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Scalars and Generating Bases for the Module of Splines with Boundary Conditions C(r;0)(I

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Scalars and Generating Bases for the Module of Splines with Boundary Conditions $C^{(r,0)}(I^\delta)$

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Abstract

Splines are piecewise polynomial functions defined over a partition of the real number line. When smoothness conditions are placed on splines defined over a given partition, they form a module over the ring of polynomials with real number coefficients. Studying and characterizing bases for these modules allow us to better characterize bases for splines defined over two-dimensional regions, which would aid in the construction of roofs for complex structures (houses, stadiums, obscurely shaped buildings, etc.) and plane wings. Over the summer, my research group was able to give a characterization for a basis of one of these modules. This project takes that basis and characterizes the scalars that would be used to generate any spline in the module, then gives a complete characterization for all bases of the module for a specific partition of intervals. This will help in giving a complete characterization for the module over any given partition of intervals, which will help in the characterization of bases for modules of splines defined over two-dimensional regions.

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1 Introduction

1.1 Splines

In this paper, we investigate bases for modules of splines over the polynomial ring $\mathbb{R}[x]$. A spline is simply a piecewise polynomial function defined over a partition of the real number line into intervals. For example, the function

$$f(x) = \begin{cases} f_1 & x \leq 0 \\ f_2 & x > 0 \end{cases}$$

is spline defined over the partition of intervals $I = (-\infty, 0) \cup (0, \infty)$ where f_1 and f_2 are both polynomials of one variable ($f_1, f_2 \in \mathbb{R}[x]$). Also, it does not matter how many intervals are in the partition. So, a spline defined over the partition $I = (-\infty, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cup \cdots \cup (a_{n-2}, a_{n-1}) \cup (a_{n-1}, \infty)$ would look like

$$f(x) = \begin{cases} f_1 & -\infty < x \leq a_1 \\ f_2 & a_1 < x \leq a_2 \\ f_3 & a_2 < x \leq a_3 \\ \vdots & \\ f_{n-1} & a_{n-2} < x \leq a_{n-1} \\ f_n & a_{n-1} < x < \infty \end{cases}$$

where $f_0, f_1, f_2, \ldots, f_{n-1}, f_n \in \mathbb{R}[x]$.

Formally, a spline is defined as follows:

Definition: Let I denote the partition of $\mathbb{R}, I_1 \cup I_2 \cup I_3 \cup \cdots \cup I_{n-1} \cup I_n \subset \mathbb{R}$ where $I_1 = (-\infty, a_1), I_2 = (a_1, a_2), I_3 = (a_2, a_3), \ldots, I_{n-1} = (a_{n-2}, a_{n-1}), I_n = (a_{n-1}, \infty)$. A spline over I is a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\mid_{I_i} = f_i$$

for all $i \in \{1, 2, \ldots, n\}$ where $f_i \in \mathbb{R}[x]$.

For a spline f defined over I, we write f as an n-tuple, $f = (f_1, f_2, f_3, \ldots, f_{n-1}, f_n)$ or as a column

$$f(x) = \begin{cases} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-1} \\ f_n \end{cases}$$

where $f_i \in \mathbb{R}[x]$ is defined over the interval I_i.

Some basic examples would be as follows:

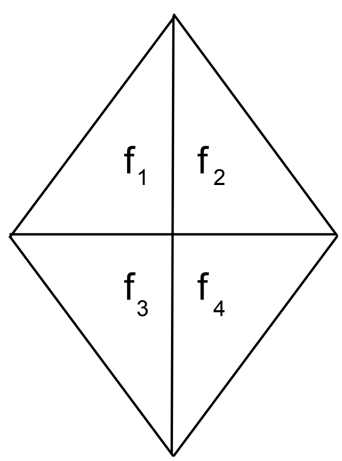
$$f(x) = \begin{cases} -x & -\infty < x \leq 0 \\ x & 0 < x < \infty \end{cases}$$

$$f(x) = \begin{cases} x & -\infty < x \leq -1 \\ x^2 & -1 < x \leq 5 \\ 5x+3 & 5 < x \leq 8 \\ 2x^2+5x+7 & 8 < x \leq \infty \end{cases}$$

$$f(x) = \begin{cases} -x^2 & -\infty < x \leq -5 \\ 3x & -5 < x \leq 0 \\ x^5+1 & 0 < x \leq 5 \\ 5 & 5 < x < \infty \end{cases}$$

For basic splines, the polynomials do not have to match up at the boundary points a_i of the partition. That is, it is not necessary for $f_i(a_i) = f_{i+1}(a_i)$ for any i such that $0 \ \leq i \leq n-1$.

The point of this research was to characterize scalars and bases for modules of splines defined over the polynomial ring. I'll get into what modules and bases are later, as well as how applying certain conditions to sets of splines causes them to have a module structure, but first, why bother to study these things?
In the single dimension case (splines defined over the number line $\mathbb{R}$) characterizing their bases has many interesting abstract applications in many different areas of mathematics.
However, the main purpose of studying the one-dimensional case is to take what we find and try to apply it to splines defined over a region in $\mathbb{R}^2$.
In the same way a spline can be defined over a subdivided region of the real number line, they can be defined in 2-dimensions over a 2-dimensional region. For example, a spline can be defined over a region like this:



where f_1, f_2, f_3, and f_4 are polynomials of two variables (i.e. $f(x, y)$) and will create a 3-dimensional graph over the given region. Thus, instead of the f_i's being in the polynomial ring $\mathbb{R}[x]$, they are in $\mathbb{R}[x, y]$. The two dimensional region the functions are defined over can be any shape and there can be any number of subregions.

Characterizing bases in the 2-dimensional case would be incredibly helpful in the construction of three dimensional objects such as roofs of complex structures (houses, stadiums, obscurely shaped buildings, etc.) or even plane wings. Splines are also helpful in approximating general functions, interpolating data, and are being found to have applications in computer graphics, image processing and computer aided design.

1.2 Smoothness

Now, let's focus on splines that are said to be C^r over a given partition of intervals I contained in $\mathbb{R}$.
The r represents a smoothness condition placed on the spline.

Definition: A C^r spline is a spline that has r continuous derivatives.

Since all polynomials are C^∞ (have infinite continuous derivatives), to see whether or not a spline is C^r we need only consider the boundary points a_i for $i \in \{1, 2, \ldots n-1\}$.

If a spline is C^0 it matches up on all boundary points a_i (i.e. $f_i(a_i) = f_{i+1}(a_i)$ for all i such that $1 \leq i \leq n-1$). If a spline is C^1 then it has one degree of smoothness, meaning it must match up on all of the boundary points for the spline and its first derivative, so

$$f_i(a_i) = f_{i+1}(a_i)$$
$$\text{and}$$
$$f_i'(a_i) = f_{i+1}'(a_i)$$

If a spline is C^2 then it matches up on all boundary points for 2 derivatives, if it is C^3, 3 derivatives, etc. So if a spline is C^r, the following are true:

$$\begin{aligned} f_i(a_i) &= f_{i+1}(a_i) \\ f_i'(a_i) &= f_{i+1}'(a_i) \\ f_i''(a_i) &= f_{i+1}''(a_i) \\ &\vdots \\ f_i^{(r-1)}(a_i) &= f_{i+1}^{(r-1)}(a_i) \\ f_i^{(r)}(a_i) &= f_{i+1}^{(r)}(a_i). \end{aligned}$$

For example, consider:

$$f(x) = \begin{cases} -x & x \leq 0 \\ x & x > 0. \end{cases}$$

Since the partition only contains 2 intervals, we need only consider smoothness at $x = 0$.
Computing, we get:

$$\begin{aligned} f_1(x) &= -x \\ f_2(x) &= x \\ \Rightarrow f_1(0) &= 0 = f_2(0) \\ \\ f_1'(x) &= -1 \\ f_2'(x) &= 1 \\ \Rightarrow f_1'(0) &= -1 \neq 1 = f_2'(0). \end{aligned}$$

Thus, this spline has 0 continuous derivatives, and is classified as C^0.
A slightly more complex example is the following:

$$f(x) = \begin{cases} -2x^3 + 16x + 10 & -\infty < x \leq -1 \\ 2x^3 + 4x + 2 & -1 < x \leq 0 \\ x^3 + 4x + 2 & 0 < x \leq 3 \\ 9x^2 - 23x + 29 & 3 < x < \infty. \end{cases}$$

Computing, we have:

$$\begin{aligned} f_1(-1) = -4 = f_2(-1),\ f_2(0) =2 = f_3(0),\ f_3(3) = 41 = f_4(3) \\ f_1'(-1) = 10 = f_2'(-1),\ f_2'(0) =4 = f_3'(0),\ f_3'(3) = 31 = f_4'(3) \\ f_1''(-1) = 12 = f_2''(-1),\ f_2''(0) =0 = f_3''(0),\ f_3''(3) = 18 = f_4''(3) \\ f_1'''(-1) = -12 \neq 12 = f_2'''(-1),\ f_2'''(0) = 12 \neq 6 = f_3'''(0),\ f_3'''(3) = 6 \neq 0 = f_4'''(3) \end{aligned}$$

This spline has 2 continuous derivatives, and is considered C^2.
This smoothness condition can also be applied to the splines defined over 2-dimensional regions mentioned in the previous section. The only difference is that for the two dimensional case, the spline, and its derivatives, must be smooth along a boundary curve, instead of a boundary point. In this paper, however, we concentrate only on splines defined over one-dimensional regions.

1.3 The Module $C^r(I)$

When considering all of the C^r splines defined over a given partition of intervals $I \subset \mathbb{R}$, we get a module structure over the ring $\mathbb{R}[x]$, and the module is written as $C^r(I)$.

To understand what this means, we must first look at the definitions of a ring and a module.

Definition: A nonempty set R is considered to be a ring with identity element 0 if for all $a, b, c \in R$ the following axioms hold:

- $(a + b) + c = a + (b + c)$
- $0 + a = a + 0 = a$ (0 is the identity)
- $a + b = b + a$ (+ is commutative)
- for each $a \in R$ there exists $-a \in R$ such that $a + (-a) = (-a) + a = 0$ ($-a$ is the inverse element of a)
- Multiplication is associative ($(a * b) * c = a * (b * c)$)
- Multiplication distributes over addition: $a * (b + c) = (a * b) + (a * c)$ and $(a + b) * c = (a * c) + (b * c)$

It is trivial to show that these characteristics hold for the set of polynomials with real number coefficients ($\mathbb{R}[x]$).

Now, on to modules. For those with some knowledge of linear algebra, a module M is basically vector space in which the scalars used for multiplication come from a ring R instead of a field. The scalars from the ring are distributive and associative with the elements of M. A key difference between modules and vector spaces, however, is that vector spaces always have bases whereas modules need not. This is because a ring R is not required to have multiplicative inverses, thus it is not always the case that elements of the module can be used to generate the rest of the module under the defined scalar multiplication. A module with a basis is said to be a free module, and the modules we deal with in this paper are all free modules.

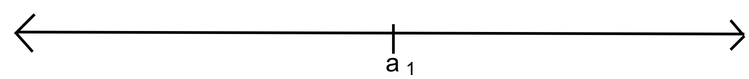
Definition: A module over a ring R (also known as an R-module) is a set M with a binary operation (normally written as addition) and scalar multiplication with scalars coming from the ring R, that satisfies the following conditions:

- M is closed under addition and scalar multiplication
- M is an abelian group under addition*
- For all $a \in R$ and all f,$g \in M$, $a(f + g) = af + ag$
- For all a,$b \in R$ and all $f \in M$, $(a + b)f = af + bf$
- For all a,$b \in R$ and all $f \in M$, $(ab)f = a(bf)$
- If 1 is the multiplicative identity in R, $1f = f$ for all $f \in M$

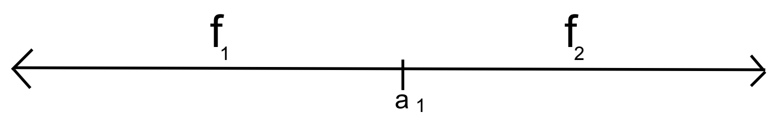
*An abelian group is simply a set A, together with a binary operator (normally addition or multiplication) such that it it is closed (for $a, b \in A$, $a + b \in A$), the elements are associative and commutative, and there exists an identity and inverse elements.

Now, we will consider a simple version of $C^r(I)$, two splines with the necessary conditions, and two polynomials from $\mathbb{R}$[x] in order to get a basic understanding of how addition and scalar multiplication work among the elements. Then, I will arbitrarily select two splines and two polynomials in order to prove that the simple example is in fact a module.

Consider the partition of intervals $I = (-\infty, a_1) \cup (a_1, \infty)$



with smoothness condition r = 1.
For $f \in C^1(I)$ we write f as a 2-tuple $f = (f_1, f_2)$ where $f_1, f_2 \in \mathbb{R}[x]$ are defined over the partition I as shown below:



and have the conditions:

$$f_1(a_1) = f_2(a_1)$$
$$f_1'(a_1) = f_2'(a_1)$$

(since there is one degree of smoothness).

Let us look at the case when $a_1 = 1$, f and g are defined such that $f = (x^2+1, 2x)$ and $g = (6x + 3, 2x^3 + 7)$, $p(x) = x^3 + 3x + 1$ and $q(x) = 3x^2 + 4x - 6$ ($p(x), q(x) \in \mathbb{R}[x]$ are the scalars).
Simple computations show that

$$f_1(1) = 2 = f_2(1)\ ,\ g_1(1) = 9 = g_2(1)$$
$$f_1'(1) = 2 = f_2'(1)\ ,\ g_1'(1) = 6 = g_2'(1)$$

so both f and g have smoothness of degree at least 1, thus $f, g \in C^1(I)$ for $I = (-\infty, 1) \cup (1, \infty)$.
Now we can play around a little with $f, g, p(x)$, and $q(x)$ in order to see how addition and scalar multiplication work amongst the elements of $C^1(I)$.

First, consider $f+g$:

$$\begin{aligned}(f+g) &= ((x^2+1)+(6x+3),(2x)+(2x^3+7))\\ &= ((x^2+6x+4,2x^3+2x+7)\end{aligned}$$

and we have

$$\begin{aligned}(f+g)_1(1) =&11 = (f+g)_2(1)\\ (f+g)'_1(1) =&8 = (f+g)'_2(1)\end{aligned}$$

so $(f+g)\in C^1(I)$ as well. Next, consider $p(x)$, f, and $p(x)f$ to look at scalar multiplication ($p(x)$ being the scalar):

$$\begin{aligned}p(x)f &= (x^3+3x+1)(x^2+1,2x)\\ &= ((x^3+3x+1)(x^2+1),(x^3+3x+1)(2x))\\ &= (x^5+4x^3+x^2+3x+1,2x^4+6x^2+2x)\end{aligned}$$

And we have:

$$\begin{aligned}p(x)f_1(1) =&10 = p(x)f_2(1)\\ p(x)f'_1(1) =&22 = p(x)f'_2(1)\end{aligned}$$

so $p(x)f\in C^1(I)$.

For $p(x)$, f, and g, we can see that

$$\begin{aligned}&p(x)(f+g) = (x^3+3x+1)(x^2+6x+4,2x^3+2x+7)\\ &\quad=((x^3+3x+1)(x^2+6x+4),(x^3+3x+1)(2x^3+2x+7))\\ &\quad=((x^3+3x+1)((x^2+1)+(6x+3)),(x^3+3x+1)((2x)+(2x^3+7)))\\ &\quad=((x^3+3x+1)(x^2+1)+(x^3+3x+1)(6x+3),\\ &\qquad(x^3+3x+1)(2x)+(x^3+3x+1)(2x^3+7))\\ &\quad=((x^3+3x+1)(x^2+1),\\ &\qquad(x^3+3x+1)(2x))+((x^3+3x+1)(6x+3),(x^3+3x+1)(2x^3+7))\\ &\quad=(x^3+3x+)(x^2+1,2x)+(x^3+3x+1)(6x+3,2x^3+7)\\ &\quad=p(x)f+p(x)g\end{aligned}$$

and for $p(x),q(x)$, and f:

$$\begin{aligned}(p(x)+q(x))f &= (x^3+3x+1+3x^2+4x-6)f\\ &= x^3f+3xf+f+3x^2f+4xf-6f\\ &= (x^3+3x+1)f+(3x^2+4x-6)f\\ &= p(x)f+q(x)f\\ &\text{and}\\ (p(x)q(x))f &= p(x)(q(x)f) \text{ simply by multiplicative associativity.}\end{aligned}$$

The additive inverses of f and g would be $-f = (-x^2 - 1, -2x)$ and $-g = (-6x - 3, -2x^3 - 7)$, and the multiplicative identity would be $(1, 1)$.

Next, let's look at arbitrarily selected $f, g \in C^1(I)$ and $p, q \in \mathbb{R}[x]$, and let a_1 be an arbitrary boundary point on the partition of intervalsl I instead of having $a_1 = 1$. Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$.
Since $f, g \in C^1(I)$ they satisfy

$$f_1(a_1) = f_2(a_1)\ ,\ g_1(a_1) = g_2(a_1)$$
$$f_1'(a_1) = f_2'(a_1)\ ,\ g_1'(a_1) = g_2'(a_1)$$

First, we will look at $(f + g) = (f_1 + g_1, f_2 + g_2)$:

$$(f_1 + g_1)(a_1) = f_1(a_1) + g_1(a_1) = f_2(a_1) + g_2(a_1) = (f_2 + g_2)(a_1)$$
$$\text{and}$$
$$\begin{aligned}(f_1 + g_1)'(a_1) &= (f_1' + g_1')(a_1)\\ &= f_1'(a_1) + g_1'(a_1) = f_2'(a_1) + g_2'(a_1)\\ &= (f_2' + g_2')(a_1) = (f_2 + g_2)'(a_1)\end{aligned}$$

so $(f + g) \in C^1(I)$ and $C^1(I)$ is closed under addition.

For scalar multiplication, consider $pf = (pf_1, pf_2)$:

$$pf_1(a_1) = p(f_1(a_1)) = p(f_2(a_1)) = pf_2(a_1)$$
$$\text{and}$$
$$pf_1'(a_1) = p(f_1'(a_1)) = p(f_2'(a_1)) = pf_2'(a_1)$$

so $pf \in C^1(I)$ and $C^1(I)$ is closed under scalar multiplication.

With p, f, and g we have:

$$\begin{aligned}p(f + g) &= p(f_1 + g_1, f_2 + g_2)\\ &= (p(f_1 + g_1), p(f_2 + g_2))\\ &= (pf_1 + pg_1, pf_2 + pg_2)\\ &= (pf_1, pf_2) + (pg_1, pg_2)\\ &= p(f_1, f_2) + p(g_1, g_2)\\ &= pf + pg\end{aligned}$$

and with p, q, and f:

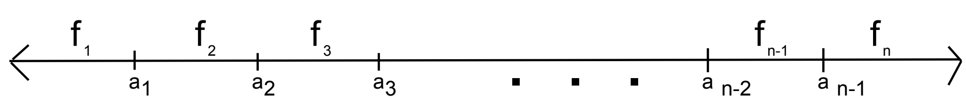
$$\begin{aligned}
(p+q)f &= (p+q)(f_1, f_2) \\
&= ((p+q)f_1, (p+q)f_2) \\
&= (pf_1 + qf_1, pf_2 + qf_2) \\
&= (pf_1, pf_2) + (qf_1, qf_2) \\
&= p(f_1, f_2) + q(f_1, f_2) \\
&= pf + qf \\
&\text{and} \\
(pq)f &= (pq)(f_1, f_2) \\
&= ((pq)f_1, (pq)f_2) \\
&= (p(qf_1), p(qf_2)) \\
&= p(qf_1, qf_2) \\
&= p(qf)
\end{aligned}$$

Thus, the elements of $\mathbb{R}[x]$ are distributive and associative with the elements of $C^1(I)$.

For the characteristics of an abelian group, we know that it is closed from above (showed $f, g \in C^r(I) \Rightarrow (f+g) \in C^r(I)$), we know the elements are associative and commutative since polynomial addition is associative and commutative, the identity element is $(1,1)$, and for all $f = (f_1, f_2) \in C^r(I)$ the inverse element is $-f = (-f_1, -f_2)$.

This all proves that $C^1(I)$ is a module over $\mathbb{R}[x]$.

As stated earlier, when considering $f \in C^r(I)$ where I is a partition of intervals such that $I = (-\infty, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cup \cdots \cup (a_{n-1}, \infty)$, we write $f = (f_1, f_2, f_3, \ldots, f_n)$ where f_i $(1 \leq i \leq n)$ is the polynomial defined over the ith interval of the partition (see picture below):



and we know that

$$
\begin{aligned}
f_i(a_i) &= f_{i+1}(a_i)\\
f_i'(a_i) &= f_{i+1}'(a_i)\\
f_i''(a_i) &= f_{i+1}''(a_i)\\
&\vdots\\
f_i^{(r-1)}(a_i) &= f_{i+1}^{(r-1)}(a_i)\\
f_i^{(r)}(a_i) &= f_{i+1}^{(r)}(a_i)
\end{aligned}
$$

for all i such that $1 \leq i \leq n-1$.

In her paper [2] Lindsey Scoppetta provides a proof showing, that $C^r(I)$ is a finitely generated module over $\mathbb{R}[x]$, and then characterizes a basis for the module.

Definition: A module M is said to be *finitely generated* if there exists a finite number of elements $s_1, s_2, s_3, \ldots, s_n$ in M such that $\{s_1, s_2, s_3, \ldots, s_n\}$ generate M.

This simply means that this set of elements can generate the rest of the elements of M by means of scalar multiplication with the elements of the ring along with addition and subtraction.

If one can find a generating set $S = \{s_1, s_2, s_3, \ldots, s_n\}$ of M, S might form a basis for the module M. A basis for a module M is simply a generating set that is linearly independent, and here we provide definitions for linear independence and bases.

Definition: A set of module elements $s_1, s_2, \ldots, s_n \in M$ is said to be linearly independent if there is no combination of non zero elements $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$ such that

$$\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n = 0.$$

If there exists such a combination of non-zero ring elements such that the above condition holds, the set is not linearly independent, and is said to be linearly dependent.

Definition: A basis for a module M is a set (denoted B) of linearly independent elements of M (denoted $\{b_1, b_2, b_3, \ldots, b_n\}$) such that B generates M.

Basically, if $B = \{b_1, b_2, \ldots, b_n\}$ is a basis for a module M, then $b_1, b_2, \ldots, b_n$ are linearly independent and there exist some $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$ such that

$$\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n = f$$

for any $f \in M$.

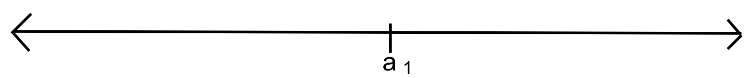
In her paper [1], Lauren Rose proves that all modules of single variable splines have bases.

1.4 Boundary Splines and the Module $C^{(r,p)}(I^\delta)$

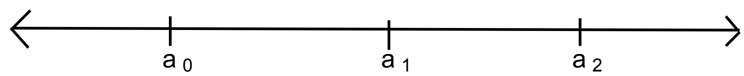
In this section, I will introduce and describe boundary splines. A boundary spline is simply a regular spline, except over the first and last intervals of the partition (I^δ) the spline is defined to be 0 and there is a minor change in the way the smoothness conditions are applied.

To construct I^δ, we look at the two infinite regions of any partition I, and we split them into two parts with new boundary points a_0 and a_n. The boundary spline defined over I^δ will be defined to be zero over the two new infinite regions of the partition. Also, the smoothness conditions at the two new boundary points may be different.

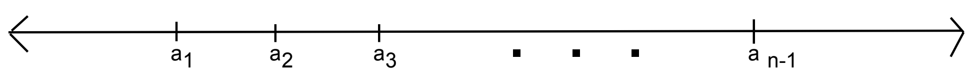
For example, first consider the partition $I = (-\infty, a_1) \cup (a_1, \infty)$ as shown below:



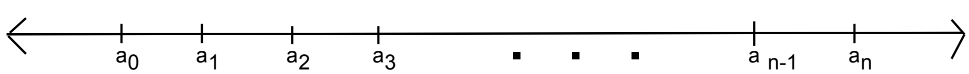
The corresponding I^δ is written $I^\delta = (-\infty, a_0) \cup (a_0, a_1) \cup (a_1, a_2) \cup (a_2, \infty)$ and will look like this:



A boundary spline defined over I^δ will be defined as 0 over $(-\infty, a_0)$ and (a_2, ∞). For a partition with an arbitrary amount of intervals $I = (-\infty, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cup \cdots \cup (a_{n-2}, a_{n-1}) \cup (a_{n-1}, \infty)$



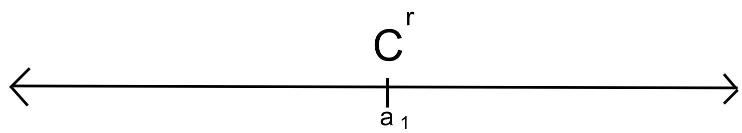
The corresponding I^δ looks like this:



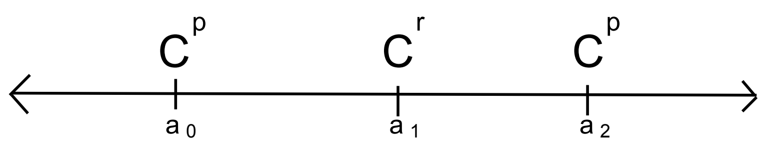
and a boundary spline defined over I^δ will be defined as 0 over $(-\infty, a_0)$ and (a_n, ∞). Also, instead requiring the splines to be C^r over I^δ, we require them to be $C^{(r,p)}$ over I^δ.

Definition: A $C^{(r,p)}$ spline over I^δ is a spline that has r-continuous derivatives (is C^r) at all interior boundary points a_i $(1 \le i \le n-1)$ and p-continuous derivatives (is C^p) at the two exterior boundary points a_0 and a_n $(p \le r)$.

For example, over the partition of intervals $I = (-\infty, a_1) \cup (a_1, \infty)$ we originally required a spline to be C^r at a_1.



A boundary spline defined over the corresponding I^δ has to be C^r at a_1, but it must also be C^p at a_0 and a_2 $(p \le r)$. So, it will have smoothness conditions that looks like this:



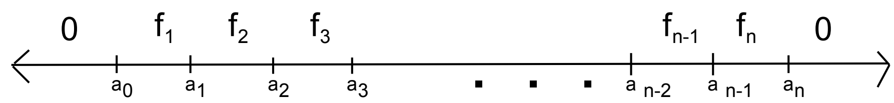
Thus, for the spline $f = (f_1, f_2)$ defined over I^δ, we will have

$$
\begin{array}{ccc}
f_1(a_0) = 0 & f_1(a_1) = f_2(a_1) & f_2(a_2) = 0 \\
f_1'(a_0) = 0 & f_1'(a_1) = f_2'(a_1) & f_2'(a_2) = 0 \\
\vdots & \vdots & \vdots \\
f_1^{(p)}(a_0) = 0 & f_1^{(p)}(a_1) = f_2^{(p)}(a_1) & f_2^{(p)}(a_2) = 0 \\
 & \vdots & \\
 & f_1^{(r-1)}(a_1) = f_2^{(r-1)}(a_1) & \\
 & f_1^{(r)}(a_1) = f_2^{(r)}(a_1). &
\end{array}
$$

We still write the splines as an n-tuple, so a spline $f \in C^{(r,p)}(I^\delta)$ for $I^\delta = (-\infty, a_0) \cup (a_0, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cup \cdots \cup (a_{n-2}, a_{n-1}) \cup (a_{n-1}, a_n) \cup (a_n, \infty)$

$$
f(x) = \begin{cases}
0 & -\infty < x \le a_0 \\
f_1 & a_0 < x \le a_1 \\
f_2 & a_1 < x \le a_2 \\
\vdots & \\
f_{n-1} & a_{n-2} < x \le a_{n-1} \\
f_n & a_{n-1} < x \le a_n \\
0 & a_n < x \le \infty
\end{cases}
$$

will be written as $f = (f_1, f_2, f_3, \ldots, f_{n-1}, f_n)$ and will look like this:



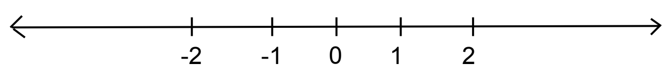
And for the arbitrary case where $f = (f_1, f_2, f_3, \ldots, f_{n-1}, f_n)$ and $I^\delta = (-\infty, a_0) \cup (a_0, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cup \cdots \cup (a_{n-2}, a_{n-1}) \cup (a_{n-1}, a_n) \cup (a_n, \infty)$ we will have

$$\begin{array}{ccc}
f_1(a_0) = 0 & f_{i-1}(a_1) = f_i(a_1) & f_n(a_n) = 0 \\
f_1'(a_0) = 0 & f_{i-1}'(a_1) = f_i'(a_1) & f_n'(a_n) = 0 \\
\vdots & \vdots & \vdots \\
f_1^{(p)}(a_0) = 0 & f_{i-1}^{(p)}(a_1) = f_i^{(p)}(a_1) & f_n^{(p)}(a_n) = 0 \\
 & \vdots & \\
 & f_{i-1}^{(r-1)}(a_1) = f_i^{(r-1)}(a_1) & \\
 & f_{i-1}^{(r)}(a_1) = f_i^{(r)}(a_1) &
\end{array}$$

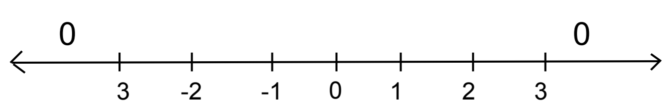
where $2 \leq i \leq n-1$.

1.5 Important Summer Results

During our workshop last summer at Bard College, my group worked to characterize bases for the module $C^{(r,p)}(I^\delta)$. An interesting part about bases for $C^r(I)$ and $C^{(r,p)}(I^\delta)$ (also proved in [1]) is that for $C^r(I)$ bases will have the same number of elements as the partition I has intervals and bases for $C^{(r,p)}(I^\delta)$ will have the same number of elements as the partition I^δ has interior intervals. Thus, a basis for $C^r(I)$ or $C^{(r,p)}(I^\delta)$ where

$I =$ 

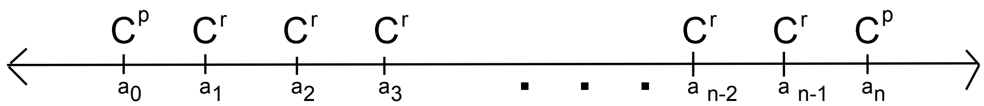
and

$I^\delta =$ 

will have 6 elements ($B = \{b_1, b_2, b_3, b_4, b_5, b_6\}$).

In order to understand my group's results as well as how we proved them, we must define the polynomial $Q \in \mathbb{R}[x]$. However, first it is necessary to define the Algebraic Condition for Continuity that holds true for all modules $C^r(I)$ and $C^{(r,p)}(I^\delta)$.

As stated before, a spline $f \in C^{(r,p)}(I^\delta)$ defined over a partition I^δ has to be p-times smooth at the exterior boundary points a_0 and a_n and r-times smooth at all interior boundary points $a_1, a_2, \ldots, a_{n-1}$.



In order to determine whether or not this is true, the following algebraic condition can be checked. Since we are working over a single-dimensional partition I^δ, we can define a linear polynomial that represents a vertical line at all of the boundary points a_i (where $0 \leq i \leq n$) as $(x - a_i)$. For simplicity, we let $l_i = (x - a_i)$.

Theorem 1: Let I be a partition with $n-1$ intervals and let $f = (f_1, f_2, f_3, \ldots, f_n)$ be a spline defined over I. Then, f is C^r if and only if

$$l_i^{r+1} \mid f_i - f_{i+1}$$

for all i such that $0 \leq i \leq n$.

For proof, see Theorem 3.0.18 of [2].

This Theorem, with a couple of small additions, holds true for $C^{(r,p)}$ splines, and we get:

For any given I^δ, $f = (f_1, f_2, f_3, \ldots, f_n)$ is $C^{(r,p)}$ if and only if

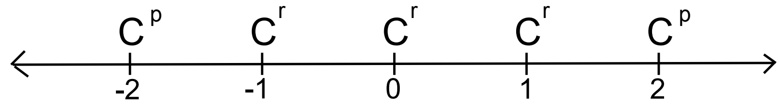
$$\begin{aligned} l_0^{p+1} &\mid 0 - f_1 = f_1 \\ l_i^{r+1} &\mid f_i - f_{i+1} \\ l_n^{p+1} &\mid f_n - 0 = f_n \end{aligned}$$

for all i such that $0 \leq i \leq n-1$. Now, we can define the polynomial Q.
(The notation $l_i^{r+1} \mid f_i - f_{i+1}$ simply means that l_i^{r+1} divides $f_i - f_{i+1}$, or more simply, $l_i^{r+1} = \alpha(f_i - f_{i+1})$ for some $\alpha \in R$ where R is the polynomial ring $\mathbb{R}[x]$).

Definition: Let I^δ be given. The polynomial Q is defined as the product

$$Q = l_0^{p+1} \left(\prod_{i=0}^{n-1} l_i^{r+1} \right) l_n^{p+1}$$

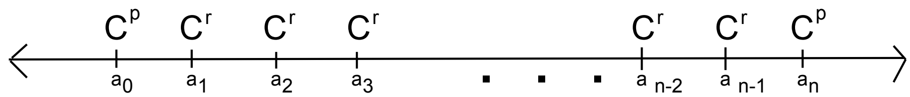
For example, consider $I^\delta = (-\infty, -2) \cup (-2, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, \infty)$ so that a $C^{(r,p)}$ spline over I^δ has the following smoothness conditions:



The polynomial Q for the module $C^{(r,p)}(I^{\delta})$ will then look like this:

$$Q = (x+2)^{p+1}(x+1)^{r+1}(x)^{r+1}(x-1)^{r+1}(x-2)^{p+1}$$

For I^{δ} with $n+1$ intervals, $(I = (-\infty, a_0) \cup (a_0, a_1) \cup (a_1, a_2) \cup, \ldots, \cup (a_{n-2}, a_{n-1}) \cup (a_{n-1}, a_n) \cup (a_n, \infty))$ the smoothness conditions will look like



and the polynomial Q for $C^{(r,p)}(I^{\delta})$ will look like this:

$$Q = (x-a_0)^{p+1}(x-a_1)^{r+1}(x-a_2)^{r+1} \ldots (x-a_{n-2})^{r+1}(x-a_{n-1})^{r+1}(x-a_n)^{p+1}.$$

Next, we must introduce determinants. A determinant is a value associated with a square $(n \times n)$ matrix which can be computed from the entries of the matrix by a specific arithmetic expression. When taking the determinant of a basis B for a module M we write the determinant as

$$\det[b_1, b_2, \ldots, b_n]$$

where the row vector b_i represents the ith column of the matrix form of B and $\det[b_1, b_2, \ldots, b_n] \in \mathbb{R}[x]$. I will go over how to compute the determinants we use a little later in this section.

We can now introduce the following proposition, a proof of which can be found on page 10 of my group's paper.

Proposition: A linearly independent set $B = \{b_1, b_2, b_3, \ldots, b_n\}$ with $b_1, b_2, b_3, \ldots b_n \in C^{(r,p)}(I^{\delta})$ forms a basis of $C^{(r,p)}(I^{\delta})$ if and only if

$$Q \mid \det[b_1, b_2, b_3, \ldots, b_n].$$

So consider $C^{(r,p)}(I^{\delta})$ where $I^{\delta} = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$. Then, the linearly independent set $B = \{b_1, b_2\}$ where $b_1, b_2 \in C^{(r,p)}(I^{\delta})$ is a basis for $C^{(r,p)}(I^{\delta})$ if and only if

$$Q = (x+1)^{p+1}(x)^{r+1}(x-1)^{p+1} \mid \det[b_1, b_2]$$

and where $I^\delta = (-\infty, a_0) \cup (a_0, a_1) \cup (a_1, a_2) \cup, \ldots, \cup (a_{n-1}, a_n) \cup (a_n, \infty)$, the linearly independent set $B = \{b_1, b_2, \ldots, b_n\}$ where $b_1, b_2, \ldots, b_n \in C^{(r,p)}(I^\delta)$ forms a basis for $C^{(r,p)}(I^\delta)$ if and only if

$$Q = (x-a_0)^{p+1}(x-a_1)^{r+1} \ldots (x-a_{n-1})^{r+1}(x-a_n)^{p+1} \mid \det[b_1, b_2, \ldots, b_n].$$

Using this Proposition, my group was able to prove the following Theorem about $C^{(r,0)}(I^\delta)$:

Theorem 2: Let I^δ be given, let

$$g_0(a_i) = (x-a_0)\big((x-a_n)x^r - (x-a_i)^{r+1}\big)$$

and let

$$g_n(a_i) = (x-a_n)\big((x-a_0)x^r - (x-a_i)^{r+1}\big).$$

Then the vectors

$$\begin{aligned}
b_1 &= \{g_0(a_1), g_n(a_1), ..., g_n(a_1)\} \\
b_2 &= \{g_0(a_2), g_0(a_2), g_n(a_2), ..., g_n(a_2)\} \\
&\vdots \\
b_i &= \{g_0(a_i), ..., g_0(a_i), g_n(a_i), ..., g_n(a_i)\} \\
&\vdots \\
b_{n-1} &= \{g_0(a_{n-1}), ..., g_0(a_{n-1}), g_n(a_{n-1})\} \\
b_n &= \{(x-a_0)(x-a_n), (x-a_0)(x-a_n), ..., (x-a_0)(x-a_n)\}
\end{aligned}$$

form a basis $B = \{b_1, b_2, ..., b_n\}$ for $C^{(r,0)}(I^\delta)$.

In order to understand how this basis works and what it ends up looking like, we will consider the simple cases of $C^{(0,0)}(I^\delta)$ and $C^{(1,0)}(I^\delta)$ where $I^\delta = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$.

We know that a basis for these modules will be a linear independent set of the form $B = \{b_1, b_2\}$. Also, we know that the polynomial Q for these modules will look like this:

$$Q = (x+1)^{p+1}(x)^{r+1}(x-1)^{p+1}.$$

So, for $C^{(0,0)}(I^\delta)$,

$$Q = x(x+1)(x-1)$$

and for $C^{(1,0)}(I^\delta)$,

$$Q = x^2(x+1)(x-1).$$

From the Theorem, we know that for $C^{(0,0)}(I^\delta)$

$$\begin{aligned}
g_0(a_i) &= (x+1)((x-1)x^0 - (x)) \\
&= (x+1)((x-1)1 - (x)) \\
&= (x+1)(x-1-x) \\
&= (x+1)(-1) \\
&= -(x+1) \\
&\text{and} \\
g_n(a_i) &= (x-1)((x+1)x^0 - (x)) \\
&= (x-1)((x+1)(1) - (x)) \\
&= (x-1)(x+1-x) \\
&= (x-1)(1) \\
&= (x-1).
\end{aligned}$$

So we have
$b_1 = (g_0(a_1), g_n(a_1)) = (-(x+1), (x-1))$, $b_2 = ((x+1)(x-1), (x+1)(x-1))$
and

$$B = \{(-(x+1), (x-1)), ((x+1)(x-1), (x+1)(x-1))\}.$$

For $C^{(1,0)}(I^\delta)$

$$\begin{aligned}
g_0(a_i) &= (x+1)((x-1)x^1 - (x)^2) \\
&= (x+1)((x-1)x - x^2) \\
&= (x+1)(x^2 - x - x^2) \\
&= (x+1)(-x) \\
&= -x(x+1) \\
&\text{and} \\
g_n(a_i) &= (x-1)((x+1)x^1 - (x)^2) \\
&= (x-1)((x+1)x - x^2) \\
&= (x-1)(x^2 + x - x^2) \\
&= (x-1)(x) \\
&= x(x-1).
\end{aligned}$$

So $b_1 = (g_0(a_1), g_n(a_1)) = (-x(x+1), x(x-1))$, $b_2 = ((x+1)(x-1), (x+1)(x-1))$
and

$$B = \{((-x(x+1), x(x-1)), ((x+1)(x-1), (x+1)(x-1))\}.$$

To show these are bases, we need only prove that $Q \mid \det[b_1, b_2]$.

For these scenarios, it is easy to compute $\det[b_1, b_2]$. First, put $B = \{b_1, b_2\}$ into matrix form, so for $C^{(0,0)}(I^\delta)$,

$$B = \begin{pmatrix} -(x+1) & (x+1)(x-1) \\ x-1 & (x+1)(x-1) \end{pmatrix}.$$

To take the determinant of a 2×2 matrix (which is the only type of determinant we will take in the entirety of this paper) where matrix A looks like

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and a, b, c, and d are elements of the ring R that the module M is defined over, we do

$$\det(A) = ad - bc.$$

Thus, the determinant of B is simply

$$\det(B) = -(x+1)(x+1)(x-1) - (x-1)(x+1)(x-1).$$

Now, we can manipulate $\det(B)$ as follows:

$$\begin{aligned}
&-(x+1)(x+1)(x-1) - (x-1)(x+1)(x-1) \\
&= -(x^2+2x+1)(x-1) - (x^2-2x+1)(x+1) \\
&= -(x^3+2x^2+x-x^2-2x-1) - (x^3-2x^2+x+x^2-2x+1) \\
&= -x^3-2x^2-x+x^2+2x+1-x^3+2x^2-x-x^2+2x-1 \\
&= -2x^3+2x \\
&= -2x(x^2-1) \\
&= -2x(x+1)(x-1)
\end{aligned}$$

and since $Q = x(x+1)(x-1)$ we get

$$\det(B)/Q = -2.$$

So it is clear that $Q \mid \det(B)$, and B is a basis.

Similarly for $C^{(1,0)}(I^\delta)$, we put B into matrix form to get

$$B = \begin{pmatrix} -x(x+1) & (x+1)(x-1) \\ x(x-1) & (x+1)(x-1) \end{pmatrix}.$$

Then compute $\det(B)$:

$$\det(B) = -x(x+1)(x+1)(x-1) - x(x-1)(x+1)(x-1)$$

and by virtually the same method, manipulate $\det(B)$ to get

$$\det(B) = -2x^2(x+1)(x-1).$$

Since $Q = x^2(x+1)(x-1)$, we have

$$\det(B)/Q = -2$$

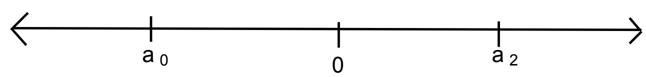
so $Q \mid \det(B)$, and B is a basis for $C^{(1,0)}(I^\delta)$.

For the remainder of this paper, the only I^δ used was such that had 2 interior intervals ($I^\delta = (-\infty, a_0) \cup (a_0, a_1) \cup (a_1, a_2) \cup (a_2, \infty)$).
Thus, the bases considered are all of the form $B = \{b_1, b_2\}$ for the module $C^{(r,0)}(I^\delta)$.
Also, without loss of generality, one can assume that one of the $a_i' s$ in any partition of intervals I^δ is equal to 0. For the partition described above, we will assume that $a_1 = 0$. Thus, we will be working with the partition of intervals $I^\delta = (-\infty, a_0) \cup (a_0, 0) \cup (0, a_2) \cup (a_2, \infty)$.

For the basis described in the Theorem above, it was possible to characterize what the scalars from $\mathbb{R}[x]$ looked like as well as provide another basis that can be used to generate bases for the module $C^{(r,0)}(I^\delta)$.

2 Scalar Characterization

We now have the partition of intervals I^δ that looks like this:



where a_0 and a_2 will either be given or represent arbitrary constants.
Thus, we know that any spline $f \in C^{(r,0)}(I^\delta)$ must be of the form $f = (F_1, F_2) = ((x-a_0)f_1(x), (x-a_2)f_2(x))$ for any $f_1(x), f_2(x) \in \mathbb{R}[x]$ such that:

$$\begin{aligned} F_1(0) &= F_2(0) \\ F_1'(0) &= F_2'(0) \\ F_1''(0) &= F_2''(0) \\ &\vdots \\ F_1^{(r-1)}(0) &= F_2^{(r-1)}(0) \\ F_1^{(r)}(0) &= F_2^{(r)}(0). \end{aligned}$$

Also, for any $f \in C^{(r,0)}(I^\delta)$, we know for any basis $B = \{b_1, b_2\}$ the following is true for the scalars $p(x), q(x) \in \mathbb{R}[x]$:

$$p(x)b_1 + q(x)b_2 = f = (F_1, F_2) = ((x-a_0)f_1(x), (x-a_2)f_2(x)).$$

With this, is it possible to completely characterize what $p(x)$ and $q(x)$ look like for the basis of $C^{(r,0)}(I^\delta)$ characterized during my group's summer research.

For a basic example, let's consider $C^{(0,0)}(I^\delta)$ where $I^\delta = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$. We know that a basis for this module is

$$B = \{(-(x+1), (x-1)), ((x+1)(x-1), (x+1)(x-1))\}.$$

So, we know for any $f \in C^{(0,0)}(I^\delta)$, we will have $f = (F_1, F_2) = ((x+1)f_1(x), (x-1)f_2(x))$ for any $f_1(x), f_2(x) \in \mathbb{R}[x]$ where

$$F_1(0) = F_2(0)$$

and the equation

$$p(x)\begin{pmatrix} -(x+1) \\ (x-1) \end{pmatrix} + q(x)\begin{pmatrix} (x-1)(x+1) \\ (x-1)(x+1) \end{pmatrix} = \begin{pmatrix} (x+1)f_1(x) \\ (x-1)f_2(x) \end{pmatrix}$$

will have a solution for $p(x), q(x) \in \mathbb{R}[x]$. This equation generates the system of two equations with two unknowns:

$$\begin{aligned} -p(x)(x+1) + q(x)(x-1)(x+1) &= (x+1)f_1(x) \\ p(x)(x-1) + q(x)(x-1)(x+1) &= (x-1)f_2(x) \end{aligned}$$

which can be simplified to

$$-p(x) + q(x)(x-1) = f_1(x) \tag{1}$$
$$p(x) + q(x)(x+1) = f_2(x). \tag{2}$$

Add equation (1) to equation (2) in order to solve for $q(x)$ and get

$$\begin{aligned} q(x)(x-1) + q(x)(x+1) &= f_1(x) + f_2(x) \\ q(x)((x-1) + (x+1)) &= f_1(x) + f_2(x) \\ q(x)(x-1+x+1) &= f_1(x) + f_2(x) \\ q(x)(2x) &= f_1(x) + f_2(x) \\ q(x) &= \frac{f_1(x) + f_2(x)}{2x} \end{aligned}$$

Since $F_1(0) = F_2(0)$, we have

$$\begin{aligned} (0+1)f_1(0) &= (0-1)f_2(x) \\ f_1(0) &= -f_2(0) \\ f_1(0) + f_2(0) &= 0 \end{aligned}$$

which shows there is no constant term in the numerator of $q(x)$. So, the x in the denominator of $q(x)$ will cancel, showing $q(x) \in \mathbb{R}[x]$.

Substitute $q(x)$ into equation (2) to get

$$\begin{aligned}
p(x) + \left(\frac{f_1(x)+f_2(x)}{2x}\right)(x+1) &= f_2(x)\\
p(x) + \frac{f_1(x)(x+1)+f_2(x)(x+1)}{2x} &= f_2(x)\\
p(x) &= f_2(x) - \left(\frac{f_1(x)(x+1)+f_2(x)(x+1)}{2x}\right)\\
p(x) &= \frac{f_2(x)(2x)-f_1(x)(x+1)-f_1(x)(x+1)}{2x}\\
p(x) &= \frac{f_2(x)(2x-x-1)-f_1(x)(x+1)}{2x}\\
p(x) &= \frac{f_2(x)(x-1)-f_1(x)(x+1)}{2x}
\end{aligned}$$

Again, since $F_1(0) = F_2(0)$, we get

$$\begin{aligned}
F_1(0) - F_2(0) &= 0\\
\Rightarrow F_2(0) - F_1(0) &= 0\\
f_2(0)(0-1) - f_1(0)(0+1) &= 0
\end{aligned}$$

showing that there is no constant term in the numerator of $p(x)$, which is enough to show $p(x) \in \mathbb{R}[x]$. So we have

$$p(x) = \frac{f_2(x)(x-1)-f_1(x)(x+1)}{2x}$$

and

$$q(x) = \frac{f_1(x)+f_2(x)}{2x}$$

To see how these work, consider the spline $f = (x+1, x^2-2x+1) \in C^{(0,0)}(I^\delta)$. $F_1 = x+1$ and $F_2 = x^2-2x+1 = (x-1)(x-1)$,
so $f_1(x) = 1,\ f_2(x) = x-1$ and

$$\begin{aligned}
p(x) &= \frac{(x-1)(x-1)-(1)(x+1)}{2x}\\
&= \frac{x^2-2x+1-x-1}{2x}\\
&= \frac{x^2-3x}{2x}\\
&= \frac{x-3}{2}
\end{aligned}$$

So $p(x) = \left(\frac{1}{2}\right)x - \frac{3}{2} \in \mathbb{R}[x]$ and

$$\begin{aligned}
q(x) &= \frac{1+x-1}{2x}\\
&= \frac{x}{2x}\\
&= \frac{1}{2}
\end{aligned}$$

so $q(x) = \frac{1}{2} \in \mathbb{R}[x]$.
So, for $f_1(x)$ and $f_2(x)$ we can see how the $2x$ in the denominator of $p(x)$ and $q(x)$ cancels out, and we have $p(x), q(x) \in \mathbb{R}[x]$.

For a more general example, consider $C^{(0,0)}(I^\delta)$ with the arbitrary $I^\delta = (-\infty, a_0) \cup (a_0, 0) \cup (0, a_2) \cup (a_2, \infty)$. Using our basis characterization, we know that

$$B = \{(-a_2(x - a_0), -a_0(x - a_2)), ((x - a_0)(x - a_2), (x - a_0)(x - a_2))\}$$

is a basis for $C^{(r,0)}(I^\delta)$, and the equation for $p(x), q(x) \in \mathbb{R}[x]$ and $f = (F_1, F_2) = ((x - a_0)f_1(x), (x - a_2)f_2(x)) \in C^{(r,p)}(I^\delta)$ will be

$$p(x)\begin{pmatrix} -a_2(x+1) \\ -a_0(x-1) \end{pmatrix} + q(x)\begin{pmatrix} (x-a_0)(x+a_2) \\ (x-a_0)(x+a_2) \end{pmatrix} = \begin{pmatrix} (x-a_0)f_1(x) \\ (x-a_2)f_2(x) \end{pmatrix}$$

This generates the system of equations

$$\begin{aligned} -p(x)(a_2)(x - a_0) + q(x)(x - a_0)(x - a_2) &= (x - a_0)f_1(x) \\ -p(x)(a_0)(x - a_2) + q(x)(x - a_0)(x - a_2) &= (x - a_2)f_2(x) \end{aligned}$$

which simplifies to

$$-p(x)(a_2) + q(x)(x - a_2) = f_1(x) \tag{3}$$
$$-p(x)(a_0) + q(x)(x - a_0) = f_2(x). \tag{4}$$

Solve equation (3) for $p(x)$ and simplify to get

$$p(x) = \frac{q(x)(x - a_2) - f_1(x)}{a_2} \tag{5}$$

Then, substitute equation (5) into equation (4) and have

$$\begin{aligned} -\left(\frac{q(x)(x - a_2) - f_1(x)}{a_2}\right)(a_0) + q(x)(x - a_0) &= f_2(x) \\ -q(x)(x - a_2)(a_0) + f_1(x)(a_0) + q(x)(x - a_0)(a_2) &= f_2(x)(a_2) \\ q(x)((x - a_0)(a_2) - (x - a_2)(a_0)) &= f_2(x)(a_2) - f_1(x)(a_0) \\ -q(x)((a_0 - a_2)(x)) &= f_2(x)(a_2) - f_1(x)(a_0) \\ q(x) &= \frac{f_1(x)a_0 - f_2(x)a_2}{(a_0 - a_2)x} \end{aligned}$$

Since we have $F_1(0) = F_2(0)$, we know

$$\begin{aligned} f_1(0)(0 - a_0) &= f_2(0)(0 - a_2) \\ -a_0 f_1(0) &= -a_2 f_2(0) \\ -a_0 f_1(0) + a_2 f_2(0) &= 0 \\ f_1(0)a_0 - f_2(0)a_2 &= 0 \end{aligned}$$

So there is no constant term in the numerator of $q(x)$, and $q(x) \in \mathbb{R}[x]$.
Next, substitute $q(x)$ into equation (5) and solve for $p(x)$:

$$\begin{aligned}
p(x) &= \frac{\left(\left(\frac{f_1(x)(a_0)-f_2(x)(a_2)}{(a_0-a_2)(x)}\right)(x-a_2) - f_1(x)\right)}{(a_2)} \\
&= \frac{f_1(x)(a_0)(x-a_2) - f_2(x)(a_2)(x-a_2) - f_1(x)(a_0-a_2)(x)}{(a_0-a_2)(x)(a_2)} \\
&= \frac{f_1(x)\left((a_0)(x-a_2) - (a_0-a_2)(x)\right) - f_2(x)(a_2)(x-a_2)}{(a_0-a_2)(x)(a_2)} \\
&= \frac{f_1(x)(x-a_0)(a_2) - f_2(x)(a_2)(x-a_2)}{(a_0-a_2)(x)(a_2)} \\
p(x) &= \frac{f_1(x)(x-a_0) - f_2(x)(x-a_2)}{(a_0-a_2)x}.
\end{aligned}$$

Again, we know $F_1(0) = F_2(0)$, so

$$\begin{aligned}
f_1(0)(0-a_0) &= f_2(0)(0-a_2) \\
f_1(0)(0-a_0) - f_2(0)(0-a_2) &= 0
\end{aligned}$$

and there is no constant term in the numerator of $p(x)$, showing $p(x) \in \mathbb{R}[x]$.
So we end up with

$$\begin{aligned}
p(x) =& \frac{f_1(x)(x-a_0) - f_2(x)(x-a_2)}{(a_0-a_2)x} \\
&\text{and} \\
q(x) =& \frac{f_1(x)a_0 - f_2(x)a_2}{(a_0-a_2)x}.
\end{aligned}$$

If we look at $C^{(1,0)}(I^\delta)$ where I^δ is the same as in the previous example, we know

$$B = \{(-a_2x(x-a_0), -a_0x(x-a_2)), ((x-a_0)(x-a_2), (x-a_0)(x-a_2))\}$$

and

$$\begin{aligned}
F_1(0) &= F_2(0) \\
F_1'(0) &= F_2'(0).
\end{aligned}$$

We use a nearly identical method to show that

$$\begin{aligned}
p(x) =& \frac{f_1(x)(x-a_0) - f_2(x)(x-a_2)}{(a_0-a_2)x^2} \\
&\text{and} \\
q(x) =& \frac{f_1(x)a_0 - f_2(x)a_2}{(a_0-a_2)x}.
\end{aligned}$$

$F_1(0) = F_2(0) \Rightarrow f_1(0)a_0 - f_2(0)a_2 = 0$, so there is no constant term in the numerator of $q(x)$ and $q(x) \in \mathbb{R}[x]$.
Also, $F_1(0) = F_2(0) \Rightarrow f_1(0)(0-a_0) - f_2(0)(0-a_2) = 0$, so there is no constant term in the numerator of $p(x)$ and we know

$$\begin{aligned} F_1'(0) &= F_2'(0) \\ F_1'(0) - F_2'(0) &= 0 \\ (f_1(0)(0-a_0))' - (f_2(0)(0-a_2))' &= 0. \end{aligned}$$

So, there is no x term in the numerator of $p(x)$ and $p(x) \in \mathbb{R}[x]$.

This allows us to then prove the following Theorem:

Theorem 3: For $C^{(r,0)}(I^\delta)$ where $I^\delta = (-\infty, a_0) \cup (a_0, 0) \cup (0, a_2) \cup (a_2, \infty)$, the basis given by Theorem 2 is

$$B = \{(-a_2x^r(x-a_0), -a_0x^r(x-a_2)), ((x-a_0)(x-a_2), (x-a_0)(x-a_2))\}$$

and will have scalars of the form

$$p(x) = \frac{f_1(x)(x-a_0) - f_2(x)(x-a_2)}{(a_0-a_2)x^{r+1}}$$

and

$$q(x) = \frac{f_1(x)a_0 - f_2(x)a_2}{(a_0-a_2)x}.$$

Proof:
Since $f = (F_1, F_2) \in C^{(r,0)}(I^\delta)$, we know

$$\begin{aligned} F_1(0) &= F_2(0) \\ F_1'(0) &= F_2'(0) \\ F_1''(0) &= F_2''(0) \\ &\vdots \\ F_1^{(r-1)}(0) &= F_2^{(r-1)}(0) \\ F_1^{(r)}(0) &= F_2^{(r)}(0). \end{aligned}$$

Start with the equation

$$p(x)\begin{pmatrix} -a_2x^r(x-a_0) \\ -a_0x^r(x-a_2) \end{pmatrix} + q(x)\begin{pmatrix} (x-a_0)(x-a_2) \\ (x-a_0)(x-a_2) \end{pmatrix} = \begin{pmatrix} (x-a_0)f_1(x) \\ (x-a_2)f_2(x) \end{pmatrix}$$

which generates the system of equations

$$-p(x)a_2x^r + q(x)(x-a_2) = f_1(x) \tag{6}$$
$$-p(x)a_0x^r + q(x)(x-a_0) = f_2(x) \tag{7}$$

Solve equation (6) for $p(x)$ and get

$$p(x) = \frac{q(x)(x-a_2) - f_1(x)}{a_2x^r} \tag{8}$$

Now, substitute equation (8) into equation (6) and we have

$$-\left(\frac{q(x)(x-a_2) - f_1(x)}{a_2x^r}\right) a_0x^r + q(x)(x-a_0) = f_2(x).$$

The x^r's cancel, so we are left with

$$\left(\frac{q(x)(x-a_2) - f_1(x)}{a_2}\right) a_0 + q(x)(x-a_0) = f_2(x)$$

and then by the exact same steps that were shown earlier in the section we end up with

$$q(x) = \frac{f_1(x)a_0 - f_2(x)a_2}{(a_0-a_2)x}.$$

Now, we substitute $q(x)$ into equation (8) and get

$$\begin{aligned}
p(x) &= \frac{\left(\frac{f_1(x)a_0 - f_2(x)a_2}{(a_0-a_2)x}\right)(x-a_2) - f_1(x)}{a_2x^r} \\
&= \frac{f_1(x)(x-a_2)a_0 - f_1(x)(a_0-a_2)x - f_2(x)(x-a_2)a_2}{(a_0-a_2)(a_2)x^{r+1}} \\
&= \frac{f_1(x)((x-a_2)a_0 - (a_0-a_2)x) - f_2(x)(x-a_2)a_2}{(a_0-a_2)(a_2)x^{r+1}} \\
&= \frac{f_1(x)(a_2(x-a_0)) - f_2(x)(x-a_2)a_2}{(a_0-a_2)(a_2)x^{r+1}} \\
p(x) &= \frac{f_1(x)(x-a_0) - f_2(x)(x-a_2)}{(a_0-a_2)x^{r+1}}
\end{aligned}$$

$F_1(0) = F_2(0) \Rightarrow f_1(0)a_0 - f_2(0)a_2 = 0$, so there is no constant term in the numerator of $q(x)$ and $q(x) \in \mathbb{R}[x]$.
Also,

$$\begin{aligned}
F_1(0) = F_2(0) &\Rightarrow f_1(0)(0-a_0) - f_2(0)(0-a_2) = 0 \\
F_1'(0) = F_2'(0) &\Rightarrow (f_1(0)(0-a_0))' - (f_2(0)(0-a_2))' = 0 \\
F_1''(0) = F_2''(0) &\Rightarrow (f_1(0)(0-a_0))'' - (f_2(0)(0-a_2))'' = 0 \\
&\vdots \\
F_1^{(r-1)}(0) = F_2^{(r-1)}(0) &\Rightarrow (f_1(0)(0-a_0))^{(r-1)} - (f_2(0)(0-a_2))^{(r-1)} = 0 \\
F_1^{(r)}(0) = F_2^{(r)}(0) &\Rightarrow (f_1(0)(0-a_0))^{(r)} - (f_2(0)(0-a_2))^{(r)} = 0.
\end{aligned}$$

This shows that there are no constant, $x, x^2, \ldots, x^{r-1}$, or x^r terms in the numerator of $p(x)$, which shows $p(x) \in \mathbb{R}[x]$, proving the Theorem.

3 The Generating Bases

Let's go back to our basis for $C^{(0,0)}(I^\delta)$ where $I^\delta = (-\infty,-1)\cup(-1,0)\cup(0,1)\cup(1,\infty)$. We have

$$B = \{b_1, b_2\} = \left\{ \begin{pmatrix} -(x+1) \\ x-1 \end{pmatrix}, \begin{pmatrix} (x+1)(x-1) \\ (x+1)(x-1) \end{pmatrix} \right\}$$

Since $b_1, b_2 \in C^{(0,0)}(I^\delta)$, if $G = \{G_1, G_2\}$ is another basis for $C^{(0,0)}(I^\delta)$ then it can generate both b_1 and b_2.
So, there must exist polynomials $\alpha, \beta, \gamma, \mu \in \mathbb{R}$ s.t.

$$b_1 = \alpha G_1 + \beta G_2 \tag{9}$$
$$b_2 = \gamma G_1 + \mu G_2. \tag{10}$$

It must be such that $G_1, G_2 \in C^{(0,0)}(I^\delta)$ in order for G to be a basis, so we know that $G_1 = (g_1(x+1), g_2(x-1))$ and $G_2 = (g_3(x+1), g_4(x-1))$ such that $g_1(0) = g_2(0)$ and $g_3(0) = g_4(0)$ for some $g_1, g_2, g_3, g_4 \in \mathbb{R}[x]$. Thus, equations (9) and (10) can be written as

$$\begin{pmatrix} -(x+1) \\ x-1 \end{pmatrix} = \alpha \begin{pmatrix} g_1(x+1) \\ g_2(x-1) \end{pmatrix} + \beta \begin{pmatrix} g_3(x+1) \\ g_4(x-1) \end{pmatrix}$$

and

$$\begin{pmatrix} (x+1)(x-1) \\ (x+1)(x-1) \end{pmatrix} = \gamma \begin{pmatrix} g_1(x+1) \\ g_2(x-1) \end{pmatrix} + \mu \begin{pmatrix} g_3(x+1) \\ g_4(x-1) \end{pmatrix}.$$

These generate the system of 4 equations with 4 unknowns:

$$\begin{aligned} -(x+1) &= \alpha g_1(x+1) + \beta g_3(x+1) \\ x-1 &= \alpha g_2(x-1) + \beta g_4(x-1) \\ (x+1)(x-1) &= \gamma g_1(x+1) + \mu g_2(x+1) \\ (x+1)(x-1) &= \gamma g_3(x-1) + \mu g_4(x-1) \end{aligned}$$

that can be simplified to

$$-1 = \alpha g_1 + \beta g_3 \tag{11}$$
$$1 = \alpha g_2 + \beta g_4 \tag{12}$$
$$(x-1) = \gamma g_1 + \mu g_3 \tag{13}$$
$$(x+1) = \gamma g_2 + \mu g_4. \tag{14}$$

To solve this, start by multiplying equation (11) by γ and equation (13) by $-\alpha$ to get

$$\begin{aligned} -\gamma &= \alpha\gamma g_1 + \beta\gamma g_3 \\ -\alpha(x-1) &= -\alpha\gamma g_1 - \alpha\mu g_3. \end{aligned}$$

Add the two equations together and get

$$\begin{aligned}
-\gamma - \alpha(x-1) &= \beta\gamma g_3 - \alpha\mu g_3 \\
-(\gamma + \alpha(x-1)) &= (\beta\gamma - \alpha\mu) g_3 \\
g_3 &= \frac{-(\gamma + \alpha(x-1))}{\beta\gamma - \alpha\mu}.
\end{aligned}$$

Next, multiply equation (12) by γ and equation (14) by $-\alpha$ and we have

$$\begin{aligned}
\gamma &= \alpha\gamma g_2 + \beta\gamma g_4 \\
-\alpha(x+1) &= -\alpha\gamma g_2 - \alpha\mu g_4
\end{aligned}$$

and add these together and solve for g_4:

$$\begin{aligned}
\gamma - \alpha(x+1) &= \beta\gamma g_4 - \alpha\mu g_4 \\
\gamma - \alpha(x+1) &= g_4(\beta\gamma - \alpha\mu) \\
g_4 &= \frac{\gamma - \alpha(x+1)}{\beta\gamma - \alpha\mu}.
\end{aligned}$$

Then, we multiply equation (11) by μ and equation (13) by β and get

$$\begin{aligned}
-\mu &= \alpha\mu g_1 + \beta\mu g_3 \\
-\beta(x-1) &= -\beta\gamma g_1 - \beta\mu g_3
\end{aligned}$$

and add them together and solve for g_1 to get

$$g_1 = \frac{-(\mu + \beta(x-1))}{\alpha\mu - \beta\gamma}.$$

Then, by looking at the pattern between g_1, g_3, and g_4 we can infer that

$$g_2 = \frac{\mu - \beta(x+1)}{\alpha\mu - \beta\gamma}.$$

This means that $G = \{G_1, G_2\}$ looks like

$$G = \left\{ \begin{pmatrix} (x+1)\left(\frac{-(\mu+\beta(x-1))}{\alpha\mu-\beta\gamma}\right) \\ (x-1)\left(\frac{\mu-\beta(x+1)}{\alpha\mu-\beta\gamma}\right) \end{pmatrix}, \begin{pmatrix} (x+1)\left(\frac{-(\gamma+\alpha(x-1))}{\beta\gamma-\alpha\mu}\right) \\ (x-1)\left(\frac{\gamma-\alpha(x+1)}{\beta\gamma-\alpha\mu}\right) \end{pmatrix} \right\}$$

where $\alpha\mu \neq \beta\gamma$, and since $g_1, g_2, g_3, g_4 \in \mathbb{R}[x]$, $\deg(\alpha\mu - \beta\gamma) = 0$. When $\alpha, \beta, \gamma, \mu$ satisty these conditions, The set $G = \{G_1, G_2\}$ will form a basis for $C^{(0,0)}(I^\delta)$.

As an example, consider the case where $\alpha = x-2, \beta = x-6, \gamma = x+2$, and $\mu = x-2$.
We have

$$\alpha\mu = (x-2)(x-2) = x^2 - 4x + 4 \neq x^2 - 4x - 12 = (x-6)(x+2) = \beta\gamma$$

and

$$\deg(\alpha\mu - \beta\gamma) = \deg(16) = 0$$

so the two conditions hold, and we end up with

$$G = \left\{ \begin{pmatrix} (x+1)\left(\frac{-x^2+6x-4}{16}\right) \\ (x-1)\left(\frac{-x^2+6x+4}{16}\right) \end{pmatrix}, \begin{pmatrix} (x+1)\left(\frac{x^2-2x+4}{16}\right) \\ (x-1)\left(\frac{x^2-2x-4}{16}\right) \end{pmatrix} \right\}.$$

It is easy to check and see that $G_1, G_2 \in C^{(0,0)}(I^\delta)$ and for $C^{(0,0)}(I^\delta)$, we know that $Q = x(x+1)(x-1)$.

So, to check that G is a basis, we compute $\det(G)$ and get

$$\begin{aligned}
\det(G) = &\left(\frac{1}{256}\right)(x+1)(-x^2+6x-4)(x-1)(x^2-2x-4) \\
&- \left(\frac{1}{256}\right)(x+1)(x^2-2x+4)(x-1)(-x^2+6x+4) \\
= &\left(\frac{1}{256}\right)(x+1)(x-1)((-x^2+6x-4)(x^2-2x-4) \\
&- (-x^2+6x+4)(x^2-2x+4)) \\
= &\left(\frac{1}{256}\right)(x+1)(x-1)(-32x) \\
= &\left(\frac{-1}{8}\right)(x)(x+1)(x-1).
\end{aligned}$$

Thus, $\det(G)/Q = -\frac{1}{8}$, G is a basis for $C^{(0,0)}(I^\delta)$ and with different α, β, γ, and μ's that satisfy the necessary conditions we can generate all bases for $C^{(0,0)}(I^\delta)$.

Now, let's look at $C^{(0,0)}(I^\delta)$ where $I^\delta = (-\infty, a_0) \cup (a_0, 0) \cup (0, a_2) \cup (a_2, \infty)$. We know that $B = \{b_1, b_2\}$ is

$$B = \left\{ \begin{pmatrix} -a_2(x-a_0) \\ -a_0(x-a_2) \end{pmatrix}, \begin{pmatrix} (x-a_0)(x-a_2) \\ (x-a_0)(x-a_2) \end{pmatrix} \right\}.$$

So, for $G = \{G_1, G_2\}$ where $G_1 = (g_1(x-a_0), g_2(x-a_2))$, $G_2 = (g_3(x-a_0), g_4(x-a_2))$ and $\alpha, \beta, \gamma, \mu \in \mathbb{R}[x]$ the equations

$$\begin{aligned}
b_1 &= \alpha G_1 + \beta G_2 \\
b_2 &= \gamma G_1 + \mu G_2
\end{aligned}$$

will generate the system of 4 equations

$$-a_2 = \alpha g_1 + \beta g_3 \tag{15}$$
$$-a_0 = \alpha g_2 + \beta g_4 \tag{16}$$
$$(x-a_2) = \gamma g_1 + \mu g_3 \tag{17}$$
$$(x-a_0) = \gamma g_2 + \mu g_4. \tag{18}$$

To solve, we go through the same method as before and start by multiplying equation (15) by γ and equation (17) by $-\alpha$ to get

$$\begin{aligned} -\gamma a_2 &= \alpha\gamma g_1 + \beta\gamma g_3 \\ -\alpha(x-a_2) &= -\alpha\gamma g_1 - \alpha\mu g_3. \end{aligned}$$

Add them together and solve for g_3 to get

$$g_3 = \frac{(\alpha-\gamma)a_2 - \alpha x}{\beta\gamma - \alpha\mu}.$$

Next, multiply equation (16) by γ and equation (18) by $-\alpha$ and we have

$$\begin{aligned} -\gamma a_0 &= \alpha\gamma g_2 + \beta\gamma g_4 \\ -\alpha(x-a_0) &= -\alpha\gamma g_2 - \alpha\mu g_4 \end{aligned}$$

which we add together and solve for g_4 to get

$$g_4 = \frac{(\alpha-\gamma)a_0 - \alpha x}{\beta\gamma - \alpha\mu}.$$

Then, multiply equation (15) by μ and equation (17) by $-\beta$ to get

$$\begin{aligned} -\mu a_2 &= \alpha\mu g_1 + \beta\mu g_3 \\ -\beta(x-a_2) &= -\beta\gamma g_1 - \beta\mu g_3 \end{aligned}$$

which we add together and solve for g_1 and have

$$g_1 = \frac{(\beta-\mu)a_2 - \beta x}{\alpha\mu - \beta\gamma}.$$

Finally, multiply equation (16) by μ and equation (18) by $-\beta$ to get

$$\begin{aligned} -\mu a_0 &= \alpha\mu g_2 + \beta\mu g_4 \\ -\beta(x-a_0) &= -\beta\gamma g_2 - \beta\mu g_4 \end{aligned}$$

which we add together and solve for g_2 to get

$$g_2 = \frac{(\beta-\mu)a_0 - \beta x}{\alpha\mu - \beta\gamma}.$$

Thus, we end up with

$$G = \left\{ \begin{pmatrix} (x-a_0)\left(\frac{(\beta-\mu)a_2-\beta x}{\alpha\mu-\beta\gamma}\right) \\ (x-a_2)\left(\frac{(\beta-\mu)a_0-\beta x}{\alpha\mu-\beta\gamma}\right) \end{pmatrix}, \begin{pmatrix} (x-a_0)\left(\frac{(\alpha-\gamma)a_2-\alpha x}{\beta\gamma-\alpha\mu}\right) \\ (x-a_2)\left(\frac{(\alpha-\gamma)a_0-\alpha x}{\beta\gamma-\alpha\mu}\right) \end{pmatrix} \right\}.$$

Once again, it is easy to check that $G_1, G_2 \in C^{(0,0)}(I^\delta)$, and we know that $\alpha\mu \neq \beta\gamma$ and $\deg(\alpha\mu - \beta\gamma) = 0$, so $\alpha\mu - \beta\gamma$ is a constant. Also, we know

$Q = x(x-a_0)(x-a_2)$ for $C^{(0,0)}(I^\delta)$.
Let $\alpha\mu - \beta\gamma = k \in \mathbb{R}$ and compute $\det(G)$:

$$\begin{aligned}
\det(G) &= (x-a_0)(x-a_2)(\frac{1}{k^2})((\beta-\mu)a_2 - \beta x)((\alpha-\gamma)a_0 - \alpha x) \\
&- (x-a_0)(x-a_2)(\frac{1}{k^2})((\beta-\mu)a_0 - \beta x)((\alpha-\gamma)a_2 - \alpha x) \\
&= (x-a_0)(x-a_2)(\frac{1}{k^2})((\beta-\mu)a_2 - \beta x)((\alpha-\gamma)a_0 - \alpha x) \\
&- ((\beta-\mu)a_0 - \beta x)((\alpha-\gamma)a_2 - \alpha x)) \\
&= (x-a_0)(x-a_2)(\frac{1}{k^2})(x(a_0-a_2)(-k)) \\
&= x(x-a_0)(x-a_2)(\frac{-1}{k})(a_0-a_2) \\
&= x(x-a_0)(x-a_2)\left(\frac{-(a_0-a_2)}{k}\right)
\end{aligned}$$

So $\det(G)/Q = \left(\frac{-(a_0-a_2)}{k}\right)$, G is a basis for $C^{(0,0)}(I^\delta)$ and with different $\alpha, \beta, \gamma, \mu \in \mathbb{R}[x]$ such that $\alpha\mu \neq \beta\gamma$ and $\deg(\alpha\mu - \beta\gamma) = 0$ it is possible to generate all bases for $C^{(0,0)}(I^\delta)$.

When considering $C^{(1,0)}(I^\delta)$ for $I^\delta = (-\infty, a_0) \cup (a_0, 0) \cup (0, a_2) \cup (a_2, \infty)$ we know that the basis from Theorem 2 is

$$B = \left\{ \begin{pmatrix} -a_2 x(x-a_0) \\ -a_0 x(x-a_2) \end{pmatrix}, \begin{pmatrix} (x-a_0)(x-a_2) \\ (x-a_0)(x-a_2) \end{pmatrix} \right\}$$

Thus, the equations

$$\begin{aligned}
b_1 &= \alpha G_1 + \beta G_2 \\
b_2 &= \gamma G_1 + \mu G_2
\end{aligned}$$

will generate the system

$$-a_2 x = \alpha g_1 + \beta g_3 \tag{19}$$
$$-a_0 x = \alpha g_2 + \beta g_4 \tag{20}$$
$$(x-a_2) = \gamma g_1 + \mu g_3 \tag{21}$$
$$(x-a_0) = \gamma g_2 + \mu g_4 \tag{22}$$

and then, by virtually the same method as in the previous two examples, we can show

$$G = \left\{ \begin{pmatrix} (x-a_0)\left(\frac{(\beta-\mu x)a_2 - \beta x}{\alpha\mu - \beta\gamma}\right) \\ (x-a_2)\left(\frac{(\beta-\mu x)a_0 - \beta x}{\alpha\mu - \beta\gamma}\right) \end{pmatrix}, \begin{pmatrix} (x-a_0)\left(\frac{(\alpha-\gamma x)a_2 - \alpha x}{\beta\gamma - \alpha\mu}\right) \\ (x-a_2)\left(\frac{(\alpha-\gamma x)a_0 - \alpha x}{\beta\gamma - \alpha\mu}\right) \end{pmatrix} \right\}.$$

Once again, it is simple to show that $G_1, G_2 \in C^{(1,0)}(I^\delta)$ (I will go through the steps to do so in the following proof) and, with $Q = x^2(x-a_0)(x-a_2)$ for

$C^{(1,0)}(I^{\delta})$, we can show that

$$\det(G)/Q = \frac{-(a_0 - a_2)}{k}$$

proving that G is a basis for $C^{(1,0)}(I^{\delta})$.

With this in mind, we can prove the following Theorem:

Theorem 4: For $C^{(r,0)}(I^{\delta})$ where $I^{\delta} = (-\infty, a_0) \cup (a_0, 0) \cup (0, a_2) \cup (a_2, \infty)$ and $\alpha, \beta, \gamma, \mu \in \mathbb{R}[x]$ such that $\alpha\mu \neq \beta\gamma$ and $\deg(\alpha\mu - \beta\gamma) = 0$, the set

$$G = \left\{ \begin{pmatrix} (x - a_0)\left(\frac{(\beta - \mu x^r)a_2 - \beta x}{\alpha\mu - \beta\gamma}\right) \\ (x - a_2)\left(\frac{(\beta - \mu x^r)a_0 - \beta x}{\alpha\mu - \beta\gamma}\right) \end{pmatrix}, \begin{pmatrix} (x - a_0)\left(\frac{(\alpha - \gamma x^r)a_2 - \alpha x}{\beta\gamma - \alpha\mu}\right) \\ (x - a_2)\left(\frac{(\alpha - \gamma x^r)a_0 - \alpha x}{\beta\gamma - \alpha\mu}\right) \end{pmatrix} \right\}$$

forms a basis for $C^{(r,0)}(I^{\delta})$. Furthermore, all bases will be generated in this way.

Proof:

We must first verify that $G_1, G_2 \in C^{(r,0)}(I^{\delta})$. Since $p = 0$, we need only check that $G_1(a_0) = 0$, $G_2(a_0) = 0$, $G_1(a_2) = 0$, and $G_2(a_2) = 0$ in order to verify that G_1 and G_2 are C^p at $x = a_0$ and $x = a_2$. Then, to prove G_1 and G_2 are C^r at $x = 0$, we use the Algebraic Condition for continuity to show that

$$(x - 0)^{r+1} = x^{r+1} \mid g_1(x - a_0) - g_2(x - a_2)$$
and
$$(x - 0)^{r+1} = x^{r+1} \mid g_3(x - a_0) - g_4(x - a_2)$$

It is clear that $g_1(x - a_0)$ and $g_3(x - a_0)$ both equal 0 at $x = a_0$ and $g_2(x - a_2)$ and $g_4(x - a_2)$ both equal 0 at $x = a_2$. Thus, it is also clear that G_1 and G_2 are C^0 at $x = a_0$ and $x = a_2$.

Since $\deg(\alpha\mu - \beta\gamma) = 0$, we let $\alpha\mu - \beta\gamma = k \in \mathbb{R}$, and we can compute

$$g_1(x - a_0) - g_2(x - a_2) = \left(\frac{-(a_2 - a_0)}{k}\right)\mu x^{r+1}$$

which is divisible by x^{r+1}, and

$$g_3(x - a_0) - g_4(x - a_2) = \left(\frac{-(a_2 - a_0)}{k}\right)\gamma x^{r+1}$$

which is also divisible by x^{r+1}. Therefore, G_1 and G_2 are C^r at $x = a_0$, and we know that $G_1, G_2 \in C^{(r,0)}(I^{\delta})$.

Now, to verify that this is a basis for $C^{(r,0)}(I^\delta)$ for all $\alpha, \beta, \gamma, \mu \in \mathbb{R}[x]$ such that $\alpha\mu \neq \beta\gamma$ and $\deg(\alpha\mu - \beta\gamma) = 0$, compute the determinant of G:

$$\begin{aligned}
\det(G) &= (x-a_0)(x-a_2)(\frac{1}{k^2})((\beta-\mu x^r)a_2 - \beta x)((\alpha-\gamma x^r)a_0 - \alpha x) \\
&- (x-a_0)(x-a_2)(\frac{1}{k^2})((\beta-\mu x^r)a_0 - \beta x)((\alpha-\gamma x^r)a_2 - \alpha x) \\
&= (x-a_0)(x-a_2)(\frac{1}{k^2})(((\beta-\mu x^r)a_2 - \beta x)((\alpha-\gamma x)a_0 - \alpha x) \\
&- ((\beta-\mu x^r)a_0 - \beta x)((\alpha-\gamma x^r)a_2 - \alpha x)) \\
&= (x-a_0)(x-a_2)(\frac{1}{k^2})((\beta-\mu x^r)(\alpha x)(a_0-a_2) - \beta x(\alpha-\gamma x^r)(a_0-a_2)) \\
&= (x-a_0)(x-a_2)(\frac{1}{k^2})(a_0-a_2)(\alpha\beta x - \alpha\mu x^{r+1} - \alpha\beta x + \beta\gamma x^{r+1}) \\
&= (x-a_0)(x-a_2)(\frac{1}{k^2})(a_0-a_2)(-k)(x^{r+1}) \\
&= x^{r+1}(x-a_0)(x-a_2)\left(\frac{-(a_0-a_2)}{k}\right)
\end{aligned}$$

We know that for $C^{(r,0)}(I^\delta)$, $Q = x^{r+1}(x-a_0)(x-a_2)$, so

$$\det(G)/Q = \left(\frac{-(a_0-a_2)}{k}\right).$$

Therefore, G is a basis for $C^{(r,0)}(I^\delta)$.

To prove the "furthermore" statement, start with our Theorem 2 basis $B = \{b_1, b_2\}$ for $C^{(r,0)}(I^\delta)$:

$$B = \left\{ \begin{pmatrix} -a_2x^r(x-a_0) \\ -a_0x^r(x-a_2) \end{pmatrix}, \begin{pmatrix} (x-a_0)(x-a_2) \\ (x-a_0)(x-a_2) \end{pmatrix} \right\}.$$

Let $G = \{G_1, G_2\}$ where $G_1 = ((g_1(x-a_0), g_2(x-a_2))$ and $G_2 = (g_3(x-a_0), g_4(x-a_2))$ be an arbitrarily selected basis for $C^{(r,0)}(I^\delta)$. Since $b_1, b_2 \in C^{(r,0)}(I^\delta)$, the equations

$$\begin{aligned}
b_1 &= \alpha G_1 + \beta G_2 \\
b_2 &= \gamma G_1 + \mu G_2
\end{aligned}$$

have a solution. These generate the system of equations

$$-a_2x^r = \alpha g_1 + \beta g_3 \tag{23}$$
$$-a_0x^r = \alpha g_2 + \beta g_4 \tag{24}$$
$$(x-a_2) = \gamma g_1 + \mu g_3 \tag{25}$$
$$(x-a_0) = \gamma g_2 + \mu g_4 \tag{26}$$

As before, multiply γ to equation (23) and $-\alpha$ to equation (25) and get

$$\begin{aligned}-a_2\gamma x^r &= \alpha\gamma g_1 + \beta\gamma g_3\\ -\alpha(x-a_2) &= -\alpha\gamma g_1 - \alpha\mu g_3\end{aligned}$$

and add them together and solve for g_3:

$$g_3 = \frac{(\alpha-\gamma x^r)a_2 - \alpha x}{\beta\gamma - \alpha\mu}.$$

Next, multiply γ to equation (24) and $-\alpha$ to equation (26) for

$$\begin{aligned}-a_0\gamma x^r &= \alpha\gamma g_2 + \beta\gamma g_4\\ -\alpha(x-a_0) &= -\alpha\gamma g_2 - \alpha\mu g_4\end{aligned}$$

and add them together and solve for g_4:

$$g_4 = \frac{(\alpha-\gamma x^r)a_0 - \alpha x}{\beta\gamma - \alpha\mu}.$$

Then, multiply equation (23) by μ and equation (25) by $-\beta$ to have

$$\begin{aligned}-a_2\mu x^r &= \alpha\mu g_1 + \beta\mu g_3\\ -\beta(x-a_2) &= -\beta\gamma g_1 - \beta\mu g_3\end{aligned}$$

and add them together and solve for g_1:

$$g_1 = \frac{(\beta-\mu x^r)a_2 - \beta x}{\alpha\mu - \beta\gamma}.$$

Finally, multiply μ to equation (24) and $-\beta$ to equation (26) for

$$\begin{aligned}-a_0\mu x^r &= \alpha\mu g_2 + \beta\mu g_4\\ -\beta(x-a_0) &= -\beta\gamma g_2 - \beta\mu g_4\end{aligned}$$

and add together and solve for g_2:

$$g_2 = \frac{(\beta-\mu x^r)a_0 - \beta x}{\alpha\mu - \beta\gamma}.$$

So we have

$$G = \left\{ \begin{pmatrix} (x-a_0)\left(\frac{(\beta-\mu x^r)a_2-\beta x}{\alpha\mu-\beta\gamma}\right) \\ (x-a_2)\left(\frac{(\beta-\mu x^r)a_0-\beta x}{\alpha\mu-\beta\gamma}\right)\end{pmatrix}, \begin{pmatrix} (x-a_0)\left(\frac{(\alpha-\gamma x^r)a_2-\alpha x}{\beta\gamma-\alpha\mu}\right) \\ (x-a_2)\left(\frac{(\alpha-\gamma x^r)a_0-\alpha x}{\beta\gamma-\alpha\mu}\right)\end{pmatrix} \right\}.$$

Since we derived the basis G from the elements of the basis $B = \{b_1, b_2\}$, we know that G can be used to characterize all bases for $C^{(r,0)}(I^\delta)$.

4 Works Cited

References

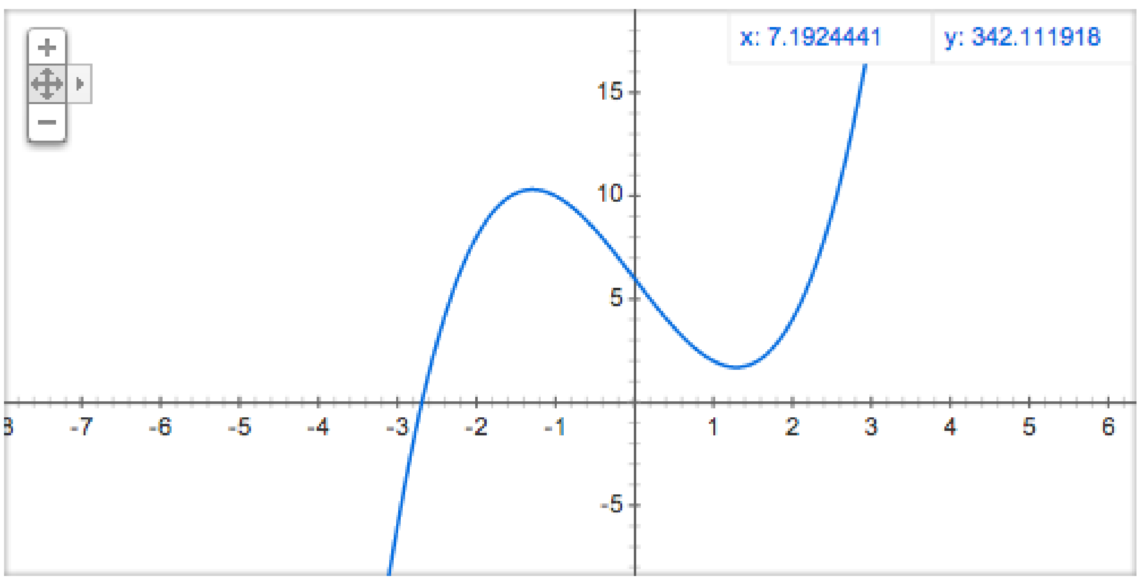
[1] Lauren Rose, *Combinatorial and Topological Invariants of Modules of Piecewise Polynomials*, Advances in Mathematics, 1995, 34-45.

[2] Lindsey Scoppetta, *Modules of Splines with Boundary Conditions*, (2012), Senior Thesis.

5 Summary

This project focuses on Module Bases for Splines with Boundary conditions. However what splines, modules, and bases are is unknown to most outside the world of mathematics. Thus, in order to explain what exactly I am doing with my project, I will start at the most basic level possible.

A function in mathematics is a relation between a set of inputs and a set of possible outputs with the characteristic that each input can have only one possible output. Most of the time, functions are viewed in the x-y-plane, which is simply a grid that has a horizontal (x) and vertical (y) axis. In this scenario, a function will typically take inputs from the x-axis, and return y-values. Which is why a function is often expressed as $f(x) = y$.
The x-value is plugged into f, f does something to x, and then returns a y.
So when viewed in a graph, the points that a function outputs have coordinates $(x, f(x))$. Meaning, the x-value represents how far to the left or right a given point will shift, whereas $f(x)$ represents how far up and down a point will shift. Frequently, functions can be executed for all values of x on the x-axis, so when viewed as a graph, a function is an infinite series of points that connect and appear as a curve. For example, pictured below is the graph of the function $f(x) = x^3 - 5x + 6$:

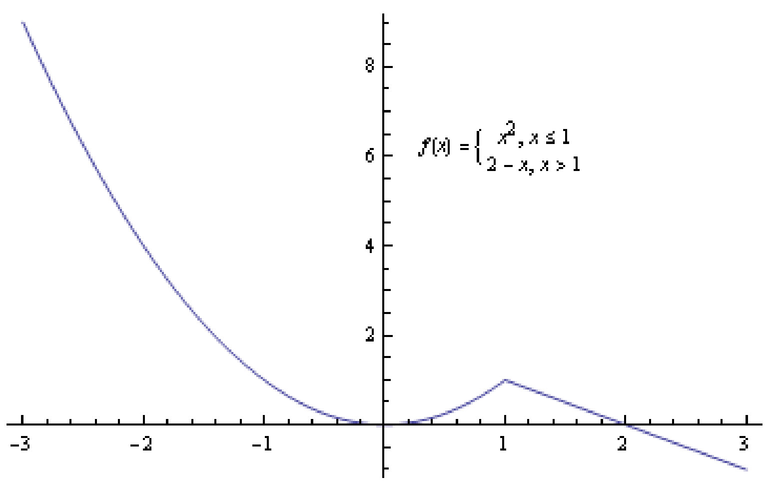


This function takes an x value from the x-axis, cubes it, subtracts 5 times the x-value, adds 6, and then moves however much the resulting value is on the y-axis, and places a point there. For example:

$$\text{When plugging in } -3\text{: } f(-3) = (-3)^3 - 5(-3) + 6 = (-27) + 15 + 6 = -6$$
$$\text{When plugging in } 0\text{: } f(0) = (0)^3 - 5(0) + 6 = 0 + 0 + 6 = 6$$
$$\text{and when plugging in } 2\text{: } f(2) = (2)^3 - 5(2) + 6 = 8 - 10 + 6 = 4$$

When doing this for every x-value, the result is the squiggly line (or curve) pictured above. Also, many different mathematical operations can be used to define functions other than polynomials ($x^3 - 3x + 6$ is a polynomial). For example $f(x) = cos(x)$ or $f(x) = |x|$ are functions.
My project focuses on sets of what are called splines. Splines are simply piecewise polynomial functions. Polynomial function means that the function must be defined as a polynomial (like $x^3 - 3x + 6$, $x^5 - 1$, $41x^5 + 32x^2 - 14x - 64$, etc.) and piecewise means defined over subdivisions of the x-axis. Thus, instead of the function being defined as one thing over all of the x-values, it will change based on where you are on the x-axis.

So consider the partition of intervals $I = (-\infty, 1) \cup (1, \infty)$ (which is simply splitting the x-axis into all numbers greater than 1 and all numbers less than 1) and the piecewise function shown below:



When x is less than 1 (in other words, x is in the interval $(-\infty, 1)$), the function used is $f(x) = x^2$, however, when x is greater than 1 (x is in the interval $(1, \infty)$), the function used is $f(x) = 2 - x$.
Also, it does not matter how many intervals are included in the partition. So, a spline defined over the partition $I = (-\infty, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cup \cdots \cup (a_{n-1}, a_n) \cup (a_n, \infty)$ would look like

$$f(x) = \begin{cases} f_1 & -\infty < x \leq a_1 \\ f_2 & a_1 < x \leq a_2 \\ f_3 & a_2 < x \leq a_3 \\ \vdots & \\ f_{n-1} & a_{n-2} < x \leq a_{n-1} \\ f_n & a_{n-1} < x < \infty \end{cases}$$

where $f_1, f_2, \ldots, f_{n-1}$, and f_n all represent polynomials. More examples of splines follow:

$$f(x) = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}$$

$$f(x) = \begin{cases} x & -\infty < x \leq -1 \\ x^2 & -1 < x \leq 5 \\ 5x+3 & 5 < x \leq 8 \\ 2x^2+5x+7 & 8 < x \leq \infty \end{cases}$$

$$f(x) = \begin{cases} -x^2 & -\infty < x \leq -5 \\ 3x & -5 < x \leq 0 \\ x^5+1 & 0 < x \leq 5 \\ 5 & 5 < x < \infty \end{cases}$$

For basic splines such as these, the y-values of the functions do not have to match up at the boundary points of the partition (i.e. $f_i(a_i) = f_{i+1}(a_i)$ does not have to be true for any a_i where a_i represents a boundary point of the partition of intervals, and $0 \leq i \leq n-1$.

My project focuses on splines that are said to be C^r over a given partition of intervals. The r is a whole number that is greater than or equal to 0 that represents a smoothness condition placed on the splines. For $r = 0$, the spline must simply match up on the boundary points of the partition (so $f_i(a_i) = f_{i+1}(a_i)$ for all i such that $0 \leq i \leq n-1$). For example, define the partition I to be $(-\infty, 0) \cup (0, \infty)$ and consider the spline:

$$f(x) = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}$$

Since the x-axis is only subdivided into 2 intervals, we need only consider what happens at $x = 0$. Since $-x$ and x are both 0 at $x = 0$, the spline is continuous, and fits the condition imposed by C^0. We can then say that $f(x) \in C^0(I)$ where I is the interval defined above.

For $r = 1$, the spline must have 1 degree of smoothness, meaning it must match up at the boundary points of the partition for the spline and its first derivative. So we will have

$$f_i(a_i) = f_{i+1}(a_i)$$
$$f_i'(a_i) = f_{i+1}'(a_i)$$

for all i such that $0 \leq i \leq n-1$ where $f'(a_i)$ denotes the derivative of the function at the boundary point a_i. The derivative of a polynomial function $f(x)$

is simply another function which graphs the slope (or rate of change) of every point of $f(x)$, and it is often written $f'(x)$. For example, for $f(x) = 3x + 4$, the slope of the function is 3 everywhere, so $f'(x) = 3$. For a function where the slope is not constant, the derivative is also a function of x. For example, when $f(x) = 3x^3 + 4x - 6$, the derivative is $f'(x) = 9x^2 + 4$.
Anyway, as r increases, so does the level of smoothness, and the number of derivatives the spline must remain connected on. So for $r = 2$ it must match up on 2 derivatives, for $r = 3$, has to match up on 3, etc.
A slightly more complex example is the following spline:

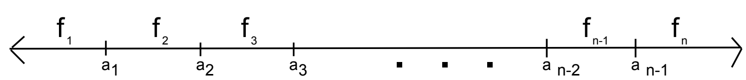
$$f(x) = \begin{cases} -2x^3 + 16x + 10 & -\infty < x \leq -1 \\ 2x^3 + 4x + 2 & -1 < x \leq 0 \\ x^3 + 4x + 2 & 0 < x \leq 3 \\ 9x^2 - 23x + 29 & 3 < x < \infty \end{cases}$$

Computing, we then have:

$$f_1(-1) = -4 = f_2(-1),\ f_2(0) = 2 = f_3(0),\ f_3(3) = 41 = f_4(3)$$
$$f_1'(-1) = 10 = f_2'(-1),\ f_2'(0) = 4 = f_3'(0),\ f_3'(3) = 31 = f_4'(3)$$
$$f_1''(-1) = 12 = f_2''(-1),\ f_2''(0) = 0 = f_3''(0),\ f_3''(3) = 18 = f_4''(3)$$
$$f_1'''(-1) = -12 \neq 12 = f_2'''(-1),\ f_2'''(0) = 12 \neq 6 = f_3'''(0),\ f_3'''(3) = 6 \neq 0 = f_4'''(3)$$

This spline matches up on 2 derivatives, and is considered C^2.

When looking at the set of splines that are considered C^r over a partition I with n intervals, it is written as $C^r(I)$ and for $f \in C^r(I)$ we write $f = (f_1, f_2, f_3, \ldots, f_n)$ where each f_i $(1 \leq i \leq n-1)$ is the polynomial function defined over the ith interval of the partition I (see picture below):



and we know that

$$f_i(a_i) = f_{i+1}(a_i)$$
$$f_i'(a_i) = f_{i+1}'(a_i)$$
$$f_i''(a_i) = f_{i+1}''(a_i)$$
$$\vdots$$
$$f_i^{(r-1)}(a_i) = f_{i+1}^{(r-1)}(a_i)$$
$$f_i^{(r)}(a_i) = f_{i+1}^{(r)}(a_i)$$

for all i such that $1 \leq i \leq n-1$. In her paper "Modules of Splines with Boundary Conditions," Lindsey Scoppetta provides a proof showing that $C^r(I)$ is a finitely generated module over the ring of polynomials with real number coefficients.

A ring R, in mathematics, is a set of numbers along with two binary operations (normally addition and multiplication) such that addition is associative and commutative (i.e. for all a, b, and $c \in R$, $(a+b)+c = a+(b+c)$ and $a+b = b+a$), there is an identity element such that $0+a = a+0 = a$ for all $a \in R$, for every $a \in R$ there is an inverse element $-a \in R$ such that $a+(-a) = (-a)+a = 0$, multiplication is also associative, and multiplication distributes over addition ($a*(b+c) = (a*b)+(a*c)$ and $(a+b)*c = (a*c)+(b*c)$).

A module M, in mathematics, is a set of tuples (like a spline where $f = (f_1, f_2, f_3, \ldots, f_n)$) such that the elements that make up the tuples come from a ring R, and scalar multiplication is defined in M where the scalars come from the ring R (scalar multiplication being when an element from the ring R is multiplied by an element of the module M). The scalars from the ring R are also distributive and associative with the elements of M, M is closed under addition and scalar multiplication (meaning if you add any number of elements of M together or multiply an element of M by a scalar from R then the result will still be an element of M), there is a multiplicative identity in M (so if the identity is 1, then for all $m \in M$, $1*m = m*1 = m$), addition for elements of M is associative and commutative, and there exists an additive identity and additive inverse elements.

A finitely generated module M is a module that has a set of elements $s_1, s_2, \ldots, s_n \in M$ such that the set can generate every element in the module by means of scalar multiplication and addition and subtraction amongst the elements.

An interesting characteristic about modules is that some have bases (and for future reference, all the modules worked with in my project have bases) and a basis is a generating set where all of the elements of the set are linearly independent.

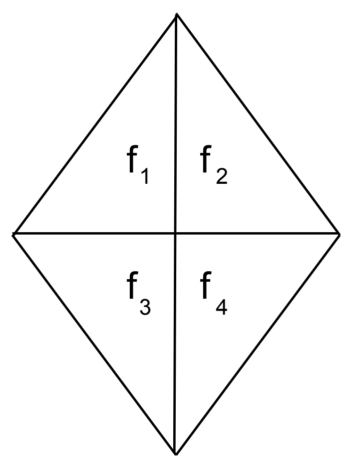
A set of module elements $s_1, s_2, \ldots, s_n \in M$ is said to be linearly independent if there is no combination of non zero ring elements $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$ such that

$$\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n = 0.$$

Over the summer, I did research with a group at Bard College and we focused mainly on the module $C^{(r,p)}(I^\delta)$ which is the same thing as $C^r(I)$, except every spline in $C^{(r,p)}(I^\delta)$ is defined to be 0 on the first and last interval in the partition ($(-\infty, a_0)$ and (a_n, ∞)) and the spline has to be p times differentiable at the boundary points a_0 and a_n and r times differentiable at all boundary points a_i in between (and $p \leq r$). During this time, my group was able to completely characterize a basis for the module $C^{(r,0)}(I^\delta)$.

When I decided to extend our research on this topic for my Capstone project, my advisor and I were able to completely characterize what the scalars would look like for the basis for $C^{(r,0)}(I^\delta)$ in the case where I^δ is partitioned into two finite intervals, as well as completely characterize all bases for $C^{(r,0)}(I^\delta)$ in the two interval case.

In the single dimension case (splines defined over subdivisions of the x-axis) characterizing their bases has many interesting abstract applications in many different areas of mathematics.
However, the main purpose of studying the one-dimensional case is to take what we find and try to apply it to splines defined over a region in $\mathbb{R}^2$.
In the same way a spline can be defined over a subdivided x-axis, they can be defined in 2-dimensions over a 2-dimensional region. For example, a spline can be defined over a region like this:



where f_1, f_2, f_3, and f_4 are polynomials of two variables (i.e. $f(x, y)$) and will create a 3-dimensional graph over the given region. Thus, instead of the f_n's being in the polynomial ring $\mathbb{R}[x]$, they are in $\mathbb{R}[x, y]$. The two dimensional region the functions are defined over can be any shape and have any number of subregions.

Characterizing bases in the 2-dimensional case would be incredibly helpful in the construction of three dimensional objects such as roofs of complex structures (houses, stadiums, obscurely shaped buildings, etc.) or even plane wings. This is because these complex structures can be difficult to characterize mathematically. However, if the area they are being constructed over is divided up into individual regions, it can greatly simplify characterization. Thus, characterizing bases for the 2-dimensional case would make it possible to mathematically generate equations for these structures.

Splines are also helpful in approximating general functions, interpolating data, and are being found to have applications in computer graphics, image processing and computer aided design.