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DIFFEOMORPHIC APPROXIMATION OF SOBOLEV HOMEOMORPHISMS

TADEUSZ IWANIEC, LEONID V. KOVALEV, AND JANI ONNINEN

ABSTRACT. Every homeomorphism $h: \mathbb{X} \rightarrow \mathbb{Y}$ between planar open sets that belongs to the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 < p < \infty$, can be approximated in the Sobolev norm by \mathcal{C}^∞ -smooth diffeomorphisms.

1. INTRODUCTION

By the very definition, the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R})$, $1 \leq p < \infty$, in a domain $\mathbb{X} \subset \mathbb{R}^n$, is the completion of \mathcal{C}^∞ -smooth real functions having finite Sobolev norm

$$\|u\|_{\mathscr{W}^{1,p}(\mathbb{X})} = \|u\|_{\mathcal{L}^p(\mathbb{X})} + \|\nabla u\|_{\mathcal{L}^p(\mathbb{X})} < \infty.$$

The question of smooth approximation becomes more intricate for Sobolev mappings, whose target is not a linear space, say a smooth manifold [11, 19, 20, 21] or even for mappings between open subsets \mathbb{X}, \mathbb{Y} of the Euclidean space \mathbb{R}^n . If a given homeomorphism $h: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is in the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ it is not obvious at all as to whether one can preserve injectivity property of the \mathcal{C}^∞ -smooth approximating mappings. It is rather surprising that this question remained unanswered after the global invertibility of Sobolev mappings became an issue in nonlinear elasticity [4, 17, 31, 35]. It was formulated and promoted by John M. Ball in the following form.

Question. [6, 7] If $h \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^n)$ is invertible, can h be approximated in $\mathscr{W}^{1,p}$ by piecewise affine invertible mappings?

J. Ball attributes this question to L.C. Evans and points out its relevance to the regularity of minimizers of neo-Hookean energy functionals [5, 9, 14, 16, 34]. Partial results toward the Ball-Evans problem were obtained in [30] (for planar bi-Sobolev mappings that are smooth outside of a finite set) and in [10] (for planar bi-Hölder mappings, with approximation in the Hölder norm). The articles [6, 33] illustrate the difficulty of preserving invertibility in the approximation process. In [24] we provided an affirmative answer to the Ball-Evans question in the planar case when $p = 2$. In the present

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paper we extend the result of [24] to all Sobolev classes $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ with $1 < p < \infty$. The case $p = 1$ still remains open.

Let \mathbb{X} be a nonempty open set in \mathbb{R}^2 . We study complex-valued functions $h = u + iv: \mathbb{X} \rightarrow \mathbb{C} \simeq \mathbb{R}^2$ of Sobolev class $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{C})$, $1 < p < \infty$. Their real and imaginary part have well defined gradient in $\mathcal{L}^p(\mathbb{X}, \mathbb{R}^2)$

$$\nabla u: \mathbb{X} \rightarrow \mathbb{R}^2 \quad \text{and} \quad \nabla v: \mathbb{X} \rightarrow \mathbb{R}^2.$$

Then we introduce the gradient mapping of h , by setting

$$(1.1) \quad \nabla h = (\nabla u, \nabla v): \mathbb{X} \rightarrow \mathbb{R}^2 \times \mathbb{R}^2.$$

The \mathcal{L}^p -norm of the gradient mapping and the p -energy of h are defined by

$$(1.2) \quad \|\nabla h\|_{\mathcal{L}^p(\mathbb{X})} = \left[\int_{\mathbb{X}} (|\nabla u|^p + |\nabla v|^p) \right]^{\frac{1}{p}}, \quad \mathbf{E}_{\mathbb{X}}[h] = \mathbf{E}_{\mathbb{X}}^p[h] = \|\nabla h\|_{\mathcal{L}^p(\mathbb{X})}^p.$$

The reader may wish to notice that this norm is slightly different from what can be found in other texts in which the authors use the differential matrix of h instead of the gradient mapping, so

$$(1.3) \quad \|Dh\|_{\mathcal{L}^p(\mathbb{X})} = \left[\int_{\mathbb{X}} (|\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}} \right]^{\frac{1}{p}}.$$

Thus our approach involves *coordinate-wise* p -harmonic mappings, which we still call p -harmonic for the sake of brevity. We shall take an advantage of the gradient mapping on numerous occasions, by exploring the associated *uncoupled* system of real p -harmonic equations for mappings with smallest p -energy. Our theorem reads as follows.

Theorem 1.1. *Let $h: \mathbb{X} \xrightarrow{\text{ont}\mathcal{Q}} \mathbb{Y}$ be an orientation-preserving homeomorphism in the Sobolev space $\mathcal{W}_{\text{loc}}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 < p < \infty$, defined for open sets $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$. Then there exist \mathcal{C}^∞ -diffeomorphisms $h_\ell: \mathbb{X} \xrightarrow{\text{ont}\mathcal{Q}} \mathbb{Y}$, $\ell = 1, 2, \dots$ such that*

- (i) $h_\ell - h \in \mathcal{W}_o^{1,p}(\mathbb{X}, \mathbb{R}^2)$, $\ell = 1, 2, \dots$
- (ii) $\lim_{\ell \rightarrow \infty} (h_\ell - h) = 0$, uniformly on \mathbb{X}
- (iii) $\lim_{\ell \rightarrow \infty} \|\nabla h_\ell - \nabla h\|_{\mathcal{L}^p(\mathbb{X})} = 0$
- (iv) $\|\nabla h_\ell\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla h\|_{\mathcal{L}^p(\mathbb{X})}$, for $\ell = 1, 2, \dots$
- (v) If h is a \mathcal{C}^∞ -diffeomorphism outside of a compact subset of \mathbb{X} , then there is a compact subset of \mathbb{X} outside which $h_\ell \equiv h$, for all $\ell = 1, 2, \dots$

A straightforward triangulation argument yields the following corollary.

Corollary 1.2. *Let $h: \mathbb{X} \xrightarrow{\text{ont}\mathcal{Q}} \mathbb{Y}$ be an orientation-preserving homeomorphism in the Sobolev space $\mathcal{W}_{\text{loc}}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 < p < \infty$, defined for open sets $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$. Then there exist piecewise affine homeomorphisms $h_\ell: \mathbb{X} \xrightarrow{\text{ont}\mathcal{Q}} \mathbb{Y}$, $\ell = 1, 2, \dots$ such that*

- (i) $h_\ell - h \in \mathcal{W}_o^{1,p}(\mathbb{X}, \mathbb{R}^2)$, $\ell = 1, 2, \dots$

- (ii) $\lim_{\ell \rightarrow \infty} (h_\ell - h) = 0$, uniformly on \mathbb{X}
- (iii) $\lim_{\ell \rightarrow \infty} \|\nabla h_\ell - \nabla h\|_{\mathcal{L}^p(\mathbb{X})} = 0$.
- (iv) If h is affine outside of a compact subset of \mathbb{X} , then there is a compact subset of \mathbb{X} outside which $h_\ell \equiv h$, for all $\ell = 1, 2, \dots$

We conclude this introduction with a sketch of the proof. The construction of an approximating diffeomorphism involves five consecutive modifications of h . Steps 1, 2, and 4 are p -harmonic replacements based on the Alessandrini-Sigalotti extension [3] of the Radó-Kneser-Choquet Theorem. The other steps involve an explicit smoothing procedure along crosscuts. For this, we adopted some lines of arguments used in J. Munkres' work [32].

2. p -HARMONIC MAPPINGS AND PRELIMINARIES

Let Ω be a bounded domain in the complex plain $\mathbb{C} \simeq \mathbb{R}^2$. A function $u: \Omega \rightarrow \mathbb{R}$ in the Sobolev class $\mathcal{W}_{\text{loc}}^{1,p}(\Omega)$, $1 < p < \infty$, is called p -harmonic if

$$(2.1) \quad \operatorname{div} |\nabla u|^{p-2} \nabla u = 0$$

meaning that

$$(2.2) \quad \int_{\Omega} \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle = 0 \quad \text{for every } \varphi \in \mathcal{C}_0^\infty(\Omega).$$

The first observation is that the gradient map $f = \nabla u: \Omega \rightarrow \mathbb{R}^2$ is K -quasiregular with $1 \leq K \leq \max\{p-1, 1/(p-1)\}$, see [12]. Consequently $u \in \mathcal{C}_{\text{loc}}^{1,\alpha}(\Omega)$ with some $0 < \alpha = \alpha(p) \leq 1$. In fact [25] the foremost regularity of a p -harmonic function ($p \neq 2$) is $\mathcal{C}_{\text{loc}}^{k,\alpha}(\Omega)$, where the integer $k \geq 1$ and the Hölder exponent $\alpha \in (0, 1]$ are determined by the equation

$$k + \alpha = \frac{7p - 6 + \sqrt{p^2 + 12p - 12}}{6p - 6} > 1 + \frac{1}{3}.$$

Thus, regardless of the exponent p , we have $u \in \mathcal{C}_{\text{loc}}^{1,\alpha}(\Omega)$ with $\alpha = 1/3$. Clearly, by elliptic regularity theory, outside the singular set

$$\mathcal{S} = \{z \in \Omega: \nabla u(z) = 0\},$$

we have $u \in \mathcal{C}^\infty(\Omega \setminus \mathcal{S})$. The singular set, being the set of zeros of a quasiregular mapping, consists of isolated points; unless $u \equiv \text{const}$. Pertaining to regularity up to the boundary, we consider a domain Ω whose boundary near a point $z_0 \in \partial\Omega$ is a \mathcal{C}^∞ -smooth arc, say $\Gamma \subset \partial\Omega$. Precisely, we assume that there exist a disk $D = D(z_0, \epsilon)$ and a \mathcal{C}^∞ -smooth diffeomorphism $\varphi: D \xrightarrow{\text{ontg}} \mathbb{C}$ such that

$$\begin{aligned} \varphi(D \cap \Omega) &= \mathbb{C}_+ = \{z: \operatorname{Im} z > 0\} \\ \varphi(\Gamma) &= \mathbb{R} = \{z: \operatorname{Im} z = 0\} \\ \varphi(D \setminus \overline{\Omega}) &= \mathbb{C}_- = \{z: \operatorname{Im} z < 0\}. \end{aligned}$$

Proposition 2.1 (Boundary Regularity). *Suppose $u \in \mathcal{W}^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is p -harmonic in Ω and \mathcal{C}^∞ -smooth when restricted to Γ . Then u is $\mathcal{C}^{1,\alpha}$ -regular up to Γ , meaning that u extends to D as a $\mathcal{C}^{1,\alpha}(D)$ -regular function, where α depends only on p .*

2.1. The Dirichlet problem. There are two formulations of the Dirichlet boundary value problem for p -harmonic equation; both are essential for our investigation. We begin with the variational formulation.

Lemma 2.2. *Let $u_\circ \in \mathcal{W}^{1,p}(\Omega)$ be a given Dirichlet data. There exists precisely one function $u \in u_\circ + \mathcal{W}_\circ^{1,p}(\Omega)$ which minimizes the p -harmonic energy:*

$$\mathcal{E}_p[u] = \inf \left\{ \int_{\Omega} |\nabla w|^p : w \in u_\circ + \mathcal{W}_\circ^{1,p}(\Omega) \right\}.$$

The solution u is certainly a p -harmonic function, so $\mathcal{C}_{\text{loc}}^{1,\alpha}(\Omega)$ -regular. However, more efficient to us will be the following classical formulation of the Dirichlet problem.

Problem 2.3. Given $u_\circ \in \mathcal{C}(\partial\Omega)$ find a p -harmonic function u in Ω which extends continuously to $\overline{\Omega}$ such that $u|_{\partial\Omega} = u_\circ$.

It is not difficult to see that such solution (if exists) is unique. However, the existence poses rather delicate conditions on $\partial\Omega$ and the data $u_\circ \in \mathcal{C}(\overline{\Omega})$. We shall confine ourselves to Jordan domains $\Omega \subset \mathbb{C}$ and the Dirichlet data $u_\circ \in \mathcal{C}(\overline{\Omega})$ of finite p -harmonic energy. In this case both formulations are valid and lead to the same solution. Indeed, the variational solution is continuous up to the boundary because each boundary point of a planar Jordan domain is a regular point for the p -Laplace operator Δ_p [18, p.418]. See [22, 6.16] for the discussion of boundary regularity and relevant capacities and [27, Lemma 2] for a capacity estimate that applies to simply connected domains.

Proposition 2.4 (Existence). *Let $\Omega \subset \mathbb{C}$ be a bounded Jordan domain and $u_\circ \in \mathcal{W}^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. There exists, unique, p -harmonic function $u \in \mathcal{W}^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $u|_{\partial\Omega} = u_\circ|_{\partial\Omega}$.*

2.2. Radó-Kneser-Choquet Theorem. Let $h = u + iv$ be a complex harmonic mapping in a Jordan domain \mathbb{U} that is continuous on $\overline{\mathbb{U}}$. Assume that the boundary mapping $h: \partial\mathbb{U} \xrightarrow{\text{ont}\circ} \Gamma$ is an orientation-preserving homeomorphism onto a convex Jordan curve. Then h is a \mathcal{C}^∞ -smooth diffeomorphism of \mathbb{U} onto the bounded component of $\mathbb{C} \setminus \Gamma$. Thus, in particular, the Jacobian determinant $J(z, h) = |h_z|^2 - |h_{\bar{z}}|^2$ is strictly positive in \mathbb{U} , see [15, p.20]. Suppose, in addition, that $\partial\mathbb{U}$ contains a \mathcal{C}^∞ -smooth arc $\gamma \subset \partial\mathbb{U}$, and h takes γ onto a \mathcal{C}^∞ -smooth subarc in Γ . Then h is \mathcal{C}^∞ -smooth up to γ and its Jacobian determinant is positive on γ as well, see [15, p.116]. Numerous presentations of the proof of Radó-Kneser-Choquet Theorem can be found, [15]. The idea that goes back to Kneser [26] and Choquet [13]

is to look at the structure of the level curves of the coordinate functions $u = \operatorname{Re} h$, $v = \operatorname{Im} h$ and their linear combinations. These ideas have been applied to more general linear and nonlinear elliptic systems of PDEs in the complex plane [8], see also [1, 2, 28, 29] for related problems concerning critical points. In the present paper we shall explore a result due to G. Alessandrini and M. Sigalotti [3] for a nonlinear system that consists of two p -harmonic equations

$$\begin{cases} \operatorname{div}|\nabla u|^{p-2}\nabla u = 0 \\ \operatorname{div}|\nabla v|^{p-2}\nabla v = 0 \end{cases}, \quad 1 < p < \infty, \quad h = u + iv.$$

Call it *uncoupled p -harmonic system*. The novelty and key element in [3] is the associated single linear elliptic PDE of divergence type (with variable coefficients) for a linear combination of u and v . Such combination represents a real part of a quasiregular mapping and, therefore, admits only isolated critical points. We shall not go into their arguments in detail, but instead extract the following p -harmonic analogue of the Radó-Kneser-Choquet Theorem.

Theorem 2.5 (G. Alessandrini and M. Sigalotti). *Let \mathbb{U} be a bounded Jordan domain and $h = u + iv: \bar{\mathbb{U}} \rightarrow \mathbb{C}$ be a continuous mapping whose coordinate functions $u, v \in \mathcal{W}^{1,p}(\mathbb{U})$, $1 < p < \infty$, are p -harmonic. Suppose that $h: \partial\mathbb{U} \xrightarrow{\text{onto}} \gamma$ is an orientation-preserving homeomorphism onto a convex Jordan curve γ . Then*

- (i) *h is a \mathcal{C}^∞ -diffeomorphism from \mathbb{U} onto the bounded component of $\mathbb{C} \setminus \gamma$. In particular,*

$$J(z, h) = |h_z|^2 - |h_{\bar{z}}|^2 > 0 \quad \text{in } \mathbb{U}.$$

- (ii) *If, in addition, $\partial\mathbb{U}$ contains a \mathcal{C}^∞ -smooth arc $\Gamma \subset \partial\mathbb{U}$ and $h(\Gamma)$ is a \mathcal{C}^∞ -smooth subarc in γ , then h is $\mathcal{C}^{1,\alpha}$ -regular up to Γ , for some $0 < \alpha = \alpha(p) < 1$ (actually \mathcal{C}^∞). Moreover $J(z, h) > 0$ on Γ as well.*

This theorem is a straightforward corollary of Theorem 5.1 in [3]. However, three remarks are in order.

- (1) In their Theorem 5.1 the authors of [3] assume that \mathbb{U} satisfies an exterior cone condition. This is needed only insofar as to ensure the existence of a continuous extension of a given homeomorphism $\Phi: \partial\mathbb{U} \rightarrow \gamma$ into \mathbb{U} whose coordinate functions are p -harmonic in \mathbb{U} . Obviously, such an extension is unique, though the p -harmonic energy need not be finite. Once we have such a mapping the exterior cone condition on \mathbb{U} for the conclusion of Theorem 5.1 is redundant, see Remark 3.2 in [3]. This is exactly the case we are dealing with in Theorem 2.5.
- (2) In regard to the statement (ii) we point out that in Theorem 5.1 of [3] the authors work with the mappings that are smooth up to the entire boundary of \mathbb{U} . Nonetheless their proof that $J(z, h) > 0$ on $\partial\mathbb{U}$ is local, so applies without any change to our case (ii).

- (3) Since $J(z, h) > 0$ in \mathbb{U} up to the arc $\Gamma \subset \partial\mathbb{U}$ the coordinate functions of h have nonvanishing gradient. This means that p -harmonic equation is uniformly elliptic up to Γ . Consequently, h is \mathcal{C}^∞ -smooth on \mathbb{U} up to Γ .

2.3. The p -harmonic replacement. Let Ω be a bounded domain in $\mathbb{R}^2 \simeq \mathbb{C}$. We consider a class $\mathcal{A}(\Omega) = \mathcal{A}^p(\Omega)$, $1 < p < \infty$, of uniformly continuous functions $h = u + iv: \Omega \rightarrow \mathbb{C}$ having finite p -harmonic energy and furnish it with the norm

$$\|h\|_{\mathcal{A}^p(\Omega)} = \|h\|_{\mathcal{C}(\Omega)} + \|\nabla h\|_{\mathcal{L}^p(\Omega)}.$$

The closure of $\mathcal{C}_\circ^\infty(\Omega)$ in $\mathcal{A}^p(\Omega)$ will be denoted by $\mathcal{A}_\circ^p(\Omega)$.

Proposition 2.6. *Let $\mathbb{U} \Subset \Omega$ be a Jordan subdomain of Ω . There exists a unique operator*

$$\mathbf{R}_\mathbb{U}: \mathcal{A}^p(\Omega) \rightarrow \mathcal{A}^p(\Omega)$$

(nonlinear if $p \neq 2$) such that for every $h \in \mathcal{A}^p(\Omega)$

$$\mathbf{R}_\mathbb{U}h = h \quad \text{in } \Omega \setminus \mathbb{U}$$

$$(2.3) \quad \mathbf{R}_\mathbb{U} \in h + \mathcal{W}_\circ^{1,p}(\mathbb{U})$$

$$\Delta_p \mathbf{R}_\mathbb{U}h = 0 \quad \text{in } \mathbb{U}$$

$$(2.4) \quad \mathbf{E}_\Omega[\mathbf{R}_\mathbb{U}h] \leq \mathbf{E}_\Omega[h]$$

Equality occurs in (2.4) if and only if h is p -harmonic in \mathbb{U} .

Proof. For $h = u + iv$ we define

$$\mathbf{R}_\mathbb{U}h = \mathbf{R}_\mathbb{U}u + i \mathbf{R}_\mathbb{U}v.$$

It is therefore enough to construct the replacement for real-valued functions. For $u \in \mathcal{A}^p(\Omega)$ real, we define

$$\mathbf{R}_\mathbb{U}u = \begin{cases} u & \text{in } \Omega \setminus \mathbb{U} \\ \tilde{u} & \text{in } \mathbb{U} \end{cases}$$

where \tilde{u} is determined uniquely as a solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} = 0 & \text{in } \mathbb{U} \\ \tilde{u} \in u + \mathcal{W}_\circ^{1,p}(\mathbb{U}) \end{cases}$$

so conditions (2.3) are fulfilled. That $\mathbf{R}_\mathbb{U}u$ is continuous in Ω is guaranteed by Proposition 2.4. The solution \tilde{u} is found as the minimizer of the p -harmonic energy in the class $u + \mathcal{W}_\circ^{1,p}(\mathbb{U})$, so we certainly have

$$\mathbf{E}_\Omega[\mathbf{R}_\mathbb{U}u] \leq \mathbf{E}_\Omega[u]$$

The same estimate holds for the imaginary part of h , so adding them up yields

$$\mathbf{E}_\Omega[\mathbf{R}_\mathbb{U}h] \leq \mathbf{E}_\Omega[h]. \quad \square$$

Remark 2.7. The reader may wish to know that the operator $\mathbf{R}_\mathbb{U}: \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$ is continuous, though we do not appeal to this fact.

2.4. Smoothing along a crosscut. Consider a bounded Jordan domain \mathbb{U} and a \mathcal{C}^∞ -smooth crosscut $\Gamma \subset \mathbb{U}$ with two distinct end-points in $\partial\mathbb{U}$. By definition, this means that there is a \mathcal{C}^∞ -diffeomorphism $\varphi: \mathbb{C} \xrightarrow{\text{onto}} \mathbb{U}$ such that $\Gamma = \varphi(\mathbb{R})$, and its distinct endpoints are given by

$$\begin{aligned} \lim_{x \rightarrow -\infty} \varphi(x) &\in \partial\mathbb{U} \\ \lim_{x \rightarrow \infty} \varphi(x) &\in \partial\mathbb{U} \end{aligned}$$

Such Γ splits \mathbb{U} into two Jordan subdomains

$$\begin{aligned} \mathbb{U}_+ &= \varphi(\mathbb{C}_+), \quad \mathbb{C}_+ = \{z: \text{Im } z > 0\} \\ \mathbb{U}_- &= \varphi(\mathbb{C}_-), \quad \mathbb{C}_- = \{z: \text{Im } z < 0\}. \end{aligned}$$

Suppose we are given a homeomorphism $f: \bar{\mathbb{U}} \rightarrow \mathbb{C}$ such that each of two mappings

$$f: \mathbb{U}_+ \rightarrow \mathbb{R}^2 \quad \text{and} \quad f: \mathbb{U}_- \rightarrow \mathbb{R}^2$$

is \mathcal{C}^∞ -smooth up to Γ . Assume that for some constant $0 < m < \infty$ we have

$$|Df(z)| \leq m \quad \text{and} \quad \det Df(z) \geq \frac{1}{m}$$

on \mathbb{U}_+ and on \mathbb{U}_- . Thus $f: \mathbb{U} \rightarrow \mathbb{R}^2$ is in fact locally bi-Lipschitz.

Proposition 2.8. *Under the above conditions there is a constant $0 < M < \infty$ such that for every open set $\mathbb{V} \subset \mathbb{U}$ containing Γ one can find a homeomorphism $g: \bar{\mathbb{U}} \xrightarrow{\text{onto}} f(\bar{\mathbb{U}})$ which is a \mathcal{C}^∞ -diffeomorphism in \mathbb{U} , with the following properties:*

$$(2.5) \quad g(z) = f(z), \quad \text{for } z \in (\bar{\mathbb{U}} \setminus \mathbb{V}) \cup \Gamma$$

$$(2.6) \quad |Dg(z)| \leq M \quad \text{and} \quad \det Dg(z) > \frac{1}{M} \quad \text{on } \mathbb{U}.$$

The key element of this smoothing device is that the constant M is independent of the neighborhood \mathbb{V} of Γ , see Figure 1. The proof is given in [24] following the ideas of [32].

We shall recall similar smoothing device for cuts along Jordan curves. Let \mathbb{U} be a simply connected domain with \mathcal{C}^∞ -regular cut along a Jordan curve $\Gamma \subset \mathbb{U}$. This means there is a diffeomorphism $\varphi: \mathbb{C} \xrightarrow{\text{onto}} \mathbb{U}$ such that $\Gamma = \varphi(\mathbb{S}^1)$, $\mathbb{S}^1 = \{z \in \mathbb{C}: |z| = 1\}$. As before Γ splits \mathbb{U} into

$$\begin{aligned} \mathbb{U}_+ &= \varphi(\mathbb{D}_+), \quad \mathbb{D}_+ = \{z: |z| < 1\} \\ \mathbb{U}_- &= \varphi(\mathbb{D}_-), \quad \mathbb{D}_- = \{z: |z| > 1\}. \end{aligned}$$

Suppose we are given a homeomorphism $f: \mathbb{U} \rightarrow \mathbb{R}^2$ such that each of two mappings

$$f: \mathbb{U}_+ \rightarrow \mathbb{R}^2 \quad \text{and} \quad f: \mathbb{U}_- \rightarrow \mathbb{R}^2$$

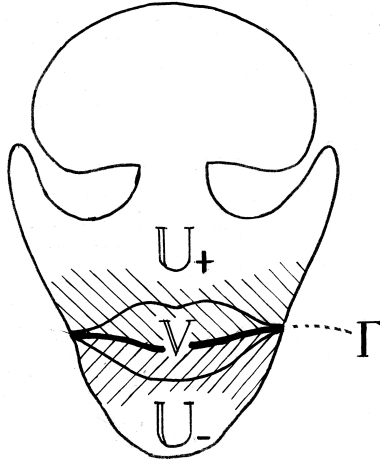


FIGURE 1. Jordan domain with a crosscut Γ and its neighborhood \mathbb{V} .

is \mathcal{C}^∞ -smooth up to Γ . Assume that for some constant $0 < m < \infty$ we have

$$|Df(z)| \leq m \quad \text{and} \quad \det Df(z) \geq \frac{1}{m}$$

on \mathbb{U}_+ and \mathbb{U}_- .

Proposition 2.9. *Under the above conditions there is a constant $0 < M < \infty$ such that for every open set $\mathbb{V} \subset \mathbb{U}$ containing Γ one can find a \mathcal{C}^∞ -diffeomorphism $g: \mathbb{U} \xrightarrow{\text{ontq}} f(\mathbb{U})$ with the following properties*

$$(2.7) \quad g(z) = f(z), \quad \text{for } z \in (\mathbb{U} \setminus \mathbb{V}) \cup \Gamma$$

$$(2.8) \quad |Dg(z)| \leq M \quad \text{and} \quad \det Dg(z) > \frac{1}{M} \quad \text{on } \mathbb{U}.$$

Having disposed of the above preliminaries we shall now proceed to the construction of the approximating sequence of diffeomorphisms.

3. THE PROOF

3.1. Scheme of the proof. Let us begin with a convention. We will often suppress the explicit dependence on the Sobolev exponent $1 < p < \infty$ in the notation, whenever it becomes selfexplanatory. For every $\epsilon > 0$ we shall construct a \mathcal{C}^∞ -diffeomorphism $\tilde{h}: \mathbb{X} \xrightarrow{\text{ontq}} \mathbb{Y}$ such that

- (A) $\tilde{h} - h \in \mathcal{A}_o(\mathbb{X})$
- (B) $\|\tilde{h} - h\|_{\mathcal{C}(\mathbb{X})} \leq \epsilon$
- (C) $\|\nabla \tilde{h} - \nabla h\|_{\mathcal{L}^p(\mathbb{X})} \leq \epsilon$
- (D) $E_{\mathbb{X}}[\tilde{h}] \leq E_{\mathbb{X}}[h]$

- (E) If h is a \mathcal{C}^∞ -diffeomorphism outside of a compact subset of \mathbb{X} , then there exist a compact subset of \mathbb{X} outside of which we have $\tilde{h} \equiv h$, for all $\epsilon > 0$.

We may and do assume that h is not a \mathcal{C}^∞ -diffeomorphism, since otherwise $\tilde{h} = h$ satisfies the desired properties. Let $x_\circ \in \mathbb{X}$ be a point such that h fails to be \mathcal{C}^∞ -diffeomorphism in any neighborhood of x_\circ .

We shall consider dyadic squares in \mathbb{Y} with respect to a selected rectangular coordinate system in \mathbb{R}^2 . By choosing the origin of the system we ensure that $h(x_\circ)$ does not lie on the boundary of any dyadic square.

Let us fix $\epsilon > 0$. The construction of \tilde{h} proceeds in 5 steps, each of which gives a homeomorphism $\tilde{h}_k: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, $k = 0, 1, \dots, 5$, in the Sobolev class $\mathcal{W}_{\text{loc}}^{1,p}(\mathbb{X}, \mathbb{Y})$ such that $\tilde{h}_0 = h$, $\tilde{h}_k \in \tilde{h}_{k-1} + \mathcal{A}_\circ(\mathbb{X})$, $k = 1, \dots, 5$ and $\tilde{h}_5 = \tilde{h}$ is the desired diffeomorphism. For each $k = 1, 2, \dots, 5$ we will secure conditions analogous to (A)-(E). Namely,

- (A_k) $\tilde{h}_k - \tilde{h}_{k-1} \in \mathcal{A}_\circ(\mathbb{X})$
- (B_k) $\|\tilde{h}_k - \tilde{h}_{k-1}\|_{\mathcal{C}(\mathbb{X})} \leq \epsilon/5$
- (C_k) $\|\nabla \tilde{h}_k - \nabla \tilde{h}_{k-1}\|_{\mathcal{L}^p(\mathbb{X})} \leq \epsilon/5$
- (D_k) $\|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla \tilde{h}_0\|_{\mathcal{L}^p(\mathbb{X})} - 2\delta$, for some $\delta > 0$;
 $\|\nabla \tilde{h}_k\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla \tilde{h}_{k-1}\|_{\mathcal{L}^p(\mathbb{X})}$, for $k = 2, 4$;
 $\|\nabla \tilde{h}_k\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla \tilde{h}_{k-1}\|_{\mathcal{L}^p(\mathbb{X})} + \delta$, for $k = 3, 5$
- (E_k) If \tilde{h}_{k-1} is a \mathcal{C}^∞ -diffeomorphism outside of a compact subset of \mathbb{X} , then there exists a compact subset in \mathbb{X} outside which we have $\tilde{h}_k \equiv \tilde{h}_{k-1}$ for all $\epsilon > 0$.

3.2. Partition of \mathbb{X} into cells. Let us distinguish one particular Whitney type partition of \mathbb{Y} and keep it fixed for the rest of our arguments.

$$\mathbb{Y} = \bigcup_{\nu=1}^{\infty} \overline{\mathbb{Y}_\nu},$$

where \mathbb{Y}_ν are mutually disjoint open dyadic squares such that

$$\text{diam } \mathbb{Y}_\nu \leq \text{dist}(\mathbb{Y}_\nu, \partial\mathbb{Y}) \leq 3 \text{diam } \mathbb{Y}_\nu \quad \text{for } \nu = 1, 2, \dots$$

unless $\mathbb{Y} = \mathbb{R}^2$, in which case \mathbb{Y}_ν are unit squares. Thus the cover of \mathbb{Y} by $\overline{\mathbb{Y}_\nu}$ is locally finite. The preimages

$$\mathbb{X}_\nu = h^{-1}(\mathbb{Y}_\nu), \quad \nu = 1, 2, \dots$$

are Jordan domains which we call *cells* in \mathbb{X} . In the forthcoming Step 1 we shall need to further divide each cell into a finite number of *daughter cells* in \mathbb{X} . Note that all but finite number of cells \mathbb{X}_ν , $\nu = 1, 2, \dots$ lie outside a given compact subset of \mathbb{X} .

STEP 1

To avoid undue indexing in the forthcoming division of cells, we shall argue in two substeps.

Step 1a. Examine one of the cells in \mathbb{X} , say $\mathfrak{X} = \mathbb{X}_\nu$, for some fixed $\nu = 1, 2, \dots$. Call it a *parent cell*. Thus $h(\mathfrak{X}) = \Upsilon$ is the corresponding Whitney square $\Upsilon = \mathbb{Y}_\nu \subset \mathbb{Y}$. To every $n = 1, 2, \dots$, there corresponds a partition of Υ into 4^n -dyadic congruent squares Υ_i , $i = 1, \dots, 4^n$

$$\overline{\Upsilon} = \overline{\Upsilon_1} \cup \dots \cup \overline{\Upsilon_{4^n}}.$$

This gives rise to a division of \mathfrak{X} into daughter cells $\mathfrak{X}_i = h^{-1}(\Upsilon_i)$

$$\overline{\mathfrak{X}} = \overline{\mathfrak{X}_1} \cup \overline{\mathfrak{X}_2} \cup \dots \cup \overline{\mathfrak{X}_{4^n}}.$$

We look at the homeomorphisms

$$h: \overline{\mathfrak{X}_i} \xrightarrow{\text{ontq}} \overline{\Upsilon_i}, \quad i = 1, 2, \dots, 4^n$$

By virtue of Proposition 2.6 we may replace them with p -harmonic homeomorphisms

$$\tilde{h}_i = \mathbf{R}_{\mathfrak{X}_i} h: \overline{\mathfrak{X}_i} \xrightarrow{\text{ontq}} \overline{\Upsilon_i}, \quad i = 1, 2, \dots, 4^n$$

which coincide with h on $\partial\mathfrak{X}_i$. This procedure may not be necessary if $h: \mathfrak{X}_i \rightarrow \Upsilon_i$ is already a \mathcal{C}^∞ -diffeomorphism. In such cases we always use the *trivial replacement* $\tilde{h}_i = h$. After all such replacements are made, we arrive at a homeomorphism

$$\tilde{h}: \overline{\mathfrak{X}} \xrightarrow{\text{ontq}} \overline{\Upsilon}$$

which is a \mathcal{C}^∞ -diffeomorphism in each cell \mathfrak{X}_i and coincides with h on $\partial\mathfrak{X}_i$. Obviously,

$$\tilde{h} = h + \sum_{i=1}^{4^n} [\tilde{h}_i - h]_0 \in h + \mathcal{A}_0(\mathfrak{X})$$

where $[\tilde{h}_i - h]_0$ stands for zero extension of $\tilde{h}_i - h$ outside \mathfrak{X}_i and, therefore, belongs to $\mathcal{A}_0(\mathfrak{X}_i)$. Furthermore, by principle of minimal p -harmonic energy, we have

$$\mathbf{E}_{\mathfrak{X}}[\tilde{h}] = \sum_{i=1}^{4^n} \mathbf{E}_{\mathfrak{X}_i}[\tilde{h}_i] \leq \sum_{i=1}^{4^n} \mathbf{E}_{\mathfrak{X}_i}[h] = \mathbf{E}_{\mathfrak{X}}[h].$$

The eventual aim is to fix the number of daughter cells in \mathfrak{X} . For this we vary n and look closely at the resulting homeomorphisms, denoted by f_n . This sequence of mappings is bounded in $\mathcal{A}(\mathfrak{X})$. It actually converges to h uniformly on $\overline{\mathfrak{X}}$. Indeed, given any point $x \in \overline{\mathfrak{X}}$, say $x \in \overline{\mathfrak{X}_i}$, for some $i = 1, 2, \dots, 4^n$, we have

$$|f_n(x) - h(x)| = |\tilde{h}_i(x) - h(x)| \leq \text{diam } \Upsilon_i = 2^{-n} \text{diam } \Upsilon.$$

Thus

$$\lim_{n \rightarrow \infty} f_n = h, \quad \text{uniformly in } \overline{\mathfrak{X}}.$$

On the other hand the mappings f_n are bounded in the Sobolev space $\mathcal{W}^{1,p}(\mathfrak{X})$, so converge to h weakly in $\mathcal{W}^{1,p}(\mathfrak{X})$. The key observation now is that

$$\|\nabla h\|_{\mathcal{L}^p(\mathfrak{X})} \leq \liminf_{n \rightarrow \infty} \|\nabla f_n\|_{\mathcal{L}^p(\mathfrak{X})} \leq \|\nabla h\|_{\mathcal{L}^p(\mathfrak{X})}$$

because of convexity of the energy functional. This gives

$$\lim_{n \rightarrow \infty} \|\nabla f_n\|_{\mathcal{L}^p(\mathfrak{X})} = \|\nabla h\|_{\mathcal{L}^p(\mathfrak{X})}$$

Then, the usual application of Clarkson's inequalities in \mathcal{L}^p -spaces, $1 < p < \infty$, yields

$$\lim_{n \rightarrow \infty} \|\nabla f_n - \nabla h\|_{\mathcal{L}^p(\mathfrak{X})} = 0$$

meaning that $f_n - h \rightarrow 0$ in the norm topology of $\mathcal{A}(\mathfrak{X})$. We can now determine the number $n = n_\nu = n(\mathfrak{X})$, simply requiring the division of \mathfrak{X} be fine enough to satisfy two conditions.

$$(3.1) \quad \begin{cases} \text{diam } \Upsilon_i = 2^{-n} \text{diam } \Upsilon \leq \epsilon/5, & i = 1, \dots, 4^n \\ \|\nabla f_n - \nabla h\|_{\mathcal{L}^p(\mathfrak{X})} \leq \frac{\epsilon}{5 \cdot 2^\nu} \end{cases}$$

where we recall that \mathfrak{X} stands for \mathbb{X}_ν .

Step 1b. Now, having $n = n_\nu$ fixed for each cell \mathfrak{X}_ν , we construct our first approximating mapping

$$\tilde{h}_1: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$$

by setting

$$\tilde{h}_1 := h + \sum_{\nu=1}^{\infty} [f_{n_\nu} - h]_{\circ} \in h + \mathcal{A}_{\circ}(\mathbb{X})$$

where, as always, $[f_{n_\nu} - h]_{\circ}$ stands for the zero extension of $f_{n_\nu} - h$ outside \mathbb{X}_ν . This mapping is a \mathcal{C}^∞ -diffeomorphism in every daughter cell. Clearly, we have the condition

$$(A_1) \quad \tilde{h}_1 - h \in \mathcal{A}_{\circ}(\mathbb{X}).$$

Moreover, by the condition in (3.1) imposed on every n_ν ,

$$(B_1) \quad \|\tilde{h}_1 - h\|_{\mathcal{C}(\mathbb{X})} \leq \sup_{\nu=1,2,\dots} \{\text{diam } \Upsilon_i: \Upsilon_i \subset \mathbb{Y}_\nu, i = 1, \dots, 4^{n_\nu}\} \leq \frac{\epsilon}{5}$$

and

$$(C_1) \quad \|\nabla \tilde{h}_1 - \nabla h\|_{\mathcal{L}^p(\mathbb{X})}^p = \sum_{\nu=1}^{\infty} \|\nabla \tilde{h}_1 - \nabla h\|_{\mathcal{L}^p(\mathbb{X}_\nu)}^p \leq \left(\frac{\epsilon}{5}\right)^p \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu p}} < \left(\frac{\epsilon}{5}\right)^p.$$

Regarding condition (D_1) , we observe that summing up the energies over all daughter cells $\mathfrak{X}_i \subset \mathbb{X}_\nu$, $i = 1, 2, \dots, 4^{n_\nu}$ and $\nu = 1, 2, \dots$, gives the total energy of \tilde{h}_1 not larger than that of h . Even more, since h fails to be a \mathcal{C}^∞ -diffeomorphism in at least one of these cells, the p -harmonic replacement takes place in this cell and, consequently, \tilde{h}_1 has strictly smaller energy. Hence

$$(D_1) \quad \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla h\|_{\mathcal{L}^p(\mathbb{X})} - 2\delta, \quad \text{for some } \delta > 0.$$

Regarding condition (E_1) , we note that under the assumption therein we made only a finite number of nontrivial (p -harmonic) replacements. The same remark will apply to the subsequent steps and will not be mentioned again. The step 1 is complete.

Before proceeding to Step 2, let us put all daughter cells in \mathbb{X} in a single sequence

$$\mathfrak{X}^1, \mathfrak{X}^2, \dots \subset \mathbb{X}.$$

Thus from now on the daughter cells from different parents are indistinguishable as far as the mapping h_1 is concerned. The point is that h_1 is a \mathcal{C}^∞ -diffeomorphism in every such cell, a property that will be pertinent to all new cells coming later either by splitting or merging the existing cells. Note that the images $\Upsilon^\alpha = h(\mathfrak{X}^\alpha)$, $\alpha = 1, 2, \dots$, form a partition of \mathbb{Y} into dyadic squares

$$\mathbb{Y} = \bigcup_{\alpha=1}^{\infty} \overline{\Upsilon^\alpha}, \quad \text{where} \quad \text{diam } \Upsilon^\alpha \leq \frac{\epsilon}{5}.$$

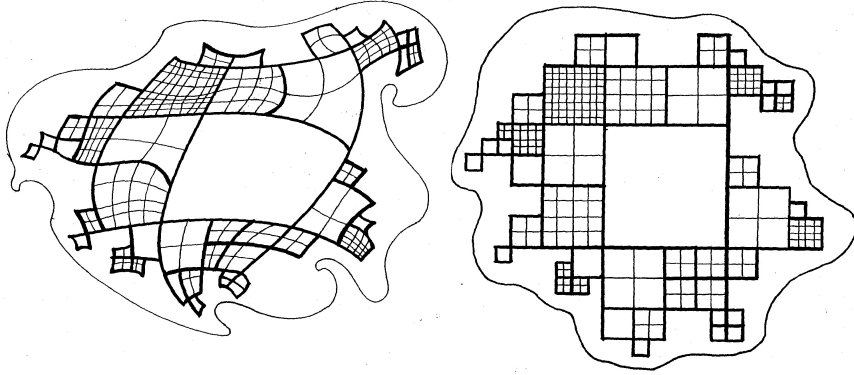


FIGURE 2. h_1 is a \mathcal{C}^∞ -diffeomorphism in each cell $\mathfrak{X}^\alpha \subset \mathbb{X}$.

STEP 2

Step 2a. (Adjacent cells) Let $\mathcal{C}(\mathbb{Y}) \subset \mathbb{Y}$ be the collection of all corners of dyadic squares Υ^α , $\alpha = 1, 2, \dots$, and $\mathcal{V}(\mathbb{X}) \subset \mathbb{X}$ denote the set of their preimages under h , called *vertices of cells*. Whenever two closed cells $\overline{\mathfrak{X}^\alpha}$ and $\overline{\mathfrak{X}^\beta}$, $\alpha \neq \beta$, intersect, their common part is either a point in $\mathcal{V}(\mathbb{X})$ or an edge, that is, a closed Jordan arc with endpoints in $\mathcal{V}(\mathbb{X})$. In this latter case we say that \mathfrak{X}^α and \mathfrak{X}^β are adjacent cells with common edge

$$\overline{C^{\alpha\beta}} = \overline{\mathfrak{X}^\alpha} \cap \overline{\mathfrak{X}^\beta}.$$

This is the closure of a Jordan open arc $C^{\alpha\beta} = \overline{C^{\alpha\beta}} \setminus \mathcal{V}(\mathbb{X})$. The mappings

$$h_1: \mathfrak{X}^\alpha \xrightarrow{\text{onto}} \Upsilon^\alpha \quad \text{and} \quad h_1: \mathfrak{X}^\beta \xrightarrow{\text{onto}} \Upsilon^\beta$$

are \mathcal{C}^∞ -diffeomorphisms but they do not necessarily match smoothly along the edges. We shall now produce a new cell $\mathfrak{X}^{\alpha\beta}$, a daughter of the adjacent cells \mathfrak{X}^α and \mathfrak{X}^β , such that

$$C^{\alpha\beta} \subset \mathfrak{X}^{\alpha\beta} \subset \mathfrak{X}^\alpha \cup C^{\alpha\beta} \cup \mathfrak{X}^\beta.$$

To construct $\mathfrak{X}^{\alpha\beta}$ we look at the adjacent dyadic squares $\overline{\Upsilon}^\alpha$ and $\overline{\Upsilon}^\beta$ in \mathbb{Y} . The intersection $\overline{\Upsilon}^\alpha \cap \overline{\Upsilon}^\beta = h(C^{\alpha\beta})$ is a closed interval. Let R be a number greater than the length of $h(C^{\alpha\beta})$ to be chosen sufficiently large later on. There exist exactly two open disks of radius R for which $h(C^{\alpha\beta})$ is a chord. Their intersection, denoted by $\mathcal{L}^{\alpha\beta}$, is a symmetric doubly convex lens of curvature R^{-1} . Thus $\mathcal{L}^{\alpha\beta}$ is enclosed between two open circular arcs $\gamma^{\alpha\beta} = \Upsilon^\alpha \cap \partial\mathcal{L}^{\alpha\beta} \subset \Upsilon^\alpha$ and $\gamma^{\beta\alpha} = \Upsilon^\beta \cap \partial\mathcal{L}^{\alpha\beta} \subset \Upsilon^\beta$. Note that $\mathcal{L}^{\alpha\beta} = \mathcal{L}^{\beta\alpha}$, but $\gamma^{\alpha\beta} \neq \gamma^{\beta\alpha}$. We call

$$(3.2) \quad \mathfrak{X}^{\alpha\beta} = \mathfrak{h}_1^{-1}(\mathcal{L}^{\alpha\beta}), \quad \text{a daughter of the adjacent cells } \mathfrak{X}^\alpha \text{ and } \mathfrak{X}^\beta.$$

As the curvature of the lens $\mathcal{L}^{\alpha\beta}$ approaches zero, the area of $\mathfrak{X}^{\alpha\beta}$ tends to 0. This allows us to choose R so that

$$(3.3) \quad \|\nabla \mathfrak{h}_1\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})} \leq \frac{\epsilon}{5 \cdot 2^{\alpha+\beta}}.$$

The lenses $\mathcal{L}^{\alpha\beta}$ are disjoint because the opening angle of each lens (the angle between arcs at their common endpoints) is at most $\pi/3$ and their long axes are either parallel or orthogonal, see Figure 3. Therefore, the cells $\mathfrak{X}^{\alpha\beta} = \mathfrak{h}_1^{-1}(\mathcal{L}^{\alpha\beta})$ are also disjoint. However, their closures may have a common point that lies in $\mathcal{V}(\mathbb{X})$. The boundary of $\mathfrak{X}^{\alpha\beta}$ consists of two open arcs

$$\Gamma^{\alpha\beta} = \mathfrak{X}^\alpha \cap \partial\mathfrak{X}^{\alpha\beta} \quad \text{and} \quad \Gamma^{\beta\alpha} = \mathfrak{X}^\beta \cap \partial\mathfrak{X}^{\alpha\beta}$$

plus their endpoints. These open arcs are \mathcal{C}^∞ -smooth because they come as images of the circular arcs enclosing the lens $\mathcal{L}^{\alpha\beta}$ under a \mathcal{C}^∞ -diffeomorphism.

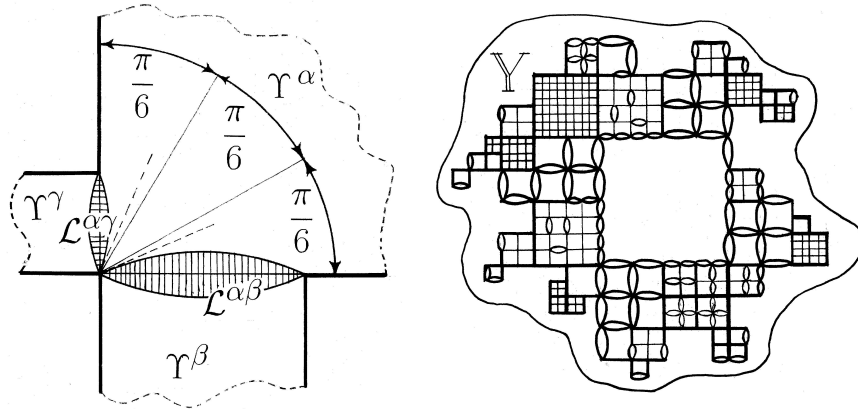


FIGURE 3. Lenses.

Remark 3.1. In what follows we shall consider only the pairs (α, β) of indices $\alpha = 1, 2, \dots$ and $\beta = 1, 2, \dots$ which correspond to adjacent cells. Such pairs will be designated the symbol $\alpha\beta$.

Step 2b. (Replacements in $\mathfrak{X}^{\alpha\beta}$) The lenses $\mathcal{L}^{\alpha\beta} \subset \mathbb{Y}$ are convex, so with the aid of Proposition 2.6 and Theorem 2.5, we may replace $\tilde{h}_1: \mathfrak{X}^{\alpha\beta} \rightarrow \mathcal{L}^{\alpha\beta}$ with the p -harmonic extension of $\tilde{h}_1: \partial\mathfrak{X}^{\alpha\beta} \rightarrow \partial\mathcal{L}^{\alpha\beta}$. We do this, and denote the result by $\tilde{h}_2^{\alpha\beta}: \mathfrak{X}^{\alpha\beta} \rightarrow \mathcal{L}^{\alpha\beta}$, only on the cells in which $\tilde{h}_1: \mathfrak{X}^{\alpha} \cup \mathfrak{X}^{\beta} \cup \mathfrak{X}^{\alpha\beta} \rightarrow \mathbb{R}^2$ is not a \mathcal{C}^∞ -diffeomorphism. In other cells we set $\tilde{h}_2^{\alpha\beta} = \tilde{h}_1$. In either case $\tilde{h}_2^{\alpha\beta} \in \tilde{h}_1 + \mathcal{A}_o(\mathfrak{X}^{\alpha\beta})$ so we define

$$\tilde{h}_2 = \tilde{h}_1 + \sum_{\alpha\beta} [\tilde{h}_2^{\alpha\beta} - \tilde{h}_1]_o.$$

Thus we have

$$(A_2) \quad \tilde{h}_2 - \tilde{h}_1 \in \mathcal{A}_o(\mathbb{X}).$$

The advantage of using \tilde{h}_2 in the next step lies in the fact that it is not only a \mathcal{C}^∞ -diffeomorphism in every cell, but also is \mathcal{C}^∞ -smooth with positive Jacobian determinant, up to each edge of the cells created here. These edges are \mathcal{C}^∞ -smooth open arcs. By cells created here we mean not only $\mathfrak{X}^{\alpha\beta}$ but also those obtained from the parent cell \mathfrak{X}^α by removing the adjacent daughters; that is,

$$\mathfrak{X}^\alpha \setminus \bigcup_{\alpha\beta} \mathfrak{X}^{\alpha\beta}, \quad \alpha = 1, 2, \dots$$

See Figure 4. The estimates of \tilde{h}_2 run as follows. By (3.1) we have,

$$(B_2) \quad \|\tilde{h}_2 - \tilde{h}_1\|_{\mathcal{C}(\mathbb{X})} \leq \sup_{\alpha\beta} \{\text{diam } \mathcal{L}^{\alpha\beta}\} \leq \sup_{\alpha} \{\text{diam } \mathbb{Y}^\alpha\} \leq \frac{\epsilon}{5}.$$

In view of the minimum p -harmonic energy principle, we have

$$\begin{aligned} \|\nabla \tilde{h}_2 - \nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})} &= \sum_{\alpha\beta} \|\nabla \tilde{h}_2 - \nabla \tilde{h}_1\|_{\mathcal{L}^p(\cup \mathfrak{X}^{\alpha\beta})} \\ &\leq \sum_{\alpha\beta} \left[\|\nabla \tilde{h}_2\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})} + \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})} \right] \\ &\leq 2 \sum_{\alpha\beta} \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathfrak{X}^{\alpha\beta})} \leq \frac{2\epsilon}{5} \sum_{\alpha\beta} 2^{-\alpha-\beta}. \end{aligned}$$

by (3.3). Hence

$$(C_2) \quad \|\nabla \tilde{h}_2 - \nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

The minimum energy principle also yields estimate

$$\begin{aligned} \|\nabla \tilde{h}_2\|_{\mathcal{L}^p(\mathbb{X})}^p &= \|\nabla \tilde{h}_2\|_{\mathcal{L}^p(\cup \mathfrak{X}^{\alpha\beta})}^p + \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X} \setminus \cup \mathfrak{X}^{\alpha\beta})}^p \\ &\leq \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\cup \mathfrak{X}^{\alpha\beta})}^p + \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X} \setminus \cup \mathfrak{X}^{\alpha\beta})}^p = \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})}^p. \end{aligned}$$

In particular

$$(D_2) \quad \|\nabla \tilde{h}_2\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla \tilde{h}_1\|_{\mathcal{L}^p(\mathbb{X})},$$

completing the proof of Step 2.

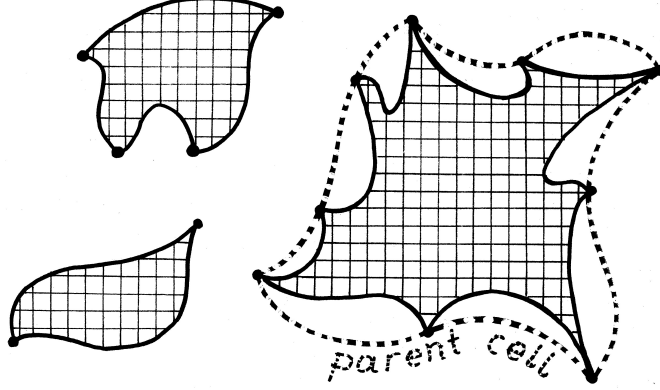


FIGURE 4. Three types of cells.

Note that \tilde{h}_2 is locally bi-Lipschitz in $\mathbb{X} \setminus \mathcal{V}(\mathbb{X})$. The exceptional set $\mathcal{V}(\mathbb{X})$ is discrete.

STEP 3

We shall now merge all the adjacent cells together, by smoothing \tilde{h}_2 around the edges $\Gamma^{\alpha\beta} \subset \mathfrak{X}^\alpha$. To achieve proper estimates we need to remove small neighborhoods of all vertices, outside which \tilde{h}_2 is certainly locally bi-Lipschitz.

Step 3a. First we cover the set $\mathcal{C}(\mathbb{Y})$ of corners of dyadic squares by disks \mathbb{D}_c centered at $c \in \mathcal{C}(\mathbb{Y})$. These disks will be chosen small enough to satisfy all the conditions listed below.

- (i) $\text{diam } \mathbb{D}_c < \epsilon/5$ for every $c \in \mathcal{C}(\mathbb{Y})$,
- (ii) $\sum_{v \in \mathcal{V}(\mathbb{X})} \int_{\mathbb{F}_v} |\nabla \tilde{h}_2|^p \leq \left(\frac{\epsilon}{20}\right)^p$, where $\mathbb{F}_v = \tilde{h}_2^{-1}(\mathbb{D}_c)$, $c = \tilde{h}_2(v) = h(v)$.

Denote by $\mathbb{X}_\circ = \mathbb{X} \setminus \bigcup \overline{\mathbb{F}_v}$. We truncate each edge $\Gamma^{\alpha\beta}$ near the endpoints by setting

$$(3.4) \quad \Gamma_\circ^{\alpha\beta} = \Gamma^{\alpha\beta} \cap \mathbb{X}_\circ.$$

These are mutually disjoint open arcs; their closures are isolated continua in $\mathbb{X} \setminus \mathcal{V}(\mathbb{X})$. This means that there are disjoint neighborhoods of them. We are actually interested in neighborhoods $\mathbb{U}^{\alpha\beta} \subset \mathfrak{X}^\alpha$ of $\Gamma_\circ^{\alpha\beta}$ that are Jordan domains in which $\Gamma_\circ^{\alpha\beta} \subset \mathbb{U}^{\alpha\beta}$ are \mathcal{C}^∞ -smooth crosscuts with two endpoints in $\partial\mathbb{U}^{\alpha\beta}$, see Section 2. It is geometrically clear that such mutually disjoint neighborhoods exist. Now the stage for next substep is established.

Step 3b. (\mathcal{C}^∞ -replacement within $\mathbb{U}^{\alpha\beta}$) It is at this stage that we will improve \tilde{h}_2 in $\mathbb{U}^{\alpha\beta}$ to a \mathcal{C}^∞ -smooth diffeomorphism with no harm to the previously established estimates for \tilde{h}_2 . The tool is Proposition 2.8. As

always, we shall make no replacement of $\hbar_2: \mathbb{U}^{\alpha\beta} \rightarrow \Upsilon^\alpha$ if it is already \mathcal{C}^∞ -diffeomorphism. Recall that we have a bi-Lipschitz mapping $\hbar_2: \mathbb{U}^{\alpha\beta} \rightarrow \hbar_2(\mathfrak{X}^\alpha) = \Upsilon^\alpha$ that takes the crosscut $\Gamma_\circ^{\alpha\beta} \subset \mathbb{U}^{\alpha\beta}$ onto a circular arc. Denote the components $\mathbb{U}_+^{\alpha\beta} = \mathbb{U}^{\alpha\beta} \setminus \overline{\mathfrak{X}^{\alpha\beta}}$ and $\mathbb{U}_-^{\alpha\beta} = \mathbb{U}^{\alpha\beta} \cap \mathfrak{X}^{\alpha\beta}$. Furthermore, we have

$$|D\hbar_2| \leq m_{\alpha\beta} \quad \text{and} \quad \det D\hbar_2 \geq \frac{1}{m_{\alpha\beta}}, \quad \text{for some } m_{\alpha\beta} > 0$$

on each component. The mappings $\hbar_2: \mathbb{U}_+^{\alpha\beta} \rightarrow \Upsilon^\alpha$ and $\hbar_2: \mathbb{U}_-^{\alpha\beta} \rightarrow \Upsilon^\alpha$ are \mathcal{C}^∞ -diffeomorphisms up to $\Gamma_\circ^{\alpha\beta}$. In accordance with Proposition 2.8 we find a constant $M_{\alpha\beta}$ such that: whenever open set $\mathbb{V}^{\alpha\beta} \subset \mathbb{U}^{\alpha\beta}$ contains the crosscut $\Gamma_\circ^{\alpha\beta}$ there exists a homeomorphism $\hbar_3^{\alpha\beta}: \overline{\mathbb{U}^{\alpha\beta}} \xrightarrow{\text{ontg}} \hbar_2(\overline{\mathbb{U}^{\alpha\beta}})$ which is a \mathcal{C}^∞ -diffeomorphism in $\mathbb{U}^{\alpha\beta}$, with the following properties

- $\hbar_3^{\alpha\beta} \equiv \hbar_2$ on $(\overline{\mathbb{U}^{\alpha\beta}} \setminus \mathbb{V}^{\alpha\beta}) \cup \Gamma_\circ^{\alpha\beta}$;
- $|\nabla \hbar_3^{\alpha\beta}| \leq M_{\alpha\beta}$ and $\det \nabla \hbar_3^{\alpha\beta} \geq \frac{1}{M_{\alpha\beta}}$ in $\mathbb{U}^{\alpha\beta}$.

Since $M_{\alpha\beta}$ does not depend on $\mathbb{V}^{\alpha\beta}$ it will be advantageous to take neighborhoods $\mathbb{V}^{\alpha\beta}$ of $\Gamma_\circ^{\alpha\beta}$ thin enough to satisfy

- $\overline{\mathbb{V}^{\alpha\beta}} \subset \mathbb{U}^{\alpha\beta} \cup \overline{\Gamma_\circ^{\alpha\beta}}$;
- $|\mathbb{V}^{\alpha\beta}| \leq \frac{1}{5^p \cdot 2^{\alpha+\beta}} \left[\frac{\epsilon}{m_{\alpha\beta} + M_{\alpha\beta}} \right]^p$ and also $|\mathbb{V}^{\alpha\beta}| \leq \frac{\delta}{2^{\alpha+\beta} M_{\alpha\beta}}$.

Note that $\hbar_3^{\alpha\beta}, \hbar_2 \in \mathcal{W}^{1,\infty}(\mathbb{U}^{\alpha\beta}) \subset \mathcal{W}^{1,p}(\mathbb{U}^{\alpha\beta})$ and $\hbar_3^{\alpha\beta} = \hbar_2$ on $\partial\mathbb{U}^{\alpha\beta}$, so we have

$$\hbar_3^{\alpha\beta} - \hbar_2 \in \mathcal{W}_\circ^{1,p}(\mathbb{U}^{\alpha\beta}).$$

Step 3c. We now define a homeomorphism $\hbar_3: \mathbb{X} \xrightarrow{\text{ontg}} \mathbb{Y}$ by the rule

$$\hbar_3 = \begin{cases} \hbar_3^{\alpha\beta} & \text{in } \mathbb{U}^{\alpha\beta} \\ \hbar_2 & \text{in } \mathbb{X} \setminus \bigcup_{\alpha\beta} \mathbb{U}^{\alpha\beta}. \end{cases}$$

Obviously, \hbar_3 is a \mathcal{C}^∞ -diffeomorphism in \mathbb{X}_\circ and $\hbar_3 - \hbar_2 \in \mathcal{W}_\circ^{1,p}(\mathbb{X}_\circ)$. Since \hbar_3 coincides with \hbar_2 outside \mathbb{X}_\circ we have $\hbar_3 = \hbar_2 + [\hbar_3 - \hbar_2]_\circ$. Hence

$$(A_3) \quad \hbar_3 - \hbar_2 \in \mathcal{A}_\circ(\mathbb{X}).$$

Then, for every $x \in \mathbb{X}$,

$$|\hbar_3(x) - \hbar_2(x)| \leq \begin{cases} \text{diam } \hbar_2(\mathbb{U}^{\alpha\beta}), & \text{for } x \in \mathbb{U}^{\alpha\beta} \\ 0, & \text{otherwise} \end{cases} \leq \text{diam } \Upsilon^\alpha \leq \frac{\epsilon}{5}$$

meaning that

$$(B_3) \quad \|\hbar_3 - \hbar_2\|_{\mathcal{C}(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

The computation of p -norms goes as follows

$$\begin{aligned}
 \|\nabla \hbar_3 - \nabla \hbar_2\|_{\mathcal{L}^p(X)}^p &= \sum_{\alpha\beta} \int_{\mathbb{V}^{\alpha\beta}} |\nabla \hbar_3 - \nabla \hbar_2|^p \\
 &\leq \sum_{\alpha\beta} |\mathbb{V}^{\alpha\beta}| \left[\|\nabla \hbar_3\|_{\mathcal{L}(\mathbb{V}^{\alpha\beta})} + \|\nabla \hbar_2\|_{\mathcal{L}(\mathbb{V}^{\alpha\beta})} \right]^p \\
 &\leq \sum_{\alpha\beta} |\mathbb{V}^{\alpha\beta}| (m_{\alpha\beta} + M_{\alpha\beta})^p \leq \sum_{\alpha\beta} \frac{\epsilon^p}{5^p 2^{\alpha+\beta}} \leq \left(\frac{\epsilon}{5}\right)^p.
 \end{aligned}$$

Hence

$$(C_3) \quad \|\nabla \hbar_3 - \nabla \hbar_2\|_{\mathcal{L}^p(X)} \leq \frac{\epsilon}{5}.$$

In the finite energy case, when $\|\nabla \hbar_2\|_{\mathcal{L}^p(\mathbb{X})} < \infty$, we observe that

$$\|\nabla \hbar_3\|_{\mathcal{L}^p(\mathbb{X} \cup \mathbb{V}^{\alpha\beta})} = \|\nabla \hbar_2\|_{\mathcal{L}^p(\mathbb{X} \cup \mathbb{V}^{\alpha\beta})} \leq \|\nabla \hbar_2\|_{\mathcal{L}^p(\mathbb{X})}.$$

Therefore, by triangle inequality,

$$\begin{aligned}
 \|\nabla \hbar_3\|_{\mathcal{L}^p(\mathbb{X})} &\leq \|\nabla \hbar_2\|_{\mathcal{L}^p(\mathbb{X})} + \sum_{\alpha\beta} \|\nabla \hbar_3\|_{\mathcal{L}^p(\mathbb{V}^{\alpha\beta})} \\
 &\leq \|\nabla \hbar_2\|_{\mathcal{L}^p(\mathbb{X})} + \sum_{\alpha\beta} |\mathbb{V}^{\alpha\beta}| \cdot \|\nabla \hbar_3\|_{\mathcal{L}(\mathbb{V}^{\alpha\beta})} \\
 &\leq \|\nabla \hbar_2\|_{\mathcal{L}^p(\mathbb{X})} + \sum_{\alpha\beta} \frac{\delta}{2^{\alpha+\beta} M_{\alpha\beta}} \cdot M_{\alpha\beta}
 \end{aligned}$$

which yields

$$(D_3) \quad \|\nabla \hbar_3\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\nabla \hbar_2\|_{\mathcal{L}^p(\mathbb{X})} + \delta.$$

The third step is completed.

STEP 4

We have already upgraded the mapping h to a homeomorphism $\hbar_3: \mathbb{X} \xrightarrow{\text{ontq}} \mathbb{Y}$ that is a \mathcal{C}^∞ -diffeomorphism in $\mathbb{X}_o = \mathbb{X} \setminus \bigcup_{v \in \mathcal{V}(\mathbb{X})} \overline{\mathbb{F}_v}$, where \mathbb{F}_v are small surroundings of the vertices of cells. Their images $\hbar_3(\mathbb{F}_v) = \hbar_2(\mathbb{F}_v) = \mathbb{D}_c$ are small disks centered at $c = h(v)$. In Step 3a, one of the preconditions on those disks was that $\text{diam } \mathbb{D}_c < \epsilon/5$. Furthermore, the closed disks $\overline{\mathbb{D}_c}$ are isolated continua in \mathbb{Y} for all $c \in \mathcal{C}(\mathbb{Y})$, so are the sets $\overline{\mathbb{F}_v}$ in \mathbb{X} . We shall now consider slightly larger concentric open disks $\mathbb{D}'_c \supset \overline{\mathbb{D}_c}$, $c \in \mathcal{C}(\mathbb{Y})$, and their preimages $\mathbb{F}'_v = \hbar_3^{-1}(\mathbb{D}'_c) \subset \mathbb{X}$, $v = h^{-1}(c) \in \mathcal{V}(\mathbb{X})$. The annulus $\mathbb{D}'_c \setminus \overline{\mathbb{D}_c}$ will be thin enough to ensure that \mathbb{D}'_c are still disjoint,

$$\text{diam } \mathbb{D}'_c < \frac{\epsilon}{5} \quad \text{for all } c \in \mathcal{C}(\mathbb{Y})$$

and

$$\sum_{v \in \mathcal{V}(\mathbb{X})} \|\nabla \hbar_3\|_{\mathcal{L}^p(\mathbb{F}'_v \setminus \mathbb{F}_v)}^p \leq \left(\frac{\epsilon}{20}\right)^p.$$

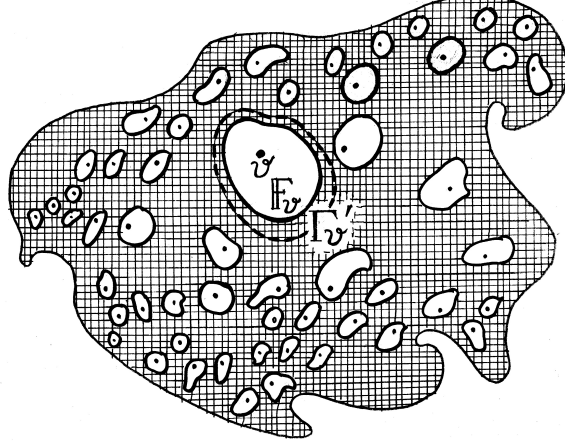


FIGURE 5. Neighborhoods of vertices.

Let Γ'_v , $v \in \mathcal{V}(\mathbb{X})$, denote the boundary of \mathbb{F}'_v . These are \mathcal{C}^∞ -smooth Jordan curves. We now define a homeomorphism $\tilde{h}_4: \mathbb{X} \xrightarrow{\text{ontq}} \mathbb{Y}$ by performing p -harmonic replacement of mappings $\tilde{h}_3: \mathbb{F}'_v \xrightarrow{\text{ontq}} \mathbb{D}'_c$, whenever such a mapping fails to be \mathcal{C}^∞ -diffeomorphism. Thus every $\tilde{h}_4: \mathbb{F}'_v \xrightarrow{\text{ontq}} \mathbb{D}'_c$ is a \mathcal{C}^∞ -diffeomorphism up to Γ'_v . Moreover $\tilde{h}_4 \in \tilde{h}_3 + \mathcal{W}_o^{1,p}(\mathbb{F}'_c)$, so

$$(A_4) \quad \tilde{h}_4 - \tilde{h}_3 \in \mathcal{A}_o(\mathbb{X}).$$

For every $x \in \mathbb{X}$, we have

$$|\tilde{h}_4(x) - \tilde{h}_3(x)| \leq \begin{cases} \text{diam } \mathbb{D}'_c & \text{in } \mathbb{F}'_v, c = h(v) \\ 0 & \text{otherwise} \end{cases} \leq \frac{\epsilon}{5}.$$

Hence

$$(B_4) \quad \|\tilde{h}_4 - \tilde{h}_3\|_{\mathcal{C}(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

By virtue of the minimum energy principle we compute the p -norms

$$\begin{aligned}
 \|\tilde{h}_4 - \tilde{h}_3\|_{\mathcal{L}^p(\mathbb{X})}^p &= \sum_{v \in \mathcal{V}(\mathbb{X})} \|\tilde{h}_4 - \tilde{h}_3\|_{\mathcal{L}^p(\mathbb{F}'_v)}^p \\
 &\leq \sum_{v \in \mathcal{V}(\mathbb{X})} [\|\tilde{h}_4\|_{\mathcal{L}^p(\mathbb{F}'_v)} + \|\tilde{h}_3\|_{\mathcal{L}^p(\mathbb{F}'_v)}]^p \\
 &\leq 2^p \sum_{v \in \mathcal{V}(\mathbb{X})} \|\tilde{h}_3\|_{\mathcal{L}^p(\mathbb{F}'_v)}^p \\
 &\leq 2^{2p-1} \sum_{v \in \mathcal{V}(\mathbb{X})} [\|\tilde{h}_3\|_{\mathcal{L}^p(\mathbb{F}'_v \setminus \mathbb{F}_v)}^p + \|\tilde{h}_3\|_{\mathcal{L}^p(\mathbb{F}_v)}^p] \\
 &\leq 2^{2p-1} \left[\left(\frac{\epsilon}{20}\right)^p + \sum_{v \in \mathcal{V}(\mathbb{X})} \|\tilde{h}_2\|_{\mathcal{L}^p(\mathbb{F}_v)}^p \right] \\
 &\leq 2^{2p} \left(\frac{\epsilon}{20}\right)^p = \left(\frac{\epsilon}{5}\right)^p.
 \end{aligned}$$

Hence

$$(C_4) \quad \|\tilde{h}_4 - \tilde{h}_3\|_{\mathcal{L}^p(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

Again by minimum energy principle we find that

$$(D_4) \quad \|\tilde{h}_4\|_{\mathcal{L}^p(\mathbb{X})}^p \leq \|\tilde{h}_3\|_{\mathcal{L}^p(\mathbb{X})}^p.$$

Just as in the previous steps, condition (E_4) remains valid, finishing Step 4.

STEP 5

The final step consists of smoothing \tilde{h}_4 in a neighborhood of each smooth Jordan curve Γ'_v , $v \in \mathcal{V}(\mathbb{X})$. We argue in much the same way as in Step 3, but this time we appeal to Proposition 2.9 instead of Proposition 2.8. By smoothing \tilde{h}_4 in a sufficiently thin neighborhood of each Γ'_v we obtain a \mathcal{C}^∞ -diffeomorphism $\tilde{h}_5: \mathbb{X} \xrightarrow{\text{ont}} \mathbb{Y}$,

$$(A_5) \quad \tilde{h}_5 - \tilde{h}_4 \in \mathcal{A}_0(\mathbb{X}).$$

$$(B_5) \quad \|\tilde{h}_5 - \tilde{h}_4\|_{\mathcal{C}(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

$$(C_5) \quad \|\tilde{h}_5 - \tilde{h}_4\|_{\mathcal{L}^p(\mathbb{X})} \leq \frac{\epsilon}{5}.$$

$$(D_5) \quad \|\tilde{h}_5\|_{\mathcal{L}^p(\mathbb{X})} \leq \|\tilde{h}_4\|_{\mathcal{L}^p(\mathbb{X})} + \delta. \quad \square$$

4. OPEN QUESTIONS

Question 4.1. Does Theorem 1.1 extend to $n = 3$?

Question 4.2. A bi-Sobolev homeomorphism $h: \mathbb{X} \xrightarrow{\text{ontq}} \mathbb{Y}$ is a mapping of class $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 \leq p < \infty$, whose inverse $h^{-1}: \mathbb{Y} \xrightarrow{\text{ontq}} \mathbb{X}$ belongs to a Sobolev class $\mathcal{W}^{1,q}(\mathbb{Y}, \mathbb{X})$, $1 \leq q < \infty$. Can h be approximated by bi-Sobolev diffeomorphisms $\{h_\ell\}$ so that $h_\ell \rightarrow h$ in $\mathcal{W}^{1,p}(\mathbb{X}, \mathbb{Y})$ and $h_\ell^{-1} \rightarrow h^{-1}$ in $\mathcal{W}^{1,q}(\mathbb{Y}, \mathbb{X})$?

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