutions in four states that offered mathematics content courses for elementary teachers. They found that 80% of the courses were taught by faculty housed in mathematics departments. Masingila, Olanoff, and Kwaka (2011) conducted a nationwide study of teachers of mathematics content courses for elementary teachers and 88.3% (n = 825) of their respondents indicated that these courses were taught in mathematics departments at their colleges or universities. While both studies had a large number of institutions report that instructors in mathematics departments collaborated with departments of education, the fact that so many of these courses are housed in mathematics departments, and thus taught by mathematics faculty, makes teacher educators of mathematics content courses a different group than teacher educators in general, who predominantly work in schools or departments of education. Researchers (e.g., Bass, 2005; Hodgson, 2001) have argued that elementary mathematics content courses may be difficult to teach for mathematicians who may not have thought about what is entailed in elementary mathematics and how best to help their students come to understand this material in deep and connected ways.

The main goal of the ME.ET project was to gather information about mathematics content courses for elementary teachers—what content is contained in them, who teaches them, and the impact they have on prospective teachers' knowledge (McCrory, Zhang, Francis, & Young, 2009). Results indicated that although most institutions have these types of courses, the requirements for prospective teachers, as well as the content of the courses varies considerably. As a result of this, as Oesterle and Liljedhal (2009) point out, these courses are significantly influenced by the people who teach them.

The ME.ET study investigated factors that influenced the knowledge development of prospective elementary teachers in mathematics content courses. They collected data from 41 instructors and 1706 students at 17 institutions, giving students pre- and post-tests based on the LMT (Learning Mathematics for Teaching) measures, as well as mathematical belief surveys. They surveyed the teacher educators about their class content, teaching methods, personal demographics, and textbook usage. The researchers found "no significance for student SES, instructor experience, instructor attitude toward the class, or class size. Significant predictors [of increase in student knowledge scores from pre- to post-test] include student CACT [SAT and/or ACT score], student attitude toward mathematics, textbook used, and method of instruction" (McCrory et al., 2009, p. 1184). The conclusions that they draw from these studies are that there are some significant ways to increase the knowledge gained by prospective teachers in elementary mathematics content courses. Particularly, working to improve students' attitudes about mathematics, using a textbook written specifically for content courses for prospective teachers, and giving the students opportunities to engage with the mathematics rather than predominantly lecturing, should all help prospective teachers gain the mathematical knowledge that they need to be effective teachers.

Looking at Teacher Educator Knowledge

While there is not a plethora of research on teacher educator knowledge, some researchers have developed frameworks and ideas about looking at what teacher educators need to know. A number of researchers have come to the conclusion that teacher educator knowledge is somehow qualitatively different than teacher knowledge (Ball, 2008; Jaworski, 2008; Perks & Prestage, 2008; Rider & Lynch-Davis, 2006; Smith, 2003; Zopf, 2010). “The recognition that there is a specialized body of knowledge and experiences that help the mathematics educator prepare preservice and inservice teachers is of great importance” (Rider & Lynch-Davis, 2006). Thus, while they may not explicate exactly what this knowledge looks like, researchers have suggested its existence, and the need for its development.

“A major difference in the professional knowledge of teachers and that of teacher educators is found in the skill of teaching different audiences, children and adults” (Smith, 2003, p. 202). While many teachers are faced with the task of introducing children to new information, teacher educators are often in the position of teaching prospective teachers information that they “already know,” or at least think they already know. As Wilson and Ball (1996) point out, “adult teachers are not children, nor should they be taught in the same ways” (p. 132). Thus, teaching teachers and prospective teachers provides different challenges for teacher educators than teaching children does. In addition, Lloyd (2006), as well as many others, found that “many preservice teachers possess weak knowledge and narrow views of mathematics and mathematics pedagogy.”

(p. 12), even if they think that they are well-prepared to teach mathematics content. Because of this, "teacher educators are faced with the task of creating opportunities for preservice teachers to develop useful, dynamic conceptions of mathematics and pedagogy" (p. 12). This task is different than teaching children mathematics, and it presents its own unique challenges.

In interviews with teacher educators about what makes a good teacher educator Smith (2003) found that "teacher educators (83.3%) believe there is much commonality in the professional expertise of teachers and teacher educators. According to the teacher educators, the main differences lie in the comprehensiveness and depth of knowledge" (p. 189). Thus, teacher educators believe that they need to know something more than teachers do. In a follow-up study, Smith (2005) asserts that "teacher educators' professional knowledge is expected to be comprehensive, rich, and deep, based on theory and testing theories in practice" (p. 190). Thus, teacher educators must study how students *and* adults (prospective teachers) learn content, and use their research to inform their practice.

A few researchers have developed frameworks for looking at teacher educator knowledge that builds on frameworks for teacher knowledge. Using Jaworski's (1992, 1994) "teaching triad," Zaslavsky and Leikin (2004) extend this idea to knowledge for teacher educators. They include Jaworski's original teaching triad as one piece of the triad for teacher educators, labeling it "challenging content for mathematics teachers" (p. 8), and add "management of mathematics teachers' learning" and "sensitivity to mathematics teachers" as the other two branches (see Figure 2.2 below.)

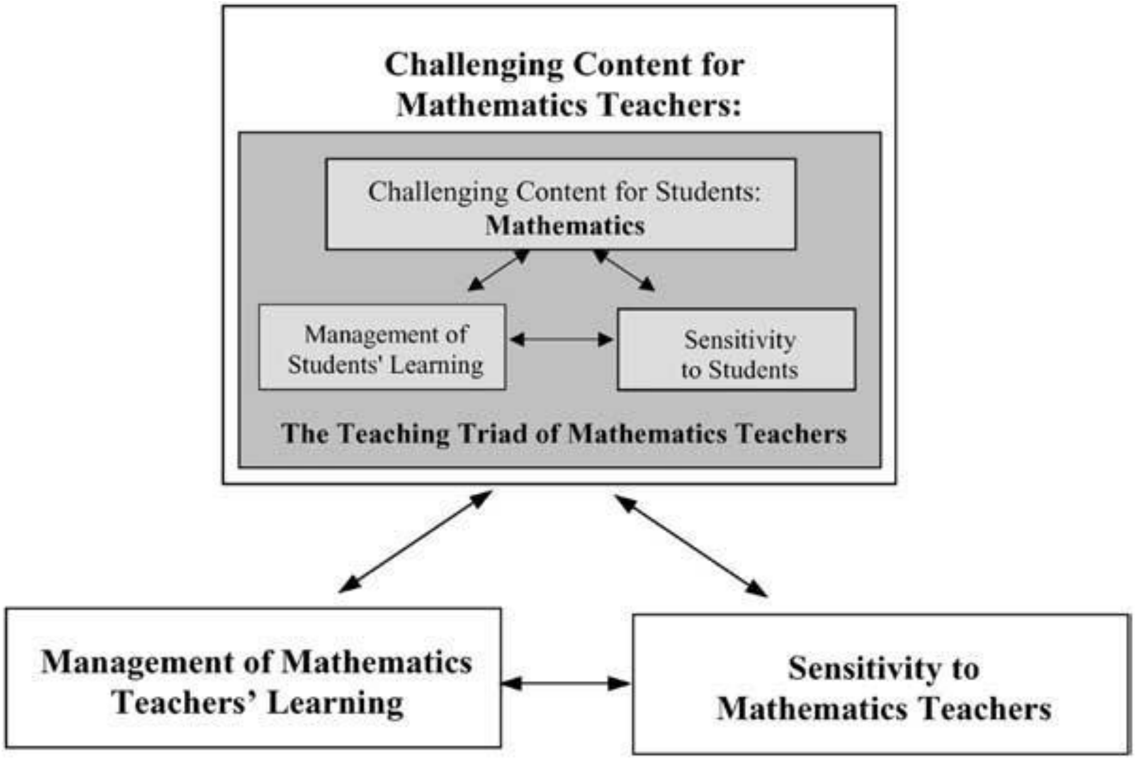


Figure 2.2. Zaslavsky and Leikin's (2004) Teacher Educator Teaching Triad

Thus, the framework for teacher educator knowledge is merely an extension of the framework for teacher knowledge, with the "students" being replaced by "mathematics teachers." Teacher educators need to be knowledgeable of everything that teachers need to know in relation to students, but also must have knowledge of teachers' learning and concerns.

Perks and Prestage (2008) give a similar model of teacher knowledge being part of the "teacher-educator-knowledge-tetrahedron" (see Figure 2.3.). Since the learners of teacher educators are teachers, then the "learner knowledge" portion of the knowledge of mathematics teacher educators is the entirety of the "teacher-knowledge tetrahedron."

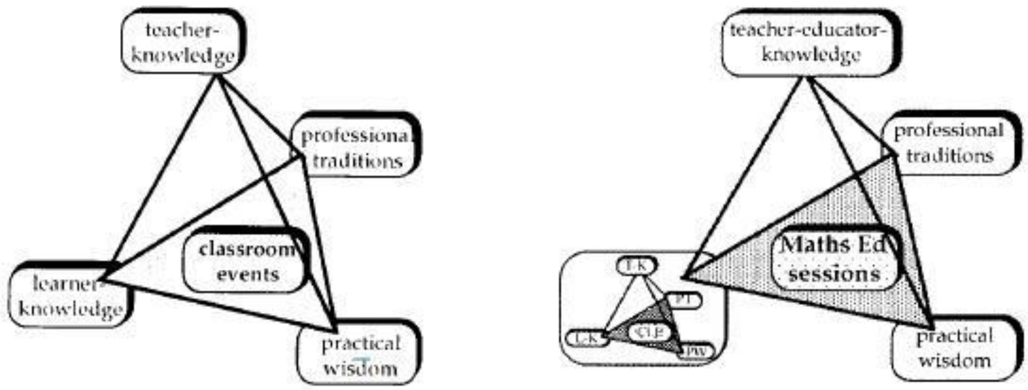


Figure 2.3. Perks and Prestage (2008) include the "Teacher Knowledge Tetrahedron" (left) as the "Learner Knoweldge" portion of their "Teacher-Educator Knowledge Tetrahedron" (p. 270-271).

In a discussion with Deborah Ball (2008), she articulated a similar idea to how her framework for teacher knowledge fits into the knowledge for teacher educators, calling her "egg framework" (Ball, Thames, & Phelps, 2008) "common content knowledge" for teacher educators, and stating that there is extended knowledge that teacher educators must have beyond that of teachers (D.L. Ball, personal communication, July 25, 2008).

Zopf (2010) studied the mathematical work of two mathematics teacher educators, one teaching a mathematics methods course for prospective elementary teachers and one running a month-long professional development session for practicing teachers, in order to investigate the mathematical knowledge needed by teacher educators. She discussed three main differences between the work of mathematics teachers and the work of mathematics teacher educators: "First the mathematical content is different" (p. 5). While the job of teachers is to teach students mathematics, the job of the teacher educator is to teach mathematical knowledge for teaching to teachers. "Second, the learners are different" (p. 5). Zopf points out the difference between teaching children and adults. Each group brings different experiences and prior knowledge to the classroom, and it is the job of their teacher to work with these experiences and help their students to construct new knowledge. “Third, the purposes of instruction are different. Children learn mathematics for their own use; teachers learn mathematical knowledge for teaching to teach mathematics to students” (p. 6). Helping teachers to unpack mathematics in a way that will help them make sense of it to present it to students requires different work than helping students make sense of mathematics. Using these three assumptions, Zopf investigated the work of the mathematics teacher educators. She concluded that “Mathematical knowledge (knowledge of and about mathematics) and mathematical knowledge for teaching are nested within MKTT [mathematical knowledge for teaching teachers]. In addition, MKTT includes mathematical knowledge of these that is more developed, more fundamental, and focused on the teaching of mathematical knowledge for teaching”(p. 192). Thus in Zopf’s mind, mathematical knowledge for teaching teachers encompasses all of mathematical knowledge for teaching, as well as something more.

Jaworski (2008) uses a Venn Diagram to model teacher knowledge and teacher educator knowledge (see figure 2.4). Her model differs slightly from those which include all of teacher knowledge in teacher educator knowledge, in that she allows for separate professional knowledge to be held by each group, as well as a shared knowledge.

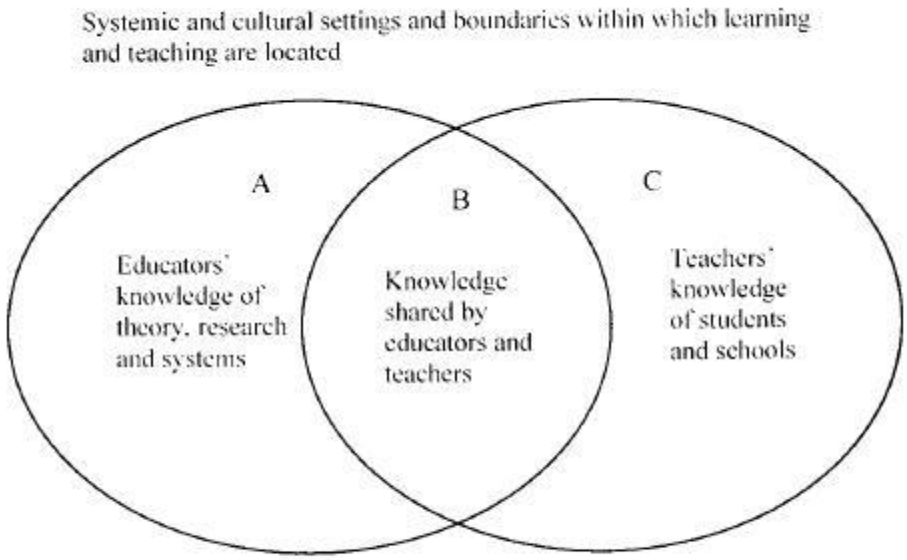


Figure 2.4. Jaworski's (2008) Venn Diagram modeling Knowledge in Teacher Education (p. 336).

In their work in science education, Abell and her colleagues (2009) suggest that there is a special type of knowledge needed by science teacher educators, which builds on pedagogical content knowledge for science teaching. "We contend that a parallel form of PCK exists for science teacher educators. In this case, the subject matter knowledge that a science teacher educator needs includes both science content and knowledge for teaching science" (Abell et al., 2009, p. 79). Included in this PCK are different orientations toward science learning, knowledge of how preservice teachers learn, knowledge of curricula for science methods courses, knowledge of instructional strategies for teaching prospective teachers, and knowledge of assessing prospective teachers. These researchers suggest adding a standard to the *Professional Knowledge Standards for Science Teacher Educators* (Lederman et al., 1997) that includes knowledge for teaching preservice teachers. This article, written by the editors of the *Journal of Science Teacher Education*, includes six standards that they say "should provide a clearly defined framework for the knowledge, skills, experiences, attitudes, and habits of mind essential for the successful science teacher educator" (p. 233). These standards include knowledge of science, science pedagogy, curriculum, instruction, and assessment, knowledge of learning and cognition, research/scholarly activity, and professional development activities. Abell and her colleagues argue that this document is incomplete, and that teacher educator knowledge must contain knowledge for teaching preservice teachers, something that is different than the knowledge outlined in the standards.

Teacher Educator Development

While researchers have suggested that there is special knowledge unique to teacher educators, they have also indicated that there is often little opportunity for novice teacher educators to acquire this knowledge (Abell et al., 2009; Chauvot, 2009; Cochran-Smith, 2003; Smith, 2005; Tzur, 2001; Zaslavsky & Leikin, 2004; Zeichner, 2004). However, researchers point out that teacher educator growth and professional development is a career-long objective (Cochran-Smith, 2003; Zaslavsky, 2008). Smith (2003) gives three main reasons in favor of teacher educator professional development: improving the profession of teacher education, maintaining interest in the profession, and professional advancement. In the next section, I discuss ways in which teacher educators can grow professionally.

Researchers have discussed various ways in which teacher educators can grow professionally. Zaslavsky and Leikin (2004) suggest that mathematics teacher educators (MTEs) develop knowledge in two ways: "*through learning*, as facilitated by a MTEE [a mathematics teacher educator educator], or *through teaching*, when they facilitate MTs learning" (Zaslavsky & Leikin, 2004, p. 9). One of the researchers' major assumptions was that the learning process for teacher educators through their teaching (of teachers) paralleled the ways in which teachers learn from their teaching. A significant feature of the teacher educators' learning has to do with the learning community that the researchers established between mathematics teacher educators (MTEs) and mathematics teacher educator educators (MTEEs). In this community of practice, the MTEs learned through interactions with various members of the community. The different types of interactions included MTE-MTE interactions, where MTEs met with each other to discuss coordinating workshops for mathematics teachers, MTs, and also where more experienced MTEs met with novice MTEs to help induct them into the community as MTEs. Other types of interactions occurred between MTEs and MTEEs, with the MTEEs helping the MTEs to articulate their goals for workshops, design activities, and reflect on their teaching. The MTEs also learned from the MTs (mathematics teachers) whom they were educating, since the MTs came up with solutions to problems that the MTEs had not expected. Finally, the MTEs learned through MTEE-MTE-MT interactions, where the MTEE facilitated a discussion with the MTs, and later on the MTEs facilitated a similar discussion with another group of MTs. The researchers note that their analysis conveyed "the iterative nature of the growth through practice of the different members of the community of mathematics educators and through their different interactions, continuously switching roles from learners to facilitators" (Zaslavsky & Leikin, 2004, p. 28). Not only were the MTEs in charge of facilitating MT learning, but also they were learning and developing their practice themselves. Participation in the community of practice was an integral part of this development.

Another important part of teacher educator development discussed by multiple researchers is the idea of reflection (Abell et al., 2009; Chauvot, 2009; Cochran-Smith, 2003; Garcia, Sanchez, & Escudero, 2007; Jaworski, 2008; Perks & Prestage, 2008; Tzur, 2001; Van Zoest, Moore, & Stockero, 2006; Zaslavsky, 2008; Zaslavsky & Leikin, 2004). Tzur states, “novice teacher educators have to become reflective practitioners. In particular, they need to learn to independently pay attention to their own ways of thinking and to be able to distinguish between their mathematics and their students’ mathematics” (p. 279). In ways similar to how teachers are encouraged to reflect on their own teaching practice (Hiebert et al., 2007; Schon, 1983), teacher educators must also develop ways of reflecting on their teaching practice. Tzur (2001) identifies his own personal process of developing from a student to a teacher of mathematics, to a teacher of mathematics teachers and ultimately to a teacher of mathematics teacher educators, and states that this can be used as a framework for how teachers can become teacher educators through reflection. At each level, teachers “may become aware of the perspectives that underlie their teacher education practice” (p. 273) by thinking about what it means first to be a learner of mathematics, then to be a teacher of mathematics, and then to educate teachers of mathematics. “Development entails a conceptual leap that results from making one’s and others’ activities and ways of thinking at a lower level the explicit focus of reflection” (Tzur, 2001, p. 275). Thus, just because one is a good teacher does not mean he or she will be a good teacher of teachers. The focus must involve moving up a step to think about and reflect on the learning and mathematical development at the lower levels.

Cochran-Smith (2003) also suggests that being reflective is an integral part in the development of teacher educators. She discusses “inquiry as stance,” as a way of looking at one’s own teaching through a critical lens and investigating and comparing it to the work and theory of others. Garcia, Sanchez, and Escudero (2007) posit a similar approach, stating "we have learnt that our practice can be a context of study and research that reveals other problems requiring additional research" (Garcia, Sanchez, & Escudero, 2007, p. 13). Thus growth in the field of teacher educator knowledge must make use of existing theories as well as the study of one's own practice.

Summary

While the study of teacher educators is relatively new, those who are in the field contend that it is essential that studying teacher educators is extremely important to the field of education as a whole. Smith (2003) states, "Professional development for teacher educators is too important not only to teacher education, but also to the educational system as a whole, to be left in a virginal state regarding research and documentation" (pp. 213-214), and others (e.g., Abell et al., 2009; Cochran-Smith, 2003; Smith, 2005) discuss the need to educate doctoral students in the process of becoming teacher educators.

However, while the need to educate teacher educators is there, there are still many questions that remain to be answered. Some of these questions are: (a) what types of professional knowledge do teacher educators need, and how are these different from the knowledge needs of teachers (Murray & Male, 2005), and (b) how should courses or programs for novice or prospective teacher educators be designed? What should be included? How much of the information should be practical versus theoretical (Smith, 2005)? In addition, researchers contend that we must investigate current knowledge and beliefs of teacher educators and look for holes in their knowledge to identify areas for growth, as well as developing frameworks for the types of content knowledge, pedagogical knowledge, and so on, that they must have to be effective teacher educators

(Sztajn, Ball, & McMahon, 2006). I attempt to begin to answer some of these questions through this study.

Research on Multiplication and Division of Fractions

While the literature in the preceding pages mainly discussed mathematical knowledge for teaching in general, my study focused specifically on knowledge of multiplication and division of fractions, so the final portion of this literature review will look at some of the research dealing with fraction knowledge. In order to look at literature dealing with multiplication and division of fractions, we must begin by exploring the fraction literature in general. The study of fractions, ratios, and proportions was called by Lamon (2007) “the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites” (p. 629). This section will look at literature dealing with these topics to attempt to describe what makes fractions, particularly multiplication and division of fractions, so difficult to learn and teach.

Rational Number Studies

The study of fractions is part of two larger areas of research, rational numbers and multiplicative structures, so I will look at the history of the study of these fields. One of the earliest and most cited works discussing rational numbers was Kieren’s (1976) essay entitled, “On the Mathematical, Cognitive, and Instructional Foundations of Rational Numbers.” In this paper, Kieren claims that in order to have a thorough understanding of rational number, one must understand and have experience with their many "interpretations," which he later renames "subconstructs" (Kieren, 1993). Kieren (1976) originally lists seven "interpretations," of rational number:

1. Rational numbers are fractions which can be compared, added, subtracted, etc.
2. Rational numbers are decimal fractions which form a natural extension (via our numeration system) to the whole numbers.
3. Rational numbers are equivalence classes of fractions. Thus {1/2, 2/4, 3/6, . . .} and {2/3, 4/6, 6/9, . . .} are rational numbers.
4. Rational numbers are numbers of the form p/q, where p, q, are integers and q $\neq$ 0. In this form, rational numbers are "ratio" numbers.
5. Rational numbers are multiplicative operators (e.g., stretchers, shrinkers, etc.).
6. Rational numbers are elements of an infinite ordered quotient field. They are numbers of the form x = p/q where x satisfies the equation qx = p.
7. Rational numbers are measures or points on a number line. (pp. 102-103)

In his later writing, he combines some of these interpretations into four general subconstructs: rational number as quotient, measure, operator, and ratio (Kieren, 1993). Other researchers have developed different ways of looking at rational numbers, but Ball (1993) summarizes that researchers have tended to agree that rational numbers "may be interpreted (a) in part-whole terms, where the whole unit may vary; (b) as a number on the number line; (c) as an operator (or scalar) that can shrink or stretch another quantity; (d) as a quotient of two integers; (e) as a rate; and (f) as a ratio" (p. 168), and that in order to have a deep understanding of rational number, students and teachers must be familiar with all of these representations, rather than merely the part-whole area models which are the ones most commonly associated with fractions and most commonly taught in schools.

While Ball (1993) and Kieren (1993) both discussed the idea of subconstructs of fractions in 1993, Lamon (2007) contends that emphasis on only the part-whole model of fractions is still a problem today. She calls it a "crisis" that "teachers are not prepared to teach content other than part-whole fractions" (p. 632), and contends that if students are going to develop deep understandings of fractions, they must develop this understanding using one of the other representations and subconstructs of rational numbers. She has referenced longitudinal studies where students were introduced to fractions using one of the subconstructs of fractions other than the part-whole model. Through studying fractions using one particular subconstruct other than part-whole, students were better able to develop “rational number sense” by studying ideas of a unit and comparing fractions, “so they can judge the relative size of fractional numbers” (p. 659). Lamon also contends that students need time to study fraction interpretations without being given rules, so that they can build on their own knowledge of partitioning and fractional reasoning, without being influenced by rules and procedures.

The studies that Lamon (2007) discusses have important implications for teacher education. She and others argue for the importance for teachers to understand the multiple subconstructs of the rational numbers. This means that teacher educators must also understand these subconstructs, as well as how students come to learn them. In addition, teachers must have the pedagogical content knowledge to know where problem areas for students in learning rational number ideas occur. Teacher educators must bring this process one step further. Not only do they need to know areas where students struggle with these ideas, they must also understand the struggles that prospective teachers have—these struggles may or may not be the same as those for students, however we can assume that helping prospective teachers who have spent years learning procedural rules for fractions see the need to understand the ideas conceptually is a struggle unique to mathematics teacher educators.

Another research area that has looked at fractions, particularly their multiplication and division, deals with literature on multiplicative structures. Vergnaud (1988) includes rational numbers as part of what he calls the *multiplicative conceptual field*, which he says "consists of all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide. . . Among these concepts are linear and *n*-linear functions, vector spaces, dimensional analysis, fraction, ratio, rate, rational number, and multiplication and division" (p. 141). The basis of a conceptual field is that it contains a set of situations that are modeled by a similar action. Movement from the additive conceptual field to the field of multiplicative structures has been shown to be difficult for students and teachers (Fischbein, Deri, Nello, & Marino, 1985; Tirosh & Graeber, 1989). The explicit nature of these difficulties will be discussed later in this chapter.

Greer (1992) describes two forms of rational numbers, fractions and decimals as "extension[s] of multiplication and division. . . considered for (a) numbers that can be traced back to counting procedures but that go outside the natural numbers through application of division at some point, and (b) numbers as measures" (p. 277). He says "numbers of the former sort are represented in the form a/b where a and b are integers" (p. 277), and calls them fractions. "Numbers of the latter sort are generally represented using decimal notation" (p. 277), which he refers to as decimals. Thus fractions and decimals are part of the multiplicative conceptual field, and are formed by some process of multiplication or division being applied to the natural numbers.

Division and Multiplication

Division of fractions has been one of the most difficult concepts in elementary mathematics for students, prospective teachers, and teachers alike (Bulgar, 2003; Flores, 2002; Sowder, Phillip, Armstrong, Schappelle, 1998). Part of the reason for this is because it lies at the intersection of two difficult concepts, division and fractions, neither of which many people are given opportunities to learn conceptually (Sowder et al., 1998). This section will discuss some of the aspects of division that make it so conceptually difficult.

Division is typically taught using two different interpretations. The *partitive* or *sharing* model involves dividing the total by the number of groups to find the number in each group (Greer, 1992). In this model, we can think of the problem $20 \div 5$ as sharing 20 things, one at a time, among 5 people, and determining the number of things that each person gets. The other model of division, called *quotitive*, *measurement*, or *repeated subtraction* division, involves dividing the total number of things by the number in each group to find the number of groups (Greer, 1992). In this model, we can think of $20 \div 5$ as handing out 5 things as many times as we can until none are left, and determining how many people got 5 things or asking the question, *How many 5's are in 20?*

The *partitive* model of division is typically the first model taught to children, and the model called the "primitive" model of division by researchers (Fischbein et al., 1985, Tirosh & Graeber, 1989). This idea is introduced as division through "fair sharing," and can be modeled for children by giving one object to each person until there are none left. For example, the problem, *I have 20 cookies and I want to share them among myself and 4 friends. How many cookies do we each get?* can be modeled by distributing a cookie to each person one at a time until each person has 4 cookies, and there are no cookies left.

The measurement type of division can be modeled by the process of repeated subtraction. The question *I have 20 cookies and I want to give 5 to each of my friends.*

How many friends can get cookies? can be modeled by repeatedly taking out groups of 5 from the 20 objects until there are no cookies left, resulting in 4 groups.

Of the two models of division, the *measurement* model is much more easily translated into situations dealing with fractions. We can think of having 5 ½ pounds of candy, giving ½ a pound to each person, and asking how many people get candy. This situation can be easily modeled by subtracting ½ from 5 ½ until there is nothing left, and we can see that there are 11 groups. Thus 5 ½ ÷ ½ = 11. However, it gets more complicated when we try to translate the *partitive* model of division into fractional situations. "The fair sharing, or partitive model is a traditional teaching model for division of whole numbers, but it can act as a barrier in the representation of division of fractions" (Rizvi & Lawson, 2007, p. 378). When we look at division of fractions using this model, the original situation that we used with whole numbers does not make sense. We cannot talk about half or a third or three-fifths of a person. The partitive situation can be modeled with a word problem, such as *I have 5 ½ pounds of candy. This is ½ of a serving of candy. How much candy in a whole serving?* We still know how much we started with and are trying to determine the size of one group, but the translation of the problem does not always make it seem like it is the same form.

A third model of division that is not discussed in much of the literature on division is called "product and factors" by Ma (1999). This model represents division as the inverse of multiplication. A word problem using this model would be something like, *A rectangular sandbox measures 6 1/2 square meters in area. If the length of the sandbox is 3 ¼ meters long, how wide is the sandbox?* This problems requires one to divide 6 ½ by 3 ¼ to come up with the answer of 2 meters for the width of the rectangle. Basically this type of question asks, what do I multiply 3 ¼ by to get 6 ½?

In a similar way to the multiple representations of division, Taber (1999, 2002) represents multiplication problems in six different forms, depending on whether the multiplier is a whole number or a fraction. Note that in a multiplication problem, the multiplier is the first number in a multiplication problem, "the factor that performs the operation" (Taber, 1999, p. 2). Taber (1999) uses multiplying the numbers 12 and ¼ as examples to demonstrate her framework. *Compare* problems exist with either the whole number or the fraction as the multiplier. If 12 is the multiplier, we are comparing two quantities, one of which is 12 times as large as the other; if ¼ is the multiplier, we are comparing two quantities, one of which is ¼ the size of the other. Note that although Taber does not talk about fractions greater than one, we can use mixed numbers in this model as well, $3\frac{1}{5} \times 4\frac{1}{7}$ would be comparing two quantities, one of which is 3 $^1\!/_5$ times the size of the other. A second type of multiplication problem, *multiplicative change*, also can be used with either a fraction or whole number multiplier. A multiplicative change problem with 12 as the multiplier would extend a quantity until it was 12 times the original size; if ¼ were the multiplier, we would decrease a quantity until it was ¼ of its original size. Again, Taber does not mention mixed numbers, but we could think of expanding a quantity until it was 3 ½ times its original size, which would be another example of this situation.

While the *compare* and *multiplicative change* models of multiplication both work, regardless of whether the multiplier in the problem is a fraction or a whole number, there are two models of multiplication that are dependent on the quality of the multiplier.

Combine problems exist when the multiplier is a whole number. For example 12 x ¼ can be thought of as combining 12 equal sized parts. This works when the multiplier is a whole number, but it does not make much sense to think about combining ¼ equal sized parts. Thus, instead of combine problems with fractional multipliers, we have *part/whole* situations, "taking ¼ of a quantity of 12" (Taber, 1999, p. 4). Thus in this situation we are taking a fractional part of a quantity. Again, while Taber does not discuss problems with mixed number multipliers, it seems that if we were to have 4 $\frac{1}{3}$ as a multiplier, we would use both the *combine* and *part/whole* models. We could think of 4 equal sized parts, and then adding another $\frac{1}{3}$ of that whatever the size of the part is to our total.

Taber's (1999) reason for breaking down multiplication of rational numbers into different structures is to look at a problem called the "multiplier effect." Taber describes this effect as "students seem to select multiplication or division as the operand that will solve the problem depending on their sense of whether the multiplicand is enlarged or reduced by the action of the problem" (p. 2). This problem was described by Fischbein, Deri, Nello, and Marino (1985) in fifth, seventh, and ninth grade students. The students were given a variety of word problems dealing with multiplication and division of rational numbers and asked to write an equation that they would use to solve the problems. In general, when the students thought that the result of the problem should be smaller than the input, they chose to divide; when they thought their result should be larger, they chose to multiply, even though in many instances, this was not the correct equation, and did not lead to the correct answer.

Graeber, Tirosh, and their colleagues did a number of studies with pre-service teachers by asking them to solve word problems involving decimal numbers similar to the Fischbein et al. (1985) study (e.g., Graeber & Tirosh, 1988; Graeber, Tirosh, & Glover, 1989; Tirosh & Graeber, 1990, 1991). They wanted to see if prospective teachers also showed evidence of the multiplier effect in dealing with multiplication and division problems. They found evidence of the same types of errors in preservice teachers that Fischbein and his colleagues had found in students. Harel and Behr (1995) used the same types of questions from the Fischbein et al. (1995) and various Graeber and Tirosh studies with practicing teachers. Their intent was to look at the strategies that teachers used to solve these, particularly those who successfully solved the problems. They anticipated that the practicing teachers would perform better on the items, since they were actively teaching multiplication and division of rational numbers, however they found that "inservice teachers, like children and college students, are also influenced by these intuitive models" (Harel & Behr, 1995, p. 32) of multiplication and division, and thus had trouble choosing the correct operation and setup to answer the questions. The researchers found that the only teachers who were successful in answering all of the questions used a "multiplicative strategy" where they reasoned about the problems proportionally to find the given unknown and algebraic equations necessary to solve them. Particularly troubling to the researchers, were teachers who used the "operation search" and "key-word" strategies. In the former strategy, teachers tested one of the four operations to see if they got an answer that was close to one they expected. If they did, this operation was chosen; if not, a different operation was performed until a "reasonable" solution was reached. In the latter strategy, choice of operation was determined purely by the existence of certain key words in the problem.

Fischbein and his colleagues, as well as Graeber, Tisosh, and their colleagues, attribute many of the problems that students and teachers have with the multiplier effect to what they call “primitive models” of multiplication and division. After giving the word problem test to over 100 preservice teachers, Graeber, Tirosh, and Glover (1989) also interviewed 33 students who had gotten at least one of the four most commonly missed problems incorrect to probe their understandings more. The conclusions that they drew from their studies were that like children, prospective teachers tend to cling to what they call “primitive models” of multiplication and division. The primitive model associated with multiplication is that of repeated addition. Thus, $3 \times \frac{1}{4}$ can be thought of as $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$. Thinking about multiplication in this way leads to the misconception that multiplication always makes larger, since adding positive quantities results in a larger outcome. This view of multiplication also poses issues when looking at a problem such as $\frac{1}{3} \times \frac{1}{4}$, where the multiplier is not a whole number. Although we can think of taking $\frac{1}{4}$ a third of a time, the repeated addition model of multiplication is difficult to extend to this type of situation.

In terms of division, the primitive model is the partitive or sharing model of division. “This primitive model, by its behavioral nature, imposes constraints on the operation of division. Two of these constraints are: the divisor ‘must’ be a whole number and the quotient ‘must’ be less than the dividend.” (Tirosh & Graeber, 1989, p. 80). These constraints come from the primitive model of division, since it does not make sense to share things among a fractional number of people, and the idea of distributing a number of things among a smaller number of people will result in the number of things that each person receives being less than the original number. This sharing idea of division can lead to the misconception that division will always make smaller.

These primitive models of multiplication and division, and the misconceptions they evoke did show up in the thinking of many of the prospective teachers who were studied. “Every interviewee, including those who had made only one error on the written work, gave evidence of holding at least one of the misconceptions” (Graeber, Tirosh, & Glover, 1989, p. 97). Even when the misconceptions were not explicit, Tirosh and Graeber (1989) found that the beliefs were held implicitly. This was evident for example, in the large number of students who, while agreeing that in a division problem the divisor can be larger than the dividend, still chose $15 \div 5$ to represent the equation to solve the problem “Fifteen friends together bought 5 pounds of cookies. If they each got the same amount, how many pounds did each get?” (Graeber & Tirosh, 1988, p. 265), representing the implicit belief that the divisor must be smaller than the dividend. Thus, we can conclude from these studies that even when prospective teachers explicitly state mathematically correct beliefs about multiplication and division, their problem solving behavior and later teaching may indicate implicitly held beliefs that are incorrect or incomplete.

Teachers’ Knowledge and Understandings of Fractions

Until recently, the research on prospective teachers’ knowledge of fractions seems to be focused mainly on division of fractions. Many researchers give justification for this focus including the facts that “division of fractions lies at the intersection of two mathematical concepts that many teachers never have had the opportunity to learn conceptually—division and fractions” (Sowder, Phillip, Armstrong, Schappelle, 1998, p.

51), and “since division with fractions is most often taught algorithmically, it is a strategic site for examining the extent to which prospective teachers understand the meaning of division itself” (Ball, 1988, p. 61). Fraction division is also a common topic taught by almost all teachers in the middle grades (3-8), and an area where students (and their teachers) often struggle with the mathematics. While the majority of practicing and prospective teachers are able to solve problems involving division of fractions, many studies show evidence that teachers are merely able to follow the “invert and multiply” procedure, which Son and Crespo (2009) say “is possibly the most mechanical and least understood procedure in the elementary mathematics curriculum” (p. 237).

Some early studies dealing with prospective teachers’ knowledge of division were conducted by Borko, Eisenhart, and their colleagues (Borko et al., 1992; Eisenhart et al., 1993) in the “Learning to Teach Mathematics” study. The two articles cited above discuss the experiences of one student teacher, Ms. Daniels, as she attempts to teach division of fractions. During the lesson, which was supposed to be review, Ms. Daniels goes over a number of problems dealing with division of fractions. She “explains” the problem to the class by discussing the invert and multiply procedure used to divide fractions, and is going along fine until a student asks her why if you are dividing fractions, you end up multiplying. Ms. Daniels is put in the position where she must give a conceptual explanation of why the division of fractions algorithm works, however she is unable to do so. She attempts to use a contextual example similar to one she learned in her mathematics methods class, but ends up giving an example that requires multiplying the two fractions that she had intended to divide. This type of error, providing a scenario that requires multiplying when the intent is to model a division situation, is not unique to

Ms. Daniels. Both Ball (1988, 1990), with preservice teachers, and Ma (1996, 1999), with practicing teachers, found the same type of errors. In fact, in these studies, the teachers were much more likely to provide scenarios that represented multiplication of fractions than those modeling division.

Ms. Daniels did realize that she had made a mistake, but she was unable to come up with a situation on the spot that modeled fraction division, so she ended up telling the students to “use the rule for right now” (Borko et al., 1992, p. 198), and that she would try to think of a better explanation later. However, when she was interviewed about the lesson, Ms. Daniels admitted that she had not tried to figure out a better explanation, and did not seem to intend to, even though the researchers pointed out that she could again be put into that situation, and would be unable to provide a conceptual explanation.

As mentioned above, Ms. Daniels’ lack of conceptual understanding of division of fractions is not unusual. When asked why they found the division of fractions problems difficult, students in Ball’s (1990) study said “that it was hard (or impossible) to relate 1 ¾ ÷ ½ to real life because, as one said, ‘you don’t think in fractions, you think more in whole numbers.’” (p. 134). While these preservice teachers considered the problem a problem of fractions, the explanations for the difficulties that teachers and others have in understanding fraction division go back to the primitive partitive, sharing model of division, which does not lend itself easily to being described when the divisor is a fraction. As stated above, an easier model of division to use to explain fraction division is the quotitive, measurement, or repeated subtraction model. Using this model, we can easily divide 3 by ½ by asking how many ½’s are contained in 3? While this model of division is much more conducive to dealing with fractions, teachers have still been shown to have trouble dealing with remainders. For example, dividing ¾ by ½ asks how many ½'s are there in ¾? The answer is 1 with ¼ left over, so a tempting answer would be 1 ¼. However, by performing the algorithm, we can see that the correct answer to the problem is 1 ½. This is because while the *remainder* is ¼, this represents half of the ½. There are 1 ½ halves in ¾. Greer (1992) points out that one aspect of multiplication and division problems that make them more difficult than addition and subtraction is the "dimensional complexity" of multiplication and division: "An intensive quantity such as miles per hour may be seen as transforming a quantity with referent 'hours' to a quantity with referent 'miles'" (p. 284). Thus when working with teachers and students on multiplicative problems, particularly those dealing with rational numbers, it is very important to understand the unit, and what the quantity refers to. In the above example, the ¼ remainder refers to ¼ of 1 unit, whereas the ½ in the answer represents ½ of the ½, the new referent (since we are asking how many ½'s are contained in ¾, we are no longer referring to the whole.) In their work with teachers on rational number understandings, Schifter (1998) and Sowder, Philipp, et al. (1998), emphasize paying careful attention to the referent or unit because it is the source of much confusion for students and teachers.

Teachers' knowledge of division of fractions continues to be a topic of much discussion in the mathematics education community. This interest may come particularly from the response to Ma's (1999) book, *Knowing and Teaching Elementary Mathematics: Teachers' Fundamental Understandings of Mathematics in China and the United States*. In her book, which came out of her dissertation, Ma conducted task-based interviews with practicing elementary teachers in the United States and China. The American teachers had the most trouble on the question which asked them to divide 1 ¾ by ½, and also to come up with a word problem describing the situation. Many of the American teachers were unable to remember how to perform the division, and only one was able to generate a story that fit the situation. The popularity of the book caused quite a stir in the mathematics education community. Mathematics educators (e.g., Askey, 1999; Howe, 1999) called for people to take action and change the way that teachers, teacher education programs, and the mathematics education community look at how elementary mathematics is taught and learned, particularly by looking at ways to enhance teachers' understandings of division of fractions.

Most of the literature dealing with teachers' knowledge of fractions deals with concepts of division. Those studies that do look at multiplication of fractions (e.g., Armstrong & Bezuk, 1995; Izsak, 2008) claim that, like with division, teachers' understanding of multiplication of fractions is mostly procedural without much depth. Izsak (2008) did a study looking at two sixth grade teachers' understanding of multiplication of fractions. His focus in the study was on the teachers' use of drawings in their representations of fraction multiplication because "discussions of pedagogical content knowledge and mathematical knowledge for teaching often make explicit reference to representations, [and] also because reform-oriented curricula in the United States are placing new demands on teachers and students to interpret and reason with a variety of inscriptions" (p. 105). Both of the teachers were using a reform curriculum for teaching where drawings and multiple representations of fractions were a large part of the curriculum, and they both reported not using drawings in their teaching prior to teaching from the new curriculum.

Izsak (2008) found that the teachers used representations of multiplication of fractions for two purposes. One of the teachers used drawings to illustrate solutions to multiplication problems which were solved using a different model (not by using the drawing), whereas the other teacher used drawings to represent the process of multiplication, and thus to come up with a solution. Izsak concludes that in order to use drawings effectively, teachers must be able to understand multiplication of fractions using three layers of nested units. Thus $2/3 \times 3/4$ must be understood as $2/3$ of $3/4$ of 1 in order to represent the action of multiplication properly using a picture. While this understanding of the three layers of nested units is necessary for teachers, Izsak claims that it is not always sufficient, as even if teachers demonstrated this knowledge in interviews, they did not always access it in dealing with multiplication of fraction problems with their classes.

While there is still a lot of current work looking at teachers' and prospective teachers' understandings of division of fractions (e.g., Li & Kulm, 2008; Lo, McCrory, & Young, 2009), a recent look at the *Proceedings of the 31st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* suggests that studying teachers' and preservice teachers' understanding of multiplication of fractions is becoming more popular (e.g., Goodson-Espy, 2009; Gulcer, Park, & McCrory, 2009; Luo, 2009; Tobias, 2009). These studies look mainly at prospective teachers' understandings of and ability to represent multiplication of fractions as word problems. Luo found that this was a difficult topic for the prospective teachers, especially when the multiplier was a fraction. Goodson-Espy found that prospective teachers were often able to write multiplication of fraction word problems, but were unable to explain why multiplication was the proper operation to answer their question. Gulcer and her colleagues found that while representing a fraction using different subconstructs (see Lamon, 2007) was relatively easy, representing *operations of fractions* using the different subconstructs was not, and it was not always evident to the researchers which subconstruct of fractions the teacher educators whom they studied were using to represent multiplication. Tobias' study looked at a class of preservice teachers' attempts to make meaning out of multiplication of fractions rules and representations. One of the challenges for the students was understanding how the whole was changing. For example, $\frac{1}{3}$ of three-fourths is the same as $\frac{1}{4}$ of one, and it was confusing for the students to talk about $\frac{1}{3}$ actually being $\frac{1}{4}$.

While the conference proceedings contain short papers, the fact that so many presentations were centered around teachers' understandings of multiplication of fractions shows that it is a topic of much current study in mathematics education, and we can expect research in this area to be forthcoming.

What Should Teachers' Understandings of Multiplication and Division of Fractions Look Like?

Much of the focus on research on teachers' understandings of multiplication and division of fractions focuses on deficiencies in knowledge (e.g., Ball, 1988, 1990; Ma, 1999, Tirosh & Graeber, 1990), but little research talks about what teachers actually should know about the subjects. The Conference Board of Mathematical Sciences (CBMS), in their report, *The Mathematical Education of Teachers* (2001), discuss some of the ideas that teachers should know regarding fraction multiplication and division. They suggest that prospective teachers must move beyond merely understanding the algorithms for multiplication and division of fractions, and make sense of why they work. They state that “the very meanings of multiplication and division must be extended beyond those derived from whole number operations” (CBMS, 2001, p. 19), where the focus is often on multiplication as repeated addition and the partitive (sharing) model of division (see Fischbein et al., 1985). The CBMS also points out that prospective teachers “are sometimes surprised to learn that there is something to understand about the usual algorithm for dividing fractions and that there are other, equivalent algorithms. Understanding division of fractions requires a deep understanding of what fractions are, and of what division means” (CBMS, 2001, p. 29). This adds to the knowledge requirements for teacher educators because they must help convince prospective teachers of the need to understand more than just “keep, change, flip” in regards to division of fractions.

Flores (2002) suggests that in order to have a “profound understanding of division of fractions,” teachers must be familiar with the multiple meanings for division and how they relate to fractions. They must understand the inverse relationship between multiplication and division—that dividing by a number is the same as multiplying by its reciprocal. Flores states that a “profound understanding of fractions will allow teachers to make sense of these procedures and help students make connections to other procedures and concepts” (p. 240). Thus teachers with this deep understanding will be able to help students’ make sense of their own strategies when dealing with fractions, and will be able to determine the mathematical validity and generalizability of solutions. In order to build up to a profound understanding of division of fractions, a person must have a deep understanding of the framework upon which these ideas rest. Thus they must have a sound understanding of multiplication and division of whole numbers as well as multiplication of fractions, which all build up to division of fraction ideas. Li (2008) points out the relationship between different algorithms for division of fractions and different interpretations of division: "the common-denominator algorithm for fraction division relates to the measurement interpretation model, whereas the invert-and-multiply algorithm relates to the equal-sharing model" (p. 549). Knowing how these algorithms correspond to the different models of division is important in understanding division of fractions thoroughly. Ott, Snook, and Gibson (1991) criticize mathematics textbooks for ignoring the partitive meaning of division of fractions, which they contend "is at least as important to modeling problem situations as the measurement concept of division of fractions" (p. 10). Thus, they seem to agree that a thorough understanding of division of fractions must contain understanding of both partitive and measurement interpretations of division. While there is not yet research defining what a "profound understanding of multiplication of fractions" might look like, we can assume that it too would be deep, broad, and thorough (Ma, 1996, 1999) and be conceptually rather than procedurally based.

Enhancing Teachers' Rational Number Knowledge

Many of the published studies involving enhancing teachers' rational number knowledge involve inservice workshops with practicing teachers. The idea behind the studies is to get teachers to think about the conceptual basis underlying the teaching and learning of fractions. "It is quite possible that the teachers do not know that a conceptual base for multiplication and division of fractions even exists. Nothing in their mathematics learning experiences would have provided a hint of that existence"

(Armstrong & Bezuk, 1995, p. 91). Researchers who worked with inservice teachers, (e.g. Armstrong & Bezuk, 1995; Schifter, 1998; Sowder, Philipp, et al., 1998), helped the teachers by enhancing their content knowledge of fractions, multiplication, and division, as well as helping to improve their pedagogical content knowledge, by having them look at how their own students viewed fractions and operations on them, and discussing ways to expand these views and clear up misconceptions. Schifter (1998) states that "teachers also needed to learn to listen and to hear their own students making sense of mathematics" (p. 83). Being able to evaluate students' understandings of the content of mathematics helps the teachers develop their pedagogical content knowledge. They see where the students are struggling and how they are interpreting the content, and through their teacher development seminars and working with colleagues, they are able to deepen their own understanding of the content, as well as ways to teach that content to students.

Studies which attempt to help preservice teachers develop knowledge of rational number concepts have mainly provided recommendations on the structure and content of undergraduate content and methods courses. Sowder, Armstrong et al. (1998) make four recommendations for the professional development of teachers. "*Recommendation 1: Instruction for teachers should provide them with opportunities to explore situations in which they have to reason explicitly in terms of quantities and qualitative relationships*" (p. 131). Many prospective teachers have never been put in situations where they are asked to *reason* quantitatively. Rather than asking them to perform procedures, classes for prospective teachers should challenge them to think about mathematical situations and relationships, determine multiple methods of solving problems, and share their reasoning with others. "*Recommendation 2. In order that teachers come to understand* *both the differences between additive and multiplicative reasoning and the ways in which students develop multiplicative reasoning, they themselves must encounter situations in which these types of reasoning are appropriate and are contrasted*" (p. 137). Many times situations involving additive reasoning are taught separately from those requiring multiplicative reasoning. The authors recommend exposing teachers to problems that cannot be solved with "key-word" strategies, and emphasizing teachers' use of proportional reasoning to deepen their understanding of solving these types of problems. This recommendation coincides with earlier discussions of the work of Harel and Behr (1995). "*Recommendation 3: Teachers should be exposed to situations that allow them to reframe proportionality from the traditional focus on the solution of missing-value problems by rote symbolic procedures to a broad, complex reasoning process that evolves over a long period of time*" (p. 141). In her review of the literature dealing with rational numbers, Lamon (2007) states that "multiplicative ideas, in particular, fractions, ratios, and proportions, are difficult and **develop over a long period of time**" (p. 651, emphasis added). Quick fixes encouraged to teach teachers to solve missing value proportion problems will only re-emphasize the procedural understandings that many teachers already have. Long periods of time are needed to develop robust understandings of proportional reasoning. "*Recommendation 4. Teachers' work with rational numbers should include connections among the forms of rational number and connections with the concepts of ratio and proportion*" (p. 145). While researchers have not necessarily been able to agree about the number of subconstructs that compose rational number, the current belief in the mathematics education community is that the traditional focus on only the part-whole area model of rational number is not adequate for anyone to develop "rational number sense" (Ball, 1993; Lamon, 2007). Teachers must be able to see rational numbers as numbers (on a number line), as operators on sets (taking 1/3 of a set of objects), as quotients (the amount of a share when 4 people share 3 of something), and as ratios, (comparing a number of one quantity to another number of a different quantity). They must experience rational numbers as concepts, rather than objects to be manipulated. Sowder, Armstrong, et al. (1998) add, "Our experience suggests that one critical aspect of teachers' knowledge of rational numbers is that they do not realize that they lack the understanding of rational numbers necessary to teach this topic in a meaningful way" (p. 145). Teachers, both pre- and in-service, must be made aware that they can understand rational numbers in meaningful ways, and they and their students should be given the opportunities to do so.

Theoretical Frameworks in the Development of Rational Number Knowledge

Earlier studies on knowledge of rational number (1980s and 1990s) were conducted by educational psychologists using a cognitive science framework. Through this framework, researchers were interested in looking at the cognitive structures students used in thinking about rational number ideas. During the same time period, researchers in the Cognitively Guided Instruction (CGI) project were having success in providing frameworks for looking a children's informal knowledge of addition and subtraction problems. As discussed above, these researchers began working with children and studying the understandings that they brought to bear in solving addition and subtraction problems. They followed by working with teachers to help them become aware of the cognitive structure of these problems, and the informal understandings that students use to solve them, even before formal instruction.

Researchers looking at rational number ideas wanted to use similar methods in order to construct frameworks of students thinking on rational number, however, Behr, Harel, Post, and Lesh (1992) cautioned, “It is not at all clear that the basic tenets of the CGI model are directly generalizable to the more complex mathematical structures embedded in rational-number usage” (p. 325). While students were familiar with the context of addition and subtraction in their every day lives, researchers argued that they may not have their own interpretations of the more complex rational number ideas. Sowder, Bezuk, and Sowder (1993) add that “rational number sense . . . seems less likely than whole number sense to develop without instructional intervention” (p. 240). However there were a number of studies that attempted to look at students’ informal understandings and cognitive development of rational number.

Kieren (1976) contended that the main substructure underlying the concept of rational numbers is the ability to partition a whole. For example, “ $x = \frac{3}{4}$ means $4x = 3$, which can have the concrete meaning ‘x is the number we attach to each part which results when we divide three crackers into four equal parts” (p. 121). Understanding the breaking up or “fracturing” of a unit is the basis for developing understandings of fractions.

Mack (1990) conducted studies with sixth grade students in an attempt to build on their informal understandings of fractions. Like Kieren, she too noted that the development of informal knowledge was based on ideas of partitioning: “All students possessed a rich store of informal knowledge of fractions that was based on partitioning units and treating the parts as whole numbers” (p. 16). However, this knowledge was not necessarily connected to the formal symbolic fraction notation. Students’ use of (often incorrect) algorithms when faced with fraction problems in symbolic form often contradicted their informal understandings of rational numbers and partitions. By using word problems and then similar symbolic problems, Mack was able to help the students bridge their informal understandings to use on the formal problems.

A conclusion of Mack's (1990) study has implications for teacher education. She claims that her "results suggest that knowledge of rote procedures interferes with students' attempts to construct meaningful algorithms" (p. 30). This same idea is echoed in Lamon (2007), and Sowder (1995). Since the majority of prospective teachers come to teacher education with knowledge of how to perform algorithms on rational numbers, these findings imply that this knowledge may hinder their ability to think through problems conceptually, or even to see a rationale behind conceptual understanding of rational numbers, since prior to their mathematics education content courses, knowledge of rational number procedures may have served them well throughout their schooling. It is the job of teacher educators to help prospective teachers see a need for developing more than a rote understanding of these concepts, and also to help facilitate this development by building on their informal understandings of fractions. Fortunately, Mack (1990) concludes that her "results, however, do not suggest that the influence of rote procedures cannot be overcome, but a great deal of time and directed effort is needed to encourage students to draw on informal knowledge rather than use rote procedures" (p. 30). Thus the job of the teacher educator is much more complicated than merely strengthening prospective teachers' knowledge of procedures.

More recent work in looking at fraction knowledge has built on cognitive science ideas and used a more constructivist view of learning (e.g., Rathouz & Rubenstein, 2009;

Simon & Blume, 1994). The idea behind constructivist teaching is explained by Simon and Blume (1994) as follows: "researchers build and modify models of students' mathematical thinking as they interact with the students and reflect on the students' behaviors and communications. Each intervention with the students is guided by the existing model. Subsequent interactions provide data that cause the model to be modified or elaborated further" (p. 475). The idea is that students construct their own knowledge of situations. The work of the teacher educator is to examine the process of student knowledge construction, and how they are putting together the pieces of knowledge. With this understanding of student knowledge construction, teachers will strategically redefine their instruction, monitoring the construction of their own mathematics knowledge for teaching in the process.

In articles discussing their teaching of fraction problems to preservice teachers, Cengiz, Flowers, Rathouz, and Rubenstein (2009), as well as Rathouz and Rubenstein (2009) describe some key aspects of their curricula. They use an investigative, problem solving approach, centered around meaningful tasks for the students around the concepts of rational numbers and multiplicative relationships. They contend that the use of good instructional tasks is essential in providing students with an opportunity to create deep, meaningful understandings of a topic. Important features of these tasks are that they cannot be solved by only implementing computational procedures, and they have a variety of ways to look at them. Another important characteristic of these classrooms is the development of classroom norms where students are expected to participate, explain their thinking, and comment on others' ideas. These norms create a classroom where the teacher acts as a facilitator of classroom discussions, letting the students do much of the talking, but asking probing questions to help move the work along. Teaching in this way provides many challenges for teacher educators, some of which I discuss in the next section.

The Implication of These Ideas for Teacher Educators

From the review of the literature above, it is evident that rational number ideas pose many challenges for practicing and prospective teachers. It is the job of mathematics teacher educators to help teachers gain better, more conceptual understandings of rational numbers, but this is not an easy job. Sowder, Bezuk, and Sowder (1993) point out that this can be very difficult for teacher educators:

> When such a course moves beyond review of content to allow prospective teachers to examine their own understanding of content and to explore and redefine the content in a manner that will allow them to teach that content, it can be perhaps *one of the most challenging courses to teach within a department of mathematics*. That some prospective teachers believe they already know the content because they are procedurally competent, whereas others believe that they are incapable of understanding the content, only increases the challenge. Many hold conceptions about the nature of mathematics, its structure, and what comprehending mathematics means that are at variance with those of the mathematics community. (pp. 242-243, emphasis added)

The ability to help teachers develop a deep understanding of rational number content, or even see the importance of having this knowledge provides challenges to teacher educators which have not been articulated so far in much of the literature.

Other challenges for teacher educators are provided by the difficult content structure of rational numbers themselves. Sztajn, Ball, and McMahon (2006) describe an eight day workshop for mathematics teacher educators. This workshop was centered around the mathematics knowledge needed for teaching, and one of the major topics of discussion was fractions. The teacher educators had a very difficult time determining what they believed was essential for prospective teachers to know about fractions, and by the end of the workshop, they still had not been able to generate a definition of fractions upon which everyone could agree.

Lamon (2007) and others argue for the importance for teachers to understand the multiple subconstructs of the rational numbers. This means that teacher educators must also understand these subconstructs, as well as how students come to learn them. In addition, teachers must have the pedagogical content knowledge to know where problem areas for students in learning rational number ideas occur. Teacher educators must bring this process one step further. Not only do they need to know areas where students struggle with these ideas, they must also understand the struggles that prospective teachers have—these struggles may or may not be the same, however we can assume that helping prospective teachers who have spent years learning procedural rules for fractions see the need to understand the ideas conceptually is a struggle unique to mathematics teacher educators.

In trying to organize classrooms as sites of discovery, teacher educators are faced with developing or finding good mathematical tasks which challenge students and can be looked at from multiple perspectives. Thus the knowledge required by teacher educators around multiplication and division of fraction concepts is indeed complex.

Chapter 3—Methods

In order to answer my research question, *What is the mathematical knowledge required by teachers of elementary mathematics content courses in the area of multiplication and division of fractions?*, I conducted a qualitative study of mathematics teacher educators in the practice of teaching. While there have been a number of frameworks developed around the topic of mathematics knowledge for teaching, the majority of the researchers agree that the knowledge needed by teachers can only be seen by looking at the actual process of teaching (e.g., Adler & Pillay, 2007; Ball & Bass, 2002; Ball, Thames, & Phelps, 2008; Davis & Simmt, 2006; Hiebert, Gallimore, & Stigler, 2002; Kazima & Adler, 2006). This is because mathematics knowledge *for teaching* is accessed by teachers during the process of teaching (Hill, Sleep, Lewis, & Ball, 2007). In terms of looking at knowledge for teacher educators, Cochran-Smith (2003) suggests "one way to conceptualize subject matter for teacher educators is in terms of the work of teacher education itself, the stuff of everyday practice—teaching courses, supervising student teachers, facilitating seminars, revising curriculum, developing assessment systems, preparing accreditation reports, admitting students, and so on" (p. 23). Thus, I attempted to do just that—get into the classroom and look at the work of teacher education. Another justification for observing teaching is that teachers (and teacher educators) "are not always able to articulate their practical knowledge" (Berliner, 2004, p. 206). Simply asking teacher educators what they know will not necessarily get at the heart of the mathematics knowledge for teacher educators. Therefore, in order to see mathematics knowledge for teaching in action, I needed to examine teacher educators in the process of teaching.

Respondents

This study involves case studies of three experienced teacher educators at different types of institutions who were teaching content courses to preservice elementary teachers. In order to identify experienced teacher educators, I used a number of criteria. First, an experienced teacher educator needed to have taught preservice teachers for a long period of time. While researchers often talk about the development of expertise over a number of years, teacher education is in a university setting, so teacher educators may teach the same courses two to three times per year over different semesters. Therefore, rather than defining a specific number of years that an experienced teacher educator should have taught, I looked for teacher educators who had taught the mathematics content course for elementary teacher for multiple semesters. As Berliner (2004) writes, "certainly, experience alone will not make a teacher an expert, but it is likely that almost every expert pedagogue has had extensive classroom experience" (p. 201). The development of expertise over a long period of time is also consistent with the findings of Ma (1999) that the teachers who showed evidence of having a profound understanding of fundamental mathematics had all been teaching for over 10 years.

Another quality I looked for in experienced teacher educators was having at least some influence over the design of the curriculum of the content course. Because many institutions offer multiple sections of their content courses for prospective teachers, often these courses are taught by people who follow a curriculum designed by someone else. The different sections may give a common exam or assignments, do the same activities during class, and so on. While there are clearly differences in the way the same course is taught in classrooms with different teachers, the teacher educators I looked for in my study were people who made the executive decisions regarding the structuring of the curriculum, the evaluation methods, and the emphasis on certain aspects of the content of the course.

A third quality that I looked for in determining an experienced teacher educator was some evidence of prior work on improving teaching, both their own and that of others. This ranged from being involved in research on preservice teacher education, by either analyzing their own work or the work of others. This way I ensured that these teacher educators had thought deeply about issues involved in teaching prospective teachers and had a background in understanding the mathematics knowledge for teaching.

A fourth quality that I looked for in teacher educators in my study was that they came from a variety of institution types. Since mathematics content courses are taught in different types of institutions, I did not want to limit myself to only looking at teacher educators from one type of institution. In their study of who teaches mathematics content courses for prospective elementary teachers, Masingila, Olanoff, and Kwaka (2011) found respondents from four different types of institutions: two-year schools, four-year schools without post-graduate programs, four-year schools with master's degrees, but not doctoral programs, and four-year schools with doctoral programs. Each of these types of institution comprised between approximately 18% and 34% of the data of schools reporting teaching mathematics content courses for prospective teachers. Therefore, in my study, I looked for teacher educators at different types of institutions. The three teacher educators in my study are from a small, four-year private college with graduate degrees in education, a four-year state university that offers master's degrees in education, and a two-year college respectively. In order to keep my study at a feasible size, I chose to limit the study to three mathematics teacher educators, so I do not have a teacher educator from a four-year school with doctoral programs.

Data Collection

Data collection for the case studies involved interviews with and classroom observations of the experienced teacher educators as they taught the multiplication and division of fractions portion of a content course for prospective teachers. I audiotaped the interviews as well as audiotaped the teacher educators while they were teaching. I also took field notes during each lesson, by attending the class, and I made summary notes following each class of the major themes and ideas presented in the class, the general questions asked by the students, where the students seemed to struggle or be successful with the material, and the actions of the teacher educator in presenting the content and interacting with the students. For the three teachers educators, I was able to observe them three, five, and three times respectively during their classes, as these were the number of class periods each spent covering fraction operations.

In addition to audiorecording and observing the lessons, I also interviewed each of the teacher educators both before and after they taught the sections on multiplication and division of fractions. Prior to the beginning of the multiplication and division of fractions section of the course, I asked the teacher educators about their goals for the section, how they planned on teaching rational number ideas, problems that they anticipated students having, their previous experiences teaching this content area, both to prospective teachers and possibly students, their views on the important ideas that their students needed to construct during the lesson, and how they planned on assessing their students' knowledge of multiplication and division of fractions. Interview questions for this interview are contained in the Appendix. The post lesson interviews focused on how the teacher educator felt the course was going, if they felt that they had met their goals or whether their goals had changed, the challenges they were facing in teaching the material, things that were going well, and things that were not going well, what they planned to do in the coming lessons, and specific instances from their teaching that either they or I found to be important. Since the questions in the follow-up interviews were dependent on what had happened in the class, I did not use a specific script for these interviews. I also met with all three of the teacher educators following the exam that they gave that included fraction multiplication and division, and with two of them following their final examination. The purpose of these meetings was to talk with the teacher educators about the assessments, see how their students had done, and see what they learned from the assessments.

By looking at multiple experienced teacher educators, I hoped to see different challenges and views of teaching multiplication and division of fractions to prospective elementary teachers, which could point to different aspects of the mathematical knowledge needed for teaching teachers. I assumed that the experienced teacher educators would show a developed knowledge base for teaching multiplication and division of fraction concepts to prospective teachers. This assumption proved true in some instances and not in others. I was also interested in seeing what knowledge each of the teacher educators developed through teaching the course and the interview sessions. While I assumed that the experienced teacher educators would have a deep knowledge base prior to participating in the project, a constructivist philosophy would say that each of the participants in the study would be continually constructing his or her own knowledge, and by reflecting on the process of teaching (Schön, 1983), each participant would develop a deeper knowledge base.

During the observations of the teachers' lessons, I focused my observations around the tasks involved in teaching. The rationale behind this was that I was trying to determine the mathematical knowledge needed by teacher educators *for teaching*, and thus the best way to determine this knowledge is by looking at the work of teacher educators. Ball and Bass (2002) worked to identify what they called "core tasks" for the work of teaching. The list that they generated can be seen below:

Mathematical Tasks of Teaching

- Presenting mathematical ideas
- Responding to students' "why" questions
- Finding an example to make a specific mathematical point
- Recognizing what is involved in using a particular representation
- Linking representations to underlying ideas and to other representations
- Connecting a topic being taught to topics from prior or future years
- Explaining mathematical goals and purposes to parents**
- Appraising and adapting the mathematical content of textbooks
- Modifying tasks to be either easier or harder
- Evaluating the plausibility of students' claims (often quickly)
- Giving or evaluating mathematical explanations
- Choosing or developing useable definitions
- Using mathematical notation and language and critiquing its use
- Asking productive mathematical questions
- Selecting representations for particular purposes
- Inspecting equivalencies

(Ball, Thames, & Phelps, 2008, p. 400).

With the exception of the task to which I assigned a double asterisk, all of these tasks also can be described as tasks of teacher educators. While teacher educators rarely explain mathematical goals and purposes to parents, they often must explain these tasks to their students, so we can add: *Explaining mathematical goals and purposes to prospective* *teachers*, as another mathematical task of teacher educators. From the research on teacher educators (Smith, 2005), we can add *Keeping up to date with current research in teacher education,* and *Doing mathematics education research*, as well as potentially *Designing curricula*.

I used this list of mathematical tasks in my observations, interviews, and data analysis. During the observations I paid particular attention to the teacher educators' use of examples, the questions they asked their students, the questions and comments that their students made and the teacher educators' responses, the structure of the lessons, and the different representations the teacher educators used. I copied everything that was written on the board and made notes of when writing on the board occurred, so that I could insert it into the appropriate point in the transcripts of the classroom sessions. During the preliminary interviews, I discussed the teacher educators' lesson plans, if and how they deviated from the text and the rationale for the setup of the lessons. In the post-lesson interviews, I asked about the rationale for the teacher educators' decisions, how they modified what they had planned during the lessons, how they selected the representations that they used in the class, and other questions based on what happened during the lessons.

A final data source I used to help answer the question of the mathematical knowledge required by teacher educators in relation to multiplication and division of fractions was the textbooks that the teacher educators used for their classes. As McCrory (2006) points out, these textbooks "define a substantial element of what students have an *opportunity* to learn" (p. 20) in their courses. Therefore they provide an insight into the necessary content knowledge that students should know, and thus would be part of the knowledge base for their teachers. Since many of the textbooks are written by mathematics educators and mathematicians, the textbooks provide a look at what these researchers consider to be important mathematics knowledge for mathematics teacher educators. In addition, authors often also include teaching tips and lesson goals for instructors in the teachers' editions. Thus, analysis of these texts can also help contribute to the knowledge base for the mathematics teacher educator.

Data Analysis

I began my data analysis by transcribing all of my interviews and audiorecordings of classroom sessions. I analyzed the interview, classroom observation, and textbook analysis data using a grounded theory approach (Strauss & Corbin, 1998). I began by using an open-coding technique of all of the data to look for common themes. From these themes, I developed categories for the data to use in the rest of my coding. I treated each of the teacher educators as one of the cases of the study. My goal was to look for themes in common in all cases, as well as evidence of knowledge that showed up in one or two experienced teacher educator but might have been missing from the knowledge base of the other teacher educators.

From the categories that I developed through coding the data, I built profiles of each of the three mathematics teacher educators. Using my interview questions as a basis, I looked at different categories such as typical classroom session, goals for multiplication and division of fractions, and knowledge of students' difficulties, in order to get an understanding of each of the three teacher educators and the principles around which they designed their instruction. Chapter Four of this paper contains descriptions of each of the teacher educators as well as their guiding principles.

In order to look at characteristics of a framework for the knowledge needed by teacher educators, I attempted to build on current frameworks of teacher knowledge (e.g., Hill, Ball, & Schilling, 2008; Shulman, 1986). The majority of the researchers who have provided basic frameworks for teacher educator knowledge use teacher knowledge frameworks as the basis for their teacher educator knowledge framework (e.g., Perks & Prestage, 2008; Zaslavsky & Leikin, 2004; Zopf, 2010), but contend that teacher educator knowledge is qualitatively more and different than the knowledge required by teachers. Since researchers contend that mathematical knowledge for teaching is shown through the work of teaching, I coded the data a second time by looking at the mathematical tasks required by the teacher educators. Using the research on both the work of teaching and fraction multiplication and division as well as the major themes I identified from the data, I was able to identify three major tasks for the work of teacher educators in teaching multiplication and division of fractions: introducing fraction multiplication, helping students make sense of fraction division, and assessing student understanding. Each of these tasks played a major role for each of my teacher educators and helped me to identify characteristics of teacher educator knowledge demonstrated by the teacher educators as well as some aspects of the knowledge base that the teacher educators may have been lacking. A description of how each of the three teacher educators dealt with these tasks and the knowledge characteristics that they brought out is contained in Chapter Five of this paper.

Chapter 4—The Teacher Educators

In order to answer my question, *What is the mathematical knowledge required by teachers of elementary mathematics content courses in the area of multiplication and division of fractions?*, I conducted case studies of three mathematics teacher educators teaching content courses to prospective elementary teachers. In this chapter I will introduce the three teacher educators in this study, describe their classroom sessions and outline the principles that guided them in their teaching and decision making.

Tom—Introduction

Tom Walker taught a mathematics content class for preservice teachers at a small, private college in the Northeastern United States. At the time of my study Tom had been teaching this course for about five years, when the dean of the college suggested that someone with experience working with elementary students and teachers should teach the course. Prior to that time, the course had been taught by faculty in the mathematics department, who may or may not have had any prior involvement with elementary teachers. In addition to the five years he spent teaching the mathematics content course, Tom has also taught elementary mathematics methods courses for 13 years as well as teaching junior high school mathematics and working as a “math lab specialist” in an elementary school. Additionally, he has taught both credit and non-credit mathematics courses at a number of colleges and universities in the area. In terms of his academic background, Tom has a bachelor’s degree in mathematics and a master’s degree in “general science,” which he said was really mathematics despite its name, but was called general science because it was funded by the National Science Foundation to help prepare people to teach mathematics and science. Because he does not have a doctoral degree, and because his position was not full-time, Tom was considered an adjunct instructor, even though he had been teaching at this college for over twenty years.

Mathematics Content Course

The course that Tom taught is somewhat different from the typical mathematics content course for elementary teachers. First, the college where Tom works only offers one course, rather than the more typical two or more (Masingila, Olanoff, & Kwaka, 2011), so Tom had to fit all of the content into one semester. Second, although the textbook that Tom uses is called *Essentials of Mathematics for Elementary Teachers* (Musser, Burger, & Peterson, 2004), many of his students were, in fact, preparing to be secondary teachers in subjects other than mathematics. Tom said that the reason for this is that the state demands that all students preparing to be teachers take some sort of mathematics course, and this course meets that requirement. In addition, many of the students preparing to be elementary teachers do not end up taking the course, because they are able to fulfill their mathematics requirement by taking a statistics course for psychology, which many of them major in. (The college where Tom worked does not allow its students to major in education.) Tom seemed disappointed by this, as he said that "many of them [elementary teachers] are out there with, . . . what I consider, somewhat less than, . . . appropriate, . . . preparation, as far as math's concerned" (First Interview, 11/10/09, lines 127-128). He explained his desire to have a second mathematics content course required, so that he could cover the material in more detail and also maybe "catch some of the psych majors" (First Interview, 11/10/09, line 156). Tom's students range in year from freshmen to graduate students, as it is necessary for the students to take the course sometime during their college career, but it is not a prerequisite for anything else in their program.

General Goals for Course

When I asked Tom about his general goals for teaching the mathematics content course, he said that his "general goals are to give the students enough confidence in their own mathematics, that they will, . . . that they'll teach good math and will model good, behavior about mathematics, I guess is the way I would say it, . . . when they're teaching kids" (First Interview, 11/10/09, lines 162-165). He believes that many of his students enter his class with negative attitudes and even fear about mathematics, and his main goal is to help change this by the time they exit his class. This goal was evident throughout my observations of Tom's classes and his interviews.

One example of Tom working to meet his goal about improving his students' attitudes about mathematics was the way that he interacted with his students during class. Tom seemed very comfortable with his students in terms of joking around with them, both about mathematics and about life in general. In response to a student who commented that we do not live very long because we spend a third of our lives sleeping, Tom said, "you're really a, down deep you're really a pessimist about this whole thing aren't you? You've gotta get out more" (Class, 11/10/09, lines 105-106), which the whole class, including the student who had commented, laughed at. This jovial, joking attitude was common throughout Tom's classes. Tom encouraged his students to ask questions, and even if it was something that he had explained multiple times before, he never seemed to make the students feel bad about asking.

In addition to the general interactions that Tom had with his students, he also tried to get them to think about the mathematics by adding humor to his teaching. He said that he “tr[ies] to have some fun with the vocabulary. You know the idea of, improper. You know, or a, how goofy would a number have to act to be called irrational” (First interview, 11/10/09, lines 352-353). In this way, he hoped the students would remember the vocabulary by thinking about the funny thing that happened during class. Overall he said, “I have fun teaching it. So, and I make sure they know that I have fun. And I try to make it fun for them too, and not, even if it’s not necessarily in the content, just goofy stuff that happens along the way” (First interview, 11/10/09, lines 358-361). This attitude and goal of making it fun was evident throughout Tom’s class sessions and his discussions in my interviews with him.

Tom’s goal of having his students feel good about mathematics seemed to be his overriding concern in terms of teaching his students. In our second interview, he relayed a story to me about meeting one of his former students out one evening: “And one of them recognized me, and said, you taught me math methods. I don’t remember one thing about it. But I do know that it was, that, that I felt good about math” (Second Interview, 11/17/09, lines 497-499). Tom takes pride in this story, because even though the student said that he did not remember the content, he was able to feel better about himself as a mathematics student, which mattered more to Tom in the end. He said, “you can make them, not afraid of it, you know. It’s nice if they pick up some skills and concepts too. . . But the idea of trying to keep them from passing their fear along to a, you know seven-hundred kids over the course of a, twenty years, I’ll take it” (Second Interview, 11/17/09, lines 509-514). This goal influenced how Tom ran his class, the problems that he chose for his students, and the ways that he tried to get them involved in the class, as will be seen in later sections of this chapter. He even said that he sees himself as a "therapist as well as a teacher" (First Interview, 11/10/09, line 602), since he is always trying to help his students overcome their anxieties and fears about mathematics.

Typical Classroom Session

When I asked Tom about a typical class session, he said that he used the textbook as a framework for the course. From my observation of his class, I would say that this was an accurate assessment. Tom usually began class by having his students discuss their problems on the homework with each other and then ask him questions that they still had. After going over the homework problems, Tom gave a short lesson to the class. These lessons seemed to involve Tom demonstrating something in the front of the class and giving some notes while he did some problems. An example of this from my first observation was Tom using Cuisinaire rods to add two-sixths and three-sixths to demonstrate why the addition algorithm for fractions makes sense.

> TW: A sixth. And, what I'm doing here, is *I'm showing you*, two of them, and all of a sudden we're into the part of fractions that are not simply [inaudible] fractions any more. One over something. I got two of them so I got two of, two over something. . . If you just start out with the notation, a perfectly sensible answer would be five-twelfths. But, quite obviously if you've got two of something and three of something you end up with five of them. And that's why I was saying the other day that, the sensible way to approach this, is by doing it like that, so that you can talk about, this, this written record, as simply being a record, a written record

> of something that you saw already happen. As opposed to, *the way things ought to be because I said so*. (Class Observation, 11/10/09, lines 146-172, emphasis added)

Rather than just telling his students a rule or a concept, Tom felt that it was important to actually show them how mathematical rules work. His demonstrations were conceptually based, rather than just showing the students a demonstration of a procedure; however, it seemed like most of the conceptual thinking was being done by Tom as he prepared and presented the demonstrations, and the students were mainly receivers of information, even if the information was conceptually based.

After he did his demonstrations and gave the students some notes, Tom usually assigned a problem set from the textbook for the students to work on. Although he seemed to stress concepts in his demonstrations, most of the assigned problems seemed to be very procedurally based, asking students to perform an algorithm to work with arithmetic operations on fractions, for example. The students worked in groups or on their own to complete the problems while Tom walked around to help them. Since most of the problems in the assignments asked the students to perform some sort of rule, most of the questions the students asked had to do with making sure that they remembered or performed the procedures correctly, for example, the order of operations or how to change a mixed number to an improper fraction.

When Tom had given the students sufficient time to work on the problems, he reconvened the class and went over any problems that they had trouble with. He then assigned homework problems that were similar to those done in class and sent the class on their way.

While I was only able to observe three of Tom's class sessions, they all seemed to run in this manner, and based on what he described in my interviews with him, I have reason to believe that the majority of his classes were very similar to those I observed.

Goals for Multiplication/Division of Fractions

When I asked Tom what his goals were for multiplication and division of fractions, his first response was "procedures." Then he added "And, truly, what those, what's behind those procedures" (First Interview, 11/10/09, lines 393-394). When I probed further into what he meant, Tom explained that he planned on showing his students what multiplication of fractions looks like pictorially and explain that he sees fraction multiplication as taking parts of parts.

> TW: I want, you know, to show rubber bands on geoboards and that stuff. Um, so I want, I want them to know more than simply the uh, the procedures cause I, you know, if everything was as straightforward as multiplication of fractions, you know, the procedure for it, then mathematics would be a snap for everybody. Yeah you multiply across the top, you multiply across the bottom, and there's your answer. Um, so I really do want them to know, that they're taking parts of parts (First Interview, 11/10/09, lines 399-406).

Originally, Tom had planned on having his students use the geoboards themselves to explore some of the multiplication concepts, however, he realized that he did not have enough geoboards for his students to use, so he ended up just showing them a model of fraction multiplication with a geoboard on the document camera.

While Tom wanted his students to know that fraction multiplication is "parts of parts," the majority of the problems that he assigned merely asked the students to perform the multiplication algorithm, which Tom believed was one of the easiest procedures in mathematics. There was one exception to this, as two of the problems from the text asked students to write a multiplication problem that was represented by an area model (see Figure 4.1).

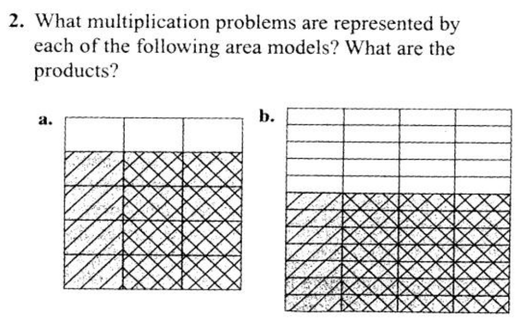


Figure 4.1. Questions from Tom's textbook using the area model of multiplication.

These questions were unlike any that Tom had demonstrated in class, and the students all asked questions on how to do them. Tom explained to the class how he would go about figuring out the problem, but the students seemed focused on the total amount that was shaded in the picture, rather than the multiplication problem itself. Ultimately, Tom ended up doing both of the problems for the students, and when a student asked during the review if Tom could "repeat the silly boxes with the lines," Tom told the class, "Well, I'm not really, um, I'm not all that excited about, making sure that you understand it that way," saying "It's a great way to explain it but it's not a way that I'm gonna go back to and ask you questions about" (Class Observation, 11/12/09, lines 547-572). While Tom stated that his goals were for the students to understand procedures *and* the concepts behind them, it seemed that while he wanted his students to be exposed to the concepts, he was mainly interested in them being able to demonstrate the procedures, rather than show that they held a conceptual understanding of what was behind the procedures.

In our interviews, Tom told me that he introduces division of fractions by asking his students to give an example of a word problem that involves dividing by a fraction. This seemed to be outside of his normal way of doing things and he gave no rationale for why he does it. My hypothesis for this is because this idea comes up in much of the literature dealing with division of fractions (e.g., Ball, 1990; Ma, 1999), so Tom saw this as something that he wanted his students to be able to do. However, he did not talk about a reason either in our interviews or during class, so this is only a guess.

When Tom posed this problem to his students in class, they were very unresponsive. One suggested a problem that involved dividing one-half by three, but Tom emphasized that he wanted a problem where they are dividing *by* a fraction. As the students remained quiet, Tom tried to motivate them.

> TW: I bet you all know how to do it. You know that flip the second one and multiply thing? I'll bet that does, you know, you may have different ways to say it, but I'll bet people know how to do it. But what is a true example of a situation where you're actually doing it? (Class Observation, 11/10/09, lines 683-686)

While Tom remained encouraging, it seemed to me that his students had never had to come up with an example like that before, and they truly did not know how.

As more students came up with examples of multiplying by fractions, rather than dividing, Tom told them that they should be encouraged that that is the most common wrong answer. This is more evidence that Tom had read literature on writing division of fraction problems, since he knew the most common error from the literature. Eventually, he asked the students to give an example of division by a whole number, and then try to modify the problem to division by fractions. A student suggested dividing 12 students into groups of three and asking how many students are in each group, but says that this cannot be changed to a fraction problem because you cannot have a fraction of a person. Tom ended up changing the problem to having 12 students and wanting to make four groups, (changing from the repeated subtraction model of division to sharing) and then asking what would happen if 12 students represented half of the group. He then suggested another problem to the class, where a person has a ten-foot board and wants to make something that requires half a foot of wood, how many can he make, which is an example of repeated subtraction division of fractions. Thus Tom did expose the students to examples of each type of division, but none of the examples were actually generated by the students. He also did not discuss the different types of division at the time of coming up with examples.

In terms of understanding the meaning behind the division algorithm, Tom said, "they don't have a clue, why they're doing that. So *I show them*, . . . using a complex fractions approach, to, you know, a fraction over a fraction, and, you can make the bottom, you know if you multiply the bottom by its inverse, then you get one in the bottom, which is what we all want" (First Interview, 11/10/09, lines 421-425, emphasis added). Similarly to his showing his students the idea behind multiplication, this was another example of Tom demonstrating for his students how something works, but it was not something that he held them accountable for. Rather, their classwork, homework, and exam problems merely asked them to perform the multiplication and division algorithms.

In addition, Tom showed his students another algorithm for dividing fractions—the divide the numerators, divide the denominators approach—which he basically described as a cool trick. A student asked why not just tell students this rule, which seems easier than inverting and multiplying, and Tom replied that it only works in specialized cases, but it is something to look out for before starting “the long way” (Class Observation, 11/12/09, line 203). While this method of fraction division can be used for any fraction division problem, Tom seemed to imply that it was only a useful method in specialized cases, however, his explanation that this method does not work all the time may have demonstrated a misconception of his, or caused a misconception for his students. Tom did not spend time talking about why this method worked; rather he just showed his students an example that used the algorithm. Thus, while his goal was for his students to understand division of fractions conceptually, this example only served to add to their procedural knowledge by giving them another algorithm for fraction division.

Tom’s final content goal for multiplication and division of fractions was for his students to know which of the mathematical properties (e.g., commutative property) hold for fractions. Since this is something that he had already covered for whole numbers, so he merely reviewed them and had the students read about them in the textbook. This was not something that he spent time on during class.

Knowledge of Students

In order to gauge Tom’s knowledge of his students, I asked him what they typically had trouble with regarding multiplication and division of fractions. Since most of Tom's goals were procedurally based, he correctly asserted that the students do not have many problems performing the algorithms for multiplication and division after a reminder:

> TW: Um, they don't have much problem with multiplication, because they don't, if they don't know what they're doing, they'll give that a shot, and you know, it's one of those things where an infinite number of monkeys and an infinite number of typewriters, you'll eventually come, you know get all the right books. . .and, most of them either remember the procedure or can drag it from memory for division (First Interview, 11/10/09, lines 446-451).

He did add, however, that his students do not always understand the problems conceptually. For example, they do not understand what he means by a part of a part, and they often have trouble coming up with problems involving division by a fraction.

When I asked him what he did to help address these problems, Tom reiterated that he shows his students models of the geoboard for multiplication and the complex fractions approach for division to expose them to the different models. Since he did not seem to ask his students to demonstrate understanding at a conceptual level, it seems like it would be difficult for him and for me to know how much they gain from his demonstrations.

Stephanie—Introduction

Stephanie Mitchell taught two different mathematics content courses for prospective elementary teachers in the mathematics department of a medium sized, state university in the Northeastern United States. She had taught the first of these content courses, which deals with number and operations, including fractions, at least six times at her current university, as well as teaching and helping to develop a similar course at her previous job at a state college. The time of my study was her first semester teaching the second mathematics content course, which deals with geometry, measurement, and probability concepts, and Stephanie much prefers teaching the course on number to the second course.

Stephanie's academic background includes a B.A. degree in mathematics and both Masters and Ph.D. degrees in mathematics education. After earning her B.A. in mathematics, Stephanie became certified to teach high school mathematics, which she did for a number of years in a very small high school where she was one of the only mathematics teachers. During the summers that she was teaching high school, Stephanie took classes toward her master's degree, and after earning her degree, she decided to leave high school teaching. A family member suggested that she try community college teaching, so she applied and was hired to teach at a state funded two-year college. She taught developmental mathematics courses for a number of years at this college, but when they began transitioning to a four year college, Stephanie was encouraged to get a Ph.D., so she took a three-year leave from teaching and went back to school.

Upon completing her dissertation work and earning her Ph.D., Stephanie went back to work at her former school which was now a four-year college. She taught developmental mathematics and calculus courses, as well as worked with a colleague to develop a sequence of two mathematics content courses for elementary teachers. However, the college that she was at did not have any mathematics majors, and Stephanie wanted an opportunity to teach more courses in line with her degree, so she decided to look for a job in mathematics education. At the time of my study, she had been working in her current position for four years, and although she was in the mathematics department, her main teaching responsibilities were teaching elementary mathematics content courses and secondary mathematics methods courses.

Mathematics Content Course

Stephanie's university requires that prospective elementary teachers take two mathematics content courses. Stephanie taught mainly the first one, which is on number and operations. This was the first semester where she also taught a section of the second course covering geometry, measurement, and probability and statistics. Stephanie's university offers several sections of each course each semester; however, most of the sections are taught by adjunct instructors. During the semester that I observed her, Stephanie was the only tenure track faculty member teaching the courses, although others in her department have done so in the past.

Because so many sections of these mathematics content courses are taught by adjunct instructors, Stephanie's department basically lets instructors of the course do their own thing, as long as they cover the course material. Consequently, the course has no official supervisor and no common syllabus, although Stephanie has provided her syllabi to some of the other instructors and offered to help them out if necessary.

The students in these courses all have elementary and special education concentrations in education. Both courses are required in the specific order, and the students must get at least a grade of C in each course to move on to their methods class.

The course that I focused on for this project is supposed to contain problem solving, number theory, numeration systems, divisibility, place value and bases, fractions, decimals and percents, and ratio and proportion, although Stephanie said that she has never had much time for the ratio and proportion ideas.

General Goals for Course

It was clear from talking to Stephanie that she enjoys teaching the first elementary mathematics content course. She said that, although she would be happy to teach other mathematics courses, she always requests the course for prospective teachers, because she likes it best. She even said that the course has "changed her life." She had some trouble describing what she likes about it so much, but she said, "I mean you'd think after teaching it for like seven times that I'd, you know once, you've seen it, you've seen it all. But I could do the same problem over and still be fascinated by it" (First Interview, 11/13/09, lines 223-225). Stephanie said that she does not get the same feelings from teaching calculus, and feels like she is always learning new things from her students in the mathematics content courses for prospective elementary teachers.

Stephanie had many related goals for her students in relation to the mathematics content course. She chose the textbook that she uses (Bassarear, 2007), because she read a statement that the author had made about owning one's mathematical knowledge rather than renting it. The statement resonated with her, and she said that it drives the way that she thinks about the course. She said, "the difference between renting and owning, and it's, that's why kids can't remember anything because they don't, they don't own it. . . the point of this course is for you to really, start owning your knowledge, not just renting it. When you're done with the class you pass it back in and, you know, it, it's not about that" (First Interview, 11/13/09, lines 421-427). So Stephanie's main goal was for her students to be able to own their mathematical knowledge at the end of taking her course.

Rather than renting it, and returning it, or "passing it back in," Stephanie wanted her students to remember the mathematics they did beyond the semester that they worked with her. In order to do this, she said that she tries to "give them time to play, to really play. Not just, pretend. They have to really, really engage" (First Interview, 11/13/09, lines 413-414) with the material. Stephanie tried to give her students opportunities to play with the mathematics and make discoveries, although she admitted that with time constraints, she was not always able to do this.

Another one of Stephanie's goals was to help expose her students to what she believes mathematics really is. She said that many of her students came from getting A's in high school, without really understanding the material. "They just, they really, really struggle. Those A students, and, because they memorized everything. And so, I guess it's part of my goal, is to, you know, take, take those students and say, you know, this is math. This other stuff you did, not so much" (First Interview, 11/13/09, lines 710-713). Stephanie talked about understanding mathematics as being able to explain concepts and use different models or manipulatives to dig deeper into what is going on, rather than relying on rules and algorithms all the time. While she stated that this was her goal for her students, and I do believe that this was her intent much of the time, she did not always seem to follow through with it. I will discuss instances of this later in the chapter.

Typical Classroom Session

Stephanie said that she tries to run her class sessions with a combination of lecture and discovery, however, she talked a lot about how difficult this is for her; "it's always a struggle of, am I lecturing too much?" (First Interview, 11/13/09, lines 512). Stephanie believes in the value of having the students make discoveries on their own, but she felt uncomfortable ending the day with students not fully understanding information that she feels is important. She described her struggles:

> There's some things they really just need to, to know. And to hear it from somebody. You know, I mean, it's nice to create these little individual minds and how they view things, but then to blurt it out and say what does this mean. . . There should be some kind of conclusion to some of this stuff. You know? And I feel like I really, I know I used to give the exploration problems in Basserear and there would be times when we just never went over them. We, they did them in groups, they handed them in. I gave them feedback. But there was no final finality to it. You know? And some of those problems were really, you know, really open-ended. (First Interview, 11/13/09, lines 516-525)

Stephanie believed that she was doing her students a disservice by not helping to explain the open-ended problems in their textbook, so she said that she felt like she spends more time lecturing than she used to, and she had gotten into a routine of "groupwork, lecture, groupwork, lecture." This matches up with what I observed from her classes.

In each of the classes that I observed, Stephanie started class off by working through some examples of the day's topic with the class. She then gave a worksheet or assignment that contained similar problems to the ones that she went over. She encouraged students to work in groups for the worksheets, but she did not usually require them too. Near the end of class, Stephanie usually went over worksheet problems with the class. At times, she had students come to the front of the class and explain what they did using the document camera, but for the most part, the full class discussions were led by Stephanie. She called on students to give input from their seats while she wrote on the worksheets on the document camera for the students to see her work. Rather than give daily homework, Stephanie assigned homework sets in the beginning of each unit that were due before each exam. I did not observe her going over homework problems with her students in class, but she did give them some time to talk about the homework before handing it in on the day it was due.

Goals for Multiplication/Division of Fractions

When I asked Stephanie about her goals for multiplication and division of fractions, she seemed to have trouble coming up with specific goals. First, she talked about how she teaches the topics. Stephanie said that she tries to relate multiplication and division of fractions to multiplication and division of whole numbers. When she taught multiplication and division, she spent time discussing various models such as the area model or cross-product model of multiplication or the partitive and quotitive models of division. Stephanie said that she tries to relate these models to multiplication and division of fractions, although she mostly uses area models with fraction multiplication and relates division to finding how many of one piece fit into another piece.

The majority of Stephanie's classwork with fractions involved the use of pattern blocks. Stephanie repeatedly exclaimed that she "loves them," and she grew very animated when she talked about them. Because pattern blocks were not the main focus in her textbook for doing multiplication and division of fractions, Stephanie did not use her book during the fractions unit except to assign homework problems. Stephanie learned about pattern blocks at a workshop she attended and fell in love with them. She said that she likes them because they are visual and because she believes using pattern blocks is more fun than the typical area model of multiplication (which would involve making a rectangular area that is one-third of a unit in length and one-half of a unit in width, for example). Stephanie had many students who told her that they are visual learners, and so she believed that the pattern blocks were helpful for these students.

When Stephanie talked about multiplication and division of whole numbers, in addition to the models, she discussed how she helps her students make sense of the algorithms, such as the one for long division: "I try to get a nice correlation between the algorithm and the, the conceptual" (First Interview, 11/13/09, lines 852-853). When she mentioned this, she implied that she tries to do the same thing for the algorithms for multiplication and division of fractions; however, she never gave clear examples of how she does this; it did not seem to appear anywhere in the planning notes that she shared with me, and she did not end up doing it during class. Therefore, it was unclear to me if she ever intended to discuss the algorithms in more detail. Part of the reason for this seems to be running out of time, but Stephanie also seems confused as to the value of going over the algorithms, as well as how she would go about doing so. The following is an excerpt from my second interview with Stephanie:

> SM: And in the past, what I normally do, is we talk about, um, uh, develop, like I develop other algorithms, say, well that's one that works. You know, I mean, there's lots of, I guess there's lots of ways you can explain it, the algorithm. Of why you invert and multiply.
>
> DO: Mm hmm.
>
> SM: But, I don't, I don't try to, and I probably should, make, you know

resolve it more for them.

DO: Mm hmm.

SM: But in the long run, is it really gonna matter that they, um, I guess they should know a mathematical reason for it. You know that, if you invert and multiply, you come up with the same answer.

DO: Right.

SM: But I'm not really sure, I'm not sure yet. My mind is not, I haven't figured out exactly, how important it is. It probably is more important than I'm, letting on. You know, because I want it to be, I want them to understand, first of all that the answer would make sense, if you inverted and multiplied.

DO: Mm hmm.

SM: Like the answer you get, by inverting and multiplying, actually conceptually, means something, using our old division model.

DO: Right.

SM: You know? So I kind of really heavily focus on that more than any, anything else, and I kind of kick myself sometimes for not, paying as much attention to, making that invert and multiply, rule, mean something. (Second Interview, 11/20/09, lines 1007-1039)

It seems like Stephanie struggled with a number of things in relation to the division of fractions algorithm. First, she believed that giving meaning to algorithms is important for her students in general, and she either believed or had been told that having meaning for the division of fractions algorithm is important for her students. However, she questioned how important this really is, wondering if it was really going to matter in the long run. The reasons behind her struggle may have to do with the fact that she did not have a clear idea of how to explain the division of fractions algorithm herself. While she states that “there’s lots of ways you can explain it, the algorithm,” she never discussed with me what any of these ways actually were, and I was left wondering if she lacked the confidence in her own ability to explain the algorithm in a way that would be clear for her students.

After discussing how she tries to teach multiplication and division of fractions, I finally got Stephanie to talk a little bit about her goals for her students:

DO: And what do you want them to be able to do, like by the end of it?

SM: Well, I do want them to work with this model [the pattern blocks]. I, you know, I don’t, one of our, we just got through writing course outlines, and one of the things that we said for topics, or one of our, not really goal, I guess it’s topics, we said let them work flexibly with fractions. So, um, I will want them to be able to do a problem like this [multiplication or division of fractions] on a test, and to be able to understand what that means, and, how to apply, pattern blocks to multiplying and dividing,

DO: Okay.

SM: fractions. Um, in terms of the, you know the meaning, the meanings of the algorithms, you know, I don’t really test on that. I think it’s more something you just discuss in class. And, you know. You know we let them use calculators too, which is kind of a, I wish we

> hadn't because I noticed that, even with the multiplication, or adding, subtracting strategies. And multiplication division strategies, they would, I think I told you this before, but they would basically, do, they'd do it on the calculator, know what answer to get, and then, do the strategy, to make it fit the answer. (First Interview, 11/13/09, lines 908-926)

Thus, Stephanie wanted her students to be able to understand and apply the models that she used in class to show multiplication and division of fractions, and to be exposed to some sort of meaning for the traditional algorithms with which they are familiar. She wanted them not to rely on their calculators or algorithms to find answers, but rather understand how the models work to show the processes of multiplication and division. She finished that part of the interview by saying, "if they left my class saying, well fractions make sense now, I'm happy" (First Interview, 11/13/09, line 1141).

Knowledge of Students

One of the key components of teacher knowledge in almost all of the current frameworks is knowledge of student difficulties and errors. Since Stephanie had taught the mathematics content course at least five times before at her current university, as well as a number of times at her previous college, I expected knowledge of students to be an area where she was strong. However, she seemed to have a difficult time relaying to me struggles that her students might have with multiplication and division of fractions. When I asked her what kinds of problems her students had with multiplication or division of fractions during our first interview, she responded, "it's hard to answer that question. I really never, I never really considered it. . . I, you know I don't, I don't know. I mean, I just, it seems like, I don't really, ask them to do it before we talk about the models, so I really can't see what the, I mean some of them just don't like this" (First Interview, 11/13/09, lines 968-976). It surprised me that this was something that Stephanie claimed not to have thought about before. It is possible that the question caught her off guard, as she was in the middle of teaching other fraction topics at the time, and had not seemed to have gone over her notes and prepared the multiplication and division of fractions sections for the current semester.

In a later discussion during the first interview, I again asked Stephanie about problems she thought her students might have with multiplication and division of fractions, and asked her to walk me through an example of how she would explain one-half of one-third. Again, she seemed to get flustered. "Oh, I knew you were gonna ask. Did you have to ask me that?" (line 1192). She then reminded me that she had another week to prepare this before she had to teach it, and she was able to explain the example, but it was clear that her understandings of both the material and the students' interactions with the material were somewhat shaky during our first interview.

To sum up Stephanie's understandings of her students prior to teaching multiplication and division of fractions this semester; Stephanie perceived that if her students struggled, it would be because they were not engaged with the pattern block model, rather than having difficulty understanding the ideas of multiplication and division of fractions with the model. Stephanie seemed to imply that if she presented the model clearly to her students, then they should be able to understand and use it, although they might be reluctant to use it if they do not see the model's usefulness in their teaching, and because it differs from what they are used to doing.

After teaching multiplication and division of fractions, Stephanie was better able to describe some student struggles:

SM: I knew that some people were gonna struggle. And I knew that they were gonna try, remember last time you said, what do you, what do you anticipate and I'm like, I don't know. I couldn't remember.

DO: Right.

SM: But after I taught the multiplication, I was thinking, oh, I betcha, I know what they do. They try to do multiplication for division problems.

DO: Mm hmm.

SM: Cause it's hard for them to, you know, and that's what usually happens on a test. (Third Interview, 11/24/09, lines 6-19)

After refreshing herself with the material, as well as seeing how her students did in the classroom with the multiplication, Stephanie was able to recall that students in the past had had trouble separating multiplication and division of fractions, especially using the pattern block model. This was evident in class when her students were working on the division problems:

s: This is what I did. I made it like this.

SM: Mm hmm.

s: We had two-thirds. And then one-quarter of that, is that.

SM: Are you multiplying or dividing?

s: Are we just multiplying?

SM: Yup.

s: Well I was good with this side [indicating the side of the worksheet with multiplication of fraction problems].

SM: Yup. I know. This side's a lot easier. (Class, 11/24/09, lines 749-763)

Stephanie attempted to deal with her students' struggles by asking them if they were multiplying or dividing, and pointed out how the division side of the worksheet was different from the multiplication side that they had worked on previously. She encouraged the students to think about division as "how many of this shape fit into the shaded region?", but it seemed clear from class that some of her students were merely trying to follow the form of the pattern block model without thinking about whether they were multiplying or dividing. From Stephanie's statement, it was clear that this is a common error that she has found, and that it usually persisted for some of her students into the exam.

After teaching division of fractions, Stephanie also talked about how students in the past have had trouble representing the remainders of division of fractions problems in fraction form, but thought that this group seemed to understand it much better than some of her classes in the past.

SM: And I was reading in our book, and it says, it talks about the remainder, you know. And I was gonna get into that, but I didn't want to confuse them. I figured well, if they're getting it,

DO: Right.

SM: and if they, and they can see that the half of two-fifths is one, you know one-fifth, then who am I to, I don't want to mess them up. (Third Interview, 11/24/09, lines 46-53)

The example that Stephanie asked in class had to do with having three quarts of water and determining how many days the water would last if each day a person drank two-fifths of a quart. The students seemed to have little trouble determining that the water would last for seven and a half days, so Stephanie decided not to bring up the solution of seven and one-fifth to avoid confusion. However, when the students were working on examples using pattern blocks, which were not given in the context of a story, they seemed to struggle more with understanding what the remainder would be. The following is part of a discussion Stephanie had with a group of students working on $\frac{1}{2}$ divided by $\frac{1}{3}$ using the pattern blocks. The students knew that the answer should be 1.5 from their calculators, but they did not understand how to relate the model to the answer.

SM: So, you want to figure out, how many, pairs of rhombuses go into the shaded region.

s: So, three.

SM: Well how many pairs of them? Not just singles.

s: Oh.

SM: How many pairs?

s: So one pair, goes into this.

SM: One pair, and a little bit left over. Right?

s: Right.

SM: Well how much left over?

s: I guess, a half of the third?

SM: A, one rhombus left over which is a half of the pair.

s: I'm confused. I can only get two in there, but then like, watch. (Class Observation, 11/24/09, lines 680-704)

Stephanie reminded the students that this was just like the example they did with the water, that one-fifth of a quart of water was equal to one-half of a day's supply and in this instance the one rhombus that they have left over is one-half of a pair of rhombuses, but it took the students longer to understand this example without the context of the day's supply of water. Since Stephanie did not end up asking a division problem with a remainder on either of the subsequent exams, it was not clear to me, and probably also not clear to her, how much her students understood this topic.

Karen—Introduction

Karen Freeny taught two different mathematics content courses for prospective elementary teachers in the mathematics department of a community college in the Northeastern United States. She has taught these courses each semester for the past ten years when she was asked to develop them by the chair of her department. Prior to this time, Karen had no prior experience teaching mathematics courses for elementary teachers, but she began learning about them at an all day session at an AMATYC (American Mathematical Association of Two-Year Colleges) conference in 2000.

Karen earned a bachelor's degree in mathematics along with certification to teach mathematics in grades 7-12. She earned a master's degree in mathematics education and taught both middle school and high school mathematics before her daughter was born. After taking a break from teaching to have children, she missed teaching and decided to try community college. She began teaching basic and intermediate algebra and general mathematics classes, which she did for five years before teaching mathematics for elementary teachers. Karen's only experience with elementary mathematics prior to developing the courses was volunteering in her children's classrooms.

In working to develop the courses, Karen said that she did a lot of reading and "went to every conference [she] could find" (KF, First Interview, 11/30/09, lines 38-39). One and a half years prior to participating in my study, Karen took a semester-long sabbatical to spend time in elementary schools. She went to kindergarten through grade six classrooms, both close to her home and across the country, in order to gain experience and knowledge about what and how mathematics was being taught in elementary school classrooms.

> KF: My students would say to me, well what kind of trouble, do students have when they're learning, lattice method of multiplication? I don't know.
>
> DO: Right.
>
> KF: When do they learn lattice method of multiplication? I don't really know. So, it was great for me to be able to get that sabbatical, to spend some time in the elementary schools. Watch the children learn. Learn how, or watch how, they're teaching it in the schools. (First Interview, 11/30/09, lines 312-321)

Karen used her sabbatical to be able to answer her students' and her own questions about how and where the mathematics she was teaching them showed up in the elementary school curriculum. I asked Karen if she had changed her teaching after her sabbatical.

> DO: Do you think, that you've changed a lot of stuff based on that?
>
> KF: Um, you know what? Not a lot of the content. But a lot of the way

that it's been able to be delivered. And, I think, it makes it easier for me to answer a lot of my students' questions.

DO: Mm hmm.

KF: That they used to have before that I, didn't know the answer to and now I do. (First Interview, 9/30/09, lines 374-382)

Karen used her sabbatical as a learning experience for herself to become much more comfortable with the elementary mathematics curriculum, and she was able to bring this information back to share with her students.

Mathematics Content Course

Karen was asked to develop mathematics content courses by the four-year colleges in her area because many of their students were beginning their college careers at the community college and transferring without having had any mathematics for teachers courses. She developed two courses, the first one on problem solving, numeration, different bases, integers, fractions, decimals, proportions, and set theory, and the second course, which focused on probability, statistics, and geometry. Since developing the courses she has taught one section of each course every semester and she has supervised another person who teaches a section of the first course. Karen said that she generally has to train a new person each year to teach the first course because, after teaching the course they realize that it is more work than teaching algebra, for which they could make the same amount of money, and they all leave after one year.

Karen's students vary from 18-year-olds fresh out of high school to people changing careers and deciding they want to be teachers. She said that she also always has a few graduate students from one of the local colleges, even though they have their own program. Her students take either one or both courses, depending on requirements where they plan to transfer after getting their two-year degree. Karen said that she has developed transfer agreements for her courses with all of the four-year colleges in the area so the students can transfer the credits to the four year schools.

General Goals for Course

Karen told me that she had a number of general goals for the course: first, she wanted to help her students understand the rules and algorithms that they know but do not understand, and to feel more comfortable with the mathematics that they are learning.

> KF: So I want them to have, a good, foundation, a good conceptual understanding of mathematics. Um, I want them to be comfortable about mathematics.
>
> DO: Mm hmm.
>
> KF: They may not grow to love math. But if they can understand it, and, not be as anxious about it, um, hopefully, when they're in the classroom, their anxieties and their dislike, are not gonna come through. (First Interview, 11/30/09, lines 191-199)

Karen knew that a teacher's positive attitude is crucial in developing positive attitudes about mathematics in her students. She believed that it was important for her students to develop a comfort level with the mathematics they were learning, and also to understand it deeply.

> KF: I talked to more students, who can remember the day, the time, what they were learning when they decided that they hated math.
>
> DO: Wow.

KF: Because the teacher, because the teacher, you know, either belittled them in class, couldn't answer their questions. Cause you know, I went over this four times and you should know this by now. Well, you know, the more you talk to the students, they went over it four times. They said it exactly the same way, those four times. They couldn't, explain what was going on. Therefore, the students never got it. They didn't understand it. (Second Interview, 12/03/09, lines 95-106)

Karen strongly believed that in order for her students to be good teachers, they must develop a deep understanding of the mathematics they will need to teach, including knowing the material in a number of different ways. Her goal for the class was to help her students do this.

Another one of Karen's goals, which is related to her first goal, was for her students to be able to see connections between what she teaches in class and what happens in elementary schools. She did this by showing her students worksheets and textbook sections from actual elementary curricula (class observation, 11/30/09, lines 26-32), and also making them observe a lesson, either in real life or on PBS Mathline and look at the mathematics that is going on.

KF: Um, but it just kind of gets them, into realizing that boy a lot of what we do in class, is used in the elementary schools.

DO: Mm hmm.

KF: Um, cause a lot of times, they never, they never used to make that connection. They would go why do I need to learn about this? Cause they're not doing that in the schools. Well yeah they are. And that's why I brought in those worksheets today. Just to quickly go through them and say, this is how they're using the fractions in the elementary school. Um, you know, and the whole idea of learning, multiplication, all these different ways.

DO: Mm hmm.

KF: You know I'd get one or two students who would say, "I don't understand, why we have to learn it this way. First, and why so many ways?"

DO: Right.

KF: Um, but if I, in the beginning of the chapter, say, these are all the different methods that they're using now, to teach multiplication, they know right up front, wow this is valuable stuff that she's gonna teach us. (First Interview, 11/30/09, lines 273-289)

Karen believed that her students would be more motivated to learn the mathematics that she was teaching them if they could see use for it in their own lives. She incorporated a service learning portion into her second content course, so students can get into classrooms and see how mathematics is being taught. Again, the purpose of this was to provide motivation to learn what she was teaching them. The students saw that she believed it was important, and she showed them how and why it would be important to them in the future.

A third goal that Karen had for her mathematics content courses is to weed out the students that were not interested in mathematics, to potentially discourage them from becoming teachers.

> KF: Cause so many students, get into elementary education because they think that they don't need to know math, or understand math. And, the very first day of class I show them a movie, first grade. And how much math is involved to teach first grade. And the commitment that they need to make, to learn the mathematics if they don't know it now, and understand it. And, so if I lose a lot of students between the first course and the second course, I don't see that as a bad thing. (First Interview, 11/30/09, lines 191-219)

Karen believed that it was her responsibility to prepare good future elementary teachers, and while she genuinely seemed to like all of her students, she stressed how important it was for them not to enter the classroom unprepared.

Typical Classroom Session

Karen told me that she organizes her class sessions by doing a combination of lecturing and group activities. This seems consistent with that I observed. She generally started class by going over homework questions. She collected and graded homework and she had each student pick one homework assignment that they would be responsible for grading during the semester. This was another way that Karen helped her students see other responsibilities they will have as teachers.

After she answered homework questions, Karen generally led the students in lecture notes, based on note templates that they downloaded before class. The templates had blanks for the students to fill in during the lesson. Most of the notes were very teacher directed—many of the class transcripts have pages of Karen talking with only brief contributions by students. However, she did ask for student explanations about why concepts work, and she helped the students use language that was mathematically correct and understandable. The following is an interaction with Karen and her students explaining why nine-fourths is the same as two and one-fourth.

KF: Why does that work?

s: Cause every four equals one.

KF: Say that again.

s: Cause every group of four equals one.

KF: Every group of four, equals one. What do you mean by that?

s: One whole number equals four.

KF: One whole number equals four. So one whole dollar equals four dollars?

s: No. Four quarters.

s: Every four quarters.

KF: Four quarters are a dol, okay. Okay. S?

s: The two represents four over four. And you have two of them.

KF: The two represents four over four. So are you saying that two is equal to four over four?

s: [Inaudible] four over four.

KF: And another four over four. (*She writes on the board.*) Cause four over four is really what number?

ss: One.

KF: So one plus one, is equal to two. All right. (Class Observation, 12/02/09, lines 65-99)

Even though the lecture was led by Karen, she stopped to ask her students why what she was doing made sense, and when they used language that was ambiguous, she helped them to clarify what they were talking about. An important aspect of being a teacher is good communication, and Karen wants her students to be able to communicate clearly when they are teaching.

Karen's notes followed the sections in her textbook; she expected her students to read the textbook and she held them responsible for the material in the textbook on exams. However, the activities that she gave were generally from other sources, as she said that she dislikes many of the activities from the workbook that comes with the textbook.

KF: I find I'm using more and more activities that I've put together.

DO: Mm hmm.

KF: rather than just right out of the workbook when I use them from the workbook. The wording's hard. My students misunderstand the directions. So I frequently have to go through and, you know, give them a, this is what he means by this and this is what we're looking for in this question. (First Interview, 11/30/09, lines 396-404)

Karen had a number of activities that she got from other sources, either from conferences or other textbooks, or activities that she had modified from these sources. She seemed to like more structure for her students than her textbook normally provided, so she tended to use activities that provided this structure.

Goals for Multiplication/Division of Fractions

Like the other teacher educators, Karen responded that she wanted her students to "understand" what multiplying and dividing fractions are. I asked her to be a bit more specific:

> KF: You know, I guess just a conceptual understanding. Um, you know, what it means to take, a third of eighteen.
>
> DO: Mm hmm.
>
> KF: That, it's really the same process as dividing eighteen by three.
>
> DO: Mm hmm.
>
> KF: Um, and that there's gonna be six, in each group. Um, and how that differs from, maybe six times a third.
>
> DO: Mm hmm.
>
> KF: Because, they have to understand, the difference between, those two problems, even though the answers are the same. (Second Interview, 12/03/09, lines 29-39) (*I think she meant the difference between a third of eighteen and eighteen times a third, since she went on to talk about how in another activity the students were concerned with modeling the answers, rather than the process of multiplying by fractions*.)
>
> KF: They need to understand, if they're gonna be in the classroom, they need to see the difference. Not only, between, a third times a fourth,

> but a fourth times a third. Because otherwise how are they gonna explain that difference? (Second Interview, 12/03/09, lines 77-80)

Karen was the only one of the teacher educators in my study who gave meaning to multiplication by talking about the difference between *A* times *B* and *B* times *A*. (Note the former is taking *B* things *A* times, while the latter is the reverse of this.) These can be two different processes, especially when fractions are involved, and one of Karen's goals was help her students understand this difference. In her worksheet on multiplication of fractions, Karen first had her students look at taking a fraction a whole number of times (e.g., four times three-fifths) and taking a fraction of a whole number (e.g., two-thirds of nine). She wanted them to understand how to model each of these types of situations and know the difference between them, before moving on to look at what it would mean to take a fraction of fraction.

Karen wanted her students to be able to make connections between multiplication of fractions and multiplication of whole numbers: "one of the things that I want them to see, is the relationship, between multiplication of fractions, and that model of multiplication as repeated addition that we talked about before" (Second Interview, 12/03/09, lines 300-302). Because of the way that she introduced fraction multiplication, by starting with multiplying a whole number times a fraction (four times three-fifths), she wanted students to think about this as taking the number three-fifths, four times. Later, she hoped they would connect this to multiplying two fractions, seeing one-half of one-fourth as taking one-fourth, one-half of a time.

In terms of division of fractions, Karen had students look at it in multiple ways.

> KF: I don't want them to think about multiplying by the reciprocal.

DO: Mm hmm.

KF: I want them, to know that basically what we're dealing with there is, equal groups. And how many times a fraction, fits into another fraction. That's what we're really looking at, when we're dividing with fractions. (Second Interview, 12/03/09, lines 403-410)

Again, this idea relates to how her students looked at division of whole numbers using the repeated subtraction idea of division. Karen's second way of looking at fraction division dealt with the idea of division as the inverse operation of multiplication.

KF: Tomorrow, let me get my worksheet out. We're gonna look at, division of fractions from an entirely different light.

DO: Is it this, activity?

KF: It is a division of fractions worksheet, yup. So we're gonna use the basic definition of division. That, if we have A divided by B is equal to C, then there exists a unique answer C, such that, the divisor multiplied by the quotient, is gonna give you your dividend. So, I'm gonna start off, relating that, to um, whole numbers. So we're gonna look at eighteen divided by three just to review. And that that translates into, three times what number is gonna be equal to eighteen. (Second Interview, 12/03/09, lines 484-495)

As Karen said, she started this activity by doing an example using whole numbers and then showing her students how to relate this example to division of fractions.

Karen admitted that she does have her students use the invert and multiply algorithm for division of fractions so that they can get some practice with it. "I don't teach it to them in class. But let's face it, they need to know it" (Second Interview, 12/03/09, lines 516-519). Karen understood that this algorithm is an important part of the elementary mathematics curriculum, so her students should know it. However, she also wanted them to be able to prove why the algorithm works. "And, I got this kind of proof thing going with them. Um, you know, how come, we multiply by the reciprocal and why that works" (Second Interview, 12/03/09, lines 558-560). In class, Karen led the students through a proof of why dividing by a fraction is the same as multiplying by its reciprocal.

> KF: Now, our third proof, is gonna deal, with this process that we just did up here. In general, how come, we can take any fraction, that we're dividing, and change it to multiply by the reciprocal? So we're gonna start out writing this, as, a complex fraction. So I'm gonna take A over B, divided by C over D. But we want to get rid of that C over D down in the denominator. So if we do, S what can we multiply it by?
>
> s: D over C.
>
> KF: D over C. And, if we multiply the denominator by D over C, S what do we multiply the numerator by?
>
> s: D over C.
>
> KF: D over C. Cause that's really what?
>
> s: One.
>
> KF: That's really our one. All right. So now our denominator goes to one, and I have AB times D over C, which will just be A, B,

> multiplied by D over C. (*She says AB, but writes A over B.*)
> Okay? So, not too involved there. Questions with that? (Class Observation, 12/04/09, lines 615-637)

Karen had a number of proofs from her textbook that she went over with her students and asked them to memorize. They were responsible for reproducing one of the proofs on the exam, though she did not tell them which one they would need to know in advance. She said that this helps her separate the A students from the B students, and I believe that she also wanted to help her students understand where the rules that they are using come from.

Karen said that she supplements the activities in her textbook on multiplication and division of fractions because believes that the book is too “traditional” (i.e., procedural and algorithmic), and she believes students need to know more. This is evident by the multiple ways that she looks at fraction multiplication and division, many of which are not in her textbook.

Knowledge of Students

When I asked Karen about problems her students had with multiplication and division of fractions, she mentioned their reliance on answer matching rather than knowing where the answer comes from. She stated that her students often modeled the answer to a problem, but were unable to show or talk about the process of getting the answer.

In order to help with this problem, the worksheets that Karen used had examples that modeled the problems at the top and then asked students to model similar examples.

However, even though the models were at the top of the page, many of the students still answered the questions merely by drawing a picture of the answer to the problem.

> KF: Now make sure, that up on your diagram, you show me, what a fourth of that set would be.
>
> s: Oh.
>
> KF: Okay. Just like they did up in the example. So don't just give me the answer.
>
> s: Do you want it five or do you want twenty over four?
>
> KF: What do you think?
>
> s: Well you said you can have improper fractions, but it's easier to write five, so.
>
> s: Yeah. We did end up circling all these.
>
> s: Does it matter?
>
> KF: Oh it's gonna matter. Cause you, I need to know, what half of twelve is gonna look like. Now you, I know you know the answer to half of twelve. Okay, six. So make sure that you can go through and show me on your diagram. (Class Observation, 12/02/09, lines 615-637)

Karen stressed to her students that it is not enough for them just to know that half of twelve is six, but they also must be able to show why this is so in a diagram. They must understand what it means to take half of a number and how to model this process, because even if they knew that half of twelve was six, they would need to be able to show their future students why this was so by dividing twelve into two groups of six.

Karen talked about the importance of the examples that she provided to her students on her worksheets, so that students could learn to learn from examples—when they are teachers, they will be in situations where they need to learn from examples, so rather than model everything directly, she wanted to give them experience with visual models and getting information from them. This goes along with her idea of helping her students develop confidence in their ability to learn mathematics on their own with guidance from the text or worksheet.

KF: I think is important that they take away from the class, is that they have, some confidence in their math abilities. Cause they're certainly not gonna remember everything that we've done in two semesters.

DO: Mm hmm.

KF: And there'll be things, there'll be new algorithms, there'll be new ways to present it, but if they don't have the confidence to look at those materials and figure out what it is they're trying to do,

DO: Mm hmm.

KF: then what good is it? (Second Interview, 12/03/09, lines 950-963)

Karen's goal was for her students to build confidence in their ability to learn from different sources, and she believed that it was her job to help them do this. Seaman and Szydlik (2007) would describe Karen's work as helping her students develop *mathematical sophistication*, which they describe as "possessing the *avenues of knowing* of the mathematical community that allow one to construct mathematics for oneself" (p. 172). Since teachers do not teach in a vacuum, they have resources available to them in order to learn the mathematics that they need to teach. However, they need to be able to construct mathematical knowledge from these resources. By providing examples at the top of her worksheets, Karen hoped to help her students develop tools to learn from the examples and construct their own mathematical understandings.

Much of the work Karen did seemed very scaffolded. Her worksheets and notes all developed concepts in a step-by-step manner. I am not sure if this is related to her personality/teaching style, or if it has to do with teaching at the community college level, though I suspect that both of these factors affected the way that she taught. She said, "I find that, students don't have, very clear direction. They get lost so easily" (Second Interview, 12/03/09, lines 923-924). Karen knew her students very well, and she developed a teaching style that she believed works well for the students that she teaches.

Summary

Each of the teacher educators in my study was motivated by the desire for their students to develop what they saw as a deeper understanding of the concepts behind the ideas of multiplication and division of fractions. Tom's main motivation was helping his students feel comfortable in his classroom. His goal was for students to see mathematics as something that is not scary and can be fun, and so he wants them to see that he has fun when he teaching. Since Tom cannot cover all of the content his students will need, especially in one course, and many of his students will not teach mathematics, his goal is for them to develop positive attitudes toward the subject, so if they have to teach it or talk about it, they will not do so negatively. Tom also seems to want his students to see that mathematics can make sense. He seemed to teach from what Simon and colleagues (2000) would call a *perception based perspective*. "Teaching [from a perception based perspective] involves creating opportunities for students to apprehend (perceive) the mathematical relationships that exist around them" (Simon et al., 2000, 594). Tom's students were not necessarily constructing their own knowledge by making sense of the material. Rather, Tom shared his own conceptual understandings with his students in the hopes that they would perceive or understand the mathematical relationships and connections. In this way, he hoped his students would see that mathematics can make sense and begin to feel less intimidated by the subject.

Stephanie was motivated by a desire for her students to be able to own rather than rent their mathematics. She had had positive experiences of her own where she discovered new ideas about mathematics that allowed her to "own her mathematics," and she wanted her students to have these same types of experiences. One of the things that helped Stephanie have meaningful experiences with mathematics was using pattern blocks to model fraction operations. By exposing her students to things that helped her understand the mathematics, Stephanie hoped her students would have the same types of experiences. However, she struggled with the idea of her students not understanding aspects of the mathematics, so there were times when she tried to expose students to her discoveries rather than letting them construct them on their own.

Karen's motivation came from her desire to prepare her students to be teachers. She was highly motivated by the fact that she was teaching future teachers, and her focus was on motivating students to learn mathematics by showing them how they would use in teaching. Many of Karen's activities focused around things that her students would do when they were teachers: using proper mathematical language, grading a homework assignment, evaluating a lesson, doing field work in the second course, and developing "mathematical sophistication" (Seaman & Szydlik, 2007) by learning from worked examples. Karen also desired to help provide clear direction to her students, because many of them came to her with weak mathematical backgrounds, and she believed that she needed to structure her lessons in a way that would help them to make sense of the mathematics.

Chapter 5—The Mathematical Tasks of Teaching Teachers

In Chapter 4, I described the three teacher educators whom I used as case studies in order to answer the question, *What is the mathematical knowledge required by teachers of elementary mathematics content courses in the area of multiplication and division of fractions?* In this chapter, I explore some of the mathematical tasks required by the teacher educators in their work teaching mathematics content courses for elementary teachers, as well as the evidence or lack of evidence of teacher educator knowledge that the tasks brought out. Because it would be impossible to highlight all of the tasks required by the work of teaching prospective teachers, I have chosen to focus on three tasks that played a major role in the data: introducing fraction multiplication, helping students make sense of fraction division, and assessing student understanding. I chose these tasks for a number of reasons. First, each is supported by the research literature as an important aspect of teacher knowledge of multiplication and division of fractions. Second, during my coding, these tasks came up as major themes in my data, and third, each of these tasks played important roles for all of the teacher educators in my study, and showed some examples of strong teacher educator knowledge as well as some gaps in the teacher educators' knowledge base.

Introducing Fraction Multiplication

Kazima, Pillay, and Adler (2008) describe designing meaningful "first encounters" with mathematical ideas as important tasks for teachers. They define these first encounters as "the first moment of the didactic process or process of study" (p. 285), and stress that these first encounters should be purposefully designed. In terms of this task for teacher educators, most of the encounters that prospective teachers have with the mathematical topics in their content courses are not “first encounters.” Thus, the job of the teacher educator is perhaps to provide a “deeper encounter” with the mathematics, or a first encounter into looking at the underlying mathematical features of a topic. I would argue, however, that introducing a mathematical topic to students, is still a task for teacher educators that requires thoughtful planning in order for the experience to be meaningful for the students. In addition, introducing mathematical concepts may provide more of a challenge for teacher educators than for teachers, since they are attempting to teach mathematics that their students “already know,” or at least think that they know (Smith, 2003; Wilson & Ball, 1996; Zopf, 2010). In this section, I discuss how each of the teacher educators in my study (re)introduced the topic of fraction multiplication to their students, and the mathematical knowledge that was required by the teacher educators in carrying out the task of providing their students with “first encounters” with the material, at least “first” as it relates to their class. To provide some context for this discussion, I begin by outlining from research some ideas of what a first encounter with fraction multiplication should look like.

There is no published literature on introducing fraction multiplication to teachers, but Mack (2000) suggests that in introducing this topic to students, it is important to build on the knowledge that students enter the classroom with. In the case of prospective teachers, the knowledge with which they enter the classroom is generally the algorithm for multiplying fractions. Therefore, helping prospective teachers to connect the fraction multiplication algorithm with new or deeper understandings of the process of multiplying fractions seems to be an important task for teacher educators. Other knowledge that prospective teachers should have before they work with fraction multiplication include knowledge of different representations of fractions such as area models, ratios, and operators, and different interpretations of multiplication, such as repeated addition multiplication or area/array models of multiplication. Therefore, building on these understandings that teachers have is important for teacher educators in introducing fraction multiplication.

Mack also suggests that partitioning ideas, that is breaking things into parts, are important building blocks for students' understanding of fraction multiplication. In her study, she introduced fraction multiplication problems in the context of stories where students were responsible for drawing a picture to interpret the problem and find the answer. Using partitioning ideas in the context of interpreting story problems helps students, and prospective teachers make sense of fraction multiplication.

In terms of what prospective teachers should know about fraction multiplication, research (e.g., Conference Board of Mathematical Sciences, 2001; Graeber, Tirosh, & Glover, 1989; Tirosh & Graber) suggests that teachers must know multiple interpretations of fraction multiplication. Since prospective teachers showed evidence of relying on the primitive model of repeated addition for multiplication (Graeber, Tirosh, & Glover, 1989), it is important that they extend their understandings beyond this model to include the compare, multiplicative change, combine, and part/whole models of multiplication outlined in Taber (1999).

While all of these ideas may not be contained in an introduction to fraction multiplication, it seems important that an introduction of this topic to prospective teachers would work to look at multiplication using one or more models and make connections to students' prior understandings.

Teacher Educators' Introduction to Fraction Multiplication

Tom began his lesson on multiplication of fractions by asking his students if it was a fair assumption to say that they know how to multiply two fractions. After much head nodding from his students, he added, "nothing in the world of mathematics, is as straightforward, as multiplying fractions. Whereby you multiply the numerators. You multiply the denominators. You grin. That's it. There it is" (TW, Class Observation, 11/10/09, lines 494-496). The idea behind doing this seemed to be in line with Tom's main goal of making his students feel comfortable with mathematics. He introduced a topic that many students find scary—fractions—by reminding them of a "straightforward" rule that they know. He did not call on any students directly, most likely in case any of them had forgotten the rule, but merely said the rule, performed a fraction multiplication problem on the board, and implied that if everything in mathematics were as easy as multiplying fractions, nobody would have problems.

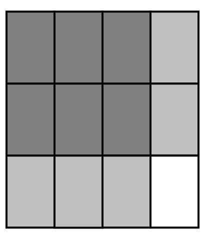
Then Tom moved on to the meatier part of his lesson: "But, anyway, but what I'd like to do for a minute, is to talk about, what this, what's really going on, when you multiply fractions" (TW, Class Observation, 11/10/09, lines 502-504). As he described to me in our interview, Tom explained to his students that they could think of multiplication of fractions as taking parts of parts, since fractions themselves could be thought of as parts of wholes.[1] He used a geoboard to model taking one-third of one-half and then one-half of one-third. He discussed the commutative property by pointing out that both of these situations resulted in one-sixth of the whole. During his discussion and

[1] One can certainly argue that defining a fraction as a "part of a whole" or fraction multiplication as "parts of parts" are incomplete definitions. While Tom did ask his students to perform fraction procedures with fractions greater than one, for example, on the exam, he asked his students to find ¼ of 8 ½, he never discussed any conceptual examples that involved multiplying fractions greater than 1, so for his demonstrations, "parts of parts" was appropriate.

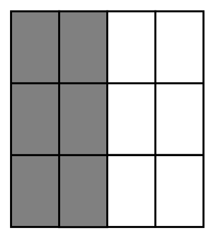
demonstration, Tom used the words "of" and "times" interchangeably, and wrote a multiplication symbol on the board while he read the problem as "one-half *of* one-third." Since Tom had only one course in which to teach all of the mathematics content that he felt necessary for his students, while this was his introduction to fraction multiplication, it also encompassed the majority of his teaching of the topic. He moved on to talking about other mathematical properties after the commutative property, and then assigned the students a number of fraction multiplication problems that were mainly procedural in nature, with the exception of the two problems represented by area models, which he later told his students not to worry about. A side note, is that while Tom said that he based his teaching on the textbook, he did not do any examples of fraction multiplication involving fractions greater than one, even though the text gives an area model for multiplying $2\frac{2}{3}$ by $3\frac{1}{2}$. This could have led to his students' confusion when asked to multiply a number greater than one by a fraction on their examination.

Like Tom, Stephanie's introduction to fraction multiplication dealt with an area problem. However, while Tom used an operator model of fraction multiplication, (i.e., finding a fraction of a given area), Stephanie's problem dealt with finding the area of a rectangular region and it was in the context of a story problem: *A group of investors purchased a rectangular parcel of land that is three-fourths of a mile long and two-thirds of a mile wide. How many square miles did they buy?* While Tom demonstrated fraction multiplication for his students, Stephanie gave an example that involved a picture, and asked her students to follow along and model the example with her. She asked her students what shape the piece of land would be, and had them all draw a rectangle in their notes. Then she related this problem to a similar problem, by asking the class the area of a rectangle that was two miles by three miles. She told the class that they knew their problem would be a multiplication problem because finding the area of the two by three rectangle involved multiplication. Next, Stephanie asked her class to draw a square, rather than a rectangle in their notes in order to model the problem. She proceeded to draw a square on the overhead and break it into fourths along the length and thirds along the width, shading in three of the fourths and two of the thirds. This resulted in six of the twelve smaller rectangles being doubly-shaded, or a rectangle that was three-fourths of a unit long by two-thirds of a unit wide (see Fig. 5.1). Stephanie showed her class how they could move some of the doubly-shaded pieces around to clearly show that one-half of the original square was shaded in.

Fig. 5.1.



Stephanie's representation of three-fourths times two-thirds



Stephanie moves the shaded two rectangles from the upper right to the lower left to show that ½ of the square is shaded

While this introduction to fraction multiplication was clear to her students as a multiplication problem, many of them had trouble understanding why, if they were finding a rectangular area, Stephanie had asked them to draw a square.

s: Is this a square now [inaudible]?

SM: Mm hmm. Well, okay, we, we kind of, we got the idea of a rectangular piece of land, right?

s: Mm hmm.

SM: That's gonna be our answer. But, in order to help us figure out, what this really means, I want you to think about the problem in terms of, three-fourths of a mile, means one whole mile, is, you're gonna, you can pace that off. . .

s: I still don't really understand how it is a square when it says it's three-fourths of the rectangle.

SM: Well, it says that the parcel's gonna be rectangular when we get done.

s: Okay.

SM: So we're gonna look at that shape. We need to figure what the whole thing is first, and then when we get done we're gonna form a rectangle.

(SM, class observation, 11/19/09, lines 123-131, 185-193)

Stephanie had trouble explaining to her students the reason that she was using a square instead of a rectangle to model the fraction multiplication. A student suggested that they were looking at measuring something out of a whole, and Stephanie agreed with this, but instead of talking about how a whole in this situation was one square mile, and hence the square, Stephanie emphasized the process of breaking up the square into pieces to form the rectangle. Thus, at first, more of the emphasis was on the procedure of demonstrating fraction multiplication using an area model than on understanding the idea of the whole being one square mile. After more confused questioning from students, Stephanie did explain that the reason that she was doing a square was because if the students had drawn a rectangle and labeled one side three-fourths and the other side two-thirds, they would not really be able to see what the area was in terms of the whole, which was a square mile. However, after this explanation, she went back to demonstrating the procedure, to show the students that the rectangle would just "pop right up in [their] faces," (line 209) once they had diagrammed it correctly.

Stephanie explained the point of modeling the multiplication to her students. She said that they all probably knew the rule to multiply fractions, where some of them multiplied first and then reduced, whereas others divided common factors out first and then multiplied, but "What we're doing today here, is to try to give you some un, basic conceptual understanding, apart, you know, kind of linking it to the rules, but not really just relying on them solely" (SM, Class Observation, 11/19/09, lines 318-320). The idea that she tried to get across was where the pieces in the rule came from—where the six-twelfths was in her diagram of the three-fourths by two-thirds rectangle, as well as how you could see that this represented one-half of the whole square mile in this case.

Once she had finished demonstrating the area model for three-fourths times two-thirds and showing the students how they could see the one-half in their answer, Stephanie continued by asking the students to try to use an area model to "find one-sixth *of* two-thirds."[2] She then went over this on the overhead and demonstrated an area model for multiplying three-fourths by two and a half. This problem was more complicated, because it involved using a three by one rectangle in order to model 2 ½ along one side. Stephanie went through this model with her students, rather than asking them to try it on their own, and she spent a lot of time discussing whether the answer was twenty out of twenty-four or fifteen-eighths (see Fig. 5.2).

[2]Although Stephanie phrased the question in this way, it does not seem like an appropriate question to use with an area model. While it is possible to model 1/6 *times* 2/3 using an area model by creating a rectangle that is 1/6 of a unit long by 2/3 of a unit wide, this does not demonstrate the concept of "one-sixth *of* two-thirds." Stephanie did not mention the difference in meaning of the two multiplication expressions, and I did not ask her about it.

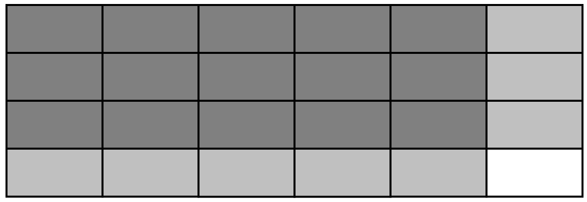


Fig. 5.2. The final picture has twenty of twenty-four rectangles double-shaded, but eight rectangles represent one-whole, since the model is 3 units long and 1 unit wide.

Stephanie spent the remainder of the lesson having her students work on modeling fraction multiplication using pattern blocks. This was different from the area model that she had her students work on in the beginning; though the students were finding fractions of areas, they were not finding rectangular areas using fractions.[3] She demonstrated one example to the class and asked them to work in groups on a worksheet of eight similar problems for the remainder of the lesson.

Karen also introduced fraction multiplication by using an area model, although she called it an array. She began by reminding her students how they used array models to multiply whole numbers, and showed the array multiplication of two times four. “Let’s go back, to what we’ve already done this semester, when we did our array model” (KF, Class Observation, 12/02/09, lines 250-251). In this way, Karen attempted to connect an idea that her students were already familiar with to multiplication of fractions. Karen’s first problem using array multiplication for fractions was four-thirds times two. Since she had already looked at two times four, she told her students that they could do exactly the same thing with the fractions. While Stephanie’s students seemed to struggle with the idea of having an answer that was greater than one, Karen seemed to get around this problem by demonstrating the entire process for her students and answering the

[3] For an explanation of the difference between the two models of multiplication, see the next section of this chapter.

questions that she asked herself. She modeled the multiplication by setting up a two by two square that she broke into pieces. Along what she called the horizontal axis (although she did not draw in actual axes), Karen divided her square into two equal sized intervals, and labeled them 1 and 2. Along the vertical axis, she broke the axis into six thirds, labeling each third up the side. Then she shaded in the figure up to $\frac{4}{3}$ on the vertical axis, and all the way to two on the horizontal axis (see Figure 5.3.) She then proceeded to explain the answer to her students: "So that's the area that we're interested in. Now, how many thirds do I have shaded in there? One, two, three, I might as well count, one, two, three, four, five, six, seven, eight. Eight-thirds. In terms of the diagram, where does that denominator come from? In your one, by one unit, you've got that broken up into thirds. So we had eight-thirds, shaded" (KF, Class Observation, 12/02/09, lines 288-292). Karen proceeded to discuss the traditional algorithm for multiplication and how this would also produce an answer of eight-thirds, and told her students that this is how the array model relates to the standard algorithm.

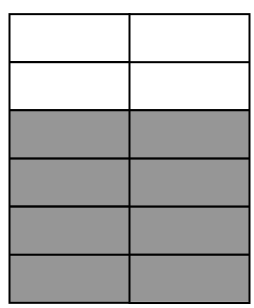


Fig. 5.3. Karen's array representation of $2 \times \frac{4}{3}$.

She then asked for questions. When there were none, she proceeded to demonstrate using the array model for $\frac{5}{6} \times \frac{3}{4}$. Again, Karen went through the whole example, modeling fourths on one axis and sixths on the other, and coming up with an answer of $\frac{15}{24}$. She explained to her students that the one-by-one square was the whole, and that it was broken into twenty-fourths, and since they had 15 of the twenty-fourths shaded in, then their answer would be $15/24$. At this point, she again asked for questions, but none of the students asked any. She then gave the class an example to work on of $4/3 \times 6/4$. At the time, I thought that this would be a difficult problem for the students to work on, since both numbers were greater than one, and the students did seem to have trouble with the problem. After walking around and talking to a few students, Karen decided to demonstrate the problem on the overhead. She wanted to show the students that since both of the numbers were greater than one, she would begin with a two by two rectangle. This time, however, Karen asked her students for contributions when she modeled the multiplication.

> KF: Now on that vertical axis, S, what number did you put on the vertical axis when you did it?
>
> s: Four-thirds.
>
> KF: Four-thirds. So we're gonna do six-fourths, along the horizontal axis. (KF, Class Observation, 12/02/09, lines 367-371)

Karen decided to go over this example with her students because she noticed that many of them were having trouble showing six-fourths on the horizontal axis, even though they were able to model four-thirds on the vertical axis.[4] After filling in the appropriate shading, Karen asked a student how many total rectangles were shaded in, and how many rectangles were in the one-by-one square. Since the answers were twenty-four and twelve respectively, the answer to the problem was $24/12$.

[4] I suspect that this was because Karen had already modeled $4/3$ along the vertical axis in the first multiplication problem, so the students merely copied this step from the previous problem.

After finishing this problem, Karen tried to relate the activity to her students' own teaching. She said:

> Now notice, if you had a worksheet, and you had your students going through, and doing the shading, and if you had them putting the answers in just like this, once they, went through, did their shading, figured out the solutions. Don't you think eventually, after doing a couple, they'd come up with their own rule? And they'd be saying to you, Miss Brown, Miss Brown, can't we just multiply these right straight across, rather than doing all this shading? So that they would come up, with a solution on their own, rather than have you tell them the solution. 'Cause then it's gonna stay with them and it's gonna make sense to what they're doing. So basically your algorithm, if you're multiplying fractions together, is to just multiply your numerators together and your denominators together. And, I am going to leave it there. (KF, Class Observation, 12/02/09, lines 400-408)

This statement follows Karen's goal of helping prepare her students to be teachers. This is one of many examples where she did and said things to get the students to think about what they would do in their own classrooms, even though her class was not a methods class. However, she did not have her students derive the algorithm themselves, since she anticipated that they were already familiar with it, but instead demonstrated how she saw the algorithm show up in the array model of multiplication, and how students who were unfamiliar with fraction multiplication could derive the formula.

After her demonstrations at the overhead, Karen assigned homework problems from the textbook and broke her students into groups to work on multiplication of fraction problems using a different model. This model built on multiplication as repeated addition, and also began with an example of multiplying a whole number by a fraction, similar to how she had done with the array model. In this way, Karen hoped to show her students how fraction multiplication ideas can build from whole number multiplication ideas with which they were already familiar.

Mathematical Knowledge in the Task of Introducing Fraction Multiplication

In terms of mathematical content knowledge, all of the teacher educators showed knowledge of the traditional multiplication of fractions algorithm, as well as how to model multiplication of fractions using a representation. Two of the teacher educators, Stephanie and Karen, demonstrated for their students how the traditional algorithm is connected to the area representation—where each of the pieces of a multiplication problem showed up in a pictorial representation of the multiplication. Tom, on the other hand, made reference to showing his students "what's really going on when you multiply fractions" (TW, Class Observation, 11/10/09, line 504), but after referring to the algorithm, he made no effort to connect his model to the algorithm. This could have to do with the type of model that Tom picked. While both Karen and Stephanie used an array, where each of the pieces of the multiplication problem are evident, Tom's model shows the process of taking a fraction of a fraction, which does not lend itself as easily to seeing the parts of the algorithm in the problem (see Fig. 5.4).

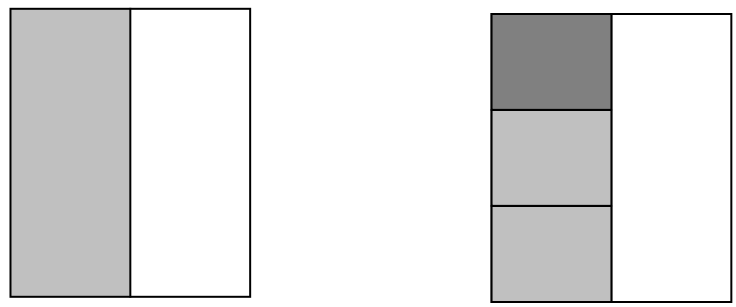


Fig.5.4. One-half of the rectangle is shaded. One-third of one-half of the rectangle is shaded.

While this model still uses an area representation of fractions, it is not an area model of fraction *multiplication*. The multiplication is an operator model, taking a fraction of a fraction.

Tom's choice of model as his only model of fraction multiplication may be seen as a lack of specialized content knowledge on his part in terms of not knowing other fraction representations that would better bring out the connections between the algorithm and the model, or it is possible that Tom decided that this connection was not important to bring out during his lesson during the time that he had. Whatever the reason, Tom missed an opportunity to help his students connect their prior knowledge to the new knowledge that he tried to share with them.

Stephanie and Karen also demonstrated other types of content knowledge by connecting their models of fraction multiplication to mathematics they had discussed earlier in the semester, specifically multiplication of whole numbers. Each related the area model of multiplication of whole numbers to their models of fraction multiplication. For Karen, this seemed to work, as she used it in later examples also, relating fraction multiplication to whole number of multiplication using both an array and the repeated addition model, and her students seemed to understand the relationship between multiplication of whole numbers and multiplication of fractions.

For Stephanie, however, this connection caused some difficulties for her students. They were used to drawing area models of whole number multiplication by drawing an *m* by *n* rectangle and figuring out how many square units were in the product. When Stephanie used a similar problem that involved multiplying fractions, she had the students start with the full square unit instead of drawing a rectangle right away. This model, that works with whole numbers, does not lend itself to showing the result of the fraction multiplication (see Fig. 5.5), which is why Stephanie switched to the square, but it led to confusion for her students.

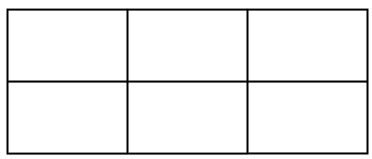


Fig.5.5. If you draw a rectangle and label the side lengths three-fourths and two-thirds, there is no way to tell how much of a square unit the area is.

Being able to choose good examples and use appropriate representations, especially to introduce a topic to students or to bring out certain subtleties, requires deep knowledge on the part of the teacher. In order to acquire this knowledge, the teacher must reflect on the model that he or she is using and determine the strengths and weaknesses of the representation. By choosing the examples and models that they did, both Stephanie and Tom may have caused misconceptions or may have been unhelpful in clearing up some misconceptions that their students had.

Choosing examples and representations that help students make mathematical connections or clear up misconceptions is part of the teachers' pedagogical content knowledge (PCK). Other aspects of PCK include knowing what concepts will be challenging for students, and anticipating what to do to help clear up confusion. Because Tom's main goal was for his students to understand the procedures behind fraction multiplication, he did not anticipate that they would have problems with the topic. "Um, they don't have much problem with multiplication, because they don't, if they don't know what they're doing, they'll give that a shot, and you know, it's one of those things where an infinite number of monkeys and an infinite number of typewriters, you'll eventually come, you know get all the right books" (TW, First Interview, 11/10/09, lines 446-450). Tom's point is that the multiplication algorithm is so easy that eventually his students will figure it out. The place where they all do struggle is with the area models of fraction multiplication, which are two problems that Tom assigns from his textbook. When the students ask about the question, Tom says, "That is, a geoboard type problem just like the kind of stuff that I showed you over here on the document camera. What, is the problem that's being illustrated here?" (TW, Class Observation, 11/10/09, lines 954-956). What Tom failed to realize is that despite the fact that what he modeled on the geoboard and the problem in the textbook both involve areas, the geoboard uses an area model of *fractions* and looks at multiplication as an operator, while the other uses an area model of *multiplication* (see Fig. 5.6). These problems both deal with areas, but do not use the same model of multiplication. Nothing that Tom did in his lesson prepared his students to understand or interpret the area model of multiplication, and therefore they struggled with the problems. It is likely that Tom had not thought about the differences between his multiplication model and the problems in the book, and did not anticipate the problems that his students would have with them.

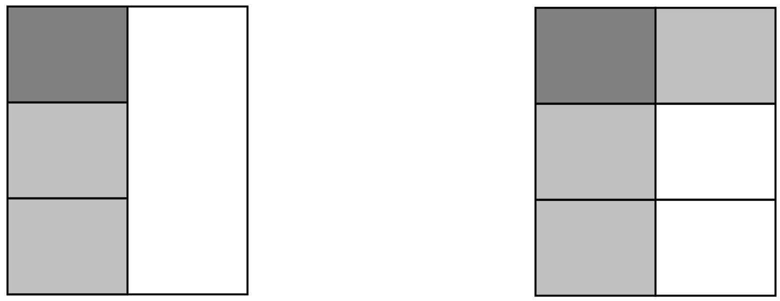


Fig. 5.6. The first picture shows Tom's model of $\frac{1}{3}$ of $\frac{1}{2}$. First, $\frac{1}{2}$ of the diagram is shaded. Then $\frac{1}{3}$ of this piece is double-shaded. The second picture shows an area model with one-half shaded vertically, one-third shaded horizontally resulting in one-sixth of the total area double shaded. While both representations result in $\frac{1}{6}$ of the total area being shaded in, the first representation involves an operator model of multiplication, taking a fraction of a fractional area, whereas the second representation involves multiplying a linear distance by a linear distance to result in an area.

Stephanie also had trouble anticipating the struggles that her students would have with her model of fraction multiplication. As stated above, she tried to relate multiplication of fractions to multiplication of whole numbers by using a rectangular area model of multiplication. She even began her multiplication problem by drawing a rectangle, but then realized that this rectangle would not help her create meaning for the multiplication and decided to draw a square. She seemed not to have anticipated the problems that this would cause for her students, and had a difficult time explaining why it was that she was doing what she was doing. Stronger knowledge of the differences between fraction multiplication and whole number multiplication using the area model could have helped Stephanie to highlight these differences for her students and caused her less discomfort during the lesson.

Karen seemed to have fewer student questions during her lesson. It is possible that she better anticipated problem areas and planned for their occurrence, however, it is more likely that the way she directed the lesson caused less outward confusion in her students. Unlike Stephanie's students, who asked questions while she was giving notes, Karen's students sat quietly and copied the notes. She said that they usually did ask questions, but they may have been intimidated because I was there. Whatever the case, it is difficult to see evidence of Karen's ability to anticipate and deal with student difficulties, because there was little evidence of difficulty by the students during the lesson.

So far, the majority of the knowledge evidenced by or lacking in the teacher educators is very similar to knowledge needed by teachers. All teachers should be familiar with the multiplication of fractions algorithm as well as different representations for representing fraction multiplication and the affordances and shortcomings of these models. Teachers should also know places where students have trouble with fraction multiplication and be able to think of ways to help clear up their students' misconceptions. So what mathematical knowledge is unique to teacher educators in terms of introducing fraction multiplication to students? First of all, the teacher educators need to be able to deal with their students' *prior knowledge* of fraction multiplication. Elementary teachers must know how to introduce a topic to children basically from scratch, but teacher educators must be able to introduce their students to *a new way of looking at fraction multiplication*. Teacher educators must be able to help their students connect what they know, which is usually the procedures, to new ideas that are behind the procedures and help facilitate the construction of a new, deeper knowledge base in their students.

Helping Students Make Sense of Fraction Division

The majority of the prior research on teachers' knowledge of fractions in general describes their lack of understanding of the division of fraction concept (e.g., Ball, 1990a; Borko et al., 1992; Eisenhart et al., 1993; Ma, 1999). Division of fractions has been called one of the most complicated topics in elementary mathematics (Lamon, 2007; Sowder et al., 1998), so it is not surprising that it is an area where many prospective and practicing teachers struggle. Thus, one of the tasks of teacher educators is to help prospective teachers to make sense of division of fractions. The literature gives us some ideas of what understanding division of fractions entails, which I describe below, and then I will look at how each of the teacher educators helped their students develop understandings of fraction division and the aspects of teacher educator knowledge that were highlighted by their teaching.

The Conference Board of Mathematical Sciences (CBMS, 2001) says that "understanding division of fractions requires a deep understanding of what fractions are, and of what division means" (p. 29). Thus, prospective teachers must be familiar with both the partitive or sharing and measurement or repeated subtraction meanings of division, and they must be able to relate these to fractions. The repeated subtraction model lends itself to using fractions much more easily. For example, $3\frac{2}{3} \div \frac{1}{6}$ can be thought of as "how many times does one-sixth go into three and two-thirds?" or "how many times can I subtract one-sixth from three and two-thirds?" The sharing model, however, which is the primitive model of division with which more people are familiar, (Fischbein et al., 1985; Graeber, Tirosh, & Glover, 1989), is much more difficult to think of using fractions. Using whole numbers, we can think of $20 \div 5$ as a sharing division problem by saying, "share 20 items among five people. How many items does each person get?" Thus, we are taking a number of items (20) and making a number of groups (5), and we want to determine how many are in each group. To translate this to fractions, we cannot think of $1\frac{3}{4} \div \frac{1}{2}$ as "take one and three-fourths items and distribute them to half a person." It does not make sense. Instead, we must think that we know the number of items ($1\frac{3}{4}$) and we are making a number of groups ($\frac{1}{2}$), and we want to know how many are in each group. In this case, the problem translates to, "one and three-fourths (pounds of chocolate) is one-half of a group (amount needed for a recipe). How much is a whole group?" While this model of division is more difficult to think about with fractions, Li (2008) points out that understanding the traditional invert-and-multiply division algorithm is easier when one looks at a partitive model of division.

In addition to the partitive and measurement models of division, Flores (2002) suggests that in order to have a "profound understanding of division of fractions," a person must understand the inverse relationship between multiplication and division, and that dividing by a number is the same as multiplying by its reciprocal. This idea also appears in Ma's (1999) description of teachers who have a profound understanding of mathematics, and having this type of understanding helps a person make connections among various topics in mathematics, something which is key in having a conceptual understanding of the topic.

Teacher Educators' Teaching of Fraction Division

Tom began his discussion of fraction division by asking the class to come up with a story problem that involved dividing by a fraction. This seemed out of character to me because it seemed to put his students on the spot and make them uncomfortable, which went against his goal of helping his students overcome their math anxiety and feel comfortable with the subject. However, I believe that Tom asked this question of his students because some of the major literature on teacher knowledge of fractions (e.g., Ball, 1990a; Ma, 1999) asks teachers to write division of fraction story problems, so he believed that it was an important concept for his students to know.

While research agrees that writing word problems is an important concept for teachers (e.g., Ma, 1999; Zopf, 2010), it is not evident that Tom's students actually acquired this skill, since none of them ever came up with a word problem that involved dividing by a fraction. The closest a student came to generating a problem was reading one from their textbook or coming up with a division of whole numbers problem. Tom provided his students with examples of both partitive and measurement fraction division problems, however, he did not discuss the differences in the two types of division.

In terms of helping his students make sense of fraction division, Tom did a few different things. After he discussed two story problems with his class involving fraction division, Tom noticed that some of his students were confused about whether the number of halves in ten was twenty or five. He went through and asked the students to divide 12 by six, three, two, one, and then one-half. The purpose of this was to show his students that "by dividing by something smaller, I'm getting something bigger" (TW class, 11/10/09, lines 834-835). Thus, if you divide by one-half, which is bigger than one, your answer should be greater than if you divide by one. This example helps dispel the misconception that many students have that dividing always results in a smaller number (Graeber, Tirosh, & Glover, 1989).

Next, Tom told his students that he wanted to "dig behind" the rule for dividing fractions. He asked students how they remembered the rule for fraction division, and got responses like, "skip, flip, multiply" and "keep, change, change." He quoted the adage "Ours is not to reason why. Just invert and multiply" (TW, class observation, 11/10/09, lines 876-877), and told the class that while most people can remember the rule, he wants them to know what it means rather than just remembering it. After this, he brought back the geoboard that he used to demonstrate fraction multiplication. He constructed a rectangle and divided it into thirds. He reminded the students what one-sixth of the whole looked like and asked them how many sixths were in one-third and how many sixths were in two-thirds. He explained that by asking how many sixths were in two-thirds, this was the same as two-thirds divided by one-sixth, just like asking how many fives are in twenty is another way of saying twenty divided by five. Tom then used the "keep, change, change" rule to figure out $\frac{2}{3} \div \frac{1}{6}$ and got $\frac{12}{3}$ which he said was "the right answer" (TW, class observation, 11/10/09, line 916). Presumably this is the right answer, because it is equivalent to four, which is the answer Tom got when he figured out how many sixths were in two-thirds using his model. While Tom said that he wanted his students to know "so much more than just the rules when you're dealing with kids" (TW, class observation, 11/10/09, line 923), he did not do anything to show the connection between his model and the algorithm beyond showing that both resulted in the same answer.

Tom spent portions of two classes talking about fraction division. He told his class that he hoped that the work they did in the first class looking at word problems and geoboard models meant something to them, and during the second class he wanted to show them another way of looking at fraction division. During this class, he showed an example of dividing two-thirds by one-fourth using a complex fraction model. Tom told his students that he wanted a denominator of one in his complex fraction, so in order to get one, he needed to multiply the denominator by its reciprocal, and then he needed to do the same thing to the numerator (see Figure 5.7), and "son of a gun, if that isn't exactly what we would do anyway, just following the, just following the directions that we normally, would have used, since the sixth grade" (TW, class observation, 11/12/09, lines 88-91). While Tom did not show that this method would work when dividing any two fractions, this one example was used to demonstrate why dividing by a fraction is the same as multiplying by the reciprocal.

$$\frac{\frac{2}{3}\times\frac{4}{1}}{\frac{1}{4}\times\frac{4}{1}}=\frac{\frac{2}{3}\times\frac{4}{1}}{1}=\frac{2}{3}\times\frac{4}{1}=\frac{8}{3}$$

Figure 5.7. Tom's complex fraction method of showing why one multiplies by the reciprocal when dividing fractions.

Tom talked about this example, ($2/3 \div 1/4$) and related it to a repeated subtraction word problem. If he was baking something that required two-thirds of a cup of something and all he had was a quarter-cup measuring cup, he would need to fill that quarter-cup two and two-thirds times to get two-thirds of a cup. Thus Tom tried to give some meaning to what $2/3 \div 1/4$ represents in a context, rather that just looking at a naked division problem without any context.

The last thing that Tom did in relation to fraction division is show his students the divide the numerators, divide the denominators "party trick" algorithm, which he calls a "gimmick." He began by asking students about the algorithm for fraction multiplication, and then said that if you can multiply across the numerators and denominators, it would make sense that you could divide across the numerators and denominators too. When a student asked why not teach this to children if it is so easy, Tom explained that it is not often when the numerators and denominators can be divided evenly, so it is a nice trick to know before you start the "long way," but it does not always make the problem easier.

Stephanie spent one class day on division of fractions. During our interviews, it seemed like she would have liked to be able to spend some more time working with division, but she had a lot of material to cover, and not enough time to do it, so she could not afford to spend more than one day on the topic. She explained to the class that she would begin with a couple of contextual problems, and then look at fraction division with pattern blocks.

Stephanie used two contextual problems that involved fractions: *Four friends decide to eat three pints of ice cream. How much ice cream does each person get?* and *Jake is stranded in the middle of the desert. He has three quarts of water and he figures that he will drink two-fifths of a quart each day. How many days' supply does he have?* The first of these problems does not involve dividing by a fraction, but the resulting quotient is a fraction. Stephanie led the class through this problem by drawing three different cylinders representing pints of ice cream and breaking each cylinder into fourths. Then she basically treated the problem as a sharing division problem, sharing $\frac{12}{4}$ pints of ice cream among three people. Each person received three of the fourths. After going over the problem, Stephanie reminded her students that the fraction that represented the answer, $\frac{3}{4}$, was another way of representing three divided by four, which was what they were doing in the problem—taking three whole pints and sharing or dividing them among four people.

Stephanie's second problem did involve dividing fractions. After she wrote the problem on the board, Stephanie asked her students to play around with it, suggesting that they work together to try to solve it, and that drawing a picture might help. The majority of the students were able to draw a diagram representing three quarts, divide each quart into fifths, and repeatedly subtract off two of the fifths to find an answer of seven and a half days' supply of water (see Figure 5.8).

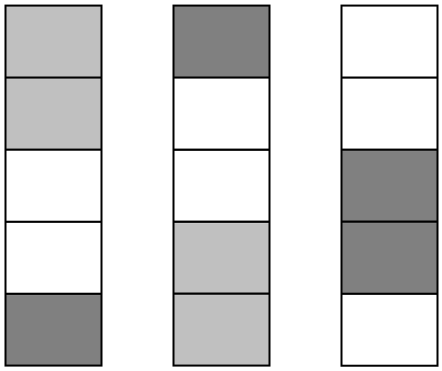


Figure 5.8. 3 quarts is divided into $\frac{2}{5}$ quart servings, showing 7 ½ servings in total.

The most common mistake that students made involved dividing the figures into sixths by drawing five lines instead of into fifths. Stephanie asked a student to present her answer to the class. Her model was very similar to the one above. Stephanie discussed the model further, reminding the students of the repeated subtraction method of division and how they were using it in this problem. There was some discussion of the remainder, whether the answer was 7 ½ or 7 $\frac{1}{5}$, but the students convinced themselves and each other that the one-fifth of a quart represented half a day's supply of water, so 7 ½ days made sense for the answer.

After determining the answer, Stephanie made sure that the students were able to translate the problem into a division number sentence. At first a student suggested that the problem was a multiplication problem, but Stephanie reminded the class that repeated subtraction was a form of division, not multiplication. Stephanie's next step was to ask the students what they remembered about the division algorithm. After reviewing the rule with her class, Stephanie implied that they would work on justifying the algorithm at a later date:

> we have this rule, that we, that we memorized, and, uh, at some point in our, our book, maybe not today, but some point we're gonna have to figure out, you know, the reason why, because, that's what this whole course is about. Why do things work the way they do? And, um, we may be able to see this later on. Um, but there, there's an actual reason for why, why it, it, works that way that we can, that we can flip. Um, so, let's just, let's just hold off on that. (SM, class observation, 11/24/09, lines 295-300).

Stephanie then told the class that what she wanted to do with them is look at using pattern block to model division. She never returned to justifying the algorithm.

In terms of using pattern blocks to model fraction division, Stephanie began with the example, $2 \div \frac{1}{6}$, and instructed her students to use the repeated subtraction method, and ask themselves, how many of the green triangles ($\frac{1}{6}$'s) go into two hexagons (2) (see Figure 5.9). Stephanie went over this example with the class and they determined that the answer was twelve. She commented that this answer was consistent with the algorithm, which gave an answer of twelve over one, and then she instructed the students to work on a worksheet with eight fraction division problems that used pattern blocks.

Stephanie walked around and helped the students, who were working in groups on the problems. The majority of the trouble that the students had stemmed from one of three things: (a) not knowing how to represent the fractional quantities using the pattern blocks; (b) trying to take a fraction of a fraction rather than asking how many of one piece fit into the other piece (multiplying instead of dividing); and (c) not understanding how to interpret the remainder as a fraction. Stephanie helped the students with their problems and encouraged them to check their answers using their division algorithm. "If I were you, I would go ahead and do the algorithm first at least to know what your answer is, and then go back and say how can I justify that answer" (SM, class observation, 11/24/09, lines 580-582.) She did not do anything to help the students relate the algorithm to the model that they were using, and the justification that she was looking for seemed to be being able to use pattern blocks to model repeated subtraction fraction division, rather than justifying the multiplying by the reciprocal division of fractions algorithm. The only connection between the two was that the answers matched.

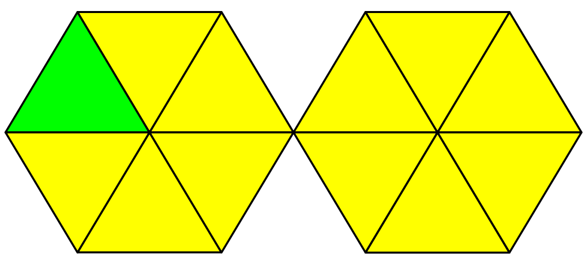


Figure 5.9. Pattern block model of $2 \div \frac{1}{6}$. How many green triangles fit into two hexagons?

Karen's group worksheet on multiplication of fractions had a section on division at the end of it. This involved using fraction bars and the repeated subtraction method of division to find out how many times one amount fit into another. Karen specifically says that this is how she wants her students to think about these problems:

KF: And then the last page, we were looking at, division with fractions. And, I don't want them to think about multiplying by the reciprocal.

DO: Mm hmm.

KF: I want them, to know that basically what we're dealing with there is, equal groups. And how many times a fraction, fits into another fraction. That's what we're really looking at, when we're dividing with fractions. (Second Interview, 12/03/09, lines 403-410)

Karen's students worked in groups on six division of fraction problems, and Karen said she was surprised at how smoothly it went. She was pleased that her students did not use the algorithm, but were able to illustrate with the fraction bars to show how many times one fraction fit into another.

Like Tom, Karen did an example with her class using a complex fraction to show why dividing by a fraction is the same as multiplying by its reciprocal. Karen used the example $\frac{\frac{15}{24}}{\frac{5}{6}}$ and worked through with the class how to get rid of the complex fraction by multiplying the $\frac{5}{6}$ by $\frac{6}{5}$, and then doing the same thing with the numerator. The reasoning for this was, as a student said, "Cause what you do to the bottom you do to the top" (KF, class observation, 12/04/09, line 197), and Karen continued to question the students to get them to explain that by multiplying the numerator and denominator by the same number, they were actually multiplying by one. After going over the numeric example, Karen showed a general example for why $\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C}$ using the same complex fraction method.

Karen's final activity with fraction division dealt with looking at division as the inverse of multiplication. She began by reminding students that to say $A \div B = C$ means that there is a unique number C such that $B \times C = A$. She used whole numbers as examples, showing that twenty-four divided by six equals four means that six times four equals twenty-four. She then moved on to look at the example $\frac{1}{2} \div \frac{1}{8}$. She said that this means that an eighth of something is going to be one-half[5]. Using a model of a rectangle as the whole, Karen divided it into eighths and labeled each eighth as $\frac{1}{2}$ (see Figure 5.10). Then she had the students count up the halves and they got that the whole would be equal to four.

Figure 5.10. This figure represents the whole. $\frac{1}{8}$ of the whole is $\frac{1}{2}$, so the whole is $8 \times \frac{1}{2}$ or 4.

$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Karen went through three examples using this method. When she presented a fourth example, a student asked if they could try one on their own. Karen gave them time to work on the problem $\frac{6}{11} \div \frac{3}{4}$, and after they had all tried it, a student presented her solution at the board. Karen gave her students a worksheet involving this method to do for homework. The end of the worksheet asked the students to solve the same problems using the division of fractions algorithm, because as Karen said, “I don’t teach it to them in class. But let’s face it, they need to know it” (MD, second interview, 12/03/09, lines 518-519).

Mathematical Knowledge in the Task of Helping Students Make Sense of Fraction Division

Helping students make sense of fraction division is a daunting task for teacher educators, which would require a lot of time in order to do in a way that was as in-depth as the research suggests is necessary. Each of the teacher educators in my study had fewer than two whole class days in order to cover fraction division, so they were not able to cover everything that they may have wanted to. Therefore, the task of making the short time they had meaningful for the students was even more difficult.

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[5] Note that in this case $\frac{1}{2}$ is a number representing $\frac{1}{2}$ of one or 0.5.

None of the teacher educators gave the opportunity for students to develop deep understanding of the sharing model of fraction division. While this is also common in elementary textbooks (Ott, Snook, & Gibson, 1991), it leaves a hole in terms of helping prospective teachers understand how fractions relate to the prominent model that people think about when they do division (Fischbein et al., 1985). Both Tom and Stephanie primarily used the repeated subtraction model to talk about fraction division. When Tom did use an example of partitive division, he did not focus on the model he was using, and Stephanie's sharing model involved division of whole numbers with a fractional quotient. Karen also did not use any examples of partitive division, but did model division as the inverse of multiplication, so her students were exposed to multiple ways of looking at division of fractions.

In terms of understanding fraction division themselves, all three of the teacher educators did seem to show aspects of deep conceptual understanding of the topic. Tom was able to write a variety of word problems on the spot, even modifying his students' examples so that they fit division by a fraction. He was also able to model fraction division with a geoboard and an algebraic complex fraction, and he was familiar with the "divide the numerators, divide the denominators" algorithm. Stephanie perhaps demonstrated the least variety in her presentation and knowledge of different ways of looking at division of fractions, as all of her examples that involved dividing by a fraction used the repeated subtraction model of division. It is unclear whether or not she had a strong knowledge of other forms of fraction division, because she did not use them in class or talk about them in our interviews. Karen showed understanding of the repeated subtraction model of fraction division, being able to look at fraction division as a complex fraction, and using the idea of division as the inverse of multiplication using models. She did not use any contextual problems, so it was unclear of either she or her students would be able to write word problems involving fraction division.

While each of the teacher educators seemed to have a deep understanding of many aspects of the content of fraction division, they all seemed to lack some of the pedagogical content knowledge that they may have needed in order to structure the lessons in a way that would help their students make connections between the algorithm and the models that they were using. Tom explicitly said that he used the geoboard model in order to help students have a deeper understanding of the "keep, change, change" rule, but he did nothing besides say this that would indicate that there was a connection between the model and the algorithm at all. Stephanie also indicated that there was a relationship between the division of fractions algorithm and the pattern block model that she used for fraction division, but the only real connection that she discussed between the two was that they produced the same answer. Karen did not discuss with her students that there was a connection between the models that she used, both "how many of this fraction fit into this fraction?" and the "division as the inverse of multiplication" idea, but she did prove that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ using a complex fraction model. She was the only one to do any sort of proof of this in general.

Neither Tom nor Stephanie proved in general why dividing by a fraction is the same as multiplying by its inverse. It can be assumed that Tom would have been able to show a general proof for this using complex fractions, but he did not seem to believe that it was necessary in the time that he had. Stephanie also questioned whether or not it was necessary to show her students a proof of why the division of fractions algorithm works.

As I discussed in Chapter 4, she seemed to be very conflicted in our interviews about how important this was, and I suspected that it may have had to do with the fact that she did not feel confident herself with any of the proofs with which she was familiar. While she stated that “there’s a lot of ways you can explain. . . why you invert and multiply” (SM, Second Interview, 11/20/09, line 1010), she never discussed any of these ways with me and did not have evidence of any proofs in her planning notes when she shared them with me. In the end, she was left with no choice but to skip a proof because she ran out of time, but it seemed like her knowledge of how she would do this, even if she had the time, was quite shaky.

In terms of the differences between teacher knowledge and teacher educator knowledge for the task of helping students make sense of fraction division, it seems that while teachers should be familiar with multiple representations of fraction division, they may not be responsible for presenting each of these representations to their students. However, teacher educators do have this responsibility because their students, future teachers, must know all of the representations. They must determine how to structure their lessons in order to expose prospective teachers to each of the different models of division, and also be able to show the relationships among the different models and the algorithm, which most prospective teachers enter the classroom knowing.

Time was also an important factor in what the teachers educators were able to present relating to division of fractions. Since it was evident that all three of the teacher educators in my study lacked the time necessary to cover everything relating to fraction division, they needed to be able to make decisions about what the most important aspects of the subject were, and build on their students’ knowledge of the division of fraction algorithm to help connect it to what they did in class. It is possible that a lack of deep knowledge of these connections caused the teacher educators to be unable to make these connections for their students.

Assessing Student Understanding

Assessing their students' understanding is a very important task for teacher educators. Without knowing what one's students know and how they know it, it is very difficult to modify one's teaching to help address misunderstandings or to continue using strategies that are resonating positively with students. Teacher educators, like teachers, have numerous opportunities to do informal assessments through reflecting in and on their actions, but when I asked the teacher educators in my study how they assessed their students' understandings, all three of them talked about exams.

Exams are the major way that many teachers and teacher educators, including those in my study, determine what and how their students are understanding the content. However, designing assessments to accurately assess teacher knowledge beyond procedures is very difficult. Researchers have spent many years and thousands of dollars in developing good assessment questions to determine the levels of teachers' content and pedagogical content knowledge (Hill, Schilling, & Ball, 2004). Teacher educators do not have an abundance of time and money to work on developing detailed assessments, but it is still a very important task for teacher educators to be able to see if and how their students are understanding their teaching, and to see if their students are meeting their goals.

Being able to assess students in a variety of ways is important in developing pedagogical content knowledge of one's students. In this section, I discuss how each of the teacher educators talked about and designed assessments, and what aspects of teacher educator knowledge are brought out by the task of assessing student understanding.

Teacher Educators' Designing and Using Assessments

When I asked Tom how he assessed his students, he said, "Exams is, is pretty much uh, is pretty much it" (First Interview, 11/10/09, lines 684-685). Since he did not collect homework or classwork, his students' grades were based on the four exams that they took during the semester. Aligning with his goal of making the students more comfortable with mathematics, Tom told me how in the past, he has allowed his students to take oral exams in his office if they claim to be bad test takers, and that if students can demonstrate that they understand a concept that they did poorly on on a later test, then he will give them points back.

I asked Tom what kinds of things he would test in regards to multiplication and division of fractions. He said,

> TW: Um, I'll probably ask for some procedure stuff. I'll probably pull some things right out of there [the text]. And, I will, probably ask them to make up, a problem, that, you know make up a problem that, is an example of, division by fractions. You know, just to see if they were paying attention when we were talking about that stuff (First Interview, 11/10/09, lines 721-724),

however, while Tom may have thought about including some conceptual based questions on his tests, his actual exam questions were very procedurally-based. There were three questions on the exam dealing with multiplication or division of fractions. The first was entirely procedurally based, asking students to divide $\frac{4}{9}$ by $\frac{4}{5}$ and write how they did it.

The other two involved performing operations after interpreting what the question was asking (How many $\frac{1}{3}$'s in $3\frac{2}{3}$? and What is $\frac{1}{4}$ of $8\frac{1}{2}$?). All of the students received full credit for the first question; although not all of them actually explained how they did the problem, they all did show some work. The majority of the students also got full credit for the second problem. The ones who missed the question generally answered $\frac{11}{3}$ rather than 11, and it was possible that they meant that there were 11 thirds in $3\frac{1}{3}$. Tom was surprised that his students had a lot of trouble interpreting the third problem. The majority of them did division instead of multiplication. Tom was not happy with this, and felt like he needed to spend more time emphasizing the meaning of words. He said, "if I had asked them what a quarter of eight was, they'd have got it" (Second Interview, 11/17/09, line 16), so he thought that they were confused by the wording of the problem when two fractions were involved.

After the exam I asked Tom if he felt that he had met his goals for multiplication and division of fractions, or if his students had met his goals. He responded,

> TW: Yeah. I guess. (*His tone suggested otherwise*). Um, that was, I've got to look back at multiplication of fractions today. That was just a, based on this, clearly that was not a, um, either I didn't ask, a good enough question to find out if they did it right, or they really don't know what they're talking about. (Second Interview, 11/17/09, lines 681-687)

Tom was clearly frustrated by his students' inability to do this problem. Procedurally, he knew that they could perform the multiplication algorithm, since they did it as part of the division, however, he wanted his students to be able to recognize a multiplication problem from the wording. When I asked him what he would do differently to help reinforce the multiplication idea, he did suggest that his students might understand the concept better if they were able to work with the conceptual ideas on their own: “it would be nice if I could round up a bunch of geoboards and have them do it themselves as opposed to, you know just, seeing me do it as a demonstration” (Second Interview, 11/17/09, lines 387-389). However, since he did not have the resources, he decided to remind his students that one of the problem-solving skills that they talked about in class was to look at a simpler problem, and ask them what one-fourth of eight was. Unfortunately, I was unable to attend the class when Tom handed back the exams and discussed the multiplication problem, but I suspect, in his eyes, that talking about simpler problems was sufficient for him to help the students understand how to do the problem.

When I asked Stephanie how she assessed her students, she answered tests and homework, as well as a comprehensive final exam. Stephanie gave three in-class exams during the semester, and had the students do test corrections as part of their grade. The students’ homework sets were due a week prior to the in-class exams, so that Stephanie could grade and return them to the students to use to study from.

I was interested in what Stephanie would include on her exams in relation to multiplication and division of fractions, so I met with her both after the third in-class exam, and also after the final exam to talk about the exam questions themselves, how the students had done, and what they had struggled with.

Stephanie had talked about intending to write a new test on the fraction unit, but she ran short on time and ended up using an old test, which only had one question on fraction multiplication and a bonus question dealing with division. The multiplication problem was as follows: *Find* $1\frac{1}{3} \times \frac{3}{4}$ *using pattern blocks. Use the diagram and shade appropriately. Explain.* The picture under the problem was of two hexagons.

Stephanie's students did not do particularly well on this multiplication problem. Only nine of the thirty students in her class received full credit on the problem. Most of the students were able to use the multiplication of fractions algorithm to get the answer correct, but very few were able to model the situation correctly with the pattern blocks. Stephanie and I discussed the students' work, and two of the most common mistakes were just shading in one hexagon, since the answer was one, or shading in three-fourths and one and one-third, but not modeling multiplication. The majority of the students who made the second error also seemed to be using two different wholes: for three-fourths, they shaded in three-fourths of the two hexagons (or one and one-half hexagons), but for one and a third, they used one hexagon as the whole. While Stephanie was able to identify the errors that her students made, as we discussed them together, she did not indicate what she would do to help the students fix their misunderstandings besides having them be responsible for test corrections. I had assumed that she would retest this topic on the final exam because of her students' poor performance, but after the final, Stephanie told me that had she purposely decided not to because she had asked it on the in-class exam.

An observation that I made in looking over the exams with Stephanie was that although nine students received full credit on the multiplication question, four of them had no explanation at all, and one other had very little explanation. These nine students all had the correct answer to the multiplication problem, and a correct model, but only four of them answered the full question, which asked them to explain their answers. This indicates that although Stephanie said that her goals were for her students to be able to explain the models of multiplication and division of fractions, she was more concerned with their ability to perform the algorithms and use the pattern block model correctly.

Because I had limited time with Stephanie, I did not get a chance to discuss the bonus problem with her. The problem asked students to divide 3 by $\frac{2}{3}$ using pattern blocks, and to shade in a diagram and explain their answer. Stephanie did tell me that her students did poorly on it. Five students left the problem completely blank; eleven were able to use the multiplying by the reciprocal division algorithm to get $4\frac{1}{2}$ as the answer, but were unable to shade in the diagram correctly or explain how division of fractions works using the pattern blocks. Two students received full bonus credit for modeling and interpreting the fraction division, while two others got partial credit for drawing an appropriate picture or having an appropriate explanation, but having an error. The remaining ten students received no credit on the problem because nothing that they wrote or drew was correct. Again, Stephanie seemed somewhat disappointed with her students' performance on this question, and at the time of my interview with her she told me that she would try to put a division problem on the final exam. However, when it came time for the final exam, Stephanie did not do this, and she did not share with me her reasoning for not doing so.

Stephanie's final exam contained eight multi-part questions, of which the students were required to choose six. Only two of the problems had anything to do with fractions, and only one part of one question involved fraction multiplication. There were no fraction division problems on the exam.

In discussing the final exam, Stephanie told me that she thought that I would find the multiplication problem interesting: *A student suggests that to multiply* $2\frac{1}{2}\times 3\frac{1}{4}$ *you can multiply* 2×3 *and* $\frac{1}{2}\times\frac{1}{4}$ *and then add the results. Do you agree with the student? Justify your answer.*

> SM: I like that question because it, it kind of got to the heart of whether or not they understand, um, (*she pauses*).
>
> DO: The distributive property?
>
> SM: the distributive property. Oh, yeah, and it was, it was something about some of the way they wrote their responses that seemed more, like, how do I say this? Like they weren't explaining the concept maybe. (Fifth Interview, 3/05/10, lines 370-377)

Out of the twenty-nine students who chose to answer this question, Stephanie only gave one full credit for it. This was because the students had all answered the question procedurally, stating that one cannot multiply in this way because you get a different answer than you would if you multiplied correctly (i.e., using the traditional multiplication algorithm.) At first Stephanie told me that she liked this question, because it gets to the heart of whether they understand, but later on in the interview, she decided that because all of her students answered procedurally, the question did not help her to understand if they really knew the concept or not.

I was somewhat surprised with Stephanie's final exam, due to the fact that she had told me that she was planning on putting more questions on it dealing with multiplication and division of fractions. Even the question that involved multiplication did not get at any of the goals that Stephanie had indicated that she wanted her students to achieve in regards to fraction multiplication or division. Because of a number of unfortunate circumstances, my post final exam interview with Stephanie occurred a number of months after she had given the final, so it was hard for her to remember her rationale for choosing or not choosing particular questions; however, she did tell me that the exam questions came from a question bank connected to her textbook, and she did not write the questions herself. Many of Stephanie's issues related to assessment have indications of her knowledge as a teacher educator, and these will be discussed later in this section.

When I asked Karen how she assessed whether or not her students had learned what she wanted them to learn, she, like the other teachers educators, responded by talking about exams. Unlike the others, however, Karen was explicit about what types of things she would test on the exams:

> KF: On their test, I will give them questions where they're gonna have to reproduce, some of the, concepts that we talked about on these worksheets.
>
> DO: Mm hmm.
>
> KF: So that they'll be given fraction bars, and they'll have to demonstrate, um, with the fraction bars maybe, you know, two-sixths divided by one-sixth, equals, what.
>
> DO: Mm hmm.
>
> KF: You know and they're gonna have to show me, why that's gonna be twice, why that's gonna be two.
>
> DO: Right.

KF: Um, and, not only do they have to go through and, use the methods that we talked about on these sheets, but they also have to explain their reasoning.

DO: Mm hmm.

KF: So they have points for diagramming it, and then points for explaining their reasoning. Um, and these all count. (Second Interview, 12/03/09, lines 1056-1079)

Karen made very clear that she designed her assessments to align with her goals for the course. She wanted her students to be able to model multiplication and division, using things like area models and fraction bars, as well as be able to explain what the models represent. Out of the three teacher educators, Karen was the person who asked the most questions on both her in-class exam and the final exam on multiplication and division of fractions (five on the in-class exam and four on the final), and she also required the most from her students in terms of modeling and explaining their models. She also asked her students to explain how to use the traditional algorithm to multiply fractions, because that is something that they will need to be able to do in order to teach fraction multiplication to their own students.

In terms of grading her students exams, Karen paid close attention not only to her students' work, but also to their explanations. If a diagram was correct, but an explanation was not, she gave little or no credit. This was because she believed that an important task of being a teacher was to be able to explain mathematics to students, and she wanted to emphasize the importance of this by making it a primary focus in terms of earning a good grade in her course.

Unfortunately, I was unable to spend much of time talking to Karen about her students' performance on their exams. She did give me copies of the exams with her grades on them, and she told me that she was not pleased with how her students had done on the in-class exam. Although her students seemed to be able to use array models and fraction bars to demonstrate multiplication and division of fractions while they were working in groups on worksheets with examples at the top of the page, the majority of her students struggled to reproduce these types of models and explanations on their own on the exam. They faired somewhat better on the final exam. Karen said that she saw improvement on the final exam, as her students were able, in many cases, to put together information from the entire course.

Because I did not conduct a full interview with Karen after her students took their exams, I was unable to ask her what kinds of things she would change in order to help her students demonstrate facility with the models on exams as well as in class, but she did indicate that she would spend some more time in class working with her students' understanding of the models, and throughout our interviews during her teaching of fraction multiplication and division, Karen talked about how she had modified her activities in the past and how she would do more changing in the future, based on what her students had and had not understood.

Mathematical Knowledge in the Task of Assessing Student Understanding

In terms of existing frameworks for teacher knowledge (e.g., Ball, Thames, & Phelps, 2008; Shulman, 1986), none of them specifically contain a category entitled knowledge of designing and using assessments. In a personal conversation with Laurie Sleep (January 28, 2011), one of the researchers on the University of Michigan's

Learning Mathematics for Teaching Project, she said that she saw assessment as fitting into a number of categories in their "egg" framework of mathematical knowledge for teaching. Part of assessment is knowing areas where students have trouble, and picking questions that help with these difficulties. This involves knowledge of content and students. Knowledge of content and teaching comes into play with designing problems that test the content that one wants to test. Specialized content knowledge is involved in terms of knowing what it is that one wants to test in the first place. While this all makes sense, there does seem to be a gap in the literature on knowledge frameworks that explicitly deals with assessing students' understandings.

While assessing their students' understanding is important for any teacher, the task becomes more complicated for teacher educators because they must assess their students' understandings of multiple models of mathematical topics. Like prospective teachers, children learning fraction multiplication and division should be exposed to a number of different models. However, unlike their teachers, children are not necessarily responsible for mastering and understanding each method. For example, it is unlikely that children would be asked to create word problems that model fraction situations, but designing problems is an important task for teachers, so it is something that they must be able to do well. For students, their teachers must provide them with opportunities to solve word problems and relate these problems to mathematical operations. Therefore, teachers must know much more. As a result, teacher educators must be knowledgeable in ways of assessing their students' relational understandings of fraction concepts, and designing and using assessments to test these deep understandings is very difficult to do (Skemp, 1976).

Neither Tom nor Stephanie designed assessments that aligned with their professed goals of having students develop a conceptual understanding of multiplication and division of fractions. It is possible that they were not sure of how to do this. Tom's questions all dealt primarily with procedures, and while the questions on Stephanie's first exam dealt with a conceptual model of fraction multiplication and division (pattern blocks), the vast majority of her students did not demonstrate abilities to work with the models or explain what the models showed, and because Stephanie did not penalize her students for not explaining their models, they may have come to understand that explanations are not as important and being able to do the work. Karen's assessments, on the other hand, did seem to align more with her teaching goals, and although I was not able to speak with her about her exams as much as I spoke with Tom and Stephanie, Karen talked in the interviews about how she would change her worksheets and activities based on areas where her students had difficulties, so it seemed that she was able to use the results of her assessments more than the other two.

Summary

It is clear that the job of teaching prospective teachers multiplication and division of fractions is not an easy one. The tasks involved in teaching these concepts for teacher educators are many, and I have chosen only three on which to focus. From these three tasks it is evident that the knowledge required for them is considerable. One of the reasons for this is that teachers need to know more than their students. Thus the prospective teachers need to develop much deeper understandings of the mathematics content than they had as students, and their teachers, mathematics teacher educators, must have the knowledge necessary to help with this development. Since mathematics teacher educators are the teachers of prospective teachers, their level of mathematical knowledge must be deeper still. They must be able to build on their students' prior understandings, which are often procedurally based and sometimes flawed, to help prospective teachers develop understandings that connect the procedures to the underlying concepts. They must be able to choose examples and representations that bring out these concepts, usually in a short period of time, making choices about what to include and what to leave out. And, teacher educators must be able to design and use assessments that test these relational understandings, which is often difficult to do. I will provide more of a summary of teacher educator knowledge as well as look at some ways that teacher educator knowledge might develop in Chapter 6.

Chapter 6—Mathematical Knowledge for Teaching Teachers: Where Do We Go From Here?

This study attempted to answer the question, *What is the mathematical knowledge required by teachers of elementary mathematics content courses in the area of multiplication and division of fractions*? In doing this, I interviewed and observed three experienced teacher educators of mathematics content courses to determine the tasks involved in the work of teaching teachers multiplication and division of fractions and the knowledge that was required by these tasks. From my data, I determined three of the major tasks involved in this work were introducing fraction multiplication, helping prospective teachers make sense of fraction division, and assessing student understanding. Through analyzing the knowledge required to perform these tasks, I determined that the mathematical knowledge for teaching teachers multiplication and division of fractions is vast and complex, and is not easily developed.

In terms of introducing fraction multiplication, as well as introducing any topic to prospective teachers, teacher educators must take into consideration the fact that their students come into their classes with preconceived understandings about the topic that are usually very procedurally based and often flawed. Teacher educators must be able to build on their students' prior understandings and help clear up misconceptions as well as introduce new ways of looking at the topics through modeling, which their students may not be familiar with from their prior schooling. Teacher educators must be able to make connections between relationships with whole numbers and fractions and help bring meaning to the fragile understandings of algorithms that many prospective teachers hold.

The topic of division of fractions is one of the most complicated in all of elementary mathematics (Lamon, 2007; Sowder et al., 1998). For each of the teacher educators in my study, the difficulty in teaching this topic was compounded by the fact that they had a very limited time in which to do so. Because a lack of time to spend on topics seems to be the norm rather than the exception for teacher educators, the knowledge required by them to determine how and what aspects of a topic to emphasize is important. Teacher educators need to be able to draw out the most important features of a topic and make connections in order to help prospective teachers deepen their understandings.

While most prospective elementary teachers enter their mathematics content courses with knowledge of the invert and multiply algorithm for fraction division, many of them have no understanding of where this algorithm comes from or why it works. For teacher educators whose goals are for their students to develop conceptual understanding of why algorithms work, they must first have these understandings themselves. While it certainly seems beneficial to model fraction division using repeated subtraction ideas, if teacher educators truly want to connect these models to the algorithm, they must know how to do so, and this connection is not a straightforward one.[6]

Being able to assess student understanding and adapt one's teaching to help bring about better understandings is important for any teacher. For mathematics teacher educators, this task is very involved because they need to assess their students' understandings of multiple ways of understanding a single topic. Two of the three teacher educators in my study, Tom and Stephanie, did not design assessments that

[6] The repeated subtraction model of the division problem $\frac{a}{b} \div \frac{c}{d}$ asks how many times does $\frac{c}{d}$ go into $\frac{a}{b}$. Using a model, we can tell that $\frac{c}{d}$ goes into one whole $\frac{d}{c}$ times. If we want to find out how many times $\frac{c}{d}$ go into $\frac{a}{b}$ of one whole, we must multiply $\frac{d}{c}$ by the fraction of the one whole that we have, $\frac{a}{b}$. Thus $\frac{a}{b} \div \frac{c}{d}$ is equivalent to $\frac{a}{b} \times \frac{d}{c}$.

helped them figure out if their students were meeting their goals of developing deep, conceptual understandings of multiplication and division of fractions. It is possible that they did not know how to design these types of assessments, as it is not an easy task (Skemp, 1976). Karen's assessments, on the other hand, were much more aligned with her teaching goals, and she was able to explain how she would use her students' performance on the exams to modify her worksheets for the following semester. In terms of designing and using assessments, Karen seemed to have a deeper mathematical knowledge than the either Tom or Stephanie, which may be due to a number of factors which I discuss later in this chapter.

Aspects of the Mathematical Knowledge Required for Teaching Teachers

Each of the three tasks above brings out aspects of a framework for teacher educator knowledge as it relates to multiplication and division of fractions. When we look at these three tasks together, we can begin to think about what might be included in a framework for teacher educator knowledge of this content. First, teacher educators must themselves understand multiple ways of representing fraction multiplication and division, as well as how these representations relate to each other, to whole number ideas, and to the algorithms. By having a vast knowledge of different ways of modeling multiplication and division of fractions, teacher educators can choose examples that help highlight the different representations that their students will need to be familiar with in their future work. Teacher educators must also be aware of how these models relate to multiplication and division of whole numbers. While there are often connections between operations with whole numbers and fractions, there are often subtleties that working with fractions brings out, that teacher educators must be aware of. For example, it is difficult to translate a sharing model of division to fractions, since it does not make sense to talk about sharing among a fraction of a person. Teacher educators must be aware that this model, also known as partitive division, gives a total amount and the number of groups to be formed and asks how many elements are in the whole group. Understanding the sharing model of division in this way would help a teacher educator to design examples that use the model of division with fractions. This in turn could be used to more easily connect to the invert and multiply algorithm for fraction division, since this model relates to the algorithm more easily than the repeated subtraction model of division.

Another important aspect that should be included in a framework for teacher educator knowledge of multiplication and division of fractions is being able to decide which aspects of a topic will help prospective teachers make the mathematical connections that they themselves will need to teach a topic. In the case of each of the teacher educators in my study, time played an important factor in what the teacher educators taught to their students, and it is reasonable to assume that this is not something that is unique to these teacher educators. Therefore, making curricular decisions about what to teach and how much time to spend on each topic is an important task for teacher educators. If teacher educators believe that prospective teachers must be able to explain why various algorithms for multiplication and division of fractions work, they must be able to dedicate enough time in their courses for these topics. They must also decide how best to go about explaining justifications for these algorithms, and the affordances and constraints of particular examples. For example, both Tom and Karen used a complex fraction approach to justify the invert and multiply algorithm for division. While this justification is accessible to their students, who have studied algebra, it will likely not be accessible to fifth and sixth grade students. So, while this justification may help prospective teachers make sense of the algorithm, it will not necessarily be useful to them in their teaching. Teacher educators must be able to understand the consequences of this justification and decide whether or not they want their students to have a different understanding of the topic. If so, they must be able to choose relevant examples to help bring out this different understanding.

Another aspect of a framework for teacher educator knowledge, is being able to set specific learning goals for one's students. Each of Tom, Stephanie, and Karen said that one of their goals was for their students to develop conceptual understanding of multiplication and division of fractions, however, none of them gave a detailed explanation of what this conceptual understanding would actually entail. By setting specific learning goals, such as, "I want my students to be able to demonstrate fraction multiplication using both an area model of multiplication and a operator model of fraction multiplication with pattern blocks. I want my students to be able to describe each model using an example, and explain how the answer to the multiplication problem relates to their models," would give the teacher educators a tangible goal toward which to work. This way, they would better be able to choose examples to help them meet their goals, as well as better assess if they are meeting their goals.

Designing and using assessments to help decide if one is achieving one's goals would be a fourth aspect of teacher educator knowledge. I have discussed above how the teacher educators in my study, particularly Tom and Stephanie, seemed to have trouble designing assessments to test their students' understandings. As a result, they were unable to determine if their students' had met their goals of developing conceptual understanding of fraction multiplication and division, because they did not test these conceptual aspects. Therefore, they were not able to learn as much about the effects of their teaching as they perhaps could have, had they designed better assessments.

While this is by no means a complete framework for the mathematical knowledge required by teacher educators of prospective elementary teachers on the topic of multiplication and division of fractions, the four aspects that I have outlined: understanding multiple representations/models of fraction multiplication and division, deciding which aspects of a topic to specifically focus on, setting specific goals of exactly what one wants one's students to know, and designing and using meaningful assessments to help decide if one is achieving one's goals; are certainly important components of such a framework. But how and when does mathematical knowledge for teaching teachers develop? Some ideas of the answers to these questions follow in the next section.

How Does One Develop MKTT?

In their nationwide study, Masingila, Olanoff, and Kwaka (2011) sent questionnaires about the characteristics of teachers of mathematics content courses for prospective elementary teachers to 1,926 two- and four-year institutions in the United States and received responses from 825 of them. The final two questions on the survey asked if there was any training or support for teacher educators of elementary mathematics content courses and what this support entailed. Only 44.3% of the institutions that answered these questions reported any kind of training for their teacher educators whatsoever, and the majority of these institutions (57%) reported only informal types of training or support. Thus, the majority of teacher educators of mathematics content courses for prospective elementary teachers are in situations where they must learn and develop the mathematical knowledge required by their jobs on their own. This seems like a difficult task, and in the case of all three teacher educators in my study, something that can result in knowledge gaps or incomplete understandings.

Researchers at the University of Delaware suggest that one way to help improve mathematical knowledge for teaching is for mathematics teacher educators to have clearly defined learning goals. “Learning goals specify the essential knowledge, skills, and dispositions that mathematics teacher educators are striving to help prospective teachers develop. In this way, learning goals shape the content of the knowledge base by determining what knowledge will be built” (Jansen, Bartell, & Berk, 2009, p. 526). In a personal conversation with James Hiebert (October, 27, 2009), he described the difference between having a general learning goal for fraction multiplication: “My goal is for preservice teachers to understand the algorithms, not just perform them,” versus a clearly defined learning goal: “My goal is for preservice teachers to understand the relationship between these three different ways of multiplying fractions.” Hiebert contended that the more specific the learning goal, the more likely it will be that teachers will be able to figure out ways to meet their goals, since they know specifically what they are striving for. With only a general goal to try to reach, it is difficult to pinpoint what it means for teachers to achieve this goal, and therefore, difficult to figure out what to do in one’s teaching to meet the goal.

The Mathematics Education program at the University of Delaware has put a strong emphasis on developing their mathematics content courses for prospective teachers and strengthening the knowledge of their teacher educators (Hiebert & Morris, 2009; Hiebert et al., 2007; Jansen, Bartell, & Berk, 2009). The professors and graduate students who work there collaboratively developed their three mathematics content courses for prospective elementary teachers through a process of modified lesson study over a number of years. In order to learn more about their program and their teacher educators, I interviewed Dawn Berk, the faculty member who led the development of the mathematics for teachers course that deals with multiplication and division of fractions, and I observed two of their graduate students teach one lesson on fraction division.

According to Berk, one of the major emphases for developing the course was in establishing concrete learning goals: “we agreed that up front we should try to be as clear as possible about, what the learning goals were, be really explicit with ourselves, about, exactly what are they supposed to learn, you know at the end of this lesson or at the end of this unit” (Dawn Berk, interview, 3/15/10, lines 95-98). Then one or more teacher educators would write a lesson plan designed to help meet these goals, and the group would modify the lesson plans before they taught the lesson in the course. After teaching the lesson, the group would meet to discuss what had gone well or not well in terms of meeting the goals of the course, and the group would again modify the lesson for the next time that they taught it. After they had developed lessons that they thought they were happy with, the group conducted short research projects to test whether or not they were meeting their goals and also whether or not they, as instructors, were getting an accurate assessment of their students’ understandings (Jansen et al., 2009).

Like the teachers in Ma’s (1999) study who showed evidence of profound understanding of fundamental mathematics (PUFM), teacher educators develop their mathematical knowledge for teaching over time, especially pedagogical content knowledge. Berk talked about how many of the examples that they use in their mathematics content courses come about from their teaching and noticing misunderstandings that their students have about certain topics. "Now that we know it is [an issue that is problematic for our students], we've actually built in examples, to make [those problems] come up" (Dawn Berk, interview, 3/15/10, line 1534). By developing knowledge of prospective teachers' misconceptions, the teacher educators are able to design activities that highlight the misconceptions, so that they can be brought to the surface and addressed. Because these examples are built into the lesson plans, future teacher educators who teach and prepare from the lessons plans are able to anticipate many of their students' misconceptions, even before teaching the lesson. In this way, the lesson plans that the group has developed work as "educative curriculum materials" (Stylianides & Stylianides, 2010) to help new teacher educators using them learn from the lesson plans.

Berk discussed that many of the teacher educators at the University of Delaware begin their teaching without having a solid grasp of multiplication and division of fractions. "We've had folks, who were high school teachers who don't necessarily come in knowing the two meanings of division (Dawn Berk, interview, 3/15/10, lines 1748-1749). Like many mathematics teacher educators, these teacher educators were not previously in a position where they needed to know a rationale for how the division of fractions algorithm works or how to represent fraction multiplication using an area model. However, at the University of Delaware, the teacher educators are in a position where they enter into a community of practice of teacher educators who all work together to understand the material: "Yeah, so we've actually had, you know I feel like we've done professional development that was mathematical, you know, training for our grad students" (Dawn Berk, interview, 3/15/10, lines 1716-1718). This training helps the group as a whole develop some of the mathematical knowledge that they need in order to help prepare future teachers.

Limitations and Future Research

This research study has a number of strengths and limitations. Its primary limitation was that in order to be a feasible study, I needed to limit my study to a small number of teacher educators teaching a small area of content. While this research provided a wealth of data, and while multiplication and division of fractions is one of the most fundamental topics in elementary mathematics, there are so many different courses for teaching prospective teachers that one qualitative study cannot possibly cover all of them. One way that I have tried to combat this limitation is to clearly describe my data gathering techniques and analysis so that other researchers can use them to observe other mathematics content courses or other content areas, to further the research process. Thus, while the study itself may not be entirely generalizable, the data gathering process and analysis will open the door for more research opportunities for others to help expand the knowledge base of mathematics teacher educators.

Another limitation to this study is that while I attempted to find "expert" teacher educators, expertise cannot necessarily be determined merely by looking at a limited number of criteria. Although Tom, Stephanie, and Karen all had a lot of experience teaching mathematics content courses for prospective teachers and all possessed a wealth of mathematical knowledge, they exhibited gaps in their knowledge bases as well. While this may be considered a flaw in the design of the study, identifying areas of mathematical knowledge for teacher educators that are troublesome for them is a way for the field of mathematics education to begin to figure out ways to help fill the knowledge gaps.

There are a number of questions that result from my study and directions for future research. A first question is why three experienced mathematics teacher educators seemed to have gaps in their knowledge in regards to teaching multiplication and division of fractions. A possible explanation lies in the fact that all three of the teacher educators that I looked at in-depth for my study basically were alone in their departments. Tom was the only person who taught the mathematics content course at his college. Stephanie was one of a number of people teaching the courses at her university, but all of the other instructors were adjunct instructors, and they were given free reign over what to do in the course. Stephanie spent little if any time with them, and she never had opportunities to discuss aspects of teaching the course in-depth with them. Of the three, Karen had the most opportunity to enhance her mathematical knowledge base for three reasons. First, she was able to take a sabbatical a few years prior to participating in my study. During this time she observed mathematics being taught in a variety of elementary classrooms, and she was able to get a much better idea for herself exactly what her students would need to know in order to be successful teachers and modify her course goals accordingly. Second, Karen's college taught two sections of the mathematics content course each semester, and Karen told me that she was responsible for training the other instructor to teach the course. Since there was a new instructor every year, Karen spent time each year reflecting on her work as a teacher educator and sharing it with a colleague, something that Tom and Stephanie did not get to do. Third, Karen told me that she attended every conference that she could find, because she was always interested in learning more about teaching prospective teachers. Neither Tom nor Stephanie mentioned attending conferences at all, so we cannot know if they took advantage of the learning opportunities that conferences provide. A direction for future research could look at how teacher educators who are alone in their departments can develop their teacher knowledge.

At the University of Delaware, teacher educators worked together to develop and modify their mathematics content courses for prospective teachers. The lessons that they designed were very detailed, and new teacher educators of these courses are able to learn from the lesson plans and also become enculturated into a learning community that meets to discuss issues involved in teaching the course each week. Similar, but smaller scaled communities of practice exist at other universities, although the number of these communities reported in the Masingila, Olanoff, and Kwaka (2011) study is small (only about 4% of the 825 institutions reported regularly scheduled, ongoing meetings of their instructors). However, it seems that having a group of people to talk to and work with greatly helps in the preparation of mathematics teacher educators (Kimani, Olanoff, & Masingila, 2008), so having more of these types of communities seems like a logical step toward helping to deepen the knowledge of mathematics teacher educators. It was evident that through working with me on my study, all three of the mathematics teacher educators were able to learn more about the mathematics that they needed to teach prospective teachers multiplication and division of fractions. Stephanie and I spent the majority of one interview working through the difference between the area model of fraction multiplication and the operator model of multiplication that uses areas, such as the pattern block model. Prior to our discussion, Stephanie had talked about how she knew these were different models, but she had trouble articulating the difference. Through our conversation, we were both able to get a better understanding of the models, and how they were different, even though both look at fractions of areas. Tom felt like he was struggling with ways to present ideas of fraction multiplication in ways that would make sense to his students, and he asked me for suggestions on other models besides the geoboard representation that he had used. Because of my study, I was able to suggest ideas that I had seen used in other classrooms, which Tom may incorporate into his future teaching. Karen already had experience talking about her course with the teacher educators that she trains, but she too articulated that it was helpful to be able to talk about her teaching of the course with someone who had experience doing so. It is clear that having people to talk to about one's teaching helps provide avenues for reflection that are less accessible when one is alone. Research that looks into how existing communities of practice of mathematics teacher educators work, and what the participants learn from them would help show possible avenues of training for mathematics teacher educators.

Another area for future research involves looking at the textbooks commonly used to teach mathematics content courses for prospective teachers, and seeing how much these act as educative curriculum materials, in areas where teacher educators may have gaps in their understandings. All three of the teacher educators in my study had a required text for their course, but only Tom actually followed his text while teaching multiplication and division of fractions, and even he supplemented the text with demonstrations. Karen used her textbook as a guide in terms of the topics her students were responsible for and the proofs of some of the concepts that were provided in the text, but all of the worksheets that she used were things that she developed on her own or picked up at conferences. Stephanie used her textbook to assign homework problems, but neither the examples that she used in class, nor the worksheets on the pattern blocks came from her text. She said this was because her textbook did not focus on the patter block model, and that was the one that she favored.

All three of these textbooks were well-used texts in mathematics content courses for elementary teachers, but all three teacher educators felt that the textbooks did not provide them with lessons designed to meet their goals. Stephanie also said that she took her final exam questions from a question bank provided with her textbook, but after discussing the exam with me, she decided that the questions on the exam did not help her to achieve her goals. Looking at to what extent the textbooks commonly used to teach mathematics content courses for prospective teachers help teachers develop deep and connected understandings of elementary mathematics is another area that is in need of research. If the textbooks do not provide adequate depth and connections, then designing textbooks that do is important if we want teacher education to improve.

In terms of helping improve the mathematical knowledge that teacher educators of mathematics content courses for prospective teachers have, my research suggests the need for professional development for mathematics teacher educators. This type of professional development is taking place already on a small scale (for example, at the University of Michigan, the University of Delaware, and Syracuse University), but there are many teacher educators like Tom, Stephanie, and Karen, who would also benefit from this type of professional development. Some ways that this could happen would be to set up some sort of forum for mathematics educators interested in improving their practice to work with each other and discuss their teaching and their successes and problem areas, designing workshops either in the summer or at conferences, such as the conference of the Association of Mathematics Teacher Educators that are designed specifically for teachers of mathematics content courses for prospective elementary teachers, and perhaps developing and distributing resources for mathematics teacher educators in some sort of an online forum. With the current technology available in our society, there is no reason why a mathematics teacher educator should have to feel alone in their teaching, even if he or she is the only member of their department teaching the course(s).

Overall my study showed that the mathematical knowledge needed to teach prospective elementary teachers multiplication and division is vast and complex. Much more research is needed in other content areas, or looking at teacher educators of mathematics methods courses or teacher educators working on professional development of practicing teachers, to understand the mathematical work entailed in their jobs and the knowledge needed in order to do the jobs well.

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Appendix—First Interview Questions

- Can you tell me about your educational background?
- Can you talk about your teaching experience?
- How did you get involved in teaching mathematics content courses for PSTs?

- Tell me about your mathematics content course. Who takes it? How many courses are required?
- What are your general goals for course?
- How do you organize your class sessions? Are they the same each time?
- Did you choose the text?
- How did you decide on this text? What do you like about it? What do you not like about it?

Now let's talk about multiplication and division of fractions:

- What are your goals for these PSTs to come away from your lessons on multiplication and division of fractions?
- In your experience, what are PSTs' general problems with multiplication and division of fractions?
- What do you do to address these issues?
- Can you walk through your lesson plan(s) with me?
- How closely are you following your textbook? Why have you made the changes that you did?
- How do you decide which examples to use?
- What do you hope that the PSTs will be able to do differently after these lessons?
- Where do you anticipate problems in this lesson?
- How will you assess whether or not the students have learned what you want them to learn?