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Complexity over Finite-Dimensional Algebras

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COMPLEXITY OVER FINITE-DIMENSIONAL ALGEBRAS

By

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ABSTRACT

In this thesis we study two types of complexity of modules over finite-dimensional algebras.

In the first part, we examine the Ω-complexity of a family of self-injective k-algebras where k is an algebraically closed field and Ω is the syzygy operator. More precisely, let T be the trivial extension of an iterated tilted algebra of type H. We prove that modules over the trivial extension T all have complexities either 0, 1, 2 or infinity, depending on the representation type of the hereditary algebra H. As part of the proof, we show that a stable equivalence between self-injective algebras preserves the complexity of modules.

In the second part, we study the τ-complexity of modules over cluster tilted algebras where τ is the Auslander-Reiten translate. We prove that modules over the cluster tilted algebra of type H all have complexities either 0, 1, 2 or infinity, depending on the representation type of the hereditary algebra H.

Contents

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Chapter 1

Introduction

The notion of complexity gives a way of measuring how ‘complicated’ a module is. We study two kinds of complexity. The first is Ω-complexity, where Ω denotes the syzygy operator, and it measures the rate of growth of the syzygies in a minimal projective resolution of a module. The second type of complexity we call τ-complexity. Here τ is the Auslander-Reiten translate and τ-complexity measures the rate of growth of the τ-orbit of a module. In Section 3.3 we see that for self-injective algebras the two notions of complexity coincide. We refer to both Ω- and τ-complexity as simply complexity when no confusion can arise.

The concept of Ω-complexity was first introduced by Alperin and Evens in the realm of group algebras in [AE]. There the authors used group cohomology to show that given a finite group G and a field k, all modules over the group algebra kG have finite complexity.

Recently the study of complexity has sparked a lot of interest for self-injective algebras. These are algebras where the projective and injective modules coincide. For example, group algebras of finite groups are self-injective. In 2004 Snashall and Solberg used Hochschild cohomology and the induced support varieties for certain self-injective algebras (those with finite generation of cohomology) to extend the results of Alperin and Evens [SS]. This has lead to a vigorous study of complexity for self-injective algebras. For recent advances see [Be, EHSST, KZ]. The study of complexity has also taken on a life of its own in the world of commutative algebra. Avramov, Buchweitz, Gasharov, Eisenbud, and Peeva have obtained a series of beautiful results on complexity for complete intersection rings [Av, AvB, AvGP, Eis]. There the complexity of the residue field is equal to the codimension of the ring which also serves as an upper bound for the complexity of any module over the ring. In [JL] Jorgensen and Leuschke obtained results on complexity over Cohen-Macaulay local rings that are not Gorenstein.

The second type of complexity, τ-complexity, gives a way of analyzing the behaviour of the Auslander-Reiten translate τ. It measures the growth of the k-dimensions of successive powers of τ applied to a nonprojective module. τ-complexity has been studied by K. Erdmann in the setting of group algebras [E]. O. Kerner and C. M. Ringel have studied the τ-complexity of AR components in [Ker, R, R2]. More recently, O. Kerner and D. Zacharia obtained results on τ-complexity over self-injective algebras [KZ]. In all of the above articles, the authors have found that there is a strong connection between the shapes of the AR components and the possible complexities of the modules in those components.

In this thesis we study the complexity of modules over certain families of algebras: trivial extensions of iterated tilted algebras and cluster tilted algebras. The thesis is organized as follows.

In Chapter 2 we provide some general background and preliminary results on finite-dimensional algebras. We introduce the path algebra of a quiver, and the Auslander-Reiten quiver (AR quiver, for short) of a finite-dimensional algebra.

In Chapter 3 we determine the complexity of modules over a family of self-injective algebras, namely the trivial extensions of iterated tilted algebras. The Chapter is divided into four sections. In Section 3.2 we introduce trivial extension algebras and analyze the shape of the Auslander-Reiten quiver of such algebras. Section 3.3 discusses the properties of complexity over self-injective algebras. Section 3.4 is dedicated to the study of complexity under stable equivalence between self-injective algebras. The key result of the section is Theorem 3.4.10 which states that a stable equivalence between self-injective algebras preserves the complexity of modules. We note that in general a stable equivalence need not preserve the complexity of modules. Finally, we combine all of our work in Section 3.5 where we calculate the complexities of all modules over trivial extensions of iterated tilted algebras. We do this in two steps. First, we compute the complexities of all modules over trivial extensions of hereditary algebras in Theorem 3.5.3 and Theorem 3.5.5. Second, we use stable equivalence to extend the results to trivial extensions of iterated tilted algebras. The main result is obtained as Corollary 3.5.14 which shows that non-projective modules over these algebras can only have complexity 1 if $\vec{\Delta}$ is a Dynkin graph, complexity 1 or 2 if $\vec{\Delta}$ is a Euclidean graph, and in all other cases the non-projective modules must have infinite complexity. Moreover, the only modules of complexity 1 are the modules with periodic resolutions.

In Chapter 4 we turn to the study of cluster tilted algebras. We begin by providing some background and introducing the main concepts. In particular, Section 4.2 is dedicated to the construction of cluster tilted algebras. We then move to the study of complexity. In the setting of cluster tilted algebras, we consider τ-complexity. In Section 4.3 we compute the complexities of modules over cluster tilted algebras. The main result is given in Theorem 4.3.7 where we show that modules over these algebras can only have τ-complexity 0, 1, 2 or ∞. Moreover, finite τ-complexity occurs if and only if the algebra is obtained from a directed graph $\vec{\Delta}$ of Dynkin or Euclidean type. Our result shows that in terms of τ-complexity the cluster tilted algebras are more closely related to the hereditary algebras than the tilted algebras.

In the Appendix we discuss the program Cpx.pl that we wrote to extract Betti numbers from the program Gröbner developed by Ed Green. In Section 5.1 we provide the Perl code for the program Cpx.pl. The Appendix ends with Section 5.2 which contains sample files that were created when we used our program to compute the Betti numbers for the algebra in Example 3.2.1.

Chapter 2

Background on Finite-Dimensional Algebras

2.1 Quivers and Path Algebras

We study finite-dimensional algebras over an algebraically closed field k. Given such an algebra Λ, we denote by Λ-mod the category of finitely generated left Λ-modules. We fix a complete set of nonisomorphic simple modules $S(1), \ldots, S(n)$. Denote their projective covers by $P(1), \ldots, P(n)$, and their injective envelopes by $I(1), \ldots, I(n)$. We assume throughout that Λ is basic, that is $P_i \ncong P_j$ for $i \neq j$. For a module M we write $|M|$ for the composition length of M i.e. the number of simple modules that appear in the composition series of M. Further, we denote by $\underline{\dim}\, M$ its dimension vector in $\mathbb{Z}^n$

$$\underline{\dim}\, M = [|\mathrm{Hom}_\Lambda(P(1), M)|\,, |\mathrm{Hom}_\Lambda(P(2), M)|\,, \ldots |\mathrm{Hom}_\Lambda(P(n), M)|]^T$$

The i^{th} entry of the dimension vector is the number of times the simple module $S(i)$ appears as a composition factor of M. We define $|\underline{\dim}\, M|$ to be the sum of the entries in the vector $\underline{\dim}\, M$. Note that $|\underline{\dim}\, M| = |M|$.

Throughout this thesis all algebras are finite-dimensional k-algebras where k is an algebraically closed field, and all modules are finitely generated.

There is a well known correspondence between finite-dimensional k-algebras and path algebras of quivers which becomes an invaluable tool in our work. For this purpose we continue by introducing quivers and defining the notion of a path algebra over a quiver. For basic notions in representation theory we refer to [ARS, ASS].

A *finite quiver* $Q = (Q_0, Q_1)$ is an oriented graph with the finite set Q_0 consisting of vertices and the finite set Q_1 consisting of arrows between the vertices. If $\alpha : i \longrightarrow j$ is an arrow from vertex i to j we write $s(\alpha) = i$ and $t(\alpha) = j$ for the *source* and *target* of the arrow respectively. A *path* in the quiver Q is an ordered sequence of arrows $p = \alpha_m \ldots \alpha_1$ where $t(\alpha_i) = s(\alpha_{i-1})$ for $1 < i \leq m$. We may thus visualize a path as a concatenation of compatible arrows

$$s(\alpha_m) \xrightarrow{\alpha_m} t(\alpha_m) = s(\alpha_{m-1}) \xrightarrow{\alpha_{m-1}} \ldots \xrightarrow{\alpha_1} t(\alpha_1)$$

The *length* of a path is defined to be the number of arrows in the path. Furthermore, we consider $e_i \in Q_0$ as the *trivial path* at vertex i and write $s(e_i) = t(e_i) = i$. A path of length at least one whose source and target coincide is called a *cycle*.

We now define the notion of a path algebra over a quiver. Given a field k we let kQ denote the k-vector space with the paths of Q as its basis. The product of two basis vectors p and q is given by the concatenated path pq if $t(p) = s(q)$ and is equal to zero whenever $t(p) \neq s(q)$. This product extends from the basis of kQ to the entire space by linearity.

The connection between finite-dimensional k-algebras and path algebras over a quiver is the following.

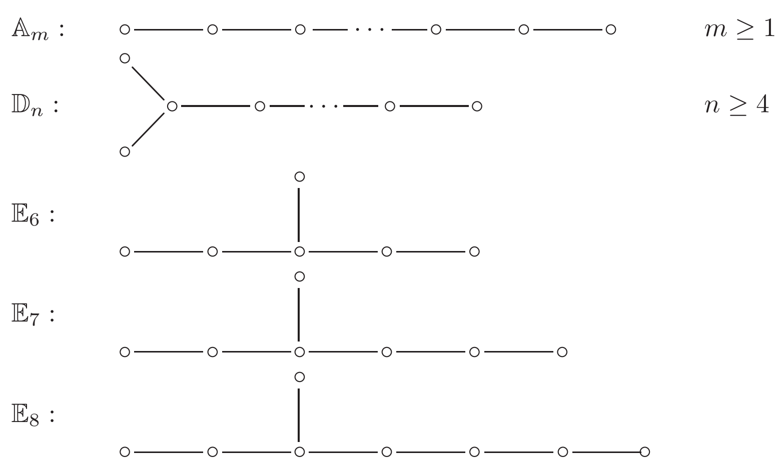
Theorem 2.1.1 (Ch.III, Section 1 in [ARS]). *Let Λ be a basic finite-dimensional algebra over an algebraically closed field k, then Λ is isomorphic to a factor algebra of a path algebra kQ for some finite quiver Q. Conversely, any factor algebra of a path algebra of a finite quiver Q without oriented cycles is a finite-dimensional algebra.*

We assume throughout that Λ is connected. An algebra Λ is said to be *connected* if Λ is not a direct product of two algebras. Note that a path algebra kQ is connected if and only if Q is connected, that is, the underlying graph of Q is connected.

Often we work with finite-dimensional hereditary k-algebras. The assumption that Λ is *hereditary* means that the global dimension of Λ is at most one. It is well known that any basic finite-dimensional hereditary algebra is isomorphic to a path algebra of a quiver Q with no oriented cycles.

The notion of representation type will play an important rôle in what follows. We say that $\Lambda = kQ$ is of *finite representation type* if the number of isomorphism classes of indecomposable Λ-modules is finite. Λ is of *infinite representation type* if it is not of finite representation type. Gabriel's [G] theorem gives a characterization of finite-dimensional hereditary algebras of finite representation type over an algebraically closed field. This characterization makes use of some very special diagrams which we introduce next. See [ASS].

The Dynkin graphs



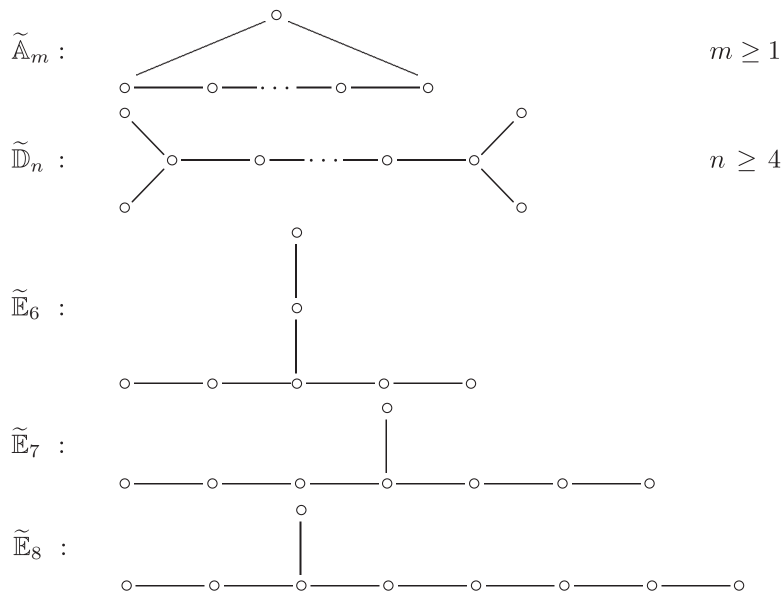
The indices $m, n \in \mathbb{N}$ count the number of vertices in the corresponding Dynkin graph.

We now give Gabriel's Theorem [G]:

Theorem 2.1.2. *Let Λ be a connected hereditary algebra over an algebraically closed field k. Then Λ is of finite representation type if and only if $\Lambda = kQ$ where the underlying graph of the quiver Q is a Dynkin diagram.*

Another family of special graphs, the Euclidean graphs, allows us to further separate the class of representation infinite hereditary algebras. See [ASS].

The Euclidean graphs



In the case of the Euclidean graphs the indices m, n refer to the number of vertices minus 1. We mention that the Euclidean graphs are also known as extended Dynkin graphs. Indeed, by removing any single vertex from an extended Dynkin graph we obtain a union of Dynkin graphs.

We separate representation infinite hereditary algebras into two disjoint classes: Λ is of *tame representation type* (or tame, for short) if the underlying graph of the quiver Q is a Euclidean diagram. In all other cases Λ is of *wild representation type* (or wild, for short). For further discussion of the tame and wild dichotomy of algebras see the paper by Y. Drozd [Dr].

2.2 The Auslander-Reiten Translation and the AR quiver

In order to introduce the Auslander-Reiten translation, we must first define the transpose Tr of a module. Given $M \in \Lambda$-mod, we construct its transpose as follows. Let $P^1 \xrightarrow{p_1} P^0 \xrightarrow{p_0} M \longrightarrow 0$ be a minimal projective presentation of M, that is, an exact sequence with $p_0 : P^0 \longrightarrow M$ and $p_1 : P^1 \longrightarrow \operatorname{Ker} p_0$ projective covers. Apply the functor $(\_)^* \overset{def}{=} \operatorname{Hom}_\Lambda(\_, \Lambda)$ to obtain an exact sequence

$$0 \longrightarrow M^* \xrightarrow{p_0^*} P^{0*} \xrightarrow{p_1^*} P^{1*} \longrightarrow \operatorname{Coker}(p_1^*) \longrightarrow 0$$

The *transpose* of M, denoted by $\operatorname{Tr} M$, is defined to be the cokernel $\operatorname{Coker}(p_1^*)$.

We denote by τ the *Auslander-Reiten translation* $\tau = \mathrm{D}\,\mathrm{Tr}$ where D is the ordinary duality $\operatorname{Hom}_k(\_, k)$ and Tr is the transpose. τ induces a bijection between the isomorphism classes of indecomposable nonprojective Λ-modules and the isomorphism classes of the indecomposable noninjective Λ-modules. The inverse of τ is $\tau^{-1} = \mathrm{Tr}\,\mathrm{D}$. An indecomposable Λ-module M is *preprojective* (*preinjective*) if $\tau^i M = 0$ for some $i > 0$ ($\tau^{-i} M = 0$ for some $i > 0$, respectively). A Λ-module is called preprojective (preinjective) if all of its direct summands are preprojective (preinjective). For instance, every projective module is preprojective.

There is a convenient way of organizing indecomposable modules and the maps between them by constructing what is called the Auslander-Reiten quiver (or AR quiver, for short). Before we give the definition of the AR quiver, we need the notions of an irreducible morphism and an almost split sequence.

A morphism $f : A \longrightarrow B$ in Λ-mod is called *irreducible* if f is neither a split monomorphism nor a split epimorphism, and if $f = ts$ for some $s : A \longrightarrow X$ and $t : X \longrightarrow B$ is a factorization of f, then s is a split monomorphism or t is a split epimorphism.

A morphism $f : A \longrightarrow B$ in Λ-mod is *left almost split* if f is not a split monomorphism, and any morphism $A \longrightarrow Y$ which is not a split monomorphism factors through f. Similarly, a morphism $g : B \longrightarrow C$ in Λ-mod is *right almost split* if g is not a split epimorphism and any morphism $Y \longrightarrow C$ which is not a split epimorphism factors through g.

A morphism $f : A \longrightarrow B$ in Λ-mod is called *left minimal* if any endomorphism $h \in \mathrm{End}(B)$ such that $hf = f$ must be an automorphism. Similarly, a morphism $g : B \longrightarrow C$ in Λ-mod is *right minimal* if any endomorphism $h \in \mathrm{End}(B)$ such that $gh = g$ must be an automorphism.

We say that a morphism $f : A \longrightarrow B$ in Λ-mod is *left minimal almost split* if it is both left minimal and left almost split. A morphism $g : B \longrightarrow C$ in Λ-mod is *right minimal almost split* if it is both right minimal and right almost split.

We are now ready to introduce almost split sequences. A short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is called an *almost split sequence* if f is left almost split and g is right almost split. This definition is equivalent to requiring the sequence

$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ to be exact and the morphism g to be minimal right almost split (equivalently, one may require that f be minimal left almost split). A consequence of these definitions is that both of the end terms of an almost split sequence are indecomposable.

Almost split sequences exist in Λ-mod. In fact, for any indecomposable nonprojective module C, there is an almost split sequence $0 \longrightarrow \tau C \longrightarrow B \longrightarrow C \longrightarrow 0$. Similarly, for any indecomposable noninjective module A, there is an almost split sequence of the form $0 \longrightarrow A \longrightarrow B \longrightarrow \tau^{-1}A \longrightarrow 0$.

Furthermore, almost split sequences are unique up to a commutative diagram. This means that whenever we have two almost split sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow A \longrightarrow B' \longrightarrow C' \longrightarrow 0$$

starting at the same term A, then the two exact sequences are isomorphic

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "\cong"] & C \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

Similarly, whenever we have two almost split sequences ending at the same term C, then the two short exact sequences are isomorphic. [ARS]

Now that we have defined irreducible morphisms and almost split sequences we can give a connection between these notions.

Lemma 2.2.1 (Thm. 5.3, Ch. V in [ARS]). *Let Λ be a finite-dimensional k-algebra.*

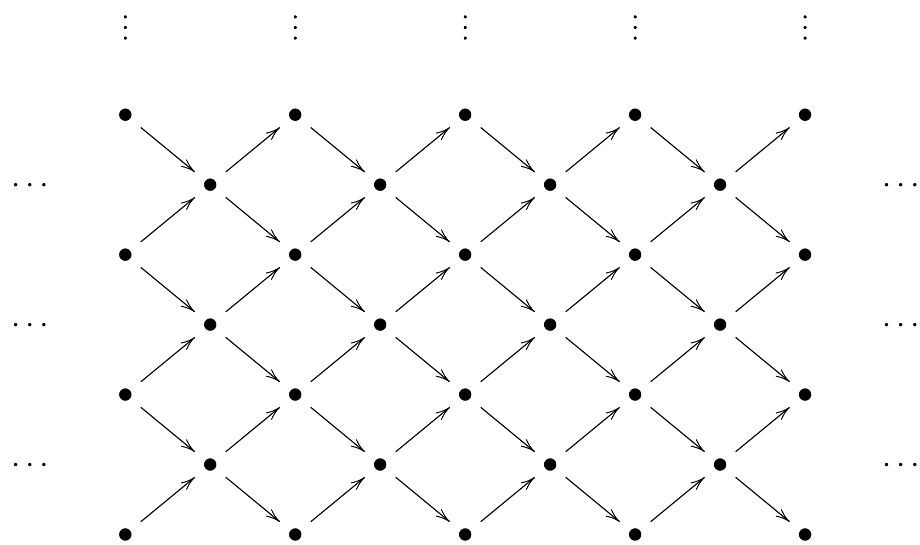
(i) Let A be an indecomposable module. Then a morphism $f : A \longrightarrow B$ is irreducible if and only if there exists some morphism $f' : A \longrightarrow B'$ such that the morphism $(f, f')^T : A \longrightarrow B \bigoplus B'$ is minimal left almost split.

(ii) Let C be an indecomposable module. Then a morphism $g : B \longrightarrow C$ is irreducible if and only if there exists some morphism $g' : B' \longrightarrow C$ such that the morphism $(g, g') : B \bigoplus B' \longrightarrow C$ is minimal right almost split.

The concepts of irreducible maps and almost split sequences lead us to the construction of the Auslander-Reiten quiver of a k-algebra Λ. The *AR quiver* of Λ, denoted $\Gamma(\Lambda)$, is a quiver that has vertices the isomorphism classes of indecomposable Λ-modules where we denote the vertex corresponding to a module X by $[X]$. There is an arrow $[X] \longrightarrow [Y]$ in $\Gamma(\Lambda)$ if and only if there is an irreducible morphism $X \longrightarrow Y$. The map $\tau = \mathrm{D}\,\mathrm{Tr}$ induces a map from the 'nonprojective vertices' to the 'noninjective vertices' where we say that a vertex $[X]$ is projective (injective) if the corresponding module X is projective (injective, respectively).

The *stable subquiver* $\Gamma_s(\Lambda)$ of $\Gamma(\Lambda)$ is the quiver obtained by deleting all vertices $[X]$ (and related arrows) for which $\tau^i(X)$ is injective or projective for some $i \in \mathbb{Z}$. Two indecomposable modules X and Y are said to be related by an irreducible morphism if there exists an irreducible morphism $X \longrightarrow Y$. An equivalence class under the equivalence relation generated by this relation is called a *component* of the AR quiver.

In the case when Λ is a connected hereditary algebra, we have some information about the components of the AR quiver. Namely, $\Gamma(\Lambda)$ has a *preprojective component* $\mathcal{P}(\Lambda)$ which contains all of the indecomposable projective Λ-modules, and a *preinjective component* $\mathcal{I}(\Lambda)$ which contains all of the indecomposable injective Λ-modules. All other components are called *regular*. For a hereditary algebra of tame representation type, all regular components of the AR quiver of Λ are stable tubes i.e. for some $n \in \mathbb{N}$ we have $\tau^n(X) \cong X$ for all X in the same regular component. For a hereditary algebra of wild representation type, all regular components of the AR quiver are of type $\mathbb{Z}A_\infty$. These components have the following shape



For proofs and further discussion see [R].

We end this section by giving a connection between the Auslander-Reiten translation and a special transformation, called the Coxeter transformation. First, we must discuss the Cartan matrix of an algebra.

The *Cartan matrix* C associated to a k-algebra Λ is the matrix with i^{th} column $\underline{\dim}\, P(i)$ i.e. $\mathrm{C} = (c_{i,j})$ where $c_{i,j} = \dim_k \mathrm{Hom}(P(i), P(j))$. If Λ is a connected hereditary k-algebra, then the determinant $\det \mathrm{C} = 1$ and therefore the Cartan matrix is invertible over $\mathbb{Z}$. The *Coxeter matrix* Φ associated to Λ is the matrix $\Phi = -\,\mathrm{C}^T\, C^{-1}$. The set of eigenvalues of the matrix Φ is called the *spectrum* of Φ and is denoted $\sigma(\Phi)$. The largest of the absolute values of these eigenvalues is the *spectral radius* $\rho = \max\{|\lambda| : \lambda \in \sigma(\Phi)\}$. The Coxeter transformation is a transformation $\mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ whose matrix with respect to the standard basis is Φ.

We now give the promised connection between the Auslander-Reiten translation τ and the Coxeter transformation. Namely, for any indecomposable nonprojective Λ-module M we have $\Phi\,(\underline{\dim}\, M) = \underline{\dim}\, \tau M$. See [R].

Chapter 3

Complexity of Trivial Extension Algebras

In this chapter we study the complexity of modules over a particular family of finite-dimensional self-injective algebras, namely the trivial extensions of iterated tilted algebras. The study of trivial extensions of artin algebras began in the work of Fossum, Griffith, and Reiten [FGR]. See also [Y, Y2, Y3]. More recently, in 2006 Skowroński used them in his study of tame self-injective algebras [Skow2]. Fernández and Platzeck presented a method in [FP] for constructing trivial extension algebras from directed graphs $\vec{\triangle}$.

The main goal of this chapter is to prove the following theorem:

Theorem. *Let* Λ *be an iterated tilted algebra from a hereditary algebra* H*. Let* X *be an indecomposable non-projective module over the trivial extension algebra* $\mathrm{T}(\Lambda)$.

Then the complexity of X satisfies the following

(i) If H is of finite representation type, then X has complexity 1.

(ii) If H is of tame representation type, then the complexity of X is 1 *or* 2. *Furthermore, there always exist modules of each complexity.*

(iii) If H is of wild representation type, then the complexity of X is infinite.

We further point out that the only indecomposable modules of complexity 1 are the periodic modules.

The proof consists of several stages. First, we use Coxeter matrices to determine the complexities of all modules over trivial extensions of hereditary algebras in Theorem 3.5.3 and Theorem 3.5.5. After obtaining this result, we proceed to show that a stable equivalence between self-injective algebras preserves the complexity of modules. Finally, in Corollary 3.5.14 we combine our previous work to obtain the Main theorem. We use a result of Tachikawa and Wakamatsu [TW] to obtain a stable equivalence between the trivial extension of a hereditary algebra and the trivial extension of an iterated tilted algebra. Information about the complexities of the modules over the first algebra allows us to determine the complexities of the modules over the second.

3.1 Preliminaries

We recall our assumptions that all algebras are finite-dimensional algebras over an algebraically closed field k. All algebras are assumed to be basic and connected. All modules are finitely generated. We write τ_Λ (or, simply τ) for the Auslander-Reiten translate (AR translate, for short) in Λ-mod. We use $\Gamma(\Lambda)$ to denote the Auslander-Reiten quiver (AR quiver, for short) of Λ-mod.

3.2 Trivial Extensions

For a k-algebra Λ we denote by T its *trivial extension algebra* $\mathrm{T} = \Lambda \ltimes \mathrm{D}(\Lambda)$. The trivial extension T has elements ordered pairs (a, b) with $a \in \Lambda, b \in \mathrm{D}(\Lambda)$ where addition is componentwise and multiplication is given by $(a, b)(a', b') = (aa', ab' + ba')$. We may view the trivial extension T as a an extension

$$0 \longrightarrow \mathrm{D}(\Lambda) \longrightarrow \mathrm{T} \longrightarrow \Lambda \longrightarrow 0$$

of Λ by the nilpotent bimodule $\mathrm{D}(\Lambda)$. We identify Λ-modules with the T-modules annihilated by $\mathrm{D}(\Lambda)$.

The trivial extension is a symmetric algebra and hence self-injective. Here we use the definition that a k-algebra Λ is *symmetric* if Λ is isomorphic to its dual $\mathrm{D}(\Lambda)$ as a two-sided Λ-module. It follows then that every artin algebra is a homomorphic image of some symmetric artin algebra.

We proceed to give a couple of useful properties of self-injective algebras. The most important of these for our purposes will be a connection between the syzygy operator Ω and the Auslander-Reiten translate $\tau = \mathrm{D}\,\mathrm{Tr}$.

We begin by recalling the notion of a *stable module category* of Λ-mod which we will denote by Λ-$\underline{\text{mod}}$. Let $\mathcal{P}$ denote the projective objects in Λ-mod. Then the objects of Λ-$\underline{\text{mod}}$ are the objects in the quotient $\Lambda\text{-mod}/\mathcal{P}$. In other words, we may think of the non-zero objects in Λ-$\underline{\text{mod}}$ as the non-projective objects in Λ-mod.

In order to describe the morphisms we need a definition. We say that a map $f : M \longrightarrow N$ *factors through a projective module* if $f = hg$ where we have two maps $g : M \longrightarrow P$ and $h : P \longrightarrow N$ with P a projective module. For Λ-modules M and N, let $\mathcal{P}(M, N)$ denote those morphisms in $\mathrm{Hom}_\Lambda(M, N)$ which factor through a projective module. We denote the quotient $\mathrm{Hom}_\Lambda(M, N)/\mathcal{P}(M, N)$ by $\underline{\mathrm{Hom}}_\Lambda(M, N)$. The morphisms of the stable module category Λ-$\underline{\text{mod}}$ are the morphisms in $\underline{\mathrm{Hom}}_\Lambda(M, N)$.

In summary, the non-zero objects in Λ-$\underline{\text{mod}}$ correspond to the non-projective objects of Λ-mod and the non-zero morphisms in Λ-$\underline{\text{mod}}$ correspond to the morphisms in Λ-mod that do not factor through a projective module. Since for self-injective algebras the projective and injective modules coincide, we may also say that the non-zero objects in Λ-$\underline{\text{mod}}$ correspond to the non-injective objects and the non-zero morphisms in Λ-$\underline{\text{mod}}$ are correspond to the morphisms in Λ-mod that do not factor through an injective module. Finally, we remark that the syzygy operator Ω is a functor on the factor category Λ-$\underline{\text{mod}}$.

Before we can provide the connection between the syzygy functor Ω and the Auslander-Reiten translate $\tau = \mathrm{D}\,\mathrm{Tr}$, we need to discuss what is known as the Nakayama automorphism. Note that if Λ is a self-injective algebra, then the functor $\mathrm{Hom}_\Lambda(\_, \Lambda) : \Lambda\text{-mod} \longrightarrow \Lambda\text{-mod}^{\mathrm{op}}$ is a duality. The Nakayama automorphism $\nu : \Lambda\text{-mod} \longrightarrow \Lambda\text{-mod}$ is an equivalence which is defined as the composition of the dualities $\mathrm{D}\,\mathrm{Hom}_\Lambda(\_, \Lambda)$ where $\mathrm{D} = \mathrm{Hom}_k(\_, k)$ as before. Furthermore, ν takes indecomposable projectives to indecomposable injectives, and preserves length.

We now give the promised connection between the syzygy functor and the Auslander-Reiten translation. Namely, for a self-injective algebra the functors $\tau = \mathrm{D}\,\mathrm{Tr}$, $\Omega^2_\Lambda \nu$, and $\nu\Omega^2_\Lambda$ from $\Lambda\text{-}\underline{\mathrm{mod}}$ to $\Lambda\text{-}\underline{\mathrm{mod}}$ are isomorphic. We point out that since ν is an automorphism, we have the equality $|\tau(X)| = |\Omega^2(X)|$ for $X \in \Lambda\text{-mod}$.

We furnish an example of a hereditary algebra and its trivial extension algebra.

Example 3.2.1. Let $\Lambda = kQ$ be the hereditary algebra given by the path algebra of the quiver Q

$$\begin{tikzcd} 1 \arrow[dr] & & 3 \\ & 2 \arrow[ur] \arrow[dl] \arrow[dr] & \\ 4 & & 5 \end{tikzcd}$$

The trivial extension T of Λ is the path algebra of the quiver

$$\begin{tikzcd} 1 \arrow[dr, "a"] & & 3 \arrow[ll, "e"'] \\ & 2 \arrow[ur, "b"] \arrow[dl, "c"] \arrow[dr, "d"] & \\ 4 \arrow[uu, "f"] & & 5 \arrow[uull, bend left=60, "g"] \end{tikzcd}$$

modulo the ideal generated by the set of relations $\{$*abea, acfa, adga, eac, fab, gab, gac, ead, fad, be-cf, cf-dg, dg-be*$\}$.

Here we have used the description of E. Fernández and M. Platzeck to construct the path algebra of the trivial extension of Λ [FP].

In the case when Λ is hereditary, there is a very nice relationship between the Auslander-Reiten quiver of Λ and that of its trivial extension algebra. Tachikawa has proved that the stable AR quiver of Λ embeds into that of T (See also [Y, Y2]).

Before stating this theorem, we provide notation and a brief description for two of the components in the stable AR subquiver of the trivial extension algebra T. The first of these is the component $\mathcal{I}_s$ which consists of all of the preinjective Λ-modules as well as the T-syzygies of the preprojective Λ-modules. The second is the component $\mathcal{P}_s$ consisting of all of the preprojective Λ-modules as well as the T-syzygies of the preinjective Λ-modules. The components $\mathcal{P}_s$ and $\mathcal{I}_s$ will play a central rôle later on. We now give the result of Tachikawa.

Lemma 3.2.2. *Let Λ be a hereditary k-algebra and let $\mathrm{T} = \Lambda \ltimes \mathrm{D}\,\Lambda$ be its trivial extension algebra.*

(i) The irreducible maps and almost split sequences in Λ-mod remain so in T-mod.

(ii) The stable AR subquiver of $\Gamma(\mathrm{T})$ is given by the disjoint union

$$\Gamma_s(\mathrm{T}) = \Gamma_s(\Lambda) \cup \Omega_{\mathrm{T}}(\Gamma_s(\Lambda)) \cup \mathcal{P}_s \cup \mathcal{I}_s$$

There is a lot of information in this lemma. First of all, we have that for any indecomposable Λ-module X, computing the AR translation of X over Λ is the same as computing the AR translation of X as a T-module i.e. $\tau_\Lambda(X) = \tau_{\mathrm{T}}(X)$. Second, any module in the stable AR quiver of T belongs to one of $\Gamma_s(\Lambda)$, $\Omega_{\mathrm{T}}(\Gamma_s(\Lambda))$, $\mathcal{P}_s$, or $\mathcal{I}_s$. Here $\Omega_{\mathrm{T}}(\Gamma_s(\Lambda))$ is obtained by applying Ω_{T} to the component $\Gamma_s(\Lambda)$ remembering that Ω_{T} preserves almost split sequences up to a projective summand of the middle term. It is interesting to note that the regular components of Λ remain so in T. Later on we will be using these observations to analyze the complexity of various T-modules. We now provide an example of an application of Tachikawa's theorem to obtain the stable AR quiver of T.

Example 3.2.3. In the case of the hereditary algebra from Example 3.2.1 the preprojective and preinjective component of Λ are shown below.

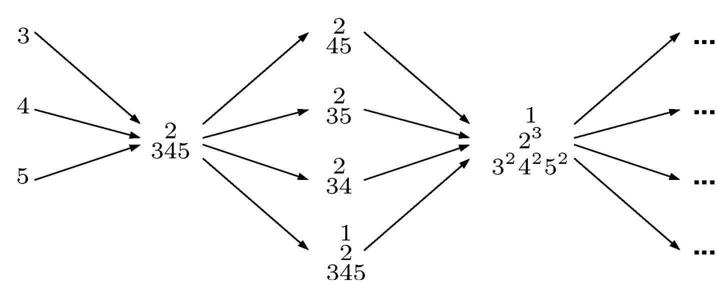


Figure 3.1: The preprojective component of $\Gamma(\Lambda)$

In Example 3.2.1 the underlying quiver of Λ is a Euclidean diagram of type $\tilde{D}_4$ and thus the hereditary algebra Λ is of tame representation type. Hence, the regular

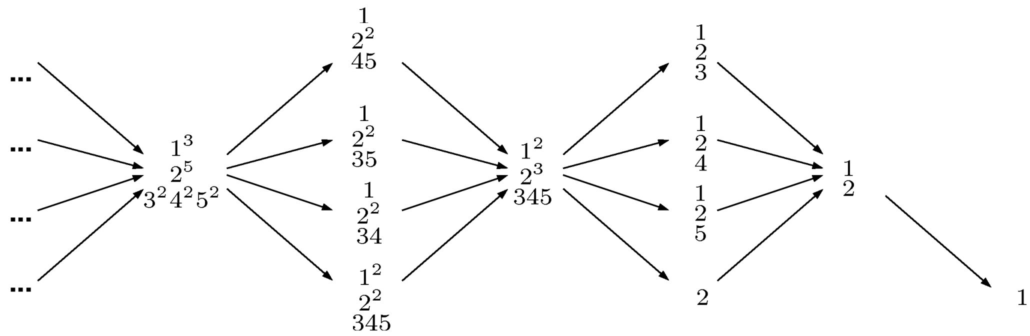


Figure 3.2: The preinjective component of $\Gamma(\Lambda)$

components of the AR quiver of Λ are tubes [R].

We now use the information about the AR quiver of Λ that we have gathered above, to construct the stable AR quiver of its trivial extension algebra T. We employ Tachikawa's result Lemma 3.2.2 which tells us that the component $\mathcal{P}_s$ of the stable AR quiver of the trivial extension T is

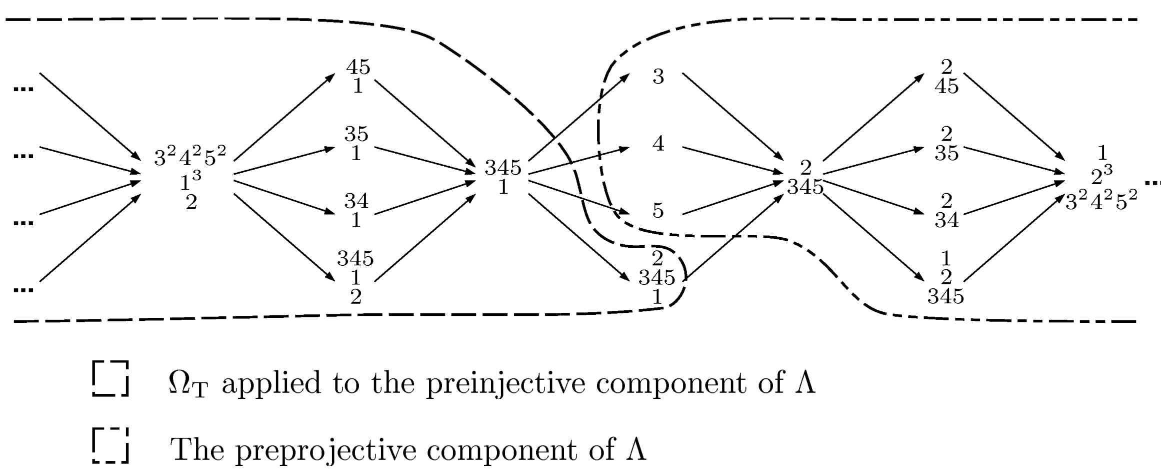
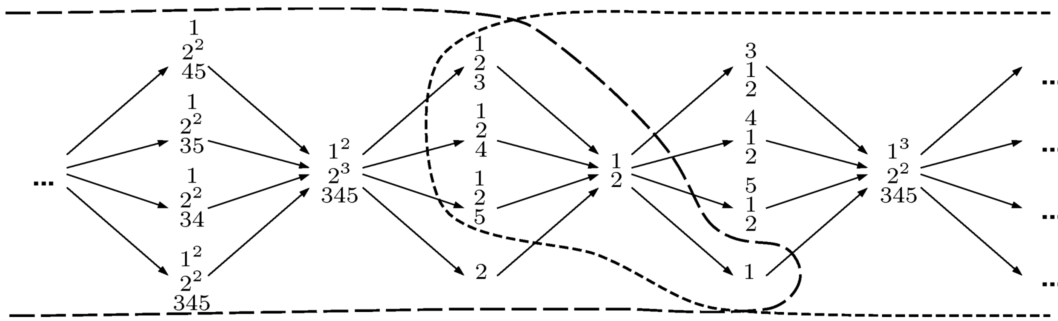


Figure 3.3: The component $\mathcal{P}_s$ of the stable AR quiver $\Gamma_s(\mathrm{T})$

The component $\mathcal{I}_s$ of the stable AR quiver of the trivial extension T is



The preinjective component of $\Gamma(\Lambda)$

Ω_{T} of the preprojective component of $\Gamma(\Lambda)$

Figure 3.4: The component $\mathcal{I}_s$ of the stable AR quiver $\Gamma_s(\mathrm{T})$

Since the regular components of the AR quiver of Λ are tubes, all other components of the stable AR quiver of T are tubes.

3.3 Complexity over Self-Injective Algebras

We are interested in the complexity of modules over a self-injective algebra Λ. Recall that if

$$\ldots \to P^n \xrightarrow{\delta_n} P^{n-1} \to \ldots \to P^0 \xrightarrow{\delta_0} M \to 0$$

is a minimal projective resolution of a finitely generated Λ-module M, then the i^{th} Betti number of M, $\beta_i(M)$, equals the number of indecomposable summands of P^i. We write Ω^i for the i^{th} syzygy of M i.e. $\Omega^i = \mathrm{Ker}(\delta_{i-1})$. We remark that β_i is equal to the number of simple modules appearing in the top of Ω^i, that is, $\beta_i = |\Omega^i/\mathrm{rad}\Omega^i|$.

Finally, the *complexity* of M over Λ is defined as

$$\mathrm{cx}_\Lambda\, M = \inf\left\{t \in \mathbb{N}_0 \mid \exists\, \alpha \in \mathbb{R} \text{ such that } \beta_i(M) \leq \alpha i^{t-1} \text{ for i} \gg 0\right\}$$

where $\mathbb{N}_0$ denotes the nonnegative integers. Thus, complexity measures how fast the sequence of Betti numbers β_i is growing. When no such $t \in \mathbb{N}_0$ exists, we say that the complexity is infinite and write $\mathrm{cx}_\Lambda\, M = \infty$. Notice that $\mathrm{cx}_\Lambda\, M = 1$ means that the Betti numbers are bounded, $\mathrm{cx}_\Lambda\, M = 0$ means that M has finite projective dimension. In the case where the algebra is self-injective, $\mathrm{cx}_\Lambda\, M = 0$ is equivalent to M being projective.

A couple of observations about complexity over self-injective algebras will be useful to us.

Lemma 3.3.1. *Let Λ be a self-injective algebra. Then*

(i) A module and its syzygies have the same complexity.

(ii) For any short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, the complexity of each term is at most the maximum of the complexity of the other two terms.

Proof. (i) The statement follows from the fact that the projective resolutions of a module and one of its syzygies only differ by finitely many terms. Complexity, however, describes the asymptotic behaviour of a projective resolution.

(ii) We begin by showing $\mathrm{cx}\, B \leq \max\{\mathrm{cx}\, A, \mathrm{cx}\, C\}$.

Let $\ldots \to P_A^2 \to P_A^1 \to P_A^0 \to A \to 0$ denote the minimal projective resolution of A and let $\ldots \to P_C^2 \to P_C^1 \to P_C^0 \to C \to 0$ be the minimal projective resolution of C. We may arrange these resolutions in a diagram which we complete to an exact commutative diagram by the Horseshoe Lemma.

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_A^1 \arrow[r] \arrow[d] & P_A^1 \oplus P_C^1 \arrow[r] \arrow[d] & P_C^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_A^0 \arrow[r] \arrow[d] & P_A^0 \oplus P_C^0 \arrow[r] \arrow[d] & P_C^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

The middle column provides a projective resolution (though not necessarily minimal) of C from which we deduce $\operatorname{cx} B \leq \max\{\operatorname{cx} A, \operatorname{cx} C\}$ since for each $i \geq 0$ we have the inequality $\beta_i(B) \leq \beta_i(A) + \beta_i(C)$.

We will now show $\operatorname{cx} C \leq \max\{\operatorname{cx} A, \operatorname{cx} B\}$. Let $P_B^0 \to B \to 0$ be the projective cover of B. We have a commutative diagram with an exact row and exact columns where Q denotes a projective module.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Omega^1(B) \arrow[d] & \Omega^1(C) \oplus Q \arrow[d] & \\
 & & P_B^0 \arrow[r, equal] \arrow[d] & P_B^0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We complete this to a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Omega^1(B) \arrow[r] \arrow[d] & \Omega^1(C) \oplus Q \arrow[d] & \\
 & & P_B^0 \arrow[r, equal] \arrow[d] & P_B^0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

and apply the Snake Lemma to obtain the short exact sequence

$$0 \longrightarrow \Omega_B \longrightarrow \Omega_C \oplus Q \longrightarrow A \longrightarrow 0$$

We already showed that the complexity of the middle term of a short exact sequence is at most the maximum of the complexity of the end terms. We apply this along with part (i) to the last sequence to obtain the inequalities

$$\operatorname{cx} C = \operatorname{cx}(\Omega C \oplus Q) \leq \max\{\operatorname{cx} A, \operatorname{cx} B\}$$

Finally, we repeat this entire process beginning with the short exact sequence $0 \longrightarrow \Omega^1(B) \longrightarrow \Omega^1(C) \oplus Q \longrightarrow A \longrightarrow 0$ to obtain yet another short exact sequence $0 \longrightarrow \Omega^2(C) \longrightarrow \Omega^1(A) \oplus Q' \longrightarrow \Omega^1(B) \longrightarrow 0$ which then yields the inequality $\operatorname{cx} A \leq \max\{\operatorname{cx} B, \operatorname{cx} C\}$.

□

We have the following well-known result for self-injective algebras. We include the proof for the reader's convenience. Recall from our earlier discussion that if Λ is self-injective, then $\tau = \nu\Omega^2 = \Omega^2\nu$ where ν is the Nakayama automorphism and ν preserves length.

Lemma 3.3.2. *Let Λ be a self-injective algebra. All modules in the same component of the stable AR quiver of Λ have the same complexity.*

Proof. Let $X \longrightarrow Y$ be an irreducible map between the indecomposable modules X and Y. We then have an almost split sequence $0 \longrightarrow \tau Y \longrightarrow X \bigoplus X' \longrightarrow Y \longrightarrow 0$ for some $X' \in \Lambda$-mod. It follows from Lemma 3.3.1 that $\operatorname{cx}_\Lambda X \leq \max\{\operatorname{cx}_\Lambda \mathrm{Y}, \operatorname{cx}_\Lambda \tau\mathrm{Y}\}$. Since Λ is self-injective $\tau Y = \nu\Omega^2 Y$ and therefore $\operatorname{cx}_\Lambda \tau Y = \operatorname{cx}_\Lambda Y$, whence the inequality $\mathrm{cx}_\Lambda X \leq \mathrm{cx}_\Lambda Y$. But we also have an irreducible map $\tau Y \longrightarrow X$ and by repeating the process we get $\mathrm{cx}_\Lambda Y = \mathrm{cx}_\Lambda \tau Y \leq \mathrm{cx}_\Lambda X$. We have obtained the two inequalities $\mathrm{cx}_\Lambda X \leq \mathrm{cx}_\Lambda Y$ and $\mathrm{cx}_\Lambda Y \leq \mathrm{cx}_\Lambda X$ which tell us that X and Y must, in fact, have the same complexity. Finally, since a component of an AR quiver is by definition connected, repeating this argument shows that all modules in the same component of the stable AR subquiver of a self-injective algebra Λ have the same complexity. $\square$

It follows that when we have a self-injective algebra, we may talk about the complexity of a component. Namely, since all modules in the same component of the stable AR quiver have the same complexity we can define this common complexity to be the *complexity of the component*.

We often make use of the following two observations.

Remark 3.3.3. Let the following be a minimal projective resolution of a finitely generated Λ-module X

$$\ldots \to P^n \xrightarrow{\delta_n} P^{n-1} \to \ldots \to P^0 \xrightarrow{\delta_0} X \to 0$$

Then the sequence of even Betti numbers grows with the same complexity as the sequence of odd Betti numbers. This follows easily from the observation that for any $n \in \mathbb{N}$, we have $\Omega^{2n}(X) \subseteq P^{2n-1}(X)$:

We recall that each $P^{2n-1}(X)$ is a finite direct sum of indecomposable projective modules where β_{2n-1} counts the number of these summands. As we mentioned earlier, $\beta_{2n} = |\operatorname{top}\Omega^{2n}(X)|$ and we may thus write

$$\begin{aligned}\beta_{2n} &= \left|\operatorname{top}\Omega^{2n}(X)\right| \leq \left|\Omega^{2n}(X)\right| \\ &\leq \left|P^{2n-1}(X)\right| \leq \beta_{2n-1}\,|\Lambda|\end{aligned}$$

Similarly,

$$\begin{aligned}\beta_{2n+1} &= \left|\operatorname{top}\Omega^{2n+1}(X)\right| \leq \left|\Omega^{2n+1}(X)\right| \\ &\leq \left|P^{2n}(X)\right| \leq \beta_{2n}\,|\Lambda|\end{aligned}$$

These calculations show that

$$\frac{\beta_{2n+1}}{|\Lambda|} \leq \beta_{2n} \leq \beta_{2n-1}\,|\Lambda|$$

and therefore the sequence of even Betti numbers has the same complexity as the sequence of odd Betti numbers.

Remark 3.3.4. For a self-injective algebra Λ the complexity of Betti numbers $\beta_n(X)$ is the same as the complexity of the sequence $|\tau^n(X)| = |\Omega^{2n}X|$ where $n \in \mathbb{N}$. This follows from the facts that $\tau = \nu\Omega^2$ and ν preserves length, and the inequalities

$$\begin{aligned} \beta_{2n} = \left|\operatorname{top}\Omega^{2n}(X)\right| &\leq \left|\Omega^{2n}(X)\right| \\ &\leq \left|P^{2n}(X)\right| \leq \beta_{2n}\,|\Lambda| \end{aligned}$$

3.4 Complexity under Stable equivalence

In this section Λ and Γ are two self-injective algebras over an algebraically closed field k. We prove that if Γ is stably equivalent to Λ, then the complexity of modules is preserved under the stable equivalence.

We begin by discussing the notion of a stable equivalence. Recall that the stable module category of Λ-mod, denoted by Λ-$\underline{\text{mod}}$, has objects in $\Lambda\text{-mod}/\mathcal{P}$ where $\mathcal{P}$ denotes the projective objects in Λ-mod. The morphisms of the stable module category are the morphisms in $\underline{\text{Hom}}_\Lambda(M,N) = \text{Hom}_\Lambda(M,N)/\mathcal{P}(M,N)$ where $\mathcal{P}(M,N)$ denotes those morphisms in $\text{Hom}_\Lambda(M,N)$ which factor through a projective module.

We can now give the definition of a stable equivalence. Let Λ-mod and Γ-mod be two module categories. We say that Λ-mod and Γ-mod are *stably equivalent* if there exists an equivalence $\mathfrak{S} : \Lambda\text{-}\underline{\text{mod}} \longrightarrow \Gamma\text{-}\underline{\text{mod}}$ where by an equivalence we mean a covariant functor that is full, faithful, and dense.

Stable equivalence has several important properties which we shall now introduce.

Theorem 3.4.1 (Prop. 1.1, Ch. X in [ARS]). *Let* $\mathfrak{S} : \Lambda\text{-}\underline{\textit{mod}} \longrightarrow \Gamma\text{-}\underline{\textit{mod}}$ *be a stable* *equivalence between self-injective k-algebras Λ and Γ. Then Λ is of finite representation type if and only if Γ is of finite representation type.*

We will only make use of the above theorem, but we mention that more is known. In fact, H. Krause has shown in [Kr] that stable equivalence always preserves the representation type of the algebra.

Stable equivalence also behaves well with regard to irreducible morphisms and almost split sequences.

Theorem 3.4.2 (Cor. 1.9, Ch. X in [ARS])**.** *Let $\mathfrak{S} : \Lambda\text{-}\underline{mod} \longrightarrow \Gamma\text{-}\underline{mod}$ be a stable equivalence between self-injective k-algebras Λ and Γ. If Λ is of infinite representation type, then Λ and Γ have isomorphic stable AR quivers.*

Before giving the next theorem, we need to introduce one more piece of notation. We will use $\Lambda\text{-mod}_{\mathcal{P}}$ to denote the full subcategory of Λ-mod consisting of modules without projective summands. Since Λ is self-injective, this coincides with the subcategory of Λ-modules without injective summands. The stable equivalence $\mathfrak{S} : \Lambda\text{-}\underline{\text{mod}} \longrightarrow \Gamma\text{-}\underline{\text{mod}}$ from the above Theorem induces a correspondence between the objects of $\Lambda\text{-mod}_{\mathcal{P}}$ and $\Gamma\text{-mod}_{\mathcal{P}}$. Moreover, $\mathfrak{S}$ has an important feature:

Theorem 3.4.3 (Prop. 1.12, Ch. X in [ARS])**.** *If Λ and Γ are self-injective, then the correspondence $\mathfrak{S}$ between objects in $\Lambda\text{-mod}_{\mathcal{P}}$ and $\Gamma\text{-mod}_{\mathcal{P}}$ from above commutes with the syzygy functor Ω.*

Remark 3.4.4. It follows that stable equivalence between self-injective algebras preserves both τ- and Ω-periodicity. We know from Thm. 3.4.2 that if a component $\mathcal{C}$

in the stable AR quiver of Λ-mod is a τ-periodic, then $\mathfrak{S}(\mathcal{C})$ is also τ-periodic in the stable AR quiver of Γ-mod. In addition, Thm. 3.4.3 says that stable equivalence between self-injective algebras also preserves Ω-periodicity. We mention that for symmetric algebras, where $\tau = \Omega^2$, τ- and Ω-periodicity coincide, but this is not the case for all self-injective algebras. For a detailed discussion of periodicity for self-injective algebras consult [Skow1].

Remark 3.4.5. In general, a stable equivalence need not preserve the complexity of modules. For example, set Λ to be the path algebra of the quiver $\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \end{tikzcd}$ modulo the ideal generated by the relation $\langle \alpha^2 \rangle$. Let Γ be the hereditary algebra given by the path algebra of the quiver $1 \longrightarrow 2$. Then Λ and Γ are stably equivalent (Section 1, Ch. X in [ARS]). Yet, the simple module over Λ has infinite complexity while all Γ-modules have finite projective resolutions and hence complexity 0.

We proceed to show that a stable equivalence $\mathfrak{S} : \Lambda\text{-}\underline{\mathrm{mod}} \longrightarrow \Gamma\text{-}\underline{\mathrm{mod}}$ between the stable module categories of two self-injective algebras Λ and Γ preserves complexity. In short, given a Λ-module M we have $\operatorname{cx}_\Lambda M = \operatorname{cx}_\Gamma \mathfrak{S}(M)$.

Every nonprojective indecomposable module over a self-injective algebra of finite representation type is periodic. In view of Remark 3.4.4, we then know that also all nonprojective Γ-modules are periodic. In the case that Λ has finite representation type, we see that the stable equivalence $\mathfrak{S} : \Lambda\text{-}\underline{\mathrm{mod}} \longrightarrow \Gamma\text{-}\underline{\mathrm{mod}}$ preserves complexity.

In what follows we assume that Λ has infinite representation type. We proceed with a couple of lemmas and a proposition to see how stable equivalence behaves with regard to k-dimensions of modules.

We are using slightly modified arguments from [KrZw]. Recall our conventions: $|M|$ denotes the k-dimension of the Λ-module M and $S_1, S_2, \ldots, S_n$ form a complete set of nonisomorphic simple Λ-modules. We write $m_i(M)$ for the multiplicity of the simple module S_i, $1 \leq i \leq n$, as a factor in the composition series of M.

Lemma 3.4.6. *Let X and Y be Λ-modules. Then*

$$|\mathrm{Hom}_\Lambda(X,Y)| \leq \sum_{i=1}^{n} m_i(X)\,|\mathrm{Hom}_\Lambda(S_i,Y)|$$

where $m_i(X)$ is the multiplicity of the simple module S_i, $1 \leq i \leq n$, as a factor in the composition series of X.

Proof. If X is a simple module, then the statement holds trivially. In general, X has a composition series:

$$0 \subset X_t \subset X_{t-1} \ldots \subset X_1 \subset X_0 = X$$

for some $t \in \mathbb{N}$.

We form short exact sequences

$$0 \longrightarrow X_{j+1} \longrightarrow X_j \longrightarrow X_j/X_{j+1} \longrightarrow 0$$

for each $0 \leq j \leq t-1$. Notice that each X_j/X_{j+1} is a simple module.

Applying the contravariant functor $\mathrm{Hom}_\Lambda(\_, Y)$ gives the exact sequences

$$0 \longrightarrow \mathrm{Hom}_\Lambda(X_j/X_{j+1}, Y) \longrightarrow \mathrm{Hom}_\Lambda(X_j, Y) \longrightarrow \mathrm{Hom}_\Lambda(X_{j+1}, Y) \longrightarrow \dots$$

From here we obtain the inequalities

$$|\mathrm{Hom}_\Lambda(X_j, Y)| \leq |\mathrm{Hom}_\Lambda(X_j/X_{j+1}, Y)| + |\mathrm{Hom}_\Lambda(X_{j+1}, Y)|$$

Writing down the above inequality for each $j = 0, \dots, t-1$ and then replacing at each step the term $|\mathrm{Hom}_\Lambda(X_{j+1}, Y)|$ on the right hand side with the next inequality

$$|\mathrm{Hom}_\Lambda(X_{j+1}, Y)| \leq |\mathrm{Hom}_\Lambda(X_{j+1}/X_{j+2}, Y)| + |\mathrm{Hom}_\Lambda(X_{j+2}, Y)|$$

yields

$$\begin{aligned} |\mathrm{Hom}_\Lambda(X, Y)| &\leq |\mathrm{Hom}_\Lambda(X_0/X_1, Y)| + |\mathrm{Hom}_\Lambda(X_1/X_2, Y)| + \dots \\ &+ |\mathrm{Hom}_\Lambda(X_{t-1}/X_t, Y)| + |\mathrm{Hom}_\Lambda(X_t, Y)| \end{aligned}$$

where X_j/X_{j+1} are exactly the composition factors of X.

We thus have

$$|\mathrm{Hom}_\Lambda(X, Y)| \leq \sum_{i=1}^{n} m_i(X)\, |\mathrm{Hom}_\Lambda(S_i, Y)| \qquad \square$$

Lemma 3.4.7. *Let X be a Λ-module without nonzero projective summands and let S be a simple Λ-module. Then* $\operatorname{Hom}_\Lambda(S, X) = \underline{\operatorname{Hom}}_\Lambda(S, X)$.

Proof. Let $X \neq 0$ be a Λ-module without projective summands. We show that if $f : S \longrightarrow X$ factors through a projective module, then $f = 0$.

Suppose for purposes of contradiction that there exists some nonzero $f : S \longrightarrow X$ that factors through a projective module. That is, we have a commutative diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & X \\ I \arrow[ur] & \end{tikzcd}$$

where we may take I to be the injective envelope of S remembering that the projective and injective modules coincide.

Notice that f is a monomorphism and maps S into $\operatorname{soc} X$. Since $S = \operatorname{soc} I$ and the diagram commutes, we have an inclusion $\operatorname{soc} I \hookrightarrow \operatorname{soc} X$ and thus an inclusion $I \hookrightarrow X$. But I is injective and must then be a direct summand of X. This contradicts our assumption that X has no nonzero injective summands. Hence, $f = 0$ and $\operatorname{Hom}_\Lambda(S, X) = \underline{\operatorname{Hom}}_\Lambda(S, X)$. □

In the following Proposition and Corollary we show that a stable equivalence $\mathfrak{S} : \Lambda\text{-}\underline{\text{mod}} \longrightarrow \Gamma\text{-}\underline{\text{mod}}$ preserves the k-dimension of modules within some finite fixed error bounds.

Proposition 3.4.8. *Let $\mathfrak{S} : \Lambda\text{-}\underline{\mathit{mod}} \longrightarrow \Gamma\text{-}\underline{\mathit{mod}}$ be a stable equivalence between self-*

injective k-algebras Λ *and* Γ*. Then there exists some* $c \in \mathbb{N}$ *such that for each non-projective* $V \in \Lambda$*-mod we have the following inequality*

$$|\mathfrak{S}(V)| \leq c\,|V|$$

Proof. This is a slight modification of Theorem 1 in [KrZw]. Take $V \in \Lambda$-$\underline{\mathrm{mod}}$ and let $W = \mathfrak{S}(V)$ be its image in Γ-$\underline{\mathrm{mod}}$. Denote by S_i, $1 \leq i \leq n$, a complete set of non-isomorphic simple Λ-modules and by T_j, $1 \leq j \leq m$, a complete set of non-isomorphic simple Γ-modules. Next, pick and fix a set of $U_j \in \Lambda$-$\underline{\mathrm{mod}}$ such that $\mathfrak{S}(U_j) = T_j$ for each $j = 1 \ldots m$. Note that the U_j may no longer be simple modules.

Set $c = \displaystyle\sum_{i,j=1}^{n,m} m_j(\Gamma) m_i(U_j)$. We prove the inequality

$$|\mathfrak{S}(V)| \leq c\,|V|$$

First, from Lemmas 3.4.6 and 3.4.7 we obtain

$$\begin{aligned} |W| &= |\mathrm{Hom}_\Gamma(\Gamma, W)| \\ &\leq \sum_{j=1}^{m} m_j(\Gamma)\,|\mathrm{Hom}_\Gamma(T_j, W)| \\ &= \sum_{j=1}^{m} m_j(\Gamma)\,|\underline{\mathrm{Hom}}_\Gamma(T_j, W)| \end{aligned}$$

Next, since $\mathfrak{S}$ is an equivalence, and $\mathfrak{S}(U_j) = T_j$ and $\mathfrak{S}(V) = W$ we have

$$\begin{aligned}
\sum_{j=1}^{m} m_j(\Gamma)\,|\underline{\mathrm{Hom}}_\Gamma(T_j, W)| &= \sum_{j=1}^{m} m_j(\Gamma)\,|\underline{\mathrm{Hom}}_\Gamma(\mathfrak{S}(U_j), \mathfrak{S}(V))| \\
&= \sum_{j=1}^{m} m_j(\Gamma)\,|\underline{\mathrm{Hom}}_\Lambda(U_j, V)|
\end{aligned}$$

The inequality $|\underline{\mathrm{Hom}}_\Lambda(U_j, V)| \leq |\mathrm{Hom}_\Lambda(U_j, V)|$ always holds. Using this and Lemma 3.4.6 we get

$$\begin{aligned}
\sum_{j=1}^{m} m_j(\Gamma)\,|\underline{\mathrm{Hom}}_\Lambda(U_j, V)| &\leq \sum_{j=1}^{m} m_j(\Gamma)\,|\mathrm{Hom}_\Lambda(U_j, V)| \\
&\leq \sum_{j=1}^{m} m_j(\Gamma) \sum_{i=1}^{n} m_i(U_j)\,|\mathrm{Hom}_\Lambda(S_i, V)| \\
&= \sum_{i,j=1}^{n,m} m_j(\Gamma) m_i(U_j)\,|\mathrm{Hom}_\Lambda(S_i, V)| \\
&\leq \sum_{i,=1}^{n,m} m_j(\Gamma) m_i(U_j)\,|\mathrm{Hom}_\Lambda(\Lambda, V)| \\
&= \sum_{i,=1}^{n,m} m_j(\Gamma) m_i(U_j)\,|V| \\
&= c\,|V|
\end{aligned}$$

remembering that $c = \sum_{i,j=1}^{n,m} m_j(\Gamma) m_i(U_j)$.

Finally, combining all of our calculations yields the desired inequality

$$|\mathfrak{S}(V)| \leq c\,|V| \qquad \square$$

Corollary 3.4.9. *Let $\mathfrak{S} : \Lambda\text{-}\underline{mod} \longrightarrow \Gamma\text{-}\underline{mod}$ be a stable equivalence between self-injective algebras. Then there exist constants c and c' such that for any nonprojective $V \in \Lambda\text{-}\underline{mod}$ we have the inequalities*

$$c' \, |V| \leq |\mathfrak{S}(V)| \leq c \, |V|$$

Proof. The inequality on the right comes directly from Prop. 3.4.8.

Let $\mathfrak{L}$ be the inverse equivalence $\mathfrak{L} : \Gamma\text{-}\underline{\mathrm{mod}} \longrightarrow \Lambda\text{-}\underline{\mathrm{mod}}$. To get the left inequality, we apply Prop. 3.4.8 to $\mathfrak{L}$ and the module $\mathfrak{S}(V) \in \Gamma\text{-}\underline{\mathrm{mod}}$. This gives us the inequality

$$|\mathfrak{L}(\mathfrak{S}(V))| \leq d \, |\mathfrak{S}(V)|$$

where $d \in \mathbb{N}$ does not depend on $\mathfrak{S}(V)$. Since $\mathfrak{S}$ and $\mathfrak{L}$ are inverse equivalences, $\mathfrak{L}(\mathfrak{S}(V)) \cong V$. Therefore $|\mathfrak{L}(\mathfrak{S}(V))| = |V|$ and the inequality above becomes

$$|V| \leq d \, |\mathfrak{S}(V)|$$

Finally, taking $c' = 1/d$ yields $c' \, |V| \leq |\mathfrak{S}(V)|$. □

We are now ready to prove that a stable equivalence between self-injective k-algebras preserves complexity.

Theorem 3.4.10. *Let $\mathfrak{S} : \Lambda\text{-}\underline{mod} \longrightarrow \Gamma\text{-}\underline{mod}$ be a stable equivalence between self-injective k-algebras Λ and Γ. Let M be a Λ-module. Then $\operatorname{cx}_\Lambda M = \operatorname{cx}_\Gamma \mathfrak{S}(M)$.*

Proof. Let M be a nonprojective Λ module, otherwise $\mathrm{cx}_\Lambda M = 0 = \mathrm{cx}_\Gamma \mathfrak{S}(M)$ holds. Apply Cor. 3.4.9 to M and its syzygies $\Omega^i(M)$ to get

$$c' \left|\Omega^i_\Lambda M\right| \leq \left|\mathfrak{S}(\Omega^i_\Lambda M)\right| \leq c \left|\Omega^i_\Lambda M\right|$$

for each $i \geq 0$.

From Thm. 3.4.3 we know that the syzygy functor Ω commutes with the stable equivalence $\mathfrak{S}$. We then have

$$c' \left|\Omega^i_\Lambda M\right| \leq \left|\Omega^i_\Gamma \mathfrak{S}(M)\right| \leq c \left|\Omega^i_\Lambda M\right|$$

for each $i \geq 0$.

Since for each $i \geq 0$ we have $\beta_i(M) \leq |\Omega^i_\Lambda M| \leq \beta_i(M)\,|\Lambda|$ we obtain

$$c'\beta_i(M) \leq \left|\Omega^i_\Gamma \mathfrak{S}(M)\right| \leq c\beta_i(M)\,|\Lambda|$$

Also, since for each $i \geq 0$ we have $\beta_i(\mathfrak{S}(M)) \leq |\Omega^i_\Gamma \mathfrak{S}(M)| \leq \beta_i(\mathfrak{S}(M))\,|\Gamma|$ we get

$$c'\beta_i(M) \leq \beta_i(\mathfrak{S}(M))\,|\Gamma| \quad \text{and} \quad \beta_i(\mathfrak{S}(M)) \leq c\beta_i(M)\,|\Lambda|$$

Rearranging these two inequalities yields

$$\frac{c'}{|\Gamma|}\beta_i(M) \leq \beta_i(\mathfrak{S}(M)) \leq \beta_i(M)c\,|\Lambda|$$

for all $i \geq 0$.

In other words, the Betti numbers of the Λ-module M and the Betti numbers of Γ-module $\mathfrak{S}(M)$ have the same rate of growth. Therefore, $\operatorname{cx}_\Lambda M = \operatorname{cx}_\Gamma \mathfrak{S}(M)$. □

3.5 Complexity of Trivial Extension Algebras

In this section we determine the complexities of all modules over trivial extensions of iterated tilted algebras. We accomplish this by first computing the complexities of all modules over trivial extensions of hereditary algebras. Then we extend our results to trivial extensions of iterated tilted algebras.

3.5.1 Trivial Extensions of Hereditary Algebras

We study the complexity of modules over the trivial extension algebra $\mathrm{T} = \Lambda \ltimes D(\Lambda)$ of hereditary algebra Λ. H. Tachikawa has shown that if Λ is of finite representation type, then so is its trivial extension T [T]. See also [Y2]. In fact, the AR quiver of T-mod consists of a single periodic component i.e. every nonprojective T-module is τ-periodic. Since T is a symmetric algebra we know that $\tau = \Omega^2$ and thus any nonprojective T-module has a periodic resolution. In terms of complexity, this means that any T-module has complexity less than or equal to 1.

Here we study the case when Λ is of infinite representation type. We will show that if Λ is tame hereditary, then any indecomposable T-module has complexity at most 2; and if Λ is wild hereditary, then any indecomposable nonprojective T-module has infinite complexity.

Proposition 3.5.1. *Let Λ be of wild representation type and let X be an indecomposable preinjective or regular Λ-module. Then viewed as a* T*-module the complexity of X is infinite.*

Proof. Let Φ be the Coxeter transformation associated with Λ and let ρ denote its spectral radius (See Section 2.2 for the definitions). Since Λ is wild the spectral radius $\rho > 1$ [A, R2]. Let $X \neq 0$ be either a preinjective or a regular indecomposable Λ-module. It has been shown that

$$\lim_{n \longrightarrow \infty} \frac{1}{\rho^n} \underline{\dim}\, \tau^n X = \vec{y}$$

where $\vec{y}$ is a strictly positive vector [A, R2] (See also [Ker]). Since $\rho > 1$ we know $|\underline{\dim}\, \tau_\Lambda^n X|$ grows exponentially. We saw in Lemma 3.2.2 that the AR quiver of Λ embeds into that of its trivial extension T and we have $\tau_\Lambda X = \tau_{\mathrm{T}} X$. So $|\underline{\dim}\, \tau_{\mathrm{T}}^n X|$ grows exponentially. Since the trivial extension algebra T is symmetric it follows from Remarks 3.3.3 and 3.3.4 that also the sequence of Betti numbers $\beta_n(X)$ grows exponentially and therefore $\mathrm{cx}_{\mathrm{T}}\, X = \infty$. $\square$

In the following proposition we obtain an upper bound for the complexity of nonprojective modules in the case when Λ is of tame representation type. We mention that we have a different proof that we present in our article [P], but here we provide a simple proof using linear algebra.

Proposition 3.5.2. *Let Λ be of tame representation type. Let X be an indecomposable nonprojective Λ-module. Then viewed as a T-module the complexity of X is at most 2.*

Proof. We assume X is an indecomposable nonprojective Λ-module. Let Φ be the Coxeter transformation associated with Λ and let ρ denote its spectral radius. R. Stekolshchik, V. Subbotin (see [S] for the results as well as a discussion of the history of this problem), J. Coleman [C], and R. B. Howlett [How] have shown that the eigenvalues λ_i of the Coxeter matrix associated with a tame algebra are roots of unity. Furthermore, if the underlying graph of Λ is a tree, then the Jordan form J of Φ contains only one 2×2 Jordan block and its eigenvalue is 1 [S]. In the case of the cycle $\tilde{A}_n$, A. J. Coleman has shown in [C] that the Jordan form J of Φ contains a 2×2 Jordan block with eigenvalue 1 and all other Jordan blocks are of size at most 2×2 [S]. Hence, we can write Φ as a product of matrices $\Phi = \mathrm{B\,J\,B}^{-1}$ where

$$J = \begin{pmatrix} J_{1,1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & J_{2,2} & & 0 & 0 & & 0 \\ 0 & 0 & & 0 & 0 & & 0 \\ & & & 1 & 1 & & 0 \\ & & & 0 & 1 & & 0 \\ \vdots & \vdots & & 0 & 0 & & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & J_{n,n} \end{pmatrix}$$

where $J_{t,t} = \left(\begin{smallmatrix} \lambda_t & 1 \\ 0 & \lambda_t \end{smallmatrix}\right)$ is a 2×2 matrix if the eigenvalue λ_t has multiplicity 2 and the corresponding eigenvector is simple i.e. has multiplicity 1, otherwise $J_{t,t} = (\lambda_t)$ is a 1×1 matrix.

Then $\Phi^i(\underline{\dim}\, X) = (\mathrm{B\,J\,B}^{-1})^i\, \underline{\dim}\, X = \mathrm{B\,J}^i\, \mathrm{B}^{-1}\, \underline{\dim}\, X$ where J^i is of the form

$$J^i = \begin{pmatrix} J^i_{1,1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & J^i_{2,2} & & 0 & 0 & & 0 \\ 0 & 0 & & 0 & 0 & & 0 \\ & & & 1 & i & \dots & 0 \\ & & & 0 & 1 & & 0 \\ \vdots & \vdots & & 0 & 0 & & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & J^i_{n,n} \end{pmatrix}$$

where $J^i_{t,t} = \left(\begin{smallmatrix} \lambda^i_t & i\lambda^{i-1}_t \\ 0 & \lambda^i_t \end{smallmatrix}\right)$ if $J_{t,t} = \left(\begin{smallmatrix} \lambda_t & 1 \\ 0 & \lambda_t \end{smallmatrix}\right)$, and $J^i_{t,t} = (\lambda^i_t)$ otherwise.

Multiplying the last equation on the left by $[1 \dots 1] \in \mathbb{Z}^n$ yields

$$\left|\underline{\dim}\, \Phi^i(X)\right| = [1 \dots 1]\, \mathrm{B\,J}^i\, \mathrm{B}^{-1}\, \underline{\dim}\, X$$

To simplify our notation, we will denote by $\|A\|$ the matrix having entries equal to the modulus of the entries of the matrix A.

We obtain the inequality

$$\left|\underline{\dim}\, \Phi^i(X)\right| \leq \|[1 \dots 1]\, \mathrm{B}\|\, \left\|\mathrm{J}^i\right\|\, \left\|\mathrm{B}^{-1}\, \underline{\dim}\, X\right\|$$

Letting C_1 and C_2 be the largest entries in the vectors $\|[1 \dots 1]\, \mathrm{B}\|$ and $\left\|\mathrm{B}^{-1}\, \underline{\dim}\, X\right\|$ respectively, allows us to write

$$\begin{aligned}\left|\underline{\dim}\,\Phi^i(X)\right| &\leq C_1\,[1\ldots1]\left\|\mathrm{J}^i\right\|C_2\,[1\ldots1]^T \\ &\leq C_1C_2\,[1\ldots1]\left\|\mathrm{J}^i\right\|[1\ldots1]^T \\ &= C_1C_2\sum j_{s,t}\end{aligned}$$

where $j_{s,t}$ are the entries in $\left\|\mathrm{J}^i\right\|$.

We recall that the non-zero entries of J^i are the powers λ_t^i of the eigenvalues λ_t for $t = 1\ldots n$ along with entries $i\lambda_s^{i-1}$ coming from the blocks of size 2×2. Using the fact that all of the eigenvalues λ_t^i are roots of unity, we see that J^i has non-zero entries only of modulus 1 and i. This tells us that the sum of the entries of $\left\|\mathrm{J}^i\right\|$ is equal to the sum, C_3, of the entries of J^i with modulus 1 plus the sum, C_4, of the entries of J^i with modulus i. We emphasize the fact that the number of entries of each type is independent of i. We can now write $\sum j_{s,t} = C_3 + C_4 i \leq C_5 i$ for some constant C_5 sufficiently large. We obtain the inequality

$$\left|\underline{\dim}\,\Phi^i(X)\right| \leq C_1C_2C_5 i$$

Denoting the constant $C_1C_2C_5$ by C' we can write

$$\left|\underline{\dim}\,\Phi^i(X)\right| \leq C' i$$

But $\underline{\dim}\, \Phi^i(X) = \underline{\dim}\, \tau_\Lambda^i(X)$ and by the result of H. Tachikawa, Lemma 3.2.2, $\tau_\Lambda(X) = \tau_{\mathrm{T}}(X)$ which then yields

$$\left|\underline{\dim}\, \tau_{\mathrm{T}}^i(X)\right| \leq C'i$$

Therefore the sequence $|\underline{\dim}\, \tau_{\mathrm{T}}^i(X)|$ grows with complexity at most 2. Finally, since T is a symmetric algebra we use Remarks 3.3.3 and 3.3.4 to conclude that the sequence of Betti numbers β_i also grows with complexity at most 2 i.e. we have the inequality $\mathrm{cx}_{\mathrm{T}}(X) \leq 2$. □

We can now use the previous results to prove that if Λ is tame hereditary, then any indecomposable T-module has complexity at most 2; and if Λ is wild hereditary, then any indecomposable nonprojective T-module has infinite complexity. The proof involves Auslander-Reiten quivers and Lemma 3.2.2

Theorem 3.5.3. *Let Λ be a hereditary k-algebra. Let X be an indecomposable nonprojective module over the trivial extension* T *of Λ.*

(i) If Λ is tame, then complexity of X is at most 2.

(ii) If Λ is wild, then X has infinite complexity.

Proof. Let X be an indecomposable nonprojective T-module. From Lemma 3.2.2 we know that the stable AR subquiver of $\Gamma(\mathrm{T})$ is given by the disjoint union

$$\Gamma_s(\mathrm{T}) = \Gamma_s(\Lambda) \cup \Omega_{\mathrm{T}}(\Gamma_s(\Lambda)) \cup \mathcal{P}_s \cup \mathcal{I}_s$$

We recall that the component $\mathcal{I}_s$ contains all of the preinjective Λ-modules as well as all of the T-syzygies of preprojective Λ-modules. The component $\mathcal{P}_s$ contains all of the preprojective Λ-modules as well as all of the T-syzygies of the preinjective Λ-modules.

We now use the above description of the stable AR subquiver of $\Gamma(\mathrm{T})$ to find the complexity of X. We recall the following two observations from Lemma 3.3.2. First, for a self-injective algebra all modules in the same component of the AR quiver have the same complexity. Second, a module and its syzygy always have the same complexity.

We first consider the case when $X \in \mathcal{P}_s \cup \mathcal{I}_s$. We observe that complexity is constant on $\mathcal{P}_s \cup \mathcal{I}_s$. Namely, if $X \in \mathcal{I}_s$ pick $M \in \mathcal{I}(\Lambda)$. Then X and M belong to the same component and $\mathrm{cx}_{\mathrm{T}}(X) = \mathrm{cx}_{\mathrm{T}}(M) = \mathrm{cx}_{\mathrm{T}}(\Omega_{\mathrm{T}}(M))$. But $\Omega_{\mathrm{T}}(M)$ is in the component $\mathcal{P}_s$ which tells us that $\mathcal{P}_s$ and $\mathcal{I}_s$ have the same complexity. Since M belongs to $\mathcal{I}(\Lambda)$, M is preinjective and we may apply Prop. 3.5.1 and Prop. 3.5.2 to conclude that if Λ is wild $\mathrm{cx}_{\mathrm{T}}(X) = \mathrm{cx}_{\mathrm{T}}(M) = \infty$, and if Λ is tame we get $\mathrm{cx}_{\mathrm{T}}(X) = \mathrm{cx}_{\mathrm{T}}(M) \leq 2$.

We now consider the case when $X \in \Gamma_s(\Lambda) \cup \Omega_{\mathrm{T}}(\Gamma_s(\Lambda))$. Since modules in $\Omega_{\mathrm{T}}(\Gamma_s(\Lambda))$ are syzygies of those in $\Gamma_s(\Lambda)$, we may pick a module $M \in \Gamma_s(\Lambda)$ with $\mathrm{cx}_{\mathrm{T}}(X) = \mathrm{cx}_{\mathrm{T}}(M)$. Since M is a regular Λ-module we can apply Prop. 3.5.1 and Prop. 3.5.2 to conclude that if Λ is wild $\mathrm{cx}_{\mathrm{T}}(X) = \mathrm{cx}_{\mathrm{T}}(M) = \infty$, and if Λ is tame $\mathrm{cx}_{\mathrm{T}}(X) = \mathrm{cx}_{\mathrm{T}}(M) \leq 2$. $\square$

We can now compute the complexity of each component of the stable AR quiver of the trivial extension algebra T of a hereditary algebra of tame representation type. One of the ingredients of the proof is the following theorem by W. Crawley-Boevey given as Corollary F in [C-B].

Theorem 3.5.4. *Let Λ be a finite-dimensional algebra over an algebraically closed field k. If Λ is of tame representation type, then every component of the Auslander-Reiten quiver of Λ contains only finitely many isomorphism classes of indecomposable modules of each dimension.*

In the case where Λ is of tame representation type, we can find the complexity of each component in the stable Auslander-Reiten quiver of the trivial extension. Once again, we use the description obtained by Tachikawa for the stable AR quiver of a trivial extension of a hereditary algebra.

Theorem 3.5.5. *Let Λ be a hereditary k-algebra of tame representation type. Then the components in the stable Auslander-Reiten quiver of its trivial extension* T *have the following complexities:* $\mathrm{cx}_{\mathrm{T}}(\mathcal{I}_s) = 2$, $\mathrm{cx}_{\mathrm{T}}(\mathcal{P}_s) = 2$, *all other components have complexity* 1.

Proof. We begin by recalling that since T is self-injective, all modules belonging to the same component of the stable AR quiver of T must have the same complexity. We proceed by exhibiting a module of the appropriate complexity in each of the components of the stable AR quiver.

We start with the component $\mathcal{I}_s$ which we recall contains all of the injective Λ-modules. In particular, since Λ is of tame representation type, there exists a simple injective Λ-module S with $\tau_\Lambda^i S \neq 0$ for any $i \in \mathbb{N}$. Also, observe that since S is injective, S is not τ-periodic that is to say $\tau_\Lambda^i S \ncong \tau_\Lambda^j S$ for $i \neq j$.

We claim that $|\tau_\Lambda^i S|$ are unbounded. Assume to the contrary that there is some $B \in \mathbb{N}$ with $|\tau_\Lambda^i S| \leq B$ for all $i \in \mathbb{N}$. Since $\tau^i S \neq 0$ for any $i \in \mathbb{N}$ and $\tau^i S \ncong \tau^j S$ for $i \neq j$, we see that $\tau_\Lambda^i S$ are infinitely many non-isomorphic indecomposable Λ-modules. By assumption $|\tau_\Lambda^i S| \leq B$. This means that for some positive integer $c \leq B$ there exist infinitely many non-isomorphic indecomposable modules of the same length c and hence of the same k-dimension. Furthermore, all of the modules lie in the same component of the Auslander-Reiten quiver of Λ. But this brings us to a contradiction with Crawley-Boevey's theorem. Therefore, $|\tau_\Lambda^i S|$ must be unbounded.

Lemma 3.2.2 tells us $\tau_\Lambda^i S = \tau_{\mathrm{T}}^i S$ and therefore the sequence $|\tau_{\mathrm{T}}^i S|$ must be unbounded. Finally, since T is symmetric Remarks 3.3.3 and 3.3.4 tell us that also the sequence of Betti numbers, β_i, is unbounded. In other words, viewed as a T-module, S has complexity greater than 1. In light of Thm. 3.5.3, where we saw that any indecomposable T-module has complexity at most 2, we conclude that S has complexity exactly 2. Therefore, also $\mathrm{cx}_{\mathrm{T}}\, \mathcal{I}_s = 2$.

Next, we consider the component $\mathcal{P}_s$ which contains the syzygies of the injective Λ-modules, $\Omega_T(\mathcal{I}(\Lambda))$. By the above, injective Λ-modules have complexity 2 when viewed as T-modules. Since a module and its syzygy always have the same complexity, we know $\mathrm{cx}_{\mathrm{T}}(\mathcal{P}_s) = 2$.

Any other component of the stable AR quiver of T is either a regular component of Λ, thus belonging to $\Gamma_s(\Lambda)$, or a component in $\Omega_{\mathrm{T}}(\Gamma_s(\Lambda))$. Since Λ is of tame representation type, the regular components of Λ are tubes. M. Auslander and I. Reiten showed in [AR] that Ω_{T} applied to a stable tube is also a stable tube. It follows that any component of the stable AR quiver of T that belongs to $\Gamma_s(\Lambda)$ or $\Omega_{\mathrm{T}}(\Gamma_s(\Lambda))$ is a stable tube. This means that any indecomposable T-module X in $\Gamma_s(\Lambda) \cup \Omega_{\mathrm{T}}(\Gamma_s(\Lambda))$ is τ-periodic. i.e. $\tau_{\mathrm{T}}^t(X) \cong X$ for some $t \in \mathbb{N}$. But $\tau_{\mathrm{T}}^i X = \Omega_{\mathrm{T}}^{2i}(X)$ for $i = 1, 2, 3, \ldots$. In particular, $X \cong \Omega_{\mathrm{T}}^{2t}(X)$, so that X has a periodic resolution over T. In terms of complexity this means that X has complexity 1. □

We point out as a separate corollary a part of what we proved above.

Corollary 3.5.6. *If Λ is of tame representation type, then the only T-modules with complexity 1 are the periodic modules.*

Note that in general it is possible to have modules of complexity 1 that are not Ω-periodic. An example was given by R. Schulz in [Sch] where he studied algebras of the form $k\langle x, y\rangle / (x^2, xy + qyx, y^2)$ where k is a field and q a nonzero element in k that is not a root of unity. Examples over commutative local rings have been provided by V. Gasharov and I. Peeva in [GP].

It is also interesting to note that if Λ is tame, then all allowed complexities occur for some T-module.

Corollary 3.5.7. *If Λ is of tame representation type, then there exist T-modules with every allowed complexity i.e. complexities 0, 1, and 2.*

Proof. Any projective T-module realizes complexity 0. Any module in the component $\mathcal{P}_s$ or $\mathcal{I}_s$ has complexity 2. All other modules have complexity 1. □

3.5.2 Trivial Extensions of Iterated Tilted Algebras

In this section Λ is an iterated tilted algebra from a finite-dimensional k-algebra where k is algebraically closed. We follow our convention that all modules are left modules. When we wish to discuss right Λ-modules, we will view them as left Λ^{op} modules instead. Let $\mathrm{T}(\Lambda)$ denote the trivial extension algebra of Λ. In this section we will use our previous work to show that the stable AR components of $\mathrm{T}(\Lambda)$ can only have complexities 1, 2, and infinity.

We begin by recalling the notions of a tilting module and a tilted algebra. Denote by $\mathrm{add}(T)$ the modules that are direct sums of direct summands of a Λ-module T. A finitely generated left Λ-module T is called a *tilting module* if it satisfies the following three conditions:

(i) $pd_\Lambda T \leq 1$

(ii) $\mathrm{Ext}^1_\Lambda(T,T) = 0$

(iii) there exists an exact sequence $0 \longrightarrow_\Lambda \Lambda \longrightarrow T' \longrightarrow T'' \longrightarrow 0$ where T' and T'' belong to $\mathrm{add}(T)$.

The first condition requires the projective dimension of T to be at most 1 i.e. the first syzygy $\Omega(T)$ has to be zero or projective. Notice that any projective Λ-module satisfies (i) and (ii). The free module Λ satisfies all three conditions. For basic notions in tilting theory we refer to [ASS, HR].

The next theorem is the fundamental theorem of tilting theory. It gives a connection between the algebra Λ and the endomorphism algebra $\Gamma = \mathrm{End}_\Lambda(T)^{op}$ where ${}_\Lambda T$ is a tilting Λ-module. We write the composition of maps as concatenation: fg means first do f, then do g for any two composable maps f and g. Then $T \cong \mathrm{Hom}_\Lambda(\Lambda, T)$ is a right $\mathrm{End}_\Lambda(T)$-module via the action defined by $t * g = tg$ for any $t \in T$ and $g \in \mathrm{End}_\Lambda(T)$. This makes T a Λ-Γ bimodule as $\lambda(tg) = (\lambda t)g$ for each $\lambda \in \Lambda \cong \mathrm{End}_\Lambda(\Lambda)$, $g \in \mathrm{End}_\Lambda(T)$, and $t \in T \cong \mathrm{Hom}_\Lambda(\Lambda, T)$.

Theorem 3.5.8 (Tilting Theorem)**.** *[BB, HR] Let T be a tilting Λ-module. Let $\Gamma = \mathrm{End}_\Lambda(T)^{op}$. Then T is a tilting Γ-module and there is an algebra isomorphism $\Lambda \cong \mathrm{End}_\Gamma(T)^{op}$.*

An algebra Λ is said to be a *tilted algebra* if there exists a hereditary algebra H and a tilting module ${}_H T$ such that $\Lambda \cong \mathrm{End}_H(T)^{op}$. Notice that all hereditary algebras are tilted algebras (just take ${}_H T = H$ in the definition).

We give an example of a tilted algebra and its AR quiver.

Example 3.5.9. Let H be the path algebra of the quiver of type $\tilde{D}_4$

$$\begin{tikzcd} 3 \arrow[dr] & & \\ & 2 \arrow[r] & 1 \\ 4 \arrow[ur] & & \end{tikzcd}$$

The AR quiver of H is then given by

$$\begin{tikzcd}[column sep=small]
1 \arrow[dr] & & 2 \arrow[dr] & & \begin{smallmatrix} 34 \\ 2 \\ 1 \end{smallmatrix} \arrow[dr] & & \\
& \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 34 \\ 2^2 \\ 1 \end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \arrow[r] & \begin{smallmatrix} 34 \\ 2 \end{smallmatrix} \arrow[r] \arrow[dr] & 3 \\
& & \begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix} \arrow[ur] & & \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \arrow[ur] & & 4
\end{tikzcd}$$

Take $T = P(4) \oplus I(1) \oplus P(4)/S(1) \oplus S(4)$. Then T is a tilting module. The tilted algebra $\Lambda = \mathrm{End}_H(T)^{op}$ is given by the path algebra of the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 4 \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] & & 1 \\ & 3 \arrow[ur, "\delta"'] & \end{tikzcd}$$

with the relations $\rho = \langle \alpha\beta - \gamma\delta \rangle$. Note that Λ is no longer a hereditary algebra.

The AR quiver of Λ is given by

$$\begin{tikzcd}
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dr] & & 3 \arrow[dr] & & \begin{smallmatrix}4\\2\end{smallmatrix} \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}23\\1\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}4\\23\\1\end{smallmatrix} \arrow[r] & \begin{smallmatrix}4\\23\end{smallmatrix} \arrow[ur] \arrow[dr] & & 4 \\
 & \begin{smallmatrix}3\\1\end{smallmatrix} \arrow[ur] & & 2 \arrow[ur] & & \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[ur] &
\end{tikzcd}$$

Once we have obtained a tilted algebra Λ from the hereditary algebra H via a tilting H-module, we may repeat the process to form yet another algebra via a tilting Λ-module. This process leads us to the notion of an iterated tilted algebra.

An algebra Λ is called an *iterated tilted algebra* if there exists a sequence of algebras $\Lambda = \Lambda_t, \Lambda_{t-1}, \dots, \Lambda_0 = H$ where H is hereditary and a sequence of tilting modules ${}_{\Lambda_i}T^{(i)}$, $0 \leq i \leq t-1$, such that $\Lambda_{i+1} = \mathrm{End}_{\Lambda_i}(T^{(i)})^{op}$ for each i. Iterated tilted algebras were introduced by I. Assem and D. Happel [AH] (See also [HR]).

Complexity over trivial extensions of iterated tilted algebras.

We now turn to the study of complexity over trivial extensions of iterated tilted algebras. Recall that for a finite-dimensional k-algebra Λ we denote by $\mathrm{T}(\Lambda)$ its trivial extension algebra $\mathrm{T}(\Lambda) = \Lambda \ltimes D(\Lambda)$. H. Tachikawa and T. Wakamatsu proved in [TW] the following theorem:

Theorem 3.5.10. *Let Λ be a finite-dimensional k-algebra and let T be a tilting* *module over Λ. Let Γ be the endomorphism algebra $\mathrm{End}_\Lambda(T)^{op}$. Then there exists an equivalence between the stable module categories of the trivial extension algebras $\mathfrak{S} : \mathrm{T}(\Lambda)\text{-}\underline{mod} \longrightarrow \mathrm{T}(\Gamma)\text{-}\underline{mod}$.*

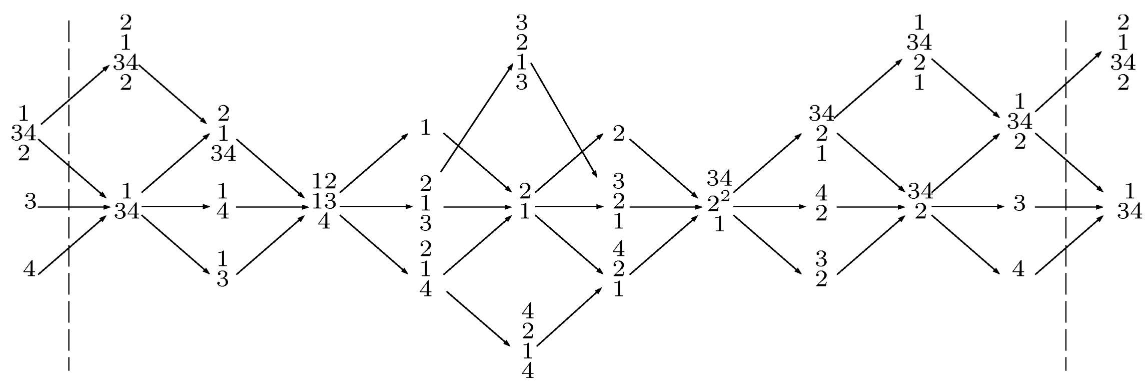
We illustrate this theorem in the following example.

Example 3.5.11. The trivial extension algebra $\mathrm{T}(H)$ of the hereditary algebra H in the previous example Ex. 3.5.9 is the path algebra of the quiver

$$\begin{tikzcd}
3 \arrow[dr, "\alpha"'] & & \\
& 2 \arrow[r, "\gamma"] & 1 \arrow[ull, bend right, "\delta"'] \arrow[dll, bend left, "\epsilon"] \\
4 \arrow[ur, "\beta"] & &
\end{tikzcd}$$

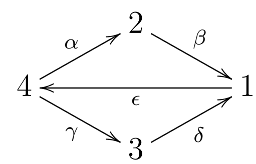
with relations $\rho = \langle \delta\alpha - \epsilon\beta, \beta\gamma\delta, \alpha\gamma\epsilon, \alpha\gamma\delta\alpha, \gamma\delta\alpha\gamma, \delta\alpha\gamma\delta \rangle$.

The AR quiver of $\mathrm{T}(H)$ is



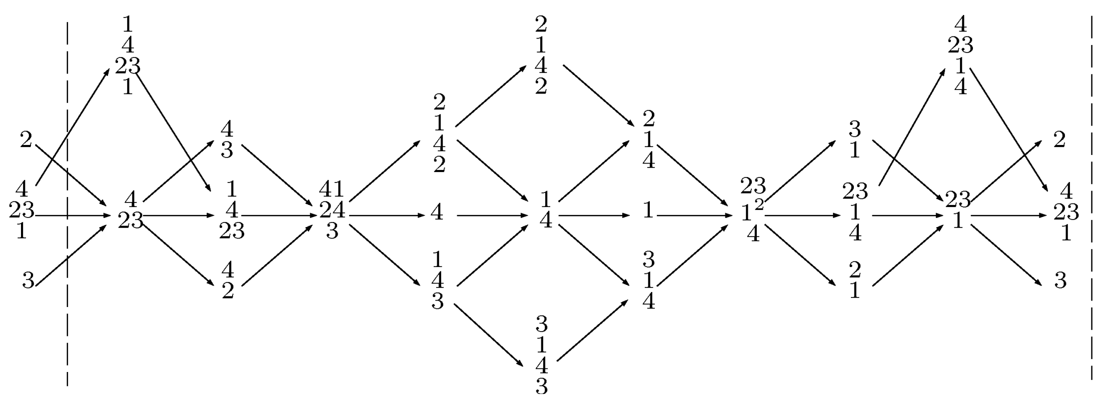
The projective $\mathrm{T}(H)$-modules are $P(1) = \begin{smallmatrix} 1 \\ 34 \\ 2 \\ 1 \end{smallmatrix}$, $P(2) = \begin{smallmatrix} 2 \\ 1 \\ 34 \\ 2 \end{smallmatrix}$, $P(3) = \begin{smallmatrix} 3 \\ 2 \\ 1 \\ 3 \end{smallmatrix}$, and $P(4) = \begin{smallmatrix} 4 \\ 2 \\ 1 \\ 4 \end{smallmatrix}$.

The trivial extension algebra $\mathrm{T}(\Lambda)$, where Λ is the tilted algebra from Example 3.5.9, is the path algebra of the quiver



with relations $\rho = \langle \alpha\beta - \gamma\delta, \alpha\beta\epsilon\alpha, \beta\epsilon\alpha\beta, \epsilon\alpha\beta\epsilon, \beta\epsilon\gamma, \delta\epsilon\alpha \rangle$.

The AR quiver of $\mathrm{T}(\Lambda)$ is



The projective $\mathrm{T}(\Lambda)$-modules are $Q(1) = \begin{smallmatrix} 1 \\ 4 \\ 23 \\ 1 \end{smallmatrix}$, $Q(2) = \begin{smallmatrix} 2 \\ 1 \\ 4 \\ 2 \end{smallmatrix}$, $Q(3) = \begin{smallmatrix} 3 \\ 1 \\ 4 \\ 3 \end{smallmatrix}$, and $Q(4) = \begin{smallmatrix} 4 \\ 23 \\ 1 \\ 4 \end{smallmatrix}$. Observe that the stable AR quiver of $\mathrm{T}(H)$ is isomorphic to the stable AR quiver of $\mathrm{T}(\Lambda)$.

Assume now that Λ is an iterated tilted algebra. We wish to relate the stable module category $\mathrm{T}(\Lambda)$-<u>mod</u> to the stable module category of a trivial extension of a hereditary algebra. We have the following proposition.

Proposition 3.5.12. *Let Λ be an iterated tilted algebra and let* $\mathrm{T}(\Lambda)$ *denote its trivial extension algebra. Then there exists a hereditary algebra H and a stable equivalence* $\mathfrak{S} : \mathrm{T}(\Lambda)\text{-}\underline{mod} \longrightarrow \mathrm{T(H)}\text{-}\underline{mod}$ *where* $\mathrm{T}(H)$ *is the trivial extension of the hereditary algebra H.*

Proof. Let Λ be an iterated tilted algebra. Then there exists a sequence of algebras $\Lambda = \Lambda_t, \Lambda_{t-1}, \ldots, \Lambda_0 = H$ where H is a hereditary algebra and a sequence of tilting modules ${}_{\Lambda_i}T^{(i)}$, $0 \leq i \leq t-1$, such that $\Lambda_{i+1} = \mathrm{End}_{\Lambda_i}(T^{(i)})^{op}$ for each i.

Applying Thm. 3.5.10 to the pairs of algebras Λ_i and Λ_{i+1}, gives a stable equivalence $\mathfrak{S}_\mathfrak{i} : \mathrm{T}(\Lambda_i)\text{-}\underline{\mathrm{mod}} \longrightarrow \mathrm{T}(\Lambda_{i+1})\text{-}\underline{\mathrm{mod}}$ for $0 \leq i \leq t-1$.

The composition $\mathfrak{S} = \mathfrak{S}_{\mathfrak{t}-1} \ldots \mathfrak{S}_1 \mathfrak{S}_0$ is the desired equivalence. In summary, we have a stable equivalence $\mathfrak{S} : \mathrm{T(H)}\text{-}\underline{\mathrm{mod}} \longrightarrow \mathrm{T}(\Lambda)\text{-}\underline{\mathrm{mod}}$ where H is a hereditary algebra. □

We proved earlier that a stable equivalence between the stable module categories of self-injective algebras preserves complexity. We can now use our previous work to analyze the complexities of modules over the trivial extension algebra of an iterated tilted algebra.

Theorem 3.5.13. *Let Λ be an iterated tilted algebra from a hereditary algebra H. Then* $\mathrm{cx}_{\mathrm{T}(\Lambda)}\, \mathfrak{S}(M) = \mathrm{cx}_{\mathrm{T}(H)}\, M$ *for each* $M \in \mathrm{T(H)}\text{-}\underline{mod}$.

Proof. This follows from the observation that $\mathrm{T}(H)$ and $\mathrm{T}(\Lambda)$ are both symmetric algebras and by Proposition 3.5.12 we know that there exists a stable equivalence $\mathfrak{G} : \mathrm{T(H)}\text{-}\underline{\mathrm{mod}} \longrightarrow \mathrm{T}(\Lambda)\text{-}\underline{\mathrm{mod}}$. By Theorem 3.4.10 stable equivalence between self-injective algebras preserves complexity of modules and we have the equality $\mathrm{cx}_{\mathrm{T}(\Lambda)}\, \mathfrak{G}(M) = \mathrm{cx}_{\mathrm{T}(H)}\, M$ for all $M \in \mathrm{T(H)}\text{-}\underline{\mathrm{mod}}$. $\square$

Corollary 3.5.14. *Let Λ be an iterated tilted algebra from a hereditary algebra H. Let $\mathcal{C}$ be a component in the stable AR quiver of the trivial extension algebra $\mathrm{T}(\Lambda)$. Then the complexity of $\mathcal{C}$ satisfies the following*

(i) If H is of finite representation type, then Λ is of finite representation type and $\mathrm{cx}_\Lambda\, \mathcal{C} = 0$.

(ii) If H is of tame representation type, then $\mathrm{cx}_{\mathrm{T}(\Lambda)}\, \mathcal{C} = 2$ if $\mathcal{C} = \mathfrak{G}(\mathcal{P}_s)$ or $\mathcal{C} = \mathfrak{G}(\mathcal{I}_s)$ where $\mathcal{P}_s$ and $\mathcal{I}_s$ are the two special components of $\mathrm{T}(H)$ that we described in the discussion preceding Lemma 3.2.2. For all other components, $\mathrm{cx}_{\mathrm{T}(\Lambda)}\, \mathcal{C} = 1$. Furthermore, the only complexity 1 components are the stable tubes.

(iii) If H is of wild representation type, then $\mathrm{cx}_{\mathrm{T}(\Lambda)}\, \mathcal{C} = \infty$.

We point out as separate corollaries the following observations.

Corollary 3.5.15. *If $\mathrm{T}(\Lambda)$ is a trivial extension of an iterated tilted algebra, then the only $\mathrm{T}(\Lambda)$-modules of complexity 1 are the periodic modules.*

Corollary 3.5.16. *If Λ is an iterated tilted algebra from a hereditary algebra of tame representation type, then there exist $\mathrm{T}(\Lambda)$-modules with every allowed complexity i.e. complexities 0, 1, and 2.*

Chapter 4

τ-Complexity of Cluster Tilted Algebras

In this chapter we study the τ-complexity of modules over cluster tilted algebras. These algebras were introduced by Buan, Marsh, Reineke, Reiten, and Todorov in their seminal papers on cluster categories [BMR, BMRRT] from 2006 and 2007. Cluster tilted algebras were inspired by the theory of cluster algebras which were introduced by Fomin and Zelevinski in [FZ]. Recently, Assem, Brüstle, and Schiffler have found connections between relation extensions of tilted algebras and cluster tilted algebras [ABS]. P. Bergh and S. Oppermann have proposed two different definitions of complexity: one for the cluster category and one for the bounded derived category of a cluster tilted algebra in [BO]. In this Chapter we study a different definition of complexity for cluster tilted algebras– the τ-complexity. While the authors of [BO] study the transjective component in the cluster category, our approach and methods are different and allow us to determine the complexity of all modules over a cluster tilted algebra.

Our goal is to prove that the behaviour of τ-complexity over a cluster tilted algebra is directly related to the representation type of the underlying hereditary algebra H. The main theorem of the chapter is the following:

Theorem. *Let $H = k\triangle$ be a finite-dimensional hereditary k-algebra where k is an algebraically closed field and $\triangle$ is a finite quiver without oriented cycles. Let T be a tilting object in the corresponding cluster category $\mathcal{C}$. Denote by $\widehat{C}$ the cluster tilted algebra $\operatorname{End}_{\mathcal{C}}(T)^{op}$. Let X be a module over $\widehat{C}$.*

(i) If H is of finite representation type, then $\operatorname{cx} X = 0$ or 1, depending on the choice of the tilting object T.

(ii) If H is of tame representation type, then $\operatorname{cx} X = 0$, 1 or 2.

(iii) If H is of wild representation type, then either $\operatorname{cx} X = 0$, or X has infinite τ-complexity.

We remark that the only modules of τ-complexity 1 are the τ-periodic modules. In the case when H is tame, there exist $\widehat{C}$-modules of each allowed complexity i.e. 0, 1, and 2. In the case when H is wild, all modules of non-zero complexity over $\widehat{C}$ have infinite complexity.

This theorem is somewhat surprising since cluster tilted algebras are a generalization of tilted algebras. Yet, the behaviour of τ-complexity over tilted algebras is not directly related to the representation type of the underlying hereditary algebra. There are many examples of hereditary algebras of infinite representation type whose tilted algebras are of finite representation type. The first algebras have modules of positive τ-complexity while the second do not. Our main result then shows that in terms of τ-complexity the cluster tilted algebras are in a way more closely related to the hereditary algebras than the tilted algebras.

We provide an example of a hereditary algebra of infinite representation type whose tilted algebra is of finite representation type. See Section 4.8 in Chapter VIII of [ASS]:

Example 4.0.17. Let H be the path algebra of the Euclidean quiver of type $\tilde{A}_3$

$$\begin{tikzcd}
 & 2 \arrow[dl] & \\
1 & & 4 \arrow[ul] \arrow[dl] \\
 & 3 \arrow[ul] &
\end{tikzcd}$$

Let T be the tilting module $T = 1 \oplus \begin{smallmatrix}4\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}4\\2\\1\end{smallmatrix} \oplus 4$. The corresponding tilted algebra $C = \mathrm{End}_H(T)^{op}$ is given by the path algebra of the quiver

$$\begin{tikzcd}
 & 2 \arrow[dl, "\beta"'] & \\
1 & & 4 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\
 & 3 \arrow[ul, "\delta"] &
\end{tikzcd}$$

modulo the relations $\langle \alpha\beta, \gamma\delta \rangle$. Here H is of tame representation type, while C is of finite representation type.

We remark that it is possible to use the work of Assem, Brüstle, and Schiffler [ABS2] to obtain some of our results concerning those AR components of the cluster tilted algebra that arise as the quotients of the transjective components in the bounded derived category of H-mod. Our approach, however, gives a direct proof and determines the τ-complexity of all AR components of the cluster tilted algebra.

4.1 Preliminaries

We recall our assumptions that all algebras are finite-dimensional algebras over an algebraically closed field k. We write τ_Λ (or, simply τ) for the Auslander-Reiten translate (AR translate, for short) in Λ-mod. We use $\Gamma(\Lambda)$ to denote the Auslander-Reiten quiver (AR quiver, for short) of Λ-mod.

Recall also that the τ-complexity of a Λ-module M measures the rate of growth of the sequence of the dimensions of $\tau^i M$,

$$\operatorname{cx} M = \inf \left\{ t \in \mathbb{N}_0 \mid \exists\, \alpha \in \mathbb{R} \text{ such that } \left|\tau^i(M)\right| \leq \alpha i^{t-1} \text{ for i} \gg 0 \right\}$$

where $\mathbb{N}_0$ denotes the nonnegative integers. When no such $t \in \mathbb{N}_0$ exists, we say that the complexity is infinite and write $\operatorname{cx} M = \infty$. Notice that $\operatorname{cx} M = 1$ means that the dimensions in the τ-orbit of a module M are bounded, and $\operatorname{cx} M = 0$ means that M

has $\tau^i M$ is projective for some $i > 0$.

We define the τ-complexity of an algebra Λ as the supremum

$$\operatorname{cx}\Lambda = \sup\{\operatorname{cx} M | M \in \Lambda\text{-mod}\}$$

We say that an AR component has complexity 0 if each module in the component has complexity 0. Furthermore, it will follow from [Ker] and our results in this chapter that all modules of non-zero τ-complexity in the same AR component have the same τ-complexity. Therefore, when a component contains modules of non-zero complexity we define this common value to be the complexity of the component. In this chapter the term *complexity* always refers to τ-complexity.

Here we study the complexity of modules over a certain family of algebras, namely the cluster tilted algebras.

4.2 Cluster Tilted Algebras

In this section we introduce cluster tilted algebras and collect some preliminary results. Cluster tilted algebras were introduced in [BMR, BMRRT] as a generalization of tilted algebras. We gave the definition of a tilted algebra in subsection 3.5.2. If the algebra H is hereditary, then a finitely generated module T over a hereditary algebra H is a *tilting module* if it satisfies the two conditions

(i) $\operatorname{Ext}^1(T, T) = 0$ and

(ii) There exists an exact sequence $0 \longrightarrow H \longrightarrow T' \longrightarrow T'' \longrightarrow 0$ where T' and T'' belong to $\mathrm{add}(T)$.

We assume throughout this chapter that T is basic i.e. the indecomposable summands of T are pairwise non-isomorphic. Condition (ii) may be replaced by the requirement that the number of indecomposable summands of T is equal to the rank of the Grothendieck group $K_0(H)$. Remark that $\mathrm{add}\, T = \mathrm{add}(T' \oplus T'')$. Otherwise we could remove a summand of T and still obtain a tilting module T_1, but then T_1 would not have $|K_0(H)|$ summands which is a contradiction.

Recall that the endomorphism algebra $C = \mathrm{End}_H(T)^{op}$ is called a *tilted algebra*, and the adjoint pair of additive functors

$$\mathrm{Hom}_H(T, \_) : \text{H-mod} \to \text{C-mod}$$

$$\_ \otimes_C T : \text{C-mod} \to \text{H-mod}$$

allows us to pass between the two module categories H-mod and C-mod.

A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of H-mod is called a *torsion theory* if the following conditions are satisfied:

(i) $\mathrm{Hom}_H(M, N) = 0$ for all $M \in \mathcal{T}$ and $N \in \mathcal{F}$

(ii) $\mathrm{Hom}_H(M, N) = 0$ for each $N \in \mathcal{F}$ implies that $M \in \mathcal{T}$

(iii) $\mathrm{Hom}_H(M, N) = 0$ for each $M \in \mathcal{T}$ implies that $N \in \mathcal{F}$

The objects in $\mathcal{T}$ are called *torsion objects* and the objects in $\mathcal{F}$ are called *torsion-free objects*.

We can always form a torsion theory from any given tilting module T over an algebra H. Namely, the full subcategory of H-mod defined by letting the torsion objects be $\mathcal{T} = \{M \mid \mathrm{Ext}^1_H(T, M) = 0\}$ is a torsion class with the corresponding torsion-free class $\mathcal{F} = \{M \mid \mathrm{Hom}_H(T, M) = 0\}$. See [ASS].

Setting tM to be the largest torsion submodule of a Λ-module M gives rise to a *canonical sequence*

$$0 \longrightarrow tM \longrightarrow M \longrightarrow M/tM \longrightarrow 0$$

a short exact sequence where the module $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$.

Lemma 4.2.1. *Let T be a tilting module over a hereditary algebra H. Then for any H-module M we have the isomorphisms*

(i) $\mathrm{Hom}_H(T, M) \cong \mathrm{Hom}_H(T, tM)$

(ii) $\mathrm{Ext}^1_H(T, M) \cong \mathrm{Ext}^1_H(T, M/tM)$.

Proof. Applying the covariant functor $\mathrm{Hom}_H(T, \_)$ to the canonical sequence yields the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_H(T, tM) \longrightarrow \mathrm{Hom}_H(T, M) \longrightarrow \mathrm{Hom}_H(T, M/tM) \longrightarrow \\ &\longrightarrow \mathrm{Ext}^1_H(T, tM) \longrightarrow \mathrm{Ext}^1_H(T, M) \longrightarrow \mathrm{Ext}^1_H(H, M/tM) \longrightarrow \\ &\longrightarrow \mathrm{Ext}^2_H(T, tM) \longrightarrow \dots \end{aligned}$$

Here $\mathrm{Hom}_H(T, M/tM) = 0$ since $M/tM \in \mathcal{F}$, $\mathrm{Ext}^1_H(T, tM) = 0$ since $tM \in \mathcal{T}$, and $\mathrm{Ext}^2_H(T, tM) = 0$ since H is hereditary. In other words, we have the isomorphisms $\mathrm{Hom}_H(T, tM) \cong \mathrm{Hom}_H(T, M)$ and $\mathrm{Ext}^1_H(T, M) \cong \mathrm{Ext}^1_H(T, M/tM)$. $\square$

For results on tilting theory we refer to [ASS, BB, HR]. In [BMR, BMRRT] the authors take tilting modules and tilted algebras to a more general setting by introducing a new category called a cluster category and a tilting object in this category. A *cluster tilted algebra* is the endomorphism algebra of this tilting object. Before we can introduce the notions of a cluster category and cluster tilted algebra, we must discuss another category, namely the bounded derived category.

4.2.1 Bounded Derived Category

Let H be a hereditary k-algebra. We construct the bounded derived category of H-mod. Recall that a *complex* $X^\bullet$ over H is a sequence of H-modules X^i and morphisms $\delta^i = X^i \to X^{i+1}$ such that $\delta^{i+1}\delta^i = 0$ for all $i \in \mathbb{Z}$. We write

$$X^\bullet : \quad \ldots\ X^{n-1} \xrightarrow{\delta^{n-1}} X^n \xrightarrow{\delta^n} X^{n+1} \xrightarrow{\delta^{n+11}} X^{n+2} \longrightarrow \ldots$$

A complex $X^\bullet$ is *bounded below* if $X^i = 0$ for all but finitely many $i < 0$ and *bounded above* if $X^i = 0$ for all but finitely many $i > 0$. A complex is *bounded* if it is both bounded below and bounded above. In this manner we obtain the *category of bounded complexes* over H. An object in this category is a bounded complex over H and a morphism between two complexes $f : X^\bullet \to Y^\bullet$ is a family of morphisms $f^i : X^i \to Y^i$ in H-mod such that $\delta^i_Y f^i = f^{i+1}\delta^i_X$. A complex $X^\bullet$ is called a *stalk complex* if there exists $s \in \mathbb{Z}$ such that $X^s \neq 0$ while $X^j = 0$ whenever $j \neq s$. The object X^s is then called the *stalk* of the complex $X^\bullet$. Note that we can view an H-module X as a bounded complex by identifying it with the stalk complex $X^\bullet$ with stalk $X^0 = X$. For each $i \in \mathbb{Z}$ the i^{th} *shift* of a complex $X^\bullet$ is a new complex $X^\bullet[i]$ whose degree j term is $X[i]^j = X^{i+j}$ for each $j \in \mathbb{Z}$.

Recall that the i^{th} *cohomology group* of the complex $X^\bullet$ is defined as

$$H^i(X^\bullet) = \operatorname{Ker}\delta^i / \operatorname{Im}\delta^{i-1}$$

for each $i \in \mathbb{Z}$.

In addition, a morphism of complexes $f : X^\bullet \to Y^\bullet$ induces group homomorphisms $H^i(f) : H^i(X) \to H^i(Y)$ for each $i \in \mathbb{Z}$. If these induced morphisms $H^i(f)$ are isomorphisms for all $i \in \mathbb{Z}$, then the morphism of complexes f is called a *quasi-isomorphism.*

Next, we need to pass to the homotopy category. First, recall that two morphisms $f^\bullet, g^\bullet : X^\bullet \longrightarrow Y^\bullet$ are called *homotopic* if there exist morphisms $h^i : X^i \longrightarrow Y^{i-1}$ for all $i \in \mathbb{Z}$ satisfying

$$f^i - g^i = \delta^i_X h^{i+1} + h^i \delta^{i-1}_Y$$

In this case we say that $f^\bullet$ and $g^\bullet$ are *homotopic* via the homotopy h. The *homotopy* *category* has the same objects, but the morphisms are obtained by defining $f^\bullet$ and $g^\bullet$ to be equivalent if they are homotopic. We may think of the objects in the homotopy category as complexes, and the maps as the maps of complexes modulo homotopy.

Finally, the *bounded derived category* of H-mod is obtained from the homotopy category by formally inverting all of the quasi-isomorphisms. By abuse of language we may say that the objects of the bounded derived category $D^b(H)$ are the bounded complexes of H-modules but now homotopic maps are equal and we have new maps obtained by localizing the category of bounded complexes by the class of quasi-isomorphisms. The indecomposable objects in $D^b(H)$ are the stalk complexes with indecomposable stalks. [H]

4.2.2 Triangulated Categories

Let us write $D^b(H)$ (or, simply $\mathcal{D}$) for the bounded derived category of H-mod. While the module category H-mod has short exact sequences, this need not be true of the bounded derived category $D^b(H)$. An important property of the category of $D^b(H)$ is that it is a triangulated category with Auslander-Reiten triangles [H]. In a triangulated category the notions of a distinguished triangle and an AR triangle replace the notions of a short exact sequence and an AR sequence in the module category.

More generally, given an additive category $\mathcal{A}$ and an auto-equivalence of the category $\Sigma : \mathcal{A} \longrightarrow \mathcal{A}$ (called, a suspension or translation), a *triangle* (X, Y, Z, u, v, w)

is a sequence of objects and morphisms of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

A *morphism of triangles* is a triple

$$(f, g, h) : (X, Y, Z, u, v, w) \longrightarrow (X', Y', Z', u', v', w')$$

such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

If the morphisms f, g and h are isomorphisms in $\mathcal{A}$, then (f, g, h) is called an *isomorphism of triangles*.

An additive category $\mathcal{A}$ with translation Σ is said to be *triangulated* if there exists a class of triangles called *distinguished triangles* satisfying the following axioms TR1 through TR4 [RS, H]:

TR1.

(i) $(X, X, 0, 1_X, 0, 0)$ is a distinguished triangle for any object X.

(ii) Every triangle isomorphic to a distinguished one is distinguished.

(iii) Every morphism $X \to Y$ in $\mathcal{A}$ can be embedded in a distinguished triangle (X, Y, Z, u, v, w).

TR2. (Rotation) A triangle (X, Y, Z, u, v, w) is distinguished if and only if (its rotation) $(Y, Z, \Sigma X, v, w, -\Sigma u)$ is distinguished.

TR3. (Morphisms) Every commutative diagram of solid arrows

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dotted, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

whose rows are distinguished triangles can be completed to a morphism of triangles by a morphism $h : Z \to Z'$.

TR4. (The octahedral axiom) Given two morphisms in $\mathcal{A}$

$$X \xrightarrow{u} Y \text{ and } Y \xrightarrow{v} Z$$

and the distinguished triangles

$$(X, Y, X', u, u', w),\ (X, Z, Y', v \circ u, w, r), \text{ and } (Y, Z, Z', v, v', t),$$

there exist morphisms $X' \xrightarrow{\alpha} Y'$ and $Y' \xrightarrow{\beta} Z'$ such that

$$(X', Y', Z', \alpha, \beta, \Sigma u' \circ t)$$

is a distinguished triangle and we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "u'"] \arrow[d, "v"] & X' \arrow[r, "s"] \arrow[d, "\alpha"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "v \circ u"] \arrow[d, "u"] & Z \arrow[r, "w"] \arrow[d, equal] & Y' \arrow[r, "r"] \arrow[d, "\beta"] & \Sigma X \arrow[d, "\Sigma u"] \\
Y \arrow[r, "v"] \arrow[d, "u'"] & Z \arrow[r, "v'"] \arrow[d, "w"] & Z' \arrow[r, "t"] \arrow[d, equal] & \Sigma Y \arrow[d, "\Sigma u'"] \\
X' \arrow[r, dotted, "\alpha"] & Y' \arrow[r, dotted, "\beta"] & Z' \arrow[r, "\Sigma u' \circ t"] & \Sigma X'
\end{tikzcd}$$

A distinguished triangle (X, Y, Z, u, v, w) is called an *Auslander-Reiten triangle* if it satisfies:

(i) X and Z are indecomposable

(ii) $w \neq 0$

(iii) If $f : W \to Z$ is not a retraction, then there exists $f' : W \to Y$ such that $vf' = f$.

Lemma 4.2.2 (Section 4, Ch. I in [H])**.** *Let (X, Y, Z, u, v, w) be an AR triangle. If $f : X \to W$ is not a section, then there exists $f' : Y \to W$ with $f = f'u$*

We say that *a triangulated category has AR triangles* if for every indecomposable object Z there exists a triangle satisfying the above conditions.

In addition to AR triangles, we also have a notion of an irreducible morphism. A morphism $h : Z' \to Z$ in an additive category is *irreducible* if h is neither a section or a retraction, but for any factorization $h = h' \circ h$ either h' is a section or h is a retraction. [H]

The following is the analogue of uniqueness of AR sequences for triangulated categories.

Proposition 4.2.3 (Section 4, Chapter I [H]). *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be an AR triangle. Then*

(i) Given Z, the AR triangle is unique up to isomorphism of triangles.

(ii) The morphisms u and v are irreducible.

As a consequence of the uniqueness of AR triangles, we may define the AR translate in a triangulated category. Given an AR triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ set $\tau Z = X$ and $\tau^{-1} X = Z$.

The concepts of AR triangles and the AR translate τ naturally lead us to the AR quiver. The AR quiver of a triangulated category has vertices $[X]$ the isomorphism classes of the indecomposable objects X in the triangulated category. There is an arrow $[X] \to [Y]$ in the AR quiver if and only if there exists an irreducible morphism $X \to Y$ in the triangulated category.

The following theorem guarantees the existence of AR triangles for the bounded derived category $D^b(H)$ of a hereditary algebra H.

Theorem 4.2.4 (Section 1, Chapter IV [H]). *Let Λ be a finite-dimensional k-algebra of finite global dimension. Then the derived category $D^b(\Lambda)$ has AR triangles.*

Here we work with the derived category of bounded complexes over a hereditary algebra H. We have seen that $D^b(H)$ is triangulated and has AR triangles. We write τ_D (or simply τ when no confusion can arise) for the AR translate and $\Gamma(D^b(H))$ for the AR quiver. For a derived category the suspension Σ is the shift of a complex. It is customary to denote the shift by [1].

We can construct the AR quiver of $D^b(H)$ from information about the hereditary algebra H. It is well known that a hereditary algebra H over algebraically closed field k is the path algebra of a quiver $\triangle$ with no oriented cycles, so we write $H = k\triangle$. Furthermore, H is of finite representation type if $\triangle$ is a Dynkin diagram, H is of tame representation type is $\triangle$ is a Euclidean diagram. In all other cases we say that H is of wild representation type and $\triangle$ is a wild diagram. [ARS]

Recall that the diagrams $\triangle$ help us construct the AR quiver of H-mod. If H is of finite representation type, then there is a unique connected component in the AR quiver containing all of the finitely many non-isomorphic indecomposable modules. If H is of infinite representation type then we have the following description of the AR quiver of H. There is a *preprojective component* of type $\mathbb{N}\triangle$ containing all of the projective H-modules. There is a *preinjective component* of type $\mathbb{N}\triangle$ containing all of the injective H-modules. All other components are called *regular*. If H is of tame representation type, then they are stable tubes and if H is of wild representation type, then they are of the form $\mathbb{Z}A_\infty$. [ARS, R]

We now turn to the components in the AR quiver of $D^b(H)$. The theorem below follows from Corollary on page 54 in [H].

Theorem 4.2.5. *Let $H = k\triangle$ be a finite-dimensional hereditary k-algebra where k is an algebraically closed field.*

(i) If H is of finite representation type, then $\Gamma(D^b(H)) = \mathbb{Z}\triangle$.

(ii) If H is of tame representation type, then $\Gamma(D^b(H))$ consists of components of the form $\mathbb{Z}\triangle$ and $\mathbb{Z}A_\infty/\tau^n$ for $n \in \mathbb{N}$.

(iii) If H is of wild representation type, then $\Gamma(D^b(H))$ consists of components of the form $\mathbb{Z}\triangle$ and $\mathbb{Z}A_\infty$

The components of type $\mathbb{Z}\triangle$ are called *transjective components* because they are formed by attaching the (shifts of) preprojective and preinjective component of H-mod. More precisely, for each $i \in \mathbb{Z}$ we add an arrow from the shifted injective object $I(a)[i]$ to the shifted projective object $P(b)[i+1]$ for each arrow from a to b in the quiver $\triangle$. All of the remaining components are called *regular*. [H]

We now collect some properties of morphisms of $D^b(H)$ that become useful to us later.

Lemma 4.2.6. *Let H be a hereditary algebra and $D^b(H)$ the bounded derived category of* H-*mod. Then for any $M, N \in$* H-*mod*

(i) $\mathrm{Hom}_{\mathcal{D}}(M[i], N[i]) = \mathrm{Hom}_{\mathcal{D}}(M, N)$ *for any $i \in \mathbb{Z}$*

(ii) $\mathrm{Hom}_{\mathcal{D}}(M, N[i]) = 0$ *for $i \neq 0, 1$*

(iii) $\mathrm{Hom}_{\mathcal{D}}(M, N[1]) = \mathrm{Ext}^1_H(M, N)$

(iv) $\mathrm{Hom}_{\mathcal{D}}(M, N[0]) = \mathrm{Hom}_H(M, N)$

4.2.3 Cluster Categories

Denote the composition of the shift [1] and the translate τ^{-1} in the bounded derived category $D^b(H)$ by $\mathbf{F} = \tau^{-1}[1] = [1]\tau^{-1}$. We may then form the factor category $D^b(H)/\mathbf{F}$. This factor category is called the *cluster category* and we denote it by $\mathcal{C}$.

The objects of $\mathcal{C}$ are the $\mathbf{F}$-orbits of objects in $D^b(H)$ and the morphisms are

$$\mathrm{Hom}_{\mathcal{C}}(\widetilde{M}, \widetilde{N}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(M, \mathbf{F}^i N)$$

where $\widetilde{M}$ and $\widetilde{N}$ denote the $\mathbf{F}$-orbits of M and N respectively. The notion of a cluster category was first introduced in [BMRRT].

In [K] it was proved that $\mathcal{C}$ has a triangulated structure induced by the triangulated structure of $D^b(H)$, and in [BMRRT] it was shown that $\mathcal{C}$ has AR triangles induced by those in $D^b(H)$. In particular, given an indecomposable object $\widetilde{X} \in \mathcal{C}$ induced by an indecomposable module $X \in$ H-mod, there is an AR triangle in $\mathcal{C}$ of the form

$$\tau\widetilde{X} \longrightarrow \widetilde{Y} \longrightarrow \widetilde{X} \longrightarrow \tau\widetilde{X}[1]$$

where

$$0 \longrightarrow \tau X \longrightarrow Y \longrightarrow X \longrightarrow 0$$

is an AR sequence in H-mod. We may thus form the AR quiver of the cluster category $\mathcal{C}$. We write τ for the translation functor in $\mathcal{C}$ (when needed, we specify $\tau = \tau_{\mathcal{C}}$).

We obtain the shapes of the components in the AR quiver of $\mathcal{C}$ by identifying all of the vertices in an $\mathbf{F}$-orbit of a vertex in $D^b(H)$. Thus, we obtain components that arise as images of components of type $\mathbb{Z}\triangle$ in $D^b(H)$; we refer to these as *transjective components*. The remaining AR components arise as images of regular components in $D^b(H)$ and are therefore called *regular components* in $\mathcal{C}$. Note that in the case when H is of finite representation type, these identifications result in a single τ-periodic component. In the case when H is of infinite representation type, the process never identifies two objects that belong to the same AR component in $D^b(H)$.

4.2.4 Cluster Tilted Algebras

A tilting object in the cluster category $\mathcal{C}$ is an object T that satisfies the following two conditions

(i) $\mathrm{Ext}^1_{\mathcal{C}}(T,T) = 0$

(ii) T has a maximal number of non-isomorphic direct summands, in other words $\mathrm{Ext}^1_{\mathcal{C}}(T\oplus X, T\oplus X) = 0$ implies that $X \in \operatorname{add} T$.

The endomorphism algebra $\mathrm{End}_{\mathcal{C}}(T)^{op} = \widehat{C}$ is called a *cluster tilted algebra.*

In [BMRRT] it was shown that one may assume that any tilting object T in $\mathcal{C}$ arises from a tilting module $T \in$ H-mod. Therefore, we will use T to denote the tilting module in H-mod as well as the tilting object in the cluster category $\mathcal{C}$.

The following theorem was proved in [BMR].

Theorem 4.2.7. *Let T be a tilting object in $\mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{C}}(T,-)$ induces an equivalence $\mathcal{C}/\operatorname{add}(\tau T) \longrightarrow \operatorname{mod}\operatorname{End}_{\mathcal{C}}(T)^{op}$.*

As a particular consequence, we have a way of constructing the AR quiver of $\widehat{C}$. We need to delete the vertices (and related arrows) corresponding to the summands of τT (and related arrows, respectively) from the AR quiver $\Gamma(\mathcal{C})$.

We provide two examples of a tilted algebra and the corresponding cluster tilted algebra.

Example 4.2.8. Let H be given by the path algebra of the quiver

$$1 \longrightarrow 2 \longrightarrow 3$$

The the AR quiver of H-mod is

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr] & & \\
 & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\
3 \arrow[ur] & & 2 \arrow[ur] & & 1
\end{tikzcd}$$

Let T be the tilting module $T = 3 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 1 = T_1 \oplus T_2 \oplus T_3$. The indecomposable torsion modules are $\mathcal{T} = \{T_1, T_2, T_3, \begin{smallmatrix}1\\2\end{smallmatrix}\}$. There is a unique indecomposable torsion-free module $\mathcal{F} = \{2\}$. The module $\begin{smallmatrix}2\\3\end{smallmatrix}$ is neither torsion nor torsion-free.

The the tilted algebra $C = \mathrm{End}_H(T)^{op}$ is the algebra of the quiver

$$1 \xleftarrow{\beta} 2 \xleftarrow{\alpha} 3$$

with the relation $\alpha\beta = 0$.

Next, we compute the images of $\mathrm{Hom}_H(T, \_)$ and $\mathrm{Ext}^1_H(T, \_)$. The summands of the tilting module T are sent to the projective C-modules: $\mathrm{Hom}_H(T, T_2) = {\begin{smallmatrix}2\\1\end{smallmatrix}}$, $\mathrm{Hom}_H(T, T_3) = {\begin{smallmatrix}3\\2\end{smallmatrix}}$, $\mathrm{Hom}_H(T, T_1) = 1$. Applying $\mathrm{Hom}_H(T, \_)$ to the short exact sequence

$$0 \longrightarrow 3 \longrightarrow {\begin{smallmatrix}1\\2\\3\end{smallmatrix}} \longrightarrow {\begin{smallmatrix}1\\2\end{smallmatrix}} \longrightarrow 0$$

gives $\mathrm{Hom}_H(T, {\begin{smallmatrix}1\\2\end{smallmatrix}}) = 2$.

Since H is hereditary, we may use the AR formula (see Thm. 2.13, Ch. IV in [ASS]) to compute

$$\mathrm{Ext}^1_H(T, 2) = D\,\mathrm{Hom}_H(2, \tau T) = D\,\mathrm{Hom}_H(2, 2) = 3$$

The AR quiver of the tilted algebra C is

$$\begin{tikzcd}[row sep=small, column sep=small]
 & {\begin{smallmatrix}2\\1\end{smallmatrix}} \arrow[dr] & & & \\
1 \arrow[ur] & & 2 \arrow[dr] & & 3 \\
 & & & {\begin{smallmatrix}3\\2\end{smallmatrix}} \arrow[ur] &
\end{tikzcd}$$

We now construct the corresponding cluster tilted algebra. The AR quiver of the triangulated category $D^b(H)$ is

$$
\begin{tikzcd}[column sep=small]
 & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr] & & 3[1] \arrow[dr] & & 2[1] \arrow[dr] & & 1[1] \\
\cdots & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2\\3\end{smallmatrix}[1] \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix}[1] \arrow[ur] \arrow[dr] & \cdots \\
3 \arrow[ur] & & 2 \arrow[ur] & & 1 \arrow[ur] & & \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1] \arrow[ur] & & 3[2]
\end{tikzcd}
$$

We can now build the AR quiver in the cluster category $\mathcal{C}$. We identify the indicated objects to obtain a Möbius band in $\mathcal{C}$.

$$
\begin{tikzcd}[column sep=small]
 & & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \arrow[dr] & & 3[1] \arrow[dr] & & 2[1] \cong 3 \arrow[dr] & & 1[1] \cong 2 \\
 & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2\\3\end{smallmatrix}[1] \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix}[1] \cong \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] \arrow[dr] & \\
3 \arrow[ur] & & 2 \arrow[ur] & & 1 \arrow[ur] & & \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1] \arrow[ur] & & 3[2] \cong \begin{smallmatrix}1\\2\\3\end{smallmatrix}
\end{tikzcd}
$$

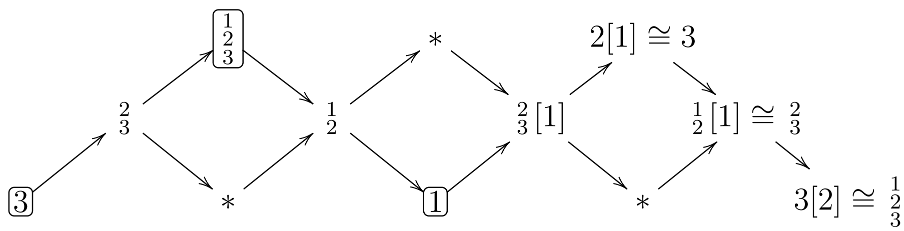
The cluster tilted algebra $\widehat{C} = \mathrm{End}_{\mathcal{C}}(T)^{op}$ where T is as before, is given by the path algebra of the quiver

$$
\begin{tikzcd}
1 \arrow[rr, bend right, "\gamma"'] & 2 \arrow[l, "\beta"'] & 3 \arrow[l, "\alpha"']
\end{tikzcd}
$$

with relations $\alpha\beta = 0$, $\beta\gamma = 0$, $\gamma\alpha = 0$.

Theorem 4.2.7 allows us to construct the AR quiver of the cluster tilted algebra $\widehat{C}$ directly from the AR quiver of the cluster category $\mathcal{C}$. The general shape of the

AR quiver of $\widehat{C}$ is obtained by simply removing the vertices corresponding to the summands of τT. In the illustration below the summands of T are boxed, the vertices that need to be deleted are marked by an asterisk $*$.



Next, we compute the image of $\mathrm{Hom}_{\mathcal{C}}(T, \_)$ to obtain the AR quiver of the cluster tilted algebra $\widehat{C}$. The summands of T are sent to the projective $\widehat{C}$-modules: $\mathrm{Hom}_{\mathcal{C}}(T, 3) = \begin{smallmatrix}1\\3\end{smallmatrix}$, $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}1\\2\\3\end{smallmatrix}) = \begin{smallmatrix}2\\1\end{smallmatrix}$, $\mathrm{Hom}_{\mathcal{C}}(T, 1) = \begin{smallmatrix}3\\2\end{smallmatrix}$. The rest of the objects are mapped as follows: $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}2\\3\end{smallmatrix}) = 1$, $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}2\\3\end{smallmatrix}[1]) = 3$, $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}1\\2\end{smallmatrix}) = 2$.

We illustrate these computations by showing $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}1\\2\end{smallmatrix}) = 2$. By the definition of morphisms in the cluster category

$$\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}1\\2\end{smallmatrix}) = \bigoplus_i \mathrm{Hom}_{\mathcal{D}}(T, ([1]\tau^{-1})^i \begin{smallmatrix}1\\2\end{smallmatrix})$$

By the properties of morphisms in the bounded derived category $D^b(H)$, the right-hand side is nonzero only for $i = 0, 1$. Hence,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(T, {\scriptstyle\begin{smallmatrix}1\\2\end{smallmatrix}}) &= \mathrm{Hom}_{\mathcal{D}}(T, {\scriptstyle\begin{smallmatrix}1\\2\end{smallmatrix}}) \oplus \mathrm{Hom}_{\mathcal{D}}(T, [1]\tau^{-1}{\scriptstyle\begin{smallmatrix}1\\2\end{smallmatrix}}) \\
&= \mathrm{Hom}_{H}(T, {\scriptstyle\begin{smallmatrix}1\\2\end{smallmatrix}}) \oplus \mathrm{Ext}^1_{H}(T, \tau^{-1}{\scriptstyle\begin{smallmatrix}1\\2\end{smallmatrix}}) \\
&= 2 \oplus \mathrm{Ext}^1_H(T, 0) = 2
\end{aligned}$$

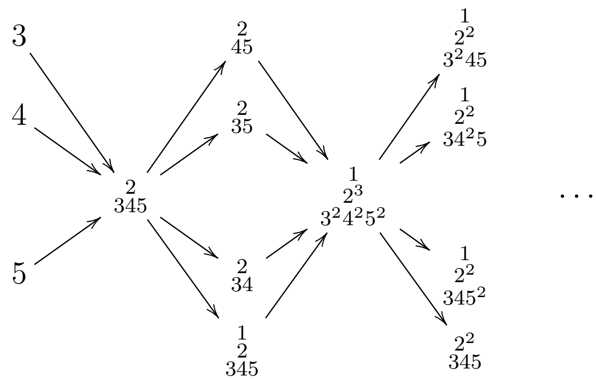
Finally, the AR quiver of the cluster tilted algebra $\widehat{C}$ is given below. Again, the ends are identified and we obtain a periodic component.

$$\begin{tikzcd}
\begin{smallmatrix}1\\3\end{smallmatrix} \arrow[dr] \arrow[rrrrrr, dash, dashed, bend left] & & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1\\3\end{smallmatrix} \\
& 1 \arrow[ur] & & 2 \arrow[ur] & & 3 \arrow[ur] &
\end{tikzcd}$$

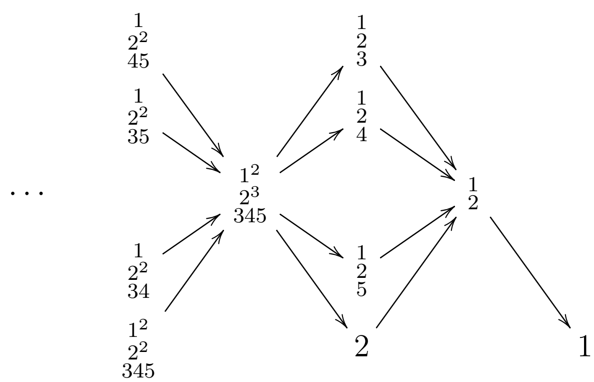
Example 4.2.9. Let H be given by the path algebra of the quiver $\vec{\triangle}$

$$\begin{tikzcd}
& & 3 \\
1 \arrow[r] & 2 \arrow[ur] \arrow[r] \arrow[dr] & 4 \\
& & 5
\end{tikzcd}$$

The the AR quiver of H-mod consists of a preprojective component $\mathcal{P}$, a preinjective component $\mathcal{I}$, and stable tubes.

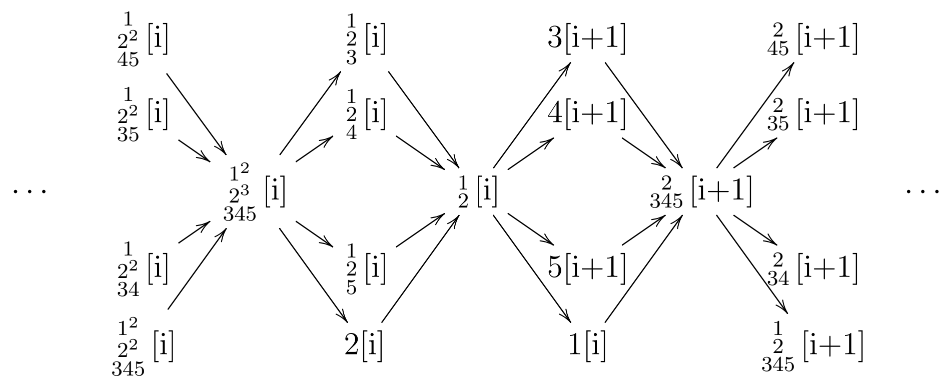


The preprojective component $\mathcal{P}$.



The preinjective component $\mathcal{I}$.

We will now construct the cluster tilted algebra and its AR quiver. First, we need the AR quiver of the bounded derived category $D^b(H)$. The transjective components are formed by gluing together the components $\mathcal{I}[i]$ and $\mathcal{P}[i+1]$ by identifying $\tau^{-1}(I_j)[i] = P_j[i+1]$ for each vertex j in the quiver $\vec{\triangle}$.



The family of transjective components of $D^b(H)$.

Each of the tubes $\mathcal{T}$ in the AR quiver of H gives rise to a family of shifted tubes $\mathcal{T}[i]$ with $i \in \mathbb{Z}$ of the same rank in the AR quiver of $D^b(H)$. Note that this is true of any situation where the hereditary algebra H is tame.

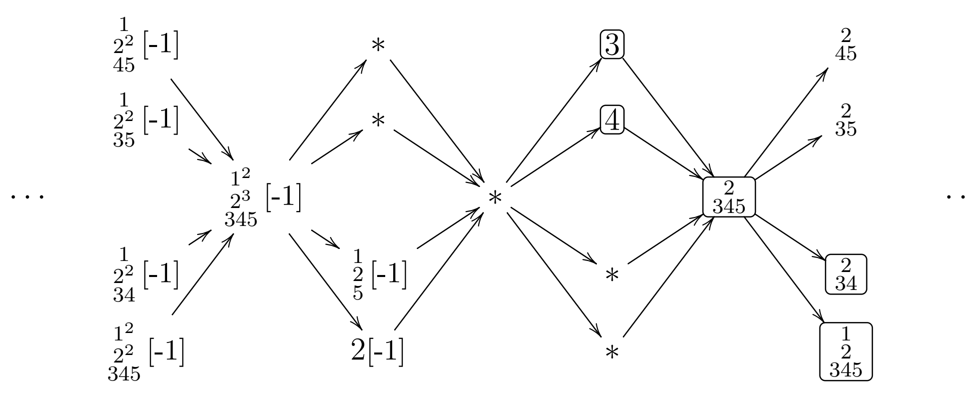
We proceed to construct the AR quiver of the cluster category $\mathcal{C}$. The identification $\tau^{-1}[1]X = X$ for all objects $X \in D^b(H)$ collapses the family of transjective components of $D^b(H)$ into a single component of $\mathcal{C}$. Objects in the cluster category $\mathcal{C}$ are then the $\tau^{-1}[1]$-orbits of objects in $D^b(H)$. Notice that for the case when H is of infinite representation type this identification process never identifies two objects that belong to the same component in $D^b(H)$. This means that the action of $\tau_{\mathcal{C}}$ in the cluster category is given exactly by the action of $\tau_{D^b(H)}$ on the representative objects in the derived category $D^b(H)$.

Similarly, each family of tubes $\mathcal{T}[i]$ in $D^b(H)$ collapses to a single tube in the cluster category $\mathcal{C}$. Furthermore, the rank of this tube in the cluster category is the same as the rank of the original tubes in H-mod.

We now proceed to build a cluster tilted algebra. Consider the tilting object T induced by the tilting module

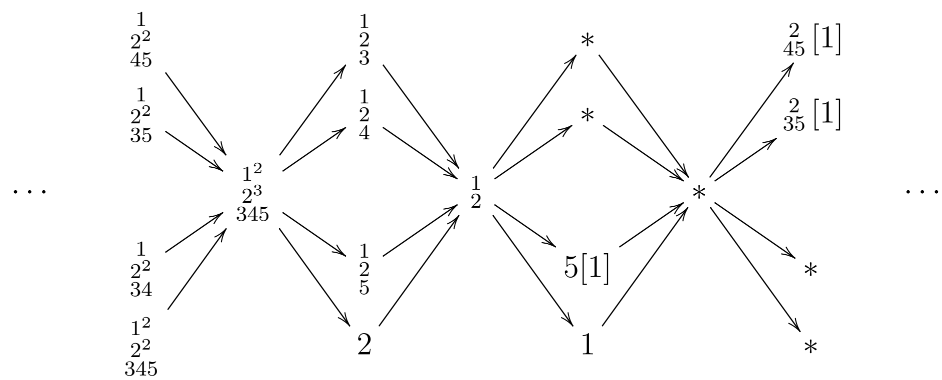
$$T = {\begin{smallmatrix}1\\2\\345\end{smallmatrix}} \oplus {\begin{smallmatrix}2\\345\end{smallmatrix}} \oplus 3 \oplus 4 \oplus {\begin{smallmatrix}2\\34\end{smallmatrix}} = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5$$

The AR quiver of the cluster tilted algebra $\widehat{C} = \mathrm{End}_{\mathcal{C}}\, T^{op}$ is obtained from the AR quiver of the cluster category $\mathcal{C}$ by deleting the vertices corresponding to the summands of τT. In the diagram below the representatives in degree 0 of the summands of the tilting object T are boxed, the vertices that are to be deleted are denoted by an asterisk.



Representatives of the objects in the transjective component of the cluster category.

We provide another diagram as a further illustration. Here the representatives in degree 1 of the summands of the tilting object T are the vertices that need to be deleted: in the cluster category $\tau T[0] = \tau^{-1}[1](\tau T[0]) = T[1]$



Representatives of the objects in the transjective component of the cluster category.

We see that the image of the transjective component splits into two disjoint AR components in $\widehat{C}$. Furthermore, the tubes in the AR quiver of $\widehat{C}$ have exactly the same rank as those in the AR quiver of the cluster category. This is easy to see since the summands of T do not live in tubes and therefore the shape of these components is unaffected when passing to the cluster tilted algebra.

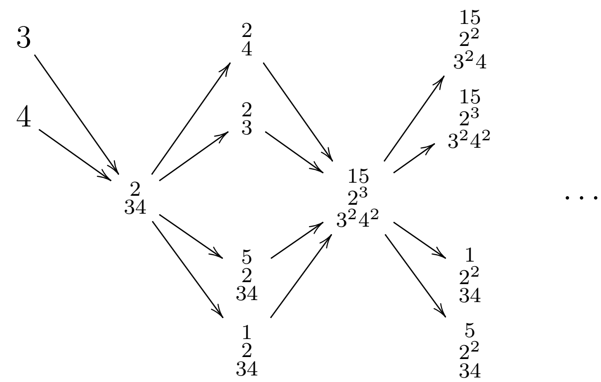
We now build the AR quiver of the cluster tilted algebra by computing the image of $\mathrm{Hom}_{\mathcal{C}}(T, \_)$. The summands of T are sent to the projective $\widehat{C}$-modules: $\mathrm{Hom}_{\mathcal{C}}(T, T_1) = \begin{smallmatrix}1\\2\\34\end{smallmatrix}$, $\mathrm{Hom}_{\mathcal{C}}(T, T_2) = \begin{smallmatrix}2\\345\end{smallmatrix}$, $\mathrm{Hom}_{\mathcal{C}}(T, T_3) = 3$, $\mathrm{Hom}_{\mathcal{C}}(T, T_4) = 4$, and $\mathrm{Hom}_{\mathcal{C}}(T, T_5) = \begin{smallmatrix}5\\2\\34\end{smallmatrix}$.

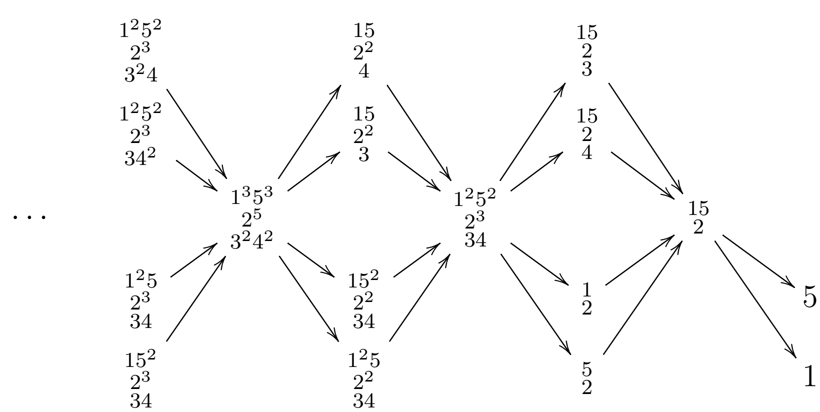
Further calculations yield: $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}2\\45\end{smallmatrix}) = \begin{smallmatrix}2\\4\end{smallmatrix}$, $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}2\\35\end{smallmatrix}) = \begin{smallmatrix}2\\3\end{smallmatrix}$, $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}1\\2^3\\3^24^25^2\end{smallmatrix}) = \begin{smallmatrix}15\\2^3\\3^24^2\end{smallmatrix}$, $\mathrm{Hom}_{\mathcal{C}}(T, 1) = 1$, $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}1\\2\\3\end{smallmatrix}) = \begin{smallmatrix}15\\2\\3\end{smallmatrix}$, $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}1\\2\\4\end{smallmatrix}) = \begin{smallmatrix}15\\2\\4\end{smallmatrix}$, $\mathrm{Hom}_{\mathcal{C}}(T, \begin{smallmatrix}1\\2\end{smallmatrix}) = \begin{smallmatrix}15\\2\end{smallmatrix}$, and $\mathrm{Hom}_{\mathcal{C}}(T, 5[1]) = 5$.

We give the details for the last calculation. By the definition of morphisms in the cluster category and by the properties in Lemma 4.2.6 we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(T, 5[1]) &= \mathrm{Hom}_{\mathcal{D}}(T, 5[1]) \oplus \mathrm{Hom}_{\mathcal{D}}(T, \tau 5[0]) \\
&= \mathrm{Ext}^1_H(T, 5) \oplus \mathrm{Hom}_H(T, \substack{1\\2\\5}[-1]) \\
&= 5 \oplus 0 = 5
\end{aligned}$$

The AR quiver of the cluster tilted algebra $\widehat{C}$ has the following components: Stable tubes of the same rank as the tubes in the original algebra, along with the two components





In this example, the transjective component of $D^b(H)$ splits into two components when passing to the cluster tilted algebra because the summands of the tilting object T form a a slice of the transjective component. This need not happen in general. When the tilting object T has summands from regular components, then the component in $\widehat{C}$ corresponding to the transjective component in $D^b(H)$ remains connected.

4.3 τ-Complexity

In this section we study the τ-complexity of modules over cluster tilted algebras. We begin by determining the k-dimension of a class of important modules over a tilted algebra. We then use our results to obtain information about the k-dimension of corresponding modules over the cluster tilted algebra. Finally, we examine the shapes of the components that occur in the AR quiver of a cluster tilted algebra and then compute the complexity of the modules in each component based on the type of the component.

The following two lemmas relate the k-dimensions of certain modules over the hereditary algebra H and the tilted algebra $C = \mathrm{End}_H(T)^{op}$.

Lemma 4.3.1. *Let T be a tilting module over a hereditary algebra H. Denote by $C = \mathrm{End}_H(T)^{op}$ the corresponding tilted algebra.*

(i) There exists a constant $c > 0$ such that for any torsion module M we have the inequality $c\,|M| \leq |\mathrm{Hom}_H(T, M)|$.

(ii) There exists a constant $c > 0$ such that for any torsion-free module M we have the inequality $c\,|M| \leq \left|\mathrm{Ext}^1_H(T, M)\right|$.

Proof. Since T is a tilting module there exists a short exact sequence of the form $0 \longrightarrow H \longrightarrow T' \longrightarrow T'' \longrightarrow 0$ where $T', T'' \in \operatorname{add} T$. Denote by t the number of indecomposable summands of $T' \oplus T''$.

Given a module M apply the contravariant functor $\mathrm{Hom}_H(\_, M)$ to the short exact sequence above to obtain the long exact sequence

$$
\begin{aligned}
0 \longrightarrow \mathrm{Hom}_H(T'', M) \longrightarrow \mathrm{Hom}_H(T', M) \longrightarrow \mathrm{Hom}_H(H, M) \longrightarrow \\
\longrightarrow \mathrm{Ext}^1_H(T'', M) \longrightarrow \mathrm{Ext}^1_H(T', M) \longrightarrow \mathrm{Ext}^1_H(H, M) \ldots
\end{aligned}
\qquad (*)
$$

Next we obtain the desired inequality depending on whether M is a torsion or a torsion-free module.

(i) Assume that M is a torsion module. Then $\mathrm{Ext}^1_H(T'', M) = 0$ in the long exact sequence because T'' is in $\mathrm{add}(T)$ and M is torsion. We obtain the short exact sequence of C-modules

$$0 \longrightarrow \mathrm{Hom}_H(T'', M) \longrightarrow \mathrm{Hom}_H(T', M) \longrightarrow \mathrm{Hom}_H(H, M) \longrightarrow 0$$

Taking k-dimensions and noting that $\mathrm{Hom}_H(H, M) \cong M$ as k-vector spaces

$$\begin{aligned} |M| + |\mathrm{Hom}_H(T'', M)| &= |\mathrm{Hom}_H(T', M)| \\ &\leq \left|\mathrm{Hom}_H(T^t, M)\right| \\ &= t\,|\mathrm{Hom}_H(T, M)| \end{aligned}$$

In summary, setting $c = 1/t$ yields $c\,|M| \leq |\mathrm{Hom}_H(T, M)|$.

(ii) Assume that M is torsion-free. In the long exact sequence $(*)$ above, we now have $\mathrm{Hom}_H(T', M) = 0$ since M is torsion-free, and $\mathrm{Ext}^1_H(H, M) = 0$ since H is projective. We thus have the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_H(H, M) \longrightarrow \mathrm{Ext}^1_H(T'', M) \longrightarrow \mathrm{Ext}^1_H(T', M) \longrightarrow 0$$

Taking k-dimensions and noting that $\mathrm{Hom}_H(H, M) \cong M$ as k-vector spaces yields

$$\begin{aligned} |M| + \left|\mathrm{Ext}^1_H(T', M)\right| &= \left|\mathrm{Ext}^1_H(T'', M)\right| \\ &\leq \left|\mathrm{Ext}^1_H(T^t, M)\right| \\ &= t\left|\mathrm{Ext}^1_H(T, M)\right| \end{aligned}$$

Thus, setting $c = 1/t$ yields $c\,|M| \leq \left|\mathrm{Ext}^1_H(T, M)\right|$. $\square$

In the next lemma we obtain a set of inequalities in the other direction. We first need a general remark.

Remark 4.3.2. For any two Λ-modules X and Y over an arbitrary k-algebra Λ we have these relationships between k-dimensions:

(i) $|\mathrm{Hom}_\Lambda(X,Y)| \leq |X|\,|Y|$. This holds since X and Y are assumed to be finitely generated modules over a finite-dimensional k-algebra.

(ii) $\left|\mathrm{Ext}^1_\Lambda(X,Y)\right| \leq |X|\,|Y|\,|\Lambda|^2$. Let $\ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be a miniminal projective resolution of the Λ-module X. $\mathrm{Ext}^1_\Lambda(X,Y)$ is by definition a quotient of a subgroup of $\mathrm{Hom}_\Lambda(P_1,Y)$. But this means that we have the set of inequalities $\left|\mathrm{Ext}^1_\Lambda(X,Y)\right| \leq |\mathrm{Hom}_\Lambda(P_1,Y)| \leq |P_1|\,|Y|$. Furthermore, we have $|P_1| \leq |\mathrm{top}\,\Omega^1(X)|\,|\Lambda| \leq |\Omega^1(X)|\,|\Lambda|$ as well as $|\Omega^1(X)| \leq |P_0| \leq |\mathrm{top}\,X|\,|\Lambda| \leq |X|\,|\Lambda|$. Assembling the above gives $\left|\mathrm{Ext}^1_\Lambda(X,Y)\right| \leq |X|\,|Y|\,|\Lambda|^2$.

Lemma 4.3.3. *Let T be a tilting module over a hereditary algebra H. Denote by $C = \mathrm{End}_H(T)^{op}$ the corresponding tilted algebra.*

(i) There exists a constant $c' > 0$ such that for any torsion module M we have the inequality $|\mathrm{Hom}_H(T,M)| \leq c'\,|M|$.

(ii) There exists a constant $c' > 0$ such that for any torsion-free module M we have the inequality $\left|\mathrm{Ext}^1_H(T,M)\right| \leq c'\,|M|$.

Proof. (i) Assume that M is a torsion module. Consider the short exact sequence

$$0 \longrightarrow H \longrightarrow T^m \longrightarrow T'' \longrightarrow 0$$

where T'' is in $\operatorname{add} T$ and $m > 0$. Fix the constant $c' = (|T''| + 1)/m$. We will demonstrate that $|\mathrm{Hom}_H(T, M)| \leq c' \, |M|$.

Apply the contravariant functor $\mathrm{Hom}_H(\_, M)$ to get the long exact sequence

$$0 \longrightarrow \mathrm{Hom}_H(T'', M) \longrightarrow \mathrm{Hom}_H(T^m, M) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_H(H, M) \longrightarrow \mathrm{Ext}^1_H(T'', M) \longrightarrow \dots$$

In this long exact sequence we have $\mathrm{Ext}^1_H(T'', M) = 0$ because M is torsion. We therefore obtain the short exact sequence of C-modules

$$0 \longrightarrow \mathrm{Hom}_H(T'', M) \longrightarrow \mathrm{Hom}_H(T^m, M) \longrightarrow \mathrm{Hom}_H(H, M) \longrightarrow 0$$

Taking k-dimensions yields

$$|M| + |\mathrm{Hom}_H(T'', M)| = |\mathrm{Hom}_H(T^m, M)|$$

which means

$$\begin{aligned} m \, |\mathrm{Hom}_H(T, M)| &= |M| + |\mathrm{Hom}_H(T'', M)| \\ &\leq |M| + |T''| \, |M| \\ &= (|T''| + 1) \, |M| \end{aligned}$$

In summary, the inequality $|\mathrm{Hom}_H(T, M)| \leq c' \, |M|$ holds.

(ii) Assume now that M is torsion-free. Consider the short exact sequence

$$0 \longrightarrow H \longrightarrow T' \longrightarrow T^m \longrightarrow 0$$

with $T' \in \operatorname{add} T$ and $m > 0$. Fix the constant $c' = (|H|^2 \, |T'| + 1)/m$. We will show that $\left|\mathrm{Ext}^1_H(T, M)\right| \leq c' \, |M|$.

Apply the contravariant functor $\mathrm{Hom}_H(\_, M)$ to the short exact sequence above to get the long exact sequence

$$\begin{aligned} \ldots \longrightarrow \mathrm{Hom}_H(T', M) \longrightarrow \mathrm{Hom}_H(H, M) \longrightarrow \mathrm{Ext}^1_H(T^m, M) \\ \longrightarrow \mathrm{Ext}^1_H(T', M) \longrightarrow \mathrm{Ext}^1_H(H, M) \longrightarrow \ldots \end{aligned}$$

In this long exact sequence we have $\mathrm{Hom}_H(T', M) = 0$ since M is torsion-free, and $\mathrm{Ext}^1_H(H, M) = 0$ since H is projective. We thus get the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_H(H, M) \longrightarrow \mathrm{Ext}^1_H(T^m, M) \longrightarrow \mathrm{Ext}^1_H(T', M) \longrightarrow 0$$

Taking k-dimensions gives $|M| + \left|\mathrm{Ext}^1_H(T', M)\right| = \left|\mathrm{Ext}^1_H(T^m, M)\right|$. We use Remark 4.3.2 to obtain

$$\begin{aligned} m\left|\mathrm{Ext}^1_H(T,M)\right| &= |M| + \left|\mathrm{Ext}^1_H(T',M)\right| \\ &\leq |M| + |H|^2\,|T'|\,|M| \\ &\leq (|H|^2\,|T'|+1)\,|M| \end{aligned}$$

Thus, $\left|\mathrm{Ext}^1_H(T,M)\right| \leq c'\,|M|$ as desired. □

The following Corollary will be of great use to us.

Corollary 4.3.4. *Let $C = \mathrm{End}_H(T)^{op}$ be a tilted algebra. Then, there exist constants $c, c' > 0$ such that for any H-module M and each $i \in \mathbb{Z}$ we have the inequalities*

$$c\left|\tau^i M\right| \leq \left|\mathrm{Hom}_H(T,\tau^i M)\right| + \left|\mathrm{Ext}^1_H(T,\tau^i M)\right| \leq c'\left|\tau^i M\right|$$

Proof. Take the canonical sequence for the module $\tau^i M$

$$0 \longrightarrow t(\tau^i M) \longrightarrow \tau^i M \longrightarrow \tau^i M/t(\tau^i M) \longrightarrow 0$$

Since k-dimension is additive on short exact sequences, we have the equality

$$\left|\tau^i M\right| = \left|t(\tau^i M)\right| + \left|\tau^i M/t(\tau^i M)\right|$$

Since for all $i \in \mathbb{Z}$ the modules $t(\tau^i M)$ are torsion and the modules $\tau^i M/t(\tau^i M)$

are torsion-free, Lemma 4.3.1 and Lemma 4.2.1 imply that there exists a $c > 0$ such that

$$\begin{aligned} c\left|\tau^i M\right| &= c\left|t(\tau^i M)\right| + c\left|\tau^i M/t(\tau^i M)\right| \\ &\leq \left|\mathrm{Hom}_H(T, t(\tau^i M))\right| + \left|\mathrm{Ext}^1_H(T, \tau^i M/t(\tau^i M))\right| \\ &= \left|\mathrm{Hom}_H(T, \tau^i M)\right| + \left|\mathrm{Ext}^1_H(T, \tau^i M)\right| \end{aligned}$$

Similarly, Lemma 4.3.3 and Lemma 4.2.1 imply that there exists a $c' > 0$ such that

$$\begin{aligned} c'\left|\tau^i M\right| &= c'\left|t(\tau^i M)\right| + c'\left|\tau^i M/t(\tau^i M)\right| \\ &\geq \left|\mathrm{Hom}_H(T, t(\tau^i M))\right| + \left|\mathrm{Ext}^1_H(T, \tau^i M/t(\tau^i M))\right| \\ &= \left|\mathrm{Hom}_H(T, \tau^i M)\right| + \left|\mathrm{Ext}^1_H(T, \tau^i M)\right| \end{aligned}$$

□

Recall that the AR quiver of the cluster tilted algebra $\widehat{C}$ is obtained from the AR quiver of the cluster category $\mathcal{C}$ by removing the vertices (and the attached arrows) corresponding to the summands of τT. Denote by $\widehat{M}$ the $\widehat{C}$-module $\mathrm{Hom}_{\mathcal{C}}(T, \widetilde{M})$ where $\widetilde{M}$ is an object in the cluster category $\mathcal{C}$. Denote by $\widehat{Z}$ an AR component of $\widehat{C}$-mod obtained from a component $\widetilde{Z}$ in the cluster category.

The following proposition provides us with the means to calculate the complexity of modules over cluster tilted algebras.

Proposition 4.3.5. *Let $\widehat{Z}$ be obtained from an AR component $\widetilde{Z}$ with infinite τ-orbits* *in the cluster category $\mathcal{C}$. Then for any module $\widehat{M}$ in $\widehat{Z}$ with positive τ-complexity we have for all $i \gg 0$*

$$\left|\tau^i_{\widehat{C}}\widehat{M}\right| = \left|\mathrm{Hom}_H(T, \tau^i_H M)\right| + \left|\mathrm{Ext}^1_H(T, \tau^{i-1}_H M)\right|$$

Proof. Let $\widehat{M}$ be a module in $\widehat{Z}$ of positive complexity. In particular, $\tau^i_{\widehat{C}} M \neq 0$ for $i \geq 0$ and the $\widehat{Z}$ extends infinitely far to the left. Therefore, moving far enough to the left in the component $\widetilde{Z}$ in the cluster category (.i.e. to the left of the summands τT that reside in the component), Thm. 4.2.7 allows us to assume that for $i \geq 0$ the vertices $\tau^i_{\widehat{C}}\widehat{M}$ in the AR-quiver of the cluster tilted algebra $\widehat{C}$ are of the form $\mathrm{Hom}_{\mathcal{C}}(\widetilde{T}, \tau^i_{\mathcal{C}}\widetilde{M})$. We then have

$$\begin{aligned}\left|\tau^i_{\widehat{C}}\,\mathrm{Hom}_{\mathcal{C}}(\widetilde{T}, \widetilde{M})\right| &= \left|\mathrm{Hom}_{\mathcal{C}}(\widetilde{T}, \tau^i_{\mathcal{C}}\widetilde{M})\right| \\ &= \left|\mathrm{Hom}_{\mathcal{C}}(\widetilde{T}, \widetilde{\tau^i_{\mathcal{D}}M})\right|\end{aligned}$$

By the definition of morphisms in the cluster category

$$\left|\mathrm{Hom}_{\mathcal{C}}(\widetilde{T}, \widetilde{\tau^i_{\mathcal{D}}M})\right| = \left|\bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(T, \mathbf{F}^j\tau^i_{\mathcal{D}}M)\right|$$

We now use properties of morphisms in the derived category $D^b(H)$ to obtain

$$\begin{aligned}
\left|\bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(T, \mathbf{F}^j \tau_{\mathcal{D}}^i M)\right| &= \left|\bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(T, (\tau_{\mathcal{D}}^{-j}[j])(\tau_{\mathcal{D}}^i M))\right| \\
&= \left|\bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(T, (\tau_H^{i-j} M)[j])\right| \\
&= \left|\mathrm{Hom}_{\mathcal{D}}(T, \tau_H^i M[0])\right| + \left|\mathrm{Hom}_{\mathcal{D}}(T, \tau_H^{i-1} M[1])\right| \\
&= \left|\mathrm{Hom}_H(T, \tau_H^i M)\right| + \left|\mathrm{Ext}_H^1(T, \tau_H^{i-1} M)\right|
\end{aligned}$$

Combining all of the above steps gives

$$\begin{aligned}
\left|\tau_{\widehat{C}}^i \widehat{M}\right| &= \left|\tau_{\widehat{C}}^i \mathrm{Hom}_{\mathcal{C}}(\widetilde{T}, \widetilde{M})\right| \\
&= \left|\mathrm{Hom}_H(T, \tau_H^i M)\right| + \left|\mathrm{Ext}_H^1(T, \tau_H^{i-1} M)\right| \qquad \square
\end{aligned}$$

We point out a consequence of the previous proposition.

Corollary 4.3.6. *Let $\widehat{C}$ be a cluster tilted algebra from a hereditary algebra H. Let $\widehat{Z}$ be obtained from an AR component $\widetilde{Z}$ with infinite τ-orbits in the cluster category $\mathcal{C}$. Then for any module $\widehat{M}$ in $\widehat{Z}$ with positive τ-complexity we have $\mathrm{cx}_{\widehat{C}} \widehat{M} \leq \mathrm{cx}_H M \leq \mathrm{cx}_{\widehat{C}} \widehat{M} + 1$.*

Proof. We first show $\mathrm{cx}_{\widehat{C}} \widehat{M} \leq \mathrm{cx}_H M$. Suppose $\mathrm{cx}_H M = t$ for some $t \in \mathbb{N} \cup \{0\}$. Then $|\tau_H^i M| \leq \alpha i^{t-1}$ for some $\alpha \in \mathbb{R}$ and for $i \gg 0$. But then by Cor. 4.3.4 also

$$\left|\mathrm{Hom}_H(T, \tau^i M)\right| \leq c' \left|\tau_H^i M\right| \leq c' \alpha i^{t-1}$$

and similarly

$$\left|\mathrm{Ext}^1_H(T,\tau^{i-1}M)\right| \leq c' \left|\tau_H^{i-1}M\right| \leq c'\alpha(i-1)^{t-1} \leq c'\alpha i^{t-1}$$

Combining these observations with Prop. 4.3.5 gives

$$\begin{aligned}\left|\tau^i_{\widehat{C}}(\mathrm{Hom}_{\mathcal{C}}(T,\widetilde{M}))\right| &= \left|\mathrm{Hom}_H(T,\tau^i_H M)\right| + \left|\mathrm{Ext}^1_H(T,\tau_H^{i-1}M)\right| \\ &\leq c'\alpha i^{t-1} + c'\alpha i^{t-1} \\ &\leq 2c'\alpha i^{t-1}\end{aligned}$$

In other words, $\mathrm{cx}_{\widehat{C}}\,\widehat{M} \leq t = \mathrm{cx}_H\, M$.

We now show the other inequality $\mathrm{cx}_H\, M \leq \mathrm{cx}_{\widehat{C}}\,\widehat{M} + 1$. Suppose $\mathrm{cx}_{\widehat{C}}\,\widehat{M} = t$ for some $t \in \mathbb{N} \cup \{0\}$. Then $\left|\tau^i_{\widehat{C}}M\right| \leq \alpha i^{t-1}$ for some $\alpha \in \mathbb{R}$ and for $i \gg 0$. But then by Prop. 4.3.5 we also have

$$\left|\mathrm{Hom}_H(T,\tau^i M)\right| \leq \alpha i^{t-1} \leq \alpha i^t$$

and

$$\left|\mathrm{Ext}^1_H(T,\tau^i M)\right| \leq \alpha(i+1)^{t-1} \leq \alpha i^t$$

for $i \gg 0$. So that Cor. 4.3.4 gives

$$\left|\tau^i M\right| \leq \left|\mathrm{Hom}_H(T,\tau^i M)\right| + \left|\mathrm{Ext}^1_H(T,\tau^i M)\right| \leq 2\alpha i^t$$

In other words, $\operatorname{cx}_H M \leq t+1 = \operatorname{cx}_{\widehat{C}} \widehat{M} + 1$. $\square$

We now use our previous results to determine the complexity of modules over the cluster tilted algebra $\widehat{C}$. Given a $\widehat{C}$-module $\widehat{M}$ we calculate its complexity based on the type of AR component in which it resides.

Before giving the next theorem, we set up some terminology and notation. Denote by $\widetilde{K}$ the transjective component in the cluster category $\mathcal{C}$. As we discussed in Example 4.2.9, passing to the cluster tilted algebra $\widehat{C}$ the quotient $\widetilde{K}/\operatorname{add}(\tau T)$ may be a single component or it may split into two disjoint components. In the latter case, one of the two components, call it $\widehat{K}'$, will contain only modules of τ-complexity 0. Therefore, in either situation we write $\widehat{K}$ for the AR component of $\widehat{C}$-mod that arises from the transjective component of $D^b(H)$ and contains modules of positive complexity.

For each regular component $\widetilde{Z}$ in the cluster category $\mathcal{C}$, denote by $\widehat{Z}$ its quotient in $\widehat{C}$-mod.

Recall our convention that we say that an AR component has complexity t if all of the modules of non-zero complexity in that component have complexity t. In case when there are no modules of non-zero complexity, we say that the component has complexity 0.

Theorem 4.3.7. *Let $H = k\triangle$ be a finite-dimensional hereditary k-algebra where k is an algebraically closed field. Then the complexities of the components of the AR quiver of $\widehat{C}$ which we described above satisfy* *(i) If H is of finite representation type,* $\operatorname{cx}\widehat{K} = 0$ *or* 1 *depending on the choice of the tilting object T.*

(ii) If H is of tame representation type, then $\operatorname{cx}\widehat{K} = 2$, $\operatorname{cx}\widehat{K}' = 0$, *and* $\operatorname{cx}\widehat{Z} = 1$.

(iii) If H is of wild representation type, then $\operatorname{cx}\widehat{K} = \infty$, $\operatorname{cx}\widehat{K}' = 0$, *and* $\operatorname{cx}\widehat{Z} = \infty$.

Proof. Recall that if H is of finite representation type, then the transjective component $\widetilde{K}$ is the unique component of $\mathcal{C}$. Furthermore, $\widetilde{K}$ is a periodic component with finitely many vertices. Thus passing to the cluster tilted algebra via Thm. 4.2.7 may result in two cases. If the objects τT do not intersect all τ-orbits, then $\widehat{K}$ is a periodic component and $\operatorname{cx} K = 1$. If the objects in $\operatorname{add}\tau T$ intersect all τ-orbits, then the corresponding component of the cluster tilted algebra has vertices with τ-complexity 0. This is the case whenever T is a complete slice and the resulting cluster tilted algebra is hereditary.

We now proceed by analyzing the transjective components arising in the case when H is of tame or wild representation type. In the first case $H = k\triangle$ where $\triangle$ is a Euclidean diagram and in the second case $H = k\triangle$ where $\triangle$ is a wild diagram. The transjective component $\widetilde{K}$ in $\mathcal{C}$ is of the form $\mathbb{Z}\triangle$. The $\widehat{C}$-modules with positive complexity reside in the component $\widehat{K}$ that looks like $\mathbb{N}\triangle$ when we look far enough to the left (i.e. to the left of the modules τT). Recall, that in the case when the quotient $\widetilde{K}/\operatorname{add}(\tau T)$ is not connected, we also obtain a component $\widetilde{K}'$ containing only modules of τ-complexity 0.

We now study the component $\widehat{K}$. Note that any $\widehat{C}$-module $\widehat{M}$ of positive complexity in $\widehat{K}$ satisfies Prop. 4.3.5 and Cor. 4.3.6. Furthermore, $\tau_H^i M$ lie in the preinjective component of H for $i \geq 0$.

When H is of tame represenentation type, then we know from the proof of Thm. 3.5.5 that modules in the preinjective component of H have complexity 2 . Therefore, Cor. 4.3.6 says that $\operatorname{cx}_{\widehat{C}} M \leq \operatorname{cx}_H M = 2$. We now show that $\operatorname{cx}_{\widehat{C}} \widehat{M} \neq 1$. Suppose on the contrary that $\operatorname{cx}_{\widehat{C}} \widehat{M} = 1$ i.e. the dimensions $\left|\tau_{\widehat{C}}^i M\right|$ are bounded for all $i \geq 0$. But then by Prop. 4.3.5 also the dimensions $|\operatorname{Hom}_H(T, \tau_H^i M)|$ and $\left|\operatorname{Ext}_H^1(T, \tau_H^{i-1} M)\right|$ are bounded for all $i \geq 0$ which by Cor. 4.3.4 means that also $|\tau_H^i M|$ are bounded i.e. $\operatorname{cx}_H M \leq 1$. This contradicts the choice of M.

O. Kerner has shown that when H is of wild representation type, then modules in the preinjective component of H have infinite complexity [Ker]. Then it follows from Prop. 4.3.5 and Cor. 4.3.6 that $\operatorname{cx}_{\widehat{C}} \widehat{M} = \infty$.

We now turn to the components $\widehat{Z}$ arising as images of regular components in $\mathcal{C}$. When H is tame the components $\widehat{Z}$ arise as images of tubes in $\mathcal{C}$. Hence, all modules in components of type $\widehat{Z}$ of positive complexity are τ-periodic and therefore $\operatorname{cx}_{\widehat{C}} \widehat{Z} = 1$.

When H is wild the components $\widehat{Z}$ arise as images of the regular components in H-mod (all are of type $\mathbb{Z}A_\infty$). Regular H-modules have infinite complexity when H is of wild representation type [Ker]. Any module $\widehat{M}$ of nonzero complexity in a component of type $\widehat{Z}$ satisfies the hypotheses of Prop. 4.3.5 and Cor. 4.3.6. Therefore

$\operatorname{cx}_{\widehat{C}} \widehat{M} = \infty$. $\square$

We point out as a separate corollary a part of what we proved in the above theorem.

Corollary 4.3.8. *Let $\widehat{C}$ be a cluster tilted algebra from a hereditary algebra of type H. Then the only $\widehat{C}$-modules of complexity 1 are the τ-periodic modules.*

Proof. Modules of complexity 1 can only occur in the case when H is of finite or tame representation type. In either case, they reside in a τ-periodic component. $\square$

Chapter 5

Appendix

We provide the Perl code for a program that we wrote to extract information on the Betti numbers of a resolution from the program Gröbner developed by Ed Green. This program requires the two output files $*$.gph and $*$.res produced by Gröbner. We provide an example at the end of this section.

5.1 Perl Code

```
# © 2008, Marju Purin

# The program requires the Groebner output files

# *.gph and *.res as its input

# See below for descriptions of what each part does.

#*********************************************************************

# This part of the program creates from the input file that it reads
```

```
# a hash called %arrowendshash whose key is the arrow's name (a letter)
# and whose value is the ending vertex for that arrow in the quiver.

#First, set initial values for the vertices
$i=1;
$j=1;
$k=1;
$recordedarrows=0;

# Make a hash that holds the alphabet
%alphabethash=(1 => "a", 2 => "b", 3 => "c", 4 => "d",
5 => "e", 6 => "f", 7  => "g", 8 => "h", 9 => "i",
10 => "j", 11 => "k", 12 => "l", 13 => "m", 14 => "n",
15 => "o", 16  => "p", 17  =>  "q", 18  => "r",
19 => "s", 20  => "t", 21  => "u", 22  => "v",
23 => "x", 24  => "y",  25 => "z");

# Get user input to determine which files to read
# The concatenation operator in perl is '.'
print "Please enter the name of the Groebner file \n";
# record the input without the newline character that <STDIN>
```

```
# always has at the end
chomp($userinput=<STDIN>);


# Add the appropriate extension for the input files
# that are to be read from Groebner
$filename1=$userinput."\.gph";
$filename2=$userinput."\.res";


# Create the default output file
$filename3=$userinput."\.cpx";


# Get user input for the output file name
# If the user presses enter, then by default the name is
#$filename3 from above.
print "Please enter a name for the output file or press Ent\n";
$userinput2=<STDIN>;


# Record the input without the newline character that <STDIN>
# always has at the end


if ($userinput2 ne "\n")
```

```
  {
  chomp($userinput2);
  $filename3=$userinput2.".cpx";
  }

# Access the required files or print an error message.
open(FILE_OUT, "> $filename3") or die "Can't open output file";
open(FILE_HANDLE, "< $filename1")  or die "Can't open $thisfile";
while (<FILE_HANDLE>)
{
  if (m/^(.+) = vert/)
  {
  # Record the number of vertices
  $vertices = $1;

  # The next line might be blank, in which case I skip it
  $line = <FILE_HANDLE>;
  while ($line eq "\n")
    {
    chomp($line);
    $line = <FILE_HANDLE>;
```

```
  } #ends nested while


# Obtain arrows from Groebner

# Input is a $vertices x $vertices matrix

# All columns end in same vertex

# Read matrix into an array called @matrix


# Force it to look on the next line in the for loop

for ($i=1; $i<=$vertices; $i++)

  {

  @matrixrow=split/\s+/, $line;


  # Process each entry in $i row of the matrix

  for ($j=1; $j<=$vertices; $j++)

    {

    # jth entry in ith row

    # Array entries start at 0 in perl

    $matentry=@matrixrow[$j-1];


    for ($k=1; $k<=$matentry; $k++)

      {
```

```
        # However many ($matentry) are in (i,j)-entry of my matrix

        # For clarity: pick ($recordedarrows+k)-th

        # entry in the alphabet from # the %alphabethash


        # Keep track of which letters are already used

        $letterentry=$alphabethash{$recordedarrows+$k};


        # Create a hash with arrows as keys, and end vertex as values

        # At this stage of the nested loops

        # all arrows end at vertex j


        # %arrowendshash contains arrows and their end vertices

        $arrowendshash{$letterentry}=$j;

        } #ends for:  for ($k=1; $k<=$matentry; $k++)


      # Keep track of the total number of arrows

      # that have been assigned

      $recordedarrows=$recordedarrows + $matentry;

      } # ends for: for ($j=1; $j<=$vertices; $j++)


    } #ends for ($i=1; $i<=$vertices; $i++)
```

```
  } # ends if

} # ends while

#*********************************************************************
# Start working with the next part of the code
# This part reads the output files
# from Groeber called *.res
# It will use information from the file *.gph
# that the above piece of code obtained
# Note that it is assumed that
# '*' in '*.gph' is the same as in '*.res'

# Initialize counters
$counter = 1;
$thismany=0;

# Prepare the output table, get size information.
# Construct a loop to see how many projectives are needed.
# Make an array for printing @printing
```

```
for ($p=1; $p<=$vertices; $p++)
{
$printing[$p]=" P".$p
}

# Print one copy to the screen, and another to the output file
# Print also the output file name inside the file $filename3

print FILE_OUT "\nFile: $filename3\n\n";
print "\nRepetition Betti#@printing\n\n";
print FILE_OUT "\nRepetition Betti#@printing\n\n";

# Use the user input we got at the very beginning of
# the previous part of the code
# to open the file *.res
# Print an error message, if unsuccessful.
$openfile = $filename2;
open(FILE_HANDLE, "< $openfile")  or die "Can't open $openfile";
while (<FILE_HANDLE>)
  {
  if (m/repetition #(\d+)/)
```

```
{
$rep=$1;
$line = <FILE_HANDLE>;
chomp($line);

# Skip blank lines
while($line ne "")
  {
  $line = <FILE_HANDLE>;
  chomp($line);
  if ($line =~ m/(\d+)(.*)(\w);/g)
    {
    $colnr = $1;
    $colend = $3;

    # Store the column ends in an array
    # One array per repetition
    # It will contain the end letters
    # Later it will have the end vertex number

    # Counting starts at 0
```

```
    # Force $colnr spot to hold that column's ending letter

    # The 0 entry will be empty for me always

    @colends[$colnr] = $colend;


    if (m/repetition #(\d+)/g)

      {

      $rep=$1;

      }


    }  #end if statement


  }  #end while statement


# This is the final array

# of column ends after repetition $rep

# in position i it will have the first column ending

# for this repetition


# Use the hash from first piece of the program to

# replace letters by column ends

# i.e. the indecomposable projective summands
```

```
# Create an array to have in position i the end of
# the arrowend recorded in  #@colends
# by checking where that arrow ends in %arrowendshash

# $pos is a position counting variable
# starts at 1
# and ends at the last entry of @colends
# We need to know the size of @colends, or rather the
# last index variable (perl #counts from 0)

$lastcol=$#colends;
# $lastcol is actually also the number of
# indecomposable projective summands
# So, this is the ($rep)-th Betti number

# Create an array of Betti numbers
# Record starting Betti 1,
# the P_1 component of the resoution
# @projectives  is a hash
# because of the way I am filling it (out #of order)
# Reset projectives at the end of each repetition
```

```
@Betti[$rep]=$lastcol;


# Print once to screen, and once to the output file.

# Print the repetition number and then the Betti number.

print "\n\n rep $rep : $Betti[$rep]";

print FILE_OUT "\n\n rep $rep : $Betti[$rep]";


for ($pos=1; $pos<=$lastcol; $pos++)

  {

  $projectives[$pos]=$arrowendshash{$colends[$pos]};

  }


# Count how many times each occurs by creating

# a hash %totalproj

# Hash key will be the proj and hash value will be

# the number of occurrences

# the variable $vertices from first part of

# program tells us the number of

# distinct indecomposables (i.e. the number of vertices).

# So, there are $vertices many choices to look for.

# Vertices are enumerated starting with 1.
```

```
# Here, $n refers to the proj at vertex $n

# The if counts the number of times projective at $n appears

# remember that string comparison needs 'eq'


for ( $n=1; $n<=$vertices; $n++)

  {

  for ($pos=1; $pos<=$lastcol; $pos++)

    {

    if ($projectives[$pos] eq $n)

      {

      #print " at position $pos $projectives[$pos] = $n  \n";

      #print " this many before $thismany \n";

      $thismany++;

      }


    } #ends nested if


  $totalproj{$n}=$thismany;


  # Print the coefficients for each projective P$n

  # Once on the screen, and once to the output file
```

```
  if ($totalproj{$n} != 0)
    {
    print " $totalproj{$n}";
    print FILE_OUT  " $totalproj{$n}";
    }

  # Print a space to make the array line up
  # correctly instead of a zero
  if ($totalproj{$n} eq "0")
    {
    print " ";
    print FILE_OUT " ";
    }

  # reset the counter $thismany
  $thismany=0;

  }  #ends for

# Reset the array to nul before starting the next repetition calculations
@colends=();
```

```
    } # finishes if

  # Clear the arrays @projectives and @Pprojectives
   @projectives=();
   @Pprojectives=();

 }  #finishes the file *.res and the big while statement

# Here all repetitions are completed.
# All Betti numbers are recorded in @Betti
# print once to screen and once to output file
# This ends the program
```

5.2 Sample Files

Let us consider the trivial extensions algebra from Example 3.2.1. The Gröbner files that we need as the input for the program Cpx.pl are

- The file trivex5b.gph containing information about the quiver of the trivial extension algebra. There is an entry in position (i, j) for every arrow in the quiver from vertex i to vertex j.

```
5 = vert
```

```
0 1 0 0 0
0 0 1 1 1
1 0 0 0 0
1 0 0 0 0
1 0 0 0 0
```

- The file trivex5b.mod contains information about the module whose resolution we are interested in. In this particular case, we have selected the simple module at the vertex 1.

```
1 1

 a;
```

- Gröbner produces the file trivex5b.gph. We only give the first 6 entries.

```
 input matrix for projective resolution #1
row1
 1    + 1 * a;




 Matrix1 at repetition #1
row1
 1    + 1 * a;


 Matrix1 at repetition #2
row1
 1    + 1 * bea;


 Matrix1 at repetition #3
row1
 1    + 1 * b;
 2    + 1 * c;
 3    + 1 * d;
```

```
 Matrix1 at repetition #4
row1
 1    - 1 * e;
 2    - 1 * e;
row2
 1    + 1 * f;
 2    ;
row3
 1    ;
 2    + 1 * g;


 Matrix1 at repetition #5
row1
 1    + 1 * ad;
 2    - 1 * ab;
 3    ;
row2
 1    ;
 2    + 1 * ab;
 3    + 1 * ac;


 Matrix1 at repetition #6
row1
 1    + 1 * ga;
 2    ;
 3    - 1 * g;
row2
 1    ;
 2    + 1 * ea;
 3    - 1 * e;
row3
 1    ;
 2    ;
 3    + 1 * f;
```

- We now run the program Cpx.pl and enter the file name trivex5b as the one to process. We choose the file name trivex5b[S1] as the file where cpx.pl records the output.

```
Please enter the name of the Groebner file

trivex5b

Please enter a name for the output file or press Ent

trivex5b[S1]
```

- Our program Cpx.pl creates the file trivex5b[S1].cpx which contains all of the information on Betti numbers.

File: trivex5b[S1].cpx

Repetition	Betti#	P1	P2	P3	P4	P5
rep 1 :	1	1				
rep 2 :	3			1	1	1
rep 3 :	3	2	1			
rep 4 :	6			2	2	2
rep 5 :	5	3	2			
rep 6 :	9			3	3	3

We can easily read off the first few terms of the projective resolution of the simple module S_1 from this chart:

$$\ldots \to P_3 \oplus P_3 \oplus P_4 \oplus P_4 \oplus P_5 \oplus P_5 \xrightarrow{\delta_2} P_1 \oplus P_1 \oplus P_2 \xrightarrow{\delta_0} P_3 \oplus P_4 \oplus P_5 \to P_1 \xrightarrow{\delta_0} S_1 \to 0$$

Bibliography

[A] N. A'Campo, *Sur les valeurs propres de la transformation de Coxeter*, Invent. Math. **33** (1976), no. 1, 61–67.

[ABS] I. Assem, T. Brüstle, and R. Schiffler, *Cluster-tilted algebras as trivial extensions*, Bull. Lond. Math. Soc. **40** no. 1 (2008), 151–162.

[ABS2] I. Assem, T. Brüstle, and R. Schiffler, *Cluster-tilted algebras and slices*, J. Algebra **319** no. 8 (2008), 3464–3479.

[AE] J. Alperin, L. Evens, *Representations, esolutions, and Quillen's dimension theorem*, J. Pure Appl. Algebra **22** (1981), 1–9.

[AH] I. Assem, D. Happel, *Generalized tilted algebras of type A_n*, An. Comm. Algebra **9** (1981), no. 20, 2101–2125.

[AR] M. Auslander, I. Reiten, R̂epresentation Theory of Artin Algebras V, Comm. Alg. **5** (1977), 519-554.

[ARS] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics, **36**, Cambridge University Press, Cambridge, (1995).

[ASS] I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras. 1: Techniques of Representation Theory*, London Mathematical Society, Student Texts, **65**, Cambridge University Press, Cambridge, (2006).

[Av] L. Avramov, *Infinite free resolutions. Six lectures on commutative algebra*, Progr. Math. **166**, Birkhauser, Basel (1998), 1–118.

[AvB] L. Avramov, R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. **142** (2000), 285–318.

[AvGP] L. Avramov, V. Gasharov, and I. Peeva *Complete Intersection Dimension*, Inst. Hautes Études Sci. Publ. Math. No. 86 (1997), 67–114.

[Be] P. Bergh, *Complexity and periodicity*, Colloq. Math., **104** (2006), no. 2, 169–191.

[BB] S. Brenner, M. C. R. Butler, *Generalizations of the Bernstein-Gel'fand-Ponomarev reflection functors*, Lecture Notes in Math., **832**, Springer, Berlin (1980), 103–169.

[BMR] A. B. Buan, R. J. Marsh, and I. Reiten, *Cluster Tilted Algebras*, Trans. Amer. Math. Soc. **359** (2007), no. 1, 323–332.

[BMRRT] A. B. Buan, R. J. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting Theory and Cluster Combinatorics*, Adv. Math. **204** (2006) no. 2, 572–618.

[BO] P. Bergh, S. Oppermann, *Cluster Tilting and Complexity*, to appear.

[C-B] W. W. Crawley-Boevey, *On tame algebras and bocses* Proc. London Math. Soc. **3** 56 (1988), no. 3, 451–483.

[C] A. J. Coleman, *Killing and the Coxeter Transformation of Kac-Moody Algebras*, Inventiones Mathematicae, **95**, Springer-Verlag (1989), 447–477.

[Dr] Y. A. Drozd, *Tame and Wild Matrix Problems*. Representation Theory II, Lecture Notes in Math., **. 832**, Springer-Verlag (1980), 242–258.

[Eis] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64.

[E] K. Erdmann, *On Auslander-Reiten components for group algebras*, J. Pure Appl. Algebra **104** (1995), no. 2, 149–160.

[EHSST] K. Erdmann, M. Holloway, N. Snashall,Ø. Solberg, R. Taillefer, *Support varieties for selfnjective algebras*, K-theory **33** (2004), 67–87.

[FP] E. Fernńdez, M. Platzeck, *Presentations of Trivial Extensions of finite-dimensional Algebras and a Theorem of Sheila Brenner*, Journal of Algebra **249** (2002), 326–344.

[FZ] S. Fomin, A. Zelevinsky, *Cluster algebras I. Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529.

[FGR] R. Fossum, P. Griffith and I. Reiten, *Trivial extensions of abelian categories. Homological algebra of trivial extensions of abelian categories with applications to ring theory*, Lecture Notes in Mathematics **456** (1975), Springer-Verlag, Berlin-New York.

[G] P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math., **6** (1972), 71–103.

[GP] V. Gasharov, I. Peeva, *Boundedness versus periodicity over commutative local rings*, Trans. Amer. Math. Soc. **320** (1990), no. 2, 569–580.

[GZ] E. Green, D. Zacharia, *Auslander Reiten Components Containing Modules with Bounded Betti numbers*, Trans. Amer. Math. Soc. **361** no.8 (2009), 4195–4214.

[H] D. Happel, *Triangulated Categories in the Represenation Theory of finite-dimensional Algebras*, London Math. Soc. Lecture Note Series **119**, Cambridge University Press, Cambridge (1988).

[HR] D. Happel, C. M. Ringel, *Tilted Algebras*, Trans. Amer. Math. Soc. **274** (1982), 399–443.

[How] R. B. Howlett, *Coxeter Groups and M-Matrices*, Bull. London Math. Soc. **14** (1982) no. 2, 137–141.

[JL] D. Jorgensen, G. Leuschke *On the growth of the Betti sequence of the canonical module*, Math. Z. **256** (2007), no. 3, 647–659.

[K] B. Keller,*On Triangulated Orbit Categories*, Doc. Math. **10** (2005), 551–581.

[Ker] O. Kerner, *Representations of Wild Quivers*, CMS Conf. Proc., **19** (1996), 65–107.

[KZ] O. Kerner, D. Zacharia, *Auslander-Reiten Theory for Modules of Finite Complexity over Selfinjective Algebras*, Bull. London Math. Soc. **43** (2011), 44–56.

[Kr] H. Krause. *Stable equivalence preserves representation type*, Comment. Math. Helv. **72** (1997), 266–284.

[KrZw] H. Krause, G. Zwara, *Stable Equivalence and Generic Modules*, Bull. London Math. Soc. **32** (2000), 615–618.

[P] M. Purin, *Complexity of Trivial Extensions of Iterated Tilted Algebras*, to appear.

[RS] M. J. Redondo, A. Solotar, *Derived Categories and their Applications*, Rev. de la U. Matemática Argentina **48** no. 3 (2007), 1–26.

[R] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. **1099**, Springer-Verlag Berlin Heidelberg, (1984).

[R2] C. M. Ringel, *The spectral radius of the Coxeter transformations for a generalized Cartan matrix*, Math. Ann. **300** (1994) no. 2, 331–339.

[Sch] R. Schulz, *Boundedness and periodicity of modules over QF rings*, J. Algebra **101** (1986) no. 2, 450–469.

[SS] D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras. 2: Tubes and Concealed Algebras of Euclidean Type*, London Mathematical Society, Student Texts, **71**, Cambridge University Press, Cambridge, (2007).

[Skow1] A. Skowroński, *Periodicity in Representation Theory of Algebras*, ICTP Advanced School and Conference on Representation Theory and Related Topics Lecture Notes (2006).

[Skow2] A. Skowroński, *Self-injective Algebras: Finite and Tame Type* , Trends in representation theory of algebras and related topics, Contemp. Math. **406** (2006), Amer. Math. Soc., 169–238.

[S] R. Stekolshchik, *Notes on Coxeter Transformations and the McKay Correspondence*, Springer Monographs in Mathematics, Springer-Verlag New York, (2008).

[T] H. Tachikawa, *Representations of Trivial Extensions of Hereditary Algebras*, Lecture Notes in Math. **832**, Springer, Berlin, (1980), 579–599.

[TW] H. Tachikawa, T. Wakamatsu, *Tilting Functors and Stable Equivalences for Self-injective Algebras*, Journal of Algebra **109** (1987), 138–165.

[Y] K. Yamagata, *Frobenius algebras*, Handbook of algebra, **1** (1996), 841–887.

[Y2] K. Yamagata, *Extensions over hereditary Artinian rings with self-dualities. I.*, J. Algebra **73** (1981), no. 2, 386–433.

[Y3] K. Yamagata, *Extensions over hereditary Artinian rings with self-dualities. II.*, J. London Math. Soc. **2** 26 (1982), no. 1, 28–36.

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