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The Higgs sector on a two-sheeted space time

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We present a general formalism based on the framework of non-commutative geometry, suitable to the study the standard model of electroweak interactions, as well as that of more general gauge theories. Left- and right-handed chiral fields are assigned to two different sheets of space-time (a discretized version of Kaluza-Klein theory). Scalar Higgs fields find themselves treated on the same footing as the gauge fields, resulting in spontaneous symmetry breaking in a natural and predictable way. We first apply the formalism to the Standard Model, where one can predict the Higgs mass and the top Yukawa coupling. We then study the left-right symmetric model, where we show that this framework imposes constraints on the type and coefficients of terms appearing in the Higgs potential.

I. INTRODUCTION

The current picture of a continuum space-time at all scales underlying a smooth manifold has proved inadequate to describe all elementary particle interactions including gravity. The mathematical frameworks of both general relativity and quantum field theory of elementary particle interactions assume such a smooth manifold, and their incompatibility is one of the main signals for the inadequacy of such a continuum picture of space-time. In search of a more general framework, Connes has proposed an alternate approach based on non-commutative geometry (NCG) [1, 2]. The basic idea of Connes is to do away with

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the precise specification of an underlying manifold as the starting point. Instead, he formulates its description in terms of an associative and involutive algebra, commutative or non-commutative. One may think of this as a generalization of the well known theorem due to Gelfand, which states that the classical topological space based on a continuum can be completely recovered from the Abelian algebra of smooth functions.

Connes' ideas have been explored in several directions. Of particular relevance is their application to the standard model and beyond. In spite of the great success of the standard model in confronting experiments, it is recognized that it is far from a fundamental theory. One of the main drawbacks is that the mechanism of spontaneous symmetry breaking depends on the *ad-hoc* addition of a Higgs sector, with a somewhat arbitrary Higgs field content and an arbitrarily chosen potential. In contrast, Connes' approach has given rise to a geometrical description of a gauge field theory in which the Higgs fields finds themselves on the same footing as the gauge fields, and where the Higgs potential has spontaneous symmetry breaking built in it.

In recent years, there has been a great deal of work beginning with Connes and Lott ([3, 4, 5, 6] are some examples). However, the approaches used so far have tended to be highly mathematical. Our approach will be somewhat heuristic and not overly concerned with mathematical details. We intend to keep close to the familiar ideas of quantum field theories and Riemannian geometry and use Connes' rigorous non-commutative geometry as a guide to construct models with interesting features from the physical point of view.

In our approach, the underlying manifold on which the theory is defined consists of the direct product between the four-dimensional (4D) Minkowski space-time and a finite number of points. Note that this is only a particular (and rather simple) realization of the more general theory of Connes. Specifically, we consider the case when the dimension of this discrete space is two; hence the name two-sheeted space time. This framework allows a simple physical interpretation, as a model in which 4D fields live on two distinct branes embedded in a higher dimensional space; in other words, a discretized version of Kaluza-Klein theory [7, 8]. As such, one can think that this theory might be derived as the 4D limit of a more fundamental theory (perhaps a string one).

We envisage the fermion left- and right-chiral fields living on two separate sheets of space time associated with the two discrete points. The generalized gauge potential $\mathbb{A}$ (known as a connection in geometrical terms) appears as a two-by-two matrix operator on this internal space. The diagonal components couple fermion fields living on the same sheet (therefore with the same chirality) and consequently are identified with the usual 4D gauge fields. The off-diagonal components couple left with right-handed fermion fields, and therefore can be identified as standard Higgs fields. We show that it is possible to build a suitable curvature operator (the field strength $\mathbb{F}$), and construct a consistent gauge theory by using these extended gauge operators. The Higgs potential terms appear as intrinsic components of the gauge invariant action for the Yang-Mills fields $*\mathbb{F}\mathbb{F}$.

We apply these ideas by first formulating the Standard Model as a $SU_C(3) \times SU_L(2) \times U_Y(1)$ gauge theory on two sheeted space-time. Due to the simplicity of the Higgs sector in this case (one Higgs field suffices), one finds that it is possible to predict the top Yukawa coupling (~ 0.8, close to but not in exact agreement with the SM value), as well as the Higgs mass. However, we think that more interesting possibilities lie in the application of this formalism to theories which may be valid at higher energy scales, such as grand unified theories based on higher gauge groups. In such theories, generally the Higgs sector can be quite complex, and in the absence of a guiding principle one must rely on additional assumptions. The approach described in this paper can provide such a guide. As an example, we formulate the left-right symmetric model in this framework.

The paper is organized as follows. In Section II, we briefly review Connes' abstract algebra approach and the concept of a spectral triple. In Section III, we adapt this formalism to a two-sheeted space-time that may be considered as a discretized version of Kaluza-Klein theory, in which the compact circle in the fifth dimension is replaced by two discrete points. Section IV and V are devoted to the construction of standard model and the left-right symmetric model in this framework. The final Section is devoted to a discussion of the results and conclusions.

II. A BRIEF REVIEW OF NON-COMMUTATIVE DIFFERENTIAL GEOMETRY.

Connes'starting point of NCG is a universal differential algebra $\Omega^*(\mathcal{A})$ constructed from an associative, commuting or non-commuting algebra $\mathcal{A}$. It is unital and involutive. This differential algebra can be thought of as being generated by the elements a, b and a symbol δ with properties $\delta 1 = 0, \delta(ab) = (\delta a)b + a\delta b$ for all $a, b \in \mathcal{A}$. The zero order forms are simply the elements of $\mathcal{A}$. Higher order forms in $\Omega^p\mathcal{A}$ are generated by 'extending' the differential δ to an operator: if $a_0\delta a_1 \dots \delta a_p$ is a $p-$form, then

$$\delta(a_0\delta a_1 \dots \delta a_p) \;=\; \delta a_0 \delta a_1 \dots \delta a_p$$

is a $(p{+}1)-$form. This also implies $\delta^2 = 0$. For a more detailed discussion at the introductory level, see, for example, [9].

In physical applications of interest to us, the abstract algebra becomes an algebra of operators acting on a Hilbert space $\mathcal{H}$. The abstract elements $a \in \mathcal{A}$ are represented as operators through a representation $\rho(a) \equiv A$ acting on $\mathcal{H}$. The abstract symbol δ becomes the exterior derivative represented by a self-adjoint operator called the Dirac operator $\mathcal{D}$. The three elements, the algebra A, the Dirac operator $\mathcal{D}$ and the Hilbert space $\mathcal{H}$ together form a *spectral triple* according to Connes.

With the help of the Dirac operator, we can now extend the representation of $\mathcal{A}$ to its algebra of differential forms through the correspondence:

$$\rho(a_0\delta a_1 \dots \delta a_p) \;=\; \rho(a_0)[\mathcal{D}, \rho(a_1)] \dots [\mathcal{D}, \rho(a_p)] \; .$$

In this sense, the Dirac operator can be used to build a 'representation' of the formal exterior derivative δ in the space of operators acting on $\mathcal{H}$; if $\phi \in \Omega(\mathcal{A})$:

$$\rho(\delta\phi) \;=\; [\mathcal{D}, \rho(\phi)] \; .$$

(Depending on the rank of the form ϕ and the grading properties of the algebra, one may also need to use anticommutators.)

One must be careful, however; if δ is a proper exterior derivative, one needs to have $\delta(\delta\phi) = 0$; but generally $[\mathcal{D}, [\mathcal{D}, \rho(\phi)]]$ is not zero. To get around this problem, one defines an equivalence relation on the spaces $\rho(\Omega^p(\mathcal{A}))$, so that all the operators of the form $[\mathcal{D}, [\mathcal{D}, \rho(\phi)]]$ are equivalent to the zero operator; technically, the space of operators corresponding to $p-$forms is $\rho(\Omega^p(\mathcal{A}))$ divided by the space of so-called junk forms $\mathcal{J} = \rho(\delta(\ker\Omega^{p-1}(\mathcal{A}))$ (this is sometimes called dividing out the junk)[1].

In our studies, $\mathcal{H}$ will consist of square integrable sections of a spinor bundle representing physical states of the fermions. Let us assume that the fermion fields form a basis for a

[1] A slightly different definition of junk forms is used in the literature. However, for our purposes, the two definitions seem to give the same results (see [9]).

representation of some group G. For theories defined on a smooth manifold, the gauge fields will then be associated with differential one-forms defined on the manifold. The field strength, which is used in the construction of the Lagrangian, will be a two-form. Generally, physically relevant quantum operators built out of gauge fields are elements of the differential algebra $\Omega(M)$ defined on the manifold.

For the purposes of illustration, consider the simple case of a gauge theory on the Minkowski manifold. The relevant algebra is $C^\infty(M)$. The Hilbert space $\mathcal{H}$ is the space of Dirac spinors; and the Dirac operator is the standard $\mathcal{D} = \not{\partial} = \gamma^\mu \partial_\mu$. Elements of $C^\infty(M)$ are in one to one correspondence with fields on the four dimensional space-time. The space of one forms is defined by:

$$\mathbb{A} \; = \; f(x)[\not{\partial}, g(x)] \; = \; f(x)\partial_\mu(g)(x)\gamma^\mu \; \equiv \; A_\mu(x)\gamma^\mu \; ,$$

with $A_\mu(x) \in C^\infty(M)$[2]. Note that the decomposition of one forms this way is similar to the usual writing of one forms as $A_\mu(x)\mathrm{d}x^\mu$, only in this case the matrices γ^μ play the role of a basis in the space of one-forms. The field strength will then be (here we use the anticommutator):

$$\begin{aligned} \mathbb{F} \; &= \; \mathrm{d}\mathbb{A} \; = \; \{\not{\partial}, A_\mu(x)\gamma^\mu\} \; = \; \gamma^\nu\gamma^\mu\partial_\nu A_\mu + \gamma^\mu\gamma^\nu A_\mu \partial_\nu \\ &= \; \gamma^\nu\gamma^\mu\partial_\nu(A_\mu) + 2\eta^{\mu\nu}A_\mu\partial_\nu \\ &= \; \gamma^\nu\gamma^\mu\frac{1}{2}[\partial_\nu(A_\mu) - \partial_\mu(A_\nu)] \; . \end{aligned} \tag{1}$$

In going from the second to the last line, we have antisymmetrized the result with respect to indices μ, ν, and dropped the symmetric terms (the ones proportional to $\eta^{\mu\nu}$). This prescription is imposed by the requirement that if $\mathbb{A} = \mathrm{d}\Phi$, $\mathrm{d}\mathbb{A} = 0$. In other words, the equivalence relation necessary to get rid of the junk forms is: two forms are equivalent to each other if their difference is symmetric in the μ, ν indices, and proportional to the identity matrix. Alternately, it is equivalent to using the wedge product of γ matrices as a basis in the space of higher forms; $\mathbb{F}$ can then be written as

$$\mathbb{F} \; = \; \partial_\nu(A_\mu)\gamma^\nu \wedge \gamma^\mu \; ,$$

with $\gamma^\nu \wedge \gamma^\mu = (\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)/2$.

[2] Note that the functions appearing in these relations are to be thought of as operators. Then, for example, $\partial g(x) = \partial(g)(x) + g(x)\partial$

III. TWO-SHEETED SPACE-TIME.

In the previous section we have given a brief description of the general formalism of Connes to build noncommutative field theories. There are several realizations of this formalism in practice [3, 4, 5, 6]. In this section (and the rest of the paper) we will discuss a model, which we call two-sheeted space time [7, 8].

In our scenario, the spectral triple consists of:

Hilbert Space $\mathcal{H}$: The direct product of the usual Minkowski spin manifold times $\mathbb{C}^2$ ($\mathbb{C}$ being the algebra of complex numbers). Formally, it can be represented as

$$\mathbf{\Psi} = \begin{bmatrix} \Psi_L \\ \Psi_R \end{bmatrix},$$

where Ψ_L, Ψ_R are each a Dirac spinor by itself; however, we can choose that each such Dirac spinor has specific helicity (either left or right-handed) and that they live on different sheets.

Algebra $\mathcal{A}$: The most general (smooth) operator acting on such a Hilbert space is a 2×2 matrix with elements C^∞ functions. However, we choose the algebra $\mathcal{A}$ to be a subset of of these matrices; specifically we require that the representations of elements of $\mathcal{A}$ be diagonal matrices:

$$\rho(a) = \begin{bmatrix} f_1(x) & 0 \\ 0 & f_2(x) \end{bmatrix}. \tag{2}$$

Dirac Operator: If we take this also to be diagonal, the resulting theory would not be very interesting; the fields living on separate sheets will not interact with each other, and in effect we will have two copies of the standard theory on a manifold. Therefore, we take the Dirac operator to have off-diagonal terms:

$$\mathcal{D} = \begin{bmatrix} i\,\partial\!\!\!/ & -\gamma^5 M \\ \gamma^5 M^\dagger & i\,\partial\!\!\!/ \end{bmatrix}, \tag{3}$$

where M is a scalar operator with dimensions of mass. With this definition, a general one-form is given by

$$\mathbb{A} = \begin{bmatrix} \gamma^\mu A_{1\mu}(x) & -\gamma^5\Phi_1(x) \\ \gamma^5\Phi_2(x) & \gamma^\mu A_{2\mu}(x) \end{bmatrix}, \tag{4}$$

where $A_{i\mu}$ are the gauge fields, and Φ_i are the Higgs fields of the theory.

The model described above is easily extended to incorporate non-abelian gauge theories. In the general case, one can take the fields Ψ_L, Ψ_R as multiplets under a fundamental representation of some gauge groups G_1, G_2. Let us denote the dimension of the Ψ_L multiplet by n and the dimension of the Ψ_R multiplet by m. Then the C^∞ functions $f_1(x), A_{1\mu}(x)$, and $f_2(x), A_{2\mu}(x)$ will became $n \times n$ matrices, and $m \times m$, respectively. They can be written in terms of the generators of the corresponding Lie algebras T_a, T'_b:

$$\begin{aligned} f_1(x) &= f_1^a(x) T_a \quad A_{1\mu}(x) = A_{1\mu}^a(x) T_a \\ f_2(x) &= f_2^b(x) T'_b \quad A_{2\mu}(x) = A_{2\mu}^b(x) T'_b \ . \end{aligned} \tag{5}$$

To support the identification of the Φ fields in (4) as Higgs fields, consider the fermionic part of the Lagrangian. With the covariant derivative defined in the usual way,

$$i\mathbb{D} \ = \ \mathcal{D} + \mathbb{A} \ , \tag{6}$$

one has

$$\begin{aligned} \mathcal{L}_\psi &= \bar{\mathbf{\Psi}} \ i\mathbb{D} \ \mathbf{\Psi} \\ &= \bar{\psi}_L(i \ \partial\!\!\!/ + \ A\!\!\!/_1)\psi_L \ + \ \bar{\psi}_R(i \ \partial\!\!\!/ + \ A\!\!\!/_2)\psi_R \\ &\quad - \left(\ \bar{\psi}_L(\Phi_1 + M)\psi_R + \bar{\psi}_R(\Phi_2 + M^\dagger)\psi_L \ \right) \ . \end{aligned} \tag{7}$$

In order for this lagrangian to be hermitian, we define $\Phi_2 = \Phi_1^\dagger$.

The field strength is then

$$\mathbb{F} \ = \ \mathrm{d}\mathbb{A} \ + \ \mathbb{A} \wedge \mathbb{A} \ , \tag{8}$$

where

$$\begin{aligned} \mathrm{d}\mathbb{A} &= \{\mathcal{D}, \mathbb{A}\} \\ &= \begin{bmatrix} i \ \partial\!\!\!/(A\!\!\!/_1) - M\Phi^\dagger - \Phi M^\dagger & \gamma^\mu\gamma^5(-i\partial_\mu(\Phi) + MA_{2\mu} - A_{1\mu}M) \\ \gamma^5\gamma^\mu(-i\partial_\mu(\Phi^\dagger) + M^\dagger A_{1\mu} - A_{2\mu}M^\dagger) & i \ \partial\!\!\!/(A\!\!\!/_2) - M^\dagger\Phi - \Phi^\dagger M \end{bmatrix} \ , \end{aligned} \tag{9}$$

and

$$\begin{aligned} \mathbb{A} \wedge \mathbb{A} &= \frac{1}{2}\{\mathbb{A}, \mathbb{A}\} \\ &= \begin{bmatrix} A\!\!\!/_1 \ A\!\!\!/_1 - \Phi\Phi^\dagger & \gamma^\mu\gamma^5(-A_{1\mu}\Phi + \Phi A_{2\mu}) \\ \gamma^5\gamma^\mu(\Phi^\dagger A_{1\mu} - A_{2\mu}\Phi^\dagger) & A\!\!\!/_2 \ A\!\!\!/_2 - \Phi^\dagger\Phi \end{bmatrix} \ . \end{aligned} \tag{10}$$

The diagonal elements of the form $\mathbb{F}$ contain $(i/2)\gamma^\mu\gamma^\nu(F_{1,2})_{\mu\nu}$, with

$$F_{1\mu\nu} \;=\; \partial_\mu A_{1\nu} - \partial_\nu A_{1\mu} - i[A_{1\mu}, A_{1\nu}] \;,$$

and the equivalent for F_2. Thus one obtains the proper covariant expressions for the field strengths of the respective gauge fields. Furthermore, it is convenient to redefine the Higgs fields as $H = \Phi + M$; then the field strength (8) can be written as:

$$\mathbb{F} \;=\; \begin{bmatrix} \frac{i}{2}\gamma^\mu\gamma^\nu F_{1\mu\nu} + (MM^\dagger - HH^\dagger) & -\gamma^\mu\gamma^5(i\partial_\mu(H) - HA_{2\mu} + A_{1\mu}H) \\ -\gamma^5\gamma^\mu(i\partial_\mu(H^\dagger) - H^\dagger A_{1\mu} + A_{2\mu}H^\dagger) & \frac{i}{2}\gamma^\mu\gamma^\nu F_{2\mu\nu} + (M^\dagger M - H^\dagger H) \end{bmatrix} \;. \qquad (11)$$

Some care has to be exercised here in treating the contributions of the Higgs field to the diagonal elements. As mentioned in the previous section, if δ is a proper exterior derivative, $\delta(\delta\alpha) = 0$ for any form α. Thus, if one takes $\mathbb{A} = \mathrm{d}\rho$, with ρ an element of the algebra as in Eq. (2), we should have $d\mathbb{A} = dd\rho = 0$. However, from Eq. (9):

$$dd\rho \;=\; \begin{bmatrix} MM^\dagger f_1 - f_1 MM^\dagger & 0 \\ 0 & M^\dagger M f_2 - f_2 M^\dagger M \end{bmatrix} \;.$$

If $MM^\dagger$ is not proportional to the unity, the diagonal elements are not generally zero. We need a prescription to deal with such cases. A suitable prescription for our purposes, which gives rise to a gauge covariant Higgs potential, is to take the trace over the Higgs field contributions [3], that is, to replace in Eq. (11), $(MM^\dagger - HH^\dagger)$ by $\mathrm{Tr}(MM^\dagger - HH^\dagger)$. However, note that here the trace is understood to be taken only when it is required for the cancellation of such terms. For example, in the case when the fermion spinor space can be split into subspaces which do not mix (like lepton and quark subspaces in the Standard Model), one should take trace on each subspace separately.

The Lagrangian for the gauge sector (including the Higgs) will then be

$$\mathcal{L}_G \;=\; -\frac{1}{4}\,\mathrm{Tr}\;\mathbb{F}\mathbb{F}\;, \qquad (12)$$

where the trace is first taken over the γ^μ matrices and then over the internal symmetry indices. The $-1/4$ factor in front of the trace insures that the kinetic energy for the gauge

[3] One could also drop these contributions completely, in effect setting the curvature associated to the Higgs fields to zero. However, the model obtained in this case is not very interesting, since it does not have a Higgs potential.

fields F_1, F_2 has the standard normalization. Then, we find

$$\mathcal{L}_G \;=\; \mathcal{L}_A \;+\; \mathcal{L}_H \;-\; V(H) \;, \tag{13}$$

where

$$\mathcal{L}_A \;=\; -\frac{1}{4}\sum_i F_{i\mu\nu}F_i^{\mu\nu} \tag{14}$$

with

$$F_{1\mu\nu} \;=\; F^a_{1\mu\nu}T_a \;,\; F_{2\mu\nu} \;=\; F^b_{2\mu\nu}T'_b \;,$$

with normalization $\mathrm{Tr}(T_aT_{a'}) = (1/2)\delta_{aa'}$. The kinetic energy for the Higgs fields has the form

$$\mathcal{L}_H \;=\; |\partial^\mu(H) + iHA_2^\mu - iA_1^\mu H|^2 + |\partial_\mu(H^\dagger) + iH^\dagger A_{1\mu} - iA_{2\mu}H^\dagger|^2 \;, \tag{15}$$

where we use the definition $|K|^2 = K^\dagger K$. Finally, the Higgs potential is given by

$$V(H) \;=\; \left(\mathrm{Tr}(MM^\dagger - HH^\dagger)\right)^2 \;+\; \left(\mathrm{Tr}(M^\dagger M - H^\dagger M)\right)^2 \;, \tag{16}$$

which requires that at least some Higgs fields have non-zero expectation values, thus insuring spontaneous symmetry breaking.

Note that Eq.(15) contains the proper covariant derivative for the Higgs fields H. To see this, note that the transformation properties of the Higgs fields are defined by the requirement that the fermionic lagrangian (7) be gauge invariant. Thus, if the fermion fields Ψ_L form a multiplet supporting a fundamental representation of the gauge group G_1, so do the fields H. That is, the transformation properties of the Higgs fields under an infinitesimal gauge transformation will be the same as for the Ψ_L fermions:

$$\Psi_L(x) \rightarrow (1 + i\alpha^a T^a)\Psi_L \;\;,\;\; H(x) \rightarrow (1 + i\alpha^a T^a)H \;.$$

Then it is easy to verify that the derivative in Eq.(15) is the appropriate covariant derivative, and the theory is gauge invariant.

To review what we have learned so far: we have started by taking the elements of the algebra $\mathcal{A}$ to be block-diagonal matrices. However, due to the non-diagonal structure of the Dirac operator, the one-forms associated with the algebra $\mathcal{A}$ are also non-diagonal - that is, non-commutative. The diagonal elements of a one-form $\mathbb{A}$ are gauge fields, while the off-diagonal elements turn out to be Higgs fields. The resulting theory turns out to be gauge invariant (one has to redefine the Higgs fields first, though). Moreover, due to the noncommutativity of one-forms, the Higgs sector aquires a quartic potential, thus ensuring spontaneous symmetry breaking. The structure of the Higgs potential is determined by the structure of the off-diagonal terms in the Dirac operator (or, since these terms are related to the fermion masses, one can say that the Higgs potential is related to the mass spectrum of the fermion sector).

We end this section with some comments on the gauge couplings and the Higgs-fermion Yukawa couplins. In the above equations we set the gauge couplings to be one. Different values can be introduced either directly (that is, setting $i\mathbb{D} = \mathcal{D} + g\mathbb{A}$ in Eq. (6) and $\mathbb{F} = \mathrm{d}\mathbb{A} + g\mathbb{A}\wedge\mathbb{A}$) or by generalizing the trace over $\mathbb{F}$ to include a gauge coupling matrix:

$$\mathcal{L}_G = -\frac{1}{4}\,\mathrm{Tr}\begin{bmatrix} 1/g_1^2 & 0 \\ 0 & 1/g_2^2 \end{bmatrix} \mathbb{F}\mathbb{F}\ . \tag{17}$$

Then one should use fields with proper normalization $A_1/g_1 \to A_1, A_2/g_2 \to A_2$, in effect setting the gauge couplings of the fields A_1, A_2 to be g_1, g_2 (one should also normalize the Higgs fields). Also, note that in the case when we have only one Higgs multiplet and one fermion multiplet, the Yukawa coupling of the Higgs with the fermions is fixed, being given by the Higgs field normalization constant. In the case when one has several fermion multiplets, one generally can (or has to) set the Yukawa coupling for each multiplet by hand; however, one still gets a sum rule relating the sum of the Yukawa couplings squared to the Higgs normalization constant. If one has several Higgs multiplets, one gets several such sum rules.

IV. STANDARD MODEL ON TWO-SHEETED SPACE TIME

As the first example of the general formalism described in the previous section, let us discuss the implementation of the Standard Model $SU_C(3)\times SU_L(2)\times U_Y(1)$ gauge theory. For simplicity, let us first restrict our analysis to a subspace of the fermion spinor space, particularly the one spanned by the u, d quark fields. Then, the components of the spinor multiplets would be

$$\Psi_L = \begin{bmatrix} u_L \\ d_L \end{bmatrix}\ ,\ \Psi_R = \begin{bmatrix} u_R \\ d_R \end{bmatrix}\ .$$

As is well known, the Ψ_L components form a $SU_L(2)$ doublet, while the individual components of both Ψ_L and Ψ_R are $U_Y(1)$ singlets.

Then, over this subspace, the gauge fields components are

$$
\begin{aligned}
A_{1\mu} &= A^a_{1\mu}\tau_a + \frac{1}{2} Y_{QL} B_\mu \;, \\
A_{2\mu} &= \frac{1}{2} Y_{QR} B_\mu \;,
\end{aligned}
\tag{18}
$$

where A^a are the $SU_L(2)$ gauge fields and B is the hypercharge field. The diagonal matrices Y_{QL}, Y_{QR} contain the hypercharges of the left and right-type quark fields:

$$
Y_{QL} = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;, \; Y_{QR} = \begin{bmatrix} 4/3 & 0 \\ 0 & -2/3 \end{bmatrix} \;.
$$

Note here the peculiar way we have introduced the hypercharge field. The matrix A_1 now contains fields associated with two gauge symmetries. This would generally require that A_1 be split into two parts: $A_1 \to \mathrm{diag}(A_1, B_1)$, but this is not necessary in this particular case, since the hypercharge matrix Y_{QL} is proportional to the identity matrix, and therefore commutes with the generators of the $SU_L(2)$ gauge transformations. This means that the $SU_L(2)$ and $U_Y(1)$ gauge fields will not mix between them when computing $\mathbb{F}$ and the trace of $\mathbb{F}\mathbb{F}$. Therefore we can add them directly in the manner of (18).

The Higgs matrix for the model has the form

$$
H = \begin{bmatrix} \bar{h}_0 & h_+ \\ -h_- & h_0 \end{bmatrix} = [\widetilde{\Phi}, \Phi] \;,
$$

where Φ is the $SU_L(2)$ doublet and $\widetilde{\Phi}$ is its charge conjugate. Under an infinitesimal $U_Y(1)$ gauge transformation $\alpha(x)$ the Higgs fields will change according to

$$
H \;\to\; (1 + i\frac{\alpha}{2} Y_{QL})\; H\; (1 - i\frac{\alpha}{2} Y_{QR}) \;\to\; H\; (1 + i\frac{\alpha}{2}(Y_{QL} - Y_{QR})) \;,
$$

since Y_{QL} is proportional to the identity matrix. The hypercharge matrix for the Higgs fields will then be

$$
Y_H = Y_{QL} - Y_{QR} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \;,
$$

that is, the doublet Φ and antidoublet $\widetilde{\Phi}$ will have definite hypercharge +1 and -1 respectively. The covariant derivative for H turns out to be

$$
D_\mu H = \partial_\mu H \;-\; i A^a_{1\mu}\tau_a H \;-\; i H \frac{1}{2} Y_H B \;,
$$

or, in terms of the $\Phi, \widetilde{\Phi}$ doublets:

$$D_\mu(\widetilde{\Phi}, \Phi) = \Big((\partial_\mu - iA^a_{1\mu}\tau_a + i\frac{1}{2}B)\widetilde{\Phi} \, , \quad (\partial_\mu - iA^a_{1\mu}\tau_a - i\frac{1}{2}B)\Phi \Big) \, . \tag{19}$$

Then,

$$\begin{aligned} \mathcal{L}_H &= 2 \ \mathrm{Tr} \ (D^\mu H^\dagger)(D_\mu H) \\ &= 2(D^\mu \widetilde{\Phi}^\dagger)(D_\mu \widetilde{\Phi}) \ + \ 2(D^\mu \Phi^\dagger)(D_\mu \Phi) \end{aligned} \tag{20}$$

will give the standard form for the SM Higgs kinetic energy terms, although it still has to be normalized (the 2 factor in front comes from the fact that there are two equal terms in Eq.(15)).

Let us assume for a moment that this is the whole extent of the theory (that is, there are no other fermions), and work out the gauge coupling constants, Yukawa couplings and normalization factors. After multiplying with the gauge coupling matrix (as in Eq. (17)), the gauge fields have to be rescaled

$$\begin{aligned} &\frac{1}{g_1} \, A^a_1 \ \rightarrow \ A^a_1 \\ &\left[\frac{1}{g_1^2} \mathrm{Tr} \left(\frac{Y^2_{QL}}{2} \right) + \frac{1}{g_2^2} \mathrm{Tr} \left(\frac{Y^2_{QR}}{2} \right) \right]^{1/2} B \ \rightarrow \ B \, , \end{aligned} \tag{21}$$

thus setting the weak coupling constant $g = g_1$ and the hypercharge coupling constant

$$\frac{1}{g'} = \left[\frac{1}{g_1^2} \mathrm{Tr} \left(\frac{Y^2_{QL}}{4} \right) + \frac{1}{g_2^2} \mathrm{Tr} \left(\frac{Y^2_{QR}}{4} \right) \right]^{1/2} .$$

If one wishes to give different masses to the up and down type fermions, one has to introduce different Yukawa couplings in the Higgs matrix

$$H = \begin{bmatrix} \bar{h}_0 & h_+ \\ -h_- & h_0 \end{bmatrix} \begin{bmatrix} \lambda_u & 0 \\ 0 & \lambda_d \end{bmatrix} = [\lambda_u \widetilde{\Phi}, \lambda_d \Phi] \, ,$$

since the vev of the $\widetilde{\Phi}$ doublet gives mass to the up-type fermions, while the vev of the Φ one gives mass to the down-type fermions. Note that this implies that the matrix M appearing in the Dirac operator has a similar structure

$$M \sim \begin{bmatrix} m\lambda_u & 0 \\ 0 & m\lambda_d \end{bmatrix} .$$

One has now to compute the rescaling factor for the Higgs field. The kinetic energy term (20) becomes

$$\mathcal{L}_H \;=\; \left(\frac{1}{g_1^2}+\frac{1}{g_2^2}\right)\lambda_u^2\;(D^\mu\widetilde{\Phi}^\dagger)(D_\mu\widetilde{\Phi})\;+\;\left(\frac{1}{g_1^2}+\frac{1}{g_2^2}\right)\lambda_d^2\;(D^\mu\Phi^\dagger)(D_\mu\Phi)\;,$$

which requires the redefinition (rescaling) of H

$$\left[\left(\frac{1}{g_1^2}+\frac{1}{g_2^2}\right)(\lambda_u^2+\lambda_d^2)\right]^{1/2}\;H\;=\;\lambda\;H\;\to\;H\;.$$

Then, the effective couplings $\lambda_{u,d}^e$ to fermions will turn out to be

$$\lambda_u^e=\lambda_u/\lambda\;,\;\lambda_d^e=\lambda_d/\lambda\;,$$

and we have the sum rule mentioned in the previous section

$$(\lambda_u^e)^2+(\lambda_d^e)^2\;=\;\frac{g_1^2g_2^2}{g_1^2+g_2^2}\;. \qquad (22)$$

Finally, let us consider the terms giving rise to the Higgs potential. Since the matrix M is not proportional to identity and it does not commute with a general zero-form, we should use the trace prescription in (9). Therefore,

$$\begin{aligned}V(H)\;&=\;\left(\frac{1}{g_1^2}+\frac{1}{g_2^2}\right)\left(\mathrm{Tr}[MM^\dagger-HH^\dagger]\right)^2\\&\to\;\left(\frac{1}{g_1^2}+\frac{1}{g_2^2}\right)^{-1}(h_0\bar{h}_0+h_-h_+-m^2)^2\;,\end{aligned} \qquad (23)$$

in terms of the rescaled fields. From this, we see that the neutral Higgs component field h_0 aquires a vacuum expectation value $< h_0 >= m\;(= v/\sqrt{2})$, and after symmetry breaking the surviving Higgs particle gets a mass

$$m_h\;=\;2m\sqrt{(\lambda_u^e)^2+(\lambda_d^e)^2}$$

(where the sum-rule (22) has been used).

We wish now to extend the previous construction to the whole Standard Model. One must then add leptons, and include color and flavor. For this purpose, we first extend the fermion space

$$\Psi_L\;=\;\left[\begin{matrix}\begin{pmatrix}u_L\\d_L\end{pmatrix}^{\alpha}\\\begin{pmatrix}\nu_L\\e_L\end{pmatrix}\end{matrix}\right]_i\;,\;\Psi_R\;=\;\left[\begin{matrix}\begin{pmatrix}u_R\\d_R\end{pmatrix}^{\alpha}\\\begin{pmatrix}0\\e_R\end{pmatrix}\end{matrix}\right]_i\;,$$

where α are the color indices, and i the flavor ones. Then the gauge one-form will split into components acting on the lepton and quark subspaces. On the lepton subspace, we have

$$\begin{aligned} A^E_{1\mu} &= \left(A^a_\mu \tau_a + \frac{1}{2} Y_{EL} B_\mu \right) \otimes \mathbf{1}_{ij} \\ A^E_{2\mu} &= \left(\frac{1}{2} Y_{ER} B_\mu \right) \otimes \mathbf{1}_{ij} \\ H^E &= \Phi \otimes \lambda^l_{ij} \ , \end{aligned} \tag{24}$$

where Y_{EL}, Y_{ER} matrices contain the hypercharge numbers associated with the left and right handed charged leptons and neutrinos, and λ^l_{ij} is the Yukawa coupling matrix for leptons. On the quark subspace

$$\begin{aligned} A^Q_{1\mu} &= \left(A^a_\mu \tau_a + \frac{1}{2} Y_{QL} B_\mu \right) \otimes (G^b_\mu \Lambda_b)_{\alpha\beta} \otimes \mathbf{1}_{ij} \\ A^Q_{2\mu} &= \left(\frac{1}{2} Y_{QR} B_\mu \right) \otimes (G^b_\mu \Lambda_b)_{\alpha\beta} \otimes \mathbf{1}_{ij} \\ H^Q &= \widetilde{\Phi} \otimes \mathbf{1}_{\alpha\beta} \otimes \lambda^u_{ij} \ + \ \Phi \otimes \mathbf{1}_{\alpha\beta} \otimes \lambda^d_{ij} \ , \end{aligned} \tag{25}$$

where G^b_μ are the $SU_C(3)$ gauge bosons, and the Λ_b are the generators of the $SU_C(3)$ gauge transformation. Also, the λ^u_{ij} and λ^d_{ij} are the Yukawa coupling matrices for the up and down-type quarks.

One can then easily verify that one obtains the Standard Model lagrangian. The gauge couplings for the electroweak sector are given by:

$$\begin{aligned} \frac{1}{g^2} &= \frac{4N}{g_1^2} \\ \frac{1}{g'^2} &= \frac{N}{g_1^2} \mathrm{Tr} \left(\frac{3Y_{QL}^2 + Y_{EL}^2}{2} \right) + \frac{N}{g_2^2} \mathrm{Tr} \left(\frac{3Y_{QR}^2 + Y_{ER}^2}{2} \right) \\ &= \frac{4N}{3g_1^2} + \frac{16N}{3g_2^2} \ . \end{aligned} \tag{26}$$

Here N stands for the number of generations. The Higgs kinetic energy has the form

$$\begin{aligned} \mathcal{L}_H &= \left(\frac{1}{g_1^2} + \frac{1}{g_2^2} \right) 3\mathrm{Tr}(\lambda^u)^2 \ (D^\mu \widetilde{\Phi}^\dagger)(D_\mu \widetilde{\Phi}) \\ &+ \left(\frac{1}{g_1^2} + \frac{1}{g_2^2} \right) \left(3\mathrm{Tr}(\lambda^d)^2 + \mathrm{Tr}(\lambda^l)^2 \right) \ (D^\mu \Phi^\dagger)(D_\mu \Phi) \ . \end{aligned} \tag{27}$$

Therefore, the constant involving in rescaling the Higgs field is

$$\lambda^2 = \left(\frac{1}{g_1^2} + \frac{1}{g_2^2} \right) \left(3\mathrm{Tr}(\lambda^u)^2 + 3\mathrm{Tr}(\lambda^d)^2 + \mathrm{Tr}(\lambda^l)^2 \right) \ \simeq \ 3\lambda_t^2 \left(\frac{1}{g_1^2} + \frac{1}{g_2^2} \right) \ ,$$

if we assume that the top Yukawa coupling dominates. The resulting sum rule will predict that the effective top Yukawa coupling is

$$\lambda_t^{e2} = \frac{1}{3}\frac{g_1^2 g_2^2}{g_1^2+g_2^2} = \frac{16N}{9}\frac{g^2 g'^2}{g^2+g'^2} ,$$

which, if one uses the values for the electroweak couplings at M_Z scale $\alpha_1(M_Z) = 1/58.97$, $\alpha_2(M_Z) = 1/29.61$ comes out to $\lambda_t^e \simeq 0.8$ (for the number of generations $N = 3$). This is to be compared with the Standard Model value, $\lambda_t^{SM} \simeq 1$.

Finally, the Higgs potential is

$$\begin{aligned} V(H) &= \left(\frac{1}{g_1^2}+\frac{1}{g_2^2}\right)\left[\left(h_0\bar{h}_0 + h_+h_- - \lambda^2 m^2\right)\left(3\mathrm{Tr}(\lambda^u)^2 + 3\mathrm{Tr}(\lambda^d)^2 + \mathrm{Tr}(\lambda^l)^2\right)\right]^2 \\ &\rightarrow \left(\frac{1}{g_1^2}+\frac{1}{g_2^2}\right)^{-1}\left(h_0\bar{h}_0 + h_+h_- - m^2\right)^2 . \end{aligned} \tag{28}$$

Then one would obtain a Higgs mass:

$$M_h \simeq 2\sqrt{3}\,\lambda_t^e m \simeq 3.5\ m_t .$$

One notes that this model predicts a rather large Higgs mass, Also, it is not exactly in agreement with the Standard Model value for the top mass (although it is not too far either)[4]. However, this is not too troubling; we do not after all expect that the Standard Model is a fundamental theory, valid at all scales. Instead, it seems probable that the $SU_C(3) \times SU_L(2) \times U_Y(1)$ gauge symmetry visible at M_Z scale is a remnant of the breaking of some larger symmetry, which is manifest at a higher scale. Hence, one should consider applying the framework used above to such extensions of the Standard Model.

V. LEFT-RIGHT SYMMETRIC MODEL ON TWO-SHEETED SPACE TIME

In spite of the great phenomenological success of the standard model as a gauge theory based on spontaneously broken symmetry $SU_C(3) \times SU_L(2) \times U_Y(1)$, it is recognized that it has several unsatisfactory features that include, besides the proliferation of free parameters, a lack of an understanding of the origin of parity violation in the low energy region. Left-right

[4] It is possible to fix the prediction for the top mass by introducing a separate coupling for the Higgs part of the field strength (as, for example, in [7]). However, this would lead to a loss of predictive power for the model.

symmetric models have always been attractive as the minimal extensions of the standard model. A natural consequence of left-right symmetric models is the existence of right-handed neutrinos, which, through the seesaw mechanism [11], give rise to small mass for the left handed neutrinos. The discovery of convincing evidence for non-zero neutrino masses gives therefore further weight to the ideea that the Standard Model may be a low energy version of such models.

To start formulating the left-right symmetric model as a theory on two-sheeted space time, we have to first specify the Hilbert space of spinors. One could use the same space as for the Standard Model; however, in that case the only Higgs multiplets one can introduce are $SU_L(2) \times SU_R(2)$ bidoublets. Since one needs triplet Higgses to break the left-right symmetry (as well as to give mass to the right handed neutrinos), we take the Hilbert space to also include the charge conjugate fields of Ψ_L, Ψ_R,

$$\Psi = \begin{bmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R^c \end{pmatrix} \\ \begin{pmatrix} \Psi_R \\ \Psi_L^c \end{pmatrix} \end{bmatrix}_i ,$$

in such a way that the left-handed fields live on one sheet and the right-handed ones live on the other. The matrix structure associated with the gauge fields will then be

$$\begin{aligned} A_{1\mu} &= \begin{bmatrix} A^a_{L\mu}\tau_a + Y_L B_\mu/2 & 0 \\ 0 & A^a_{R\mu}\tau_a - Y_R B_\mu/2 \end{bmatrix} \\ A_{2\mu} &= \begin{bmatrix} A^a_{R\mu}\tau_a + Y_R B_\mu/2 & 0 \\ 0 & A^a_{L\mu}\tau_a - Y_L B_\mu/2 \end{bmatrix} . \end{aligned} \tag{29}$$

The A^a_L, A^a_R are the gauge fields associated with the $SU_L(2)$, $SU_R(2)$ group transformations, while B is the one associated with the $U(1)_{B-L}$ group. The $Y_{L,R}$ numbers are the charges associated with $U(1)_{B-L}$ transformations for the left and right-handed fields. They take different values on lepton and quark spaces; for leptons $Y_{EL} = Y_{ER} = -1$, and for quarks $Y_{QL} = Y_{QR} = 1/3$.

Since charge conjugate fermions are part of the spinor space, one can introduce triplet Higgses in the gauge one-form. Thus, on the lepton subspace we define

$$\mathbb{H} = \begin{bmatrix} H_1 \otimes \lambda^{1l}_{ij} & \Delta_L \otimes \lambda^L_{ij} \\ \Delta^\dagger_R \otimes \lambda^{R\dagger}_{ij} & H^c_2 \otimes \lambda^{2l}_{ij} \end{bmatrix} , \tag{30}$$

where H_1 is a (2,2,0) bidoublet and H_2 its conjugate,

$$H_1 = \begin{bmatrix} h_1^0 & h_2^+ \\ h_1^- & h_2^0 \end{bmatrix} \ , \quad H_2 = \begin{bmatrix} h_2^{0*} & -h_1^+ \\ -h_2^- & h_1^{0*} \end{bmatrix} \ , \quad H_2^c = \sigma_2 H_2^T \sigma_2 \ ,$$

and $\Delta_{L,R}$ are $SU(2)$ triplets which couple to the left and right-handed leptons

$$\Delta_{L,R} = \begin{bmatrix} \delta^-/\sqrt{2} & \delta^0 \\ \delta^{--} & -\delta^-/\sqrt{2} \end{bmatrix}_{L.R} .$$

The $\lambda_{ij}^{1,2l}, \lambda_{ij}^{R,L}$ matrices acting on the flavor space are the Yukawa couplings associated with the Dirac mass term for both the charged leptons and neutrinos (the $\lambda^{1,2l}$ matrices), and the Majorana mass terms for the right- and left-handed neutrinos (the $\lambda^{R,L}$ ones)[5]. On the other hand the Higgs matrix (30) on the quark subspace will contain only the diagonal H_1, H_2^c fields (with Yukawa matrices $\lambda^{1,2q}$ associated with the quarks) since the Δ_L, Δ_R triplets do not couple to quarks. Note also that $H_2^c = H_1^\dagger$; we use this notation to remind ourselves that the mass term $\bar{\psi}_R^c H_2^c \psi_L^c$ is usually written as $\bar{\psi}_L H_2 \psi_R$.

Substituting (29) and (30) in Eq. (4) (with $\Phi_1 = \mathbb{H}, \Phi_2 = \mathbb{H}^\dagger$), one obtains the corresponding expressions for the field strength $\mathbb{F}$, which will give the standard Lagrangian terms for the gauge fields A_L, A_R, and B. It can easily be verified that Eq. (15) will give the right kinetic energy terms for the Higgs multiplets

$$\begin{aligned} \mathcal{L}_H &= |\partial_\mu H + iHA^a_{R\mu}\tau_a - iA^a_{L\mu}\tau_a H|^2 \\ \mathcal{L}_\Delta &= |\partial_\mu \Delta_{L,R} + i\Delta_{L,R}A^a_{L,R\mu}\tau_a - iA^a_{L,R\mu}\tau_a\Delta_{L,R} - iY_E B_\mu \Delta_{L,R}|^2 \ , \end{aligned} \tag{31}$$

where, due to left-right symmetry, $Y_{EL} = Y_{ER} = Y_E$. The coupling constants for the $SU(2)$ and $U(1)$ gauge fields are given by

$$\begin{aligned} \frac{1}{g^2} &= 4N\left(\frac{1}{g_2^2} + \frac{1}{g_1^2}\right) \\ \frac{1}{g'^2} &= 2N\left(\frac{1}{g_1^2} + \frac{1}{g_2^2}\right)\left(3Y_Q^2 + Y_L^2\right) \ . \end{aligned} \tag{32}$$

It is interesting to note that, due to left-right symmetry on each sheet, the ratio of the two coupling constants is fixed: $g^2/g'^2 = 2/3$ (of course, this is taken to be valid at the energy

[5] Note that in order to have left-right symmetry in the fermion sector, one should also require that $\lambda^R = \lambda^L, \lambda^l = \lambda^{l\dagger}$.

where this description holds). This symmetry could be broken by considering more complex structure for the gauge coupling matrix in (17). Also, note that the rescaling constants for the Higgs fields are

$$\begin{aligned}
\lambda_{\Delta R}^2 &= \mathrm{Tr}\left[\lambda^R \lambda^{R\dagger}\right]/(4Ng^2) \\
\lambda_{\Delta L}^2 &= \mathrm{Tr}\left[\lambda^L \lambda^{L\dagger}\right]/(4Ng^2) \\
\lambda_H^2 &= \mathrm{Tr}\left[\lambda^{1l}\lambda^{1l\dagger} + \lambda^{2l}\lambda^{2l\dagger} + 3(\lambda^{1q}\lambda^{1q\dagger} + \lambda^{2q}\lambda^{2q\dagger})\right]/(4Ng^2)\ . \qquad (33)
\end{aligned}$$

Before continuing, let us briefly review the salient features of gauge symmetry breaking in the left-right symmetric models. The theory has five gauge bosons, two charged ones (A_L and A_R) and three neutral: B, A_{0L}, A_{0R}. At some high energy, the symmetry is unbroken, and all gauge bosons are massless. However, at some large scale, the right-handed triplet Higgs aquires a vacuum expectation value $< \Delta_R >= v_R$, and breaks left-right symmetry. One of the charged bosons (more precisely, a combination of the A_L and A_R) and one of the neutral ones aquire masses of order v_R. Since these massive bosons have not been observed at present day colliders, one infers that v_R should be above the TeV scale[6]. Furthermore, the right handed neutrinos also aquire Majorana mass through the vev of Δ_R. If one accepts that the smallness of left handed neutrino masses is due to the seesaw mechanism, this would point to the scale v_R to be around 10^{14} GeV. The remaining symmetry (that of the Standard Model $SU_L(2) \times U_Y(1)$) is broken at the electroweak scale by vacuum expectations for the bidoublet Higgses $< h_1^0 >= v_1, < h_2^0 >= v_2$. Note that the left-handed triplet Δ_L may also aquire a vacuum expectation value v_L; however, since this vev gives Majorana mass to the left-handed neutrinos, v_L should be below eV scale.

A very interesting question then arises: is this pattern of symmetry breaking consistent with our formulation of the left-right symmetric model on a two-sheeted space time? Note that, in our scenario, the Higgs potential is fixed (up to maybe an overall multiplicative constant) once the matrix M appearing in the Dirac operator has been given. Therefore one can potentially hope to predict how the gauge symmetry is broken. We shall investigate this in what follows, under different sets of assumptions concerning the structure of the M matrix.

[6] Constraints on the scale of v_R can also be inferred from weak interaction data; see for example [12] and references therein.

A) We first choose a form for the matrix M suggested by the pattern of fermion masses (that is, choose a nonzero entry in the mass matrix at places where fermion masses would naturally appear). Thus, one can choose

$$M = \begin{bmatrix} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \otimes \lambda^{1l}_{ij} \ , & \begin{pmatrix} 0 & m_L \\ 0 & 0 \end{pmatrix} \otimes \lambda^{L}_{ij} \\ \begin{pmatrix} 0 & 0 \\ m_R & 0 \end{pmatrix} \otimes \lambda^{R}_{ij} \ , & \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \otimes \lambda^{2l}_{ij} \end{bmatrix} , \tag{34}$$

on the lepton subspace. On the quark subspace, we will have a similar structure (with the fermion Yukawa couplings $\lambda^{1l}, \lambda^{2l}$ replaced by the quark ones $\lambda^{1q}, \lambda^{2q}$), with the difference that the off diagonal matrices will be zero (since one generally does not introduce triplets in the quark sector, and thus one does not get a Majorana mass for the quarks).

Before computing the Higgs potential, one has to insure the $\mathrm{d}\mathrm{d}\rho = 0$ condition by using the trace prescription discussed in section I. Following Eq. (2), if we set $f_1 = \mathrm{diag}\{f_{1L}, f_{1R}\}$, we obtain on the left-handed sheet

$$(\mathrm{d}\mathrm{d}\rho)_{11} = \begin{pmatrix} (MM^\dagger)_{11} f_{1L} - f_{1L}(MM^\dagger)_{11} & (MM^\dagger)_{12} f_{1R} - f_{1L}(MM^\dagger)_{12} \\ (MM^\dagger)_{21} f_{1L} - f_{1R}(MM^\dagger)_{22} & (MM^\dagger)_{22} f_{1R} - f_{1R}(MM^\dagger)_{22} \end{pmatrix} , \tag{35}$$

with a similar form on the right-handed sheet. In order to satisfy $\mathrm{d}\mathrm{d}\rho = 0$, one then should drop the off-diagonal terms and take the trace of the diagonal ones. The contribution of the Higgs terms to the diagonal elements of $\mathbb{F}$ is then

$$\begin{gathered} MM^\dagger - \mathbb{H}\mathbb{H}^\dagger \ \rightarrow \\ \begin{bmatrix} \mathrm{Tr}[H_1 H_1^\dagger + \Delta_L \Delta_L^\dagger - (MM^\dagger)_{11}] & 0 \\ 0 & \mathrm{Tr}[H_2^c H_2^{c\dagger} + \Delta_R^\dagger \Delta_R - (MM^\dagger)_{22}] \end{bmatrix} . \end{gathered} \tag{36}$$

Now, if we assume that by suitable transformations, one can take the vacuum expectation values of the Higgs fields to be

$$< H_1 > = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} , \quad < \Delta_{L,R} > = \begin{bmatrix} 0 & v_{L,R} \\ 0 & 0 \end{bmatrix} ,$$

the Higgs potential will become

$$\begin{aligned} V(< \mathbb{H} >) = & \left[(v_1^2 + v_2^2 - m_1^2 - m_2^2)|\lambda^{1l}|^2 + (v_L^2 - m_L^2)|\lambda^L|^2 \right]^2 \\ & + \left[(v_1^2 + v_2^2 - m_1^2 - m_2^2)|\lambda^{2l}|^2 + (v_R^2 - m_R^2)|\lambda^L|^2 \right]^2 \\ & + 3\left(v_1^2 + v_2^2 - m_1^2 - m_2^2\right)^2 \left(|\lambda^{1q}|^4 + |\lambda^{2q}|^4\right) , \end{aligned} \tag{37}$$

where the first two lines are contributions coming from the lepton subspace and the last line is the contribution due to the quark subspace. (Here we have used $|\lambda|^2$ as a shorthand for $\mathrm{Tr}[\lambda\lambda^\dagger]$.) One then sees that this potential has a minimum at $v_L = m_L,\ v_R = m_R$ and $v_1^2 + v_2^2 = m_1^2 + m_2^2$. In other words, the vevs of the triplets Δ_R, Δ_L can be fixed to m_R, m_L, but there is a degeneracy in determining the individual values of the bidoublet Higgs vevs. If, however, one takes $m_1^2 + m_2^2$ to be at electroweak scale, both v_1 and v_2 are at this scale, and this does not create any problem with respect to the symmetry breaking pattern. So we see that we can obtain the desired symmetry breaking pattern, although one has to put in by hand a large value for m_R, very small value for m_L and electroweak scale for m_1, m_2.

Note that this result is by no means trivial. For example, if there would be no quark sector in the theory (drop the third line in Eq. (37)), one could not separate the vevs of the bidoublets from the vevs of the triplets. Hence there would be no way to make sure that H does not aquire a v_R scale vev. This is due to the highly degenerate structure of the potential (there are several flat directions). In particular, note that if one adopts the pattern (34) for the mass matrix M (where nonzero entries correspond to Higgs fields that one expects to get vacuum expectation values), it would be tempting to say that $< \mathbb{H} >= M$ will give a minimum of the potential. One could then say that the fields Φ which appear initially in the gauge one-form (4) are the Higgs fields in the broken symmetry phase of the theory [6]. However, while it is true that $V(< \mathbb{H} >= M) = 0$, this does not make it a true vacuum, since degeneracies could exist.

B) One might then also consider structures for the M matrix which are not linked to the fermion masses. Actually, the parameters which seem to have physical relevance are the matrices $MM^\dagger$ and $M^\dagger M$, since they appear in both, the Higgs potential and the evaluation of the $\mathrm{d}\mathrm{d}\rho = 0$ condition. In the most general case, the Higgs potential can be written therefore in terms of four parameters: $\mu_1^{l2} = \mathrm{Tr}(MM^\dagger)_{11},\ \mu_2^{l2} = \mathrm{Tr}(MM^\dagger)_{22}$ on the lepton space, and $\mu_1^{q2} = \mathrm{Tr}(MM^\dagger)_{11},\ \mu_2^{q2} = \mathrm{Tr}(MM^\dagger)_{22}$ on the quark space. One would then obtain

$$
\begin{aligned}
V(< \mathbb{H} >) \ = \ & \left[(v_1^2 + v_2^2)|\lambda_e^{1l}|^2 + v_L^2 - \mu_1^{l2}\right]^2 \\
& + \left[(v_1^2 + v_2^2)|\lambda_e^{2l}|^2 + v_R^2 - \mu_2^{l2}\right]^2 \\
& +3\left[(v_1^2 + v_2^2)|\lambda_e^{1q}|^2 - \mu_1^{q2}\right]^2 + 3\left[(v_1^2 + v_2^2)|\lambda_e^{2q}|^2 - \mu_2^{q2}\right]^2 \ , \qquad (38)
\end{aligned}
$$

where we have used the vevs rescaled by the corresponding Higgs fields factors (33), $v_{1,2} \rightarrow$

$v_{1,2}/\lambda_H$, $v_{L,R} \to v_{L,R}/\lambda_{\Delta L,R}$, and the constants $\lambda_e^{kl,q} = \lambda^{kl,q}/\lambda_H$. Again we see that the scale of v_1, v_2 is set by the μ_1^q, μ_2^q parameters, so these have to be at electroweak scale. Interestingly, for this potential one does not have to necessarily fine tune the vev of the Δ_L field anymore; it will be naturally driven to zero for a whole range of values for the parameter μ_1^l. Indeed, let us call v_0^2 the value of $v_1^2 + v_2^2$ for which the last line of Eq. (38) is minimised; one then sees that if $v_0^2|\lambda_e^{1l}|^2 > \mu_1^{l2}$, then the minimum of the potential requires $v_L = 0$.

C) In the two examples shown above, the Higgs potential has a relatively simple structure (being the sum of several perfect squares). More complex potentials can be obtained in the case when the matrix M has some symmetries. For example, if M is proportional to identity in the generation space, one need not take trace in (37) over the generation indices. Then, for example, if one would compute the contribution of the $\mathbb{H}_{11}$ part on the lepton subspace, one would obtain

$$V_{11}(v_1, v_2, v_L) \;=\; \mathrm{Tr}\left[\left|\; (v_1^2 + v_2^2)\lambda_e^{1l}\lambda_e^{1l\dagger} + v_L^2\lambda^L\lambda^{L\dagger} - \mu_1^{l2} \;\right|^2\right] \;.$$

One sees that while the coefficients of the potential terms quadratic in the Higgs fields are proportional to $\mathrm{Tr}(\lambda\lambda^\dagger)$, the coefficients of the quartic terms are proportional to $\mathrm{Tr}(\lambda\lambda^\dagger\lambda\lambda^\dagger)$, and therefore somewhat independent. One can even break the symmetry between the vacuum expectation values of the ϕ_1^0 and ϕ_2^0 fields; if the M_{11} and/or M_{22} elements are taken to be diagonal in SU(2) space, then the diagonal elements of (35) will be zero, and no trace is necessary. In this case, the (partial) Higgs potential will be

$$V_{11}(H) \;=\; \mathrm{Tr}\left[\left|\; H_1 H_1^\dagger \otimes \lambda_e^{1l}\lambda_e^{1l\dagger} + \Delta_L\Delta_L^\dagger \otimes \lambda^L\lambda^{L\dagger} - \mu_1^{l2}\mathbf{1}\otimes\mathbf{1} \;\right|^2\right] \;,$$

or

$$V_{11}(v_1, v_2, v_L) \;=\; \mathrm{Tr}\left[\left|v_1^2\lambda_e^{1l}\lambda_e^{1l\dagger} + v_L^2\lambda^L\lambda^{L\dagger} - \mu_1^{l2}\right|^2 + \left|v_2^2\lambda_e^{1l}\lambda_e^{1l\dagger} - \mu_1^{l2}\right|^2\right] \;.$$

Finally, if one sets the off-diagonal elements of the M matrix to zero ($M_{12} = M_{21} = 0$), one can keep the off-diagonal elements in $\mathbb{H}\mathbb{H}^\dagger - MM^\dagger$, which will contribute to the Higgs potential a term

$$V_{12}(H) \;=\; 2\mathrm{Tr}\left[\left|\; H_1\Delta_R \otimes \lambda_e^{1l}\lambda^R + \Delta_L H_1 \otimes \lambda^L\lambda_e^{2l\dagger} \;\right|^2\right] \;. \tag{39}$$

It is instructive to compare the Higgs potential obtained in our model with the general Higgs potential discussed in [11, 12]. We see that our model can give rise to most of the terms found in [11]. One might note that only $H_1, H_1^\dagger$ appears in our potential; however, this is due to the particular choice of the Higgs matrix (30); a more general choice

$$\mathbb{H} = \begin{bmatrix} H_1 \otimes \lambda^{1l} + H_2 \otimes \lambda^{3l} & \Delta_L \otimes \lambda^L \\ \Delta_R^\dagger \otimes \lambda^{R\dagger} & H_2^c \otimes \lambda^{2l} + H_1^c \otimes \lambda^{4l} \end{bmatrix} \quad (40)$$

can be made, which will introduce $H_2, H_2^\dagger$ in the potential (if desirable). The only important difference seems to be that the terms coupling the vacuum expectation values of the left and right triplets $\mathrm{Tr}(\Delta_L^\dagger \Delta_L)\mathrm{Tr}(\Delta_R^\dagger \Delta_R)$ are missing in our model. In the absence of such a term, one cannot take the Higgs potential left-right symmetric ($\mu_1^{l2} = \mu_2^{l2}, \lambda^L = \lambda^R$). Indeed, it turns out that for a left-right symmetric potential (as in in [11]) what drives v_L to zero while keeping v_R at GUT scale is just such a term $\sim v_L^2 v_R^2$, that couples the two vacuum expectation values. Lacking such a term in our model, we are forced to take μ_2^l at GUT scale, and μ_1^l at electroweak scale.

VI. CONCLUSIONS

We have developed a formalism based on the framework of Connes' non-commutative geometry (NCG), with the purpose of studying the standard model of electroweak interactions and beyond. Our model is based on a two-sheeted space-time that can be thought of as a discretized version of Kaluza-Klein theory, in which the compact fifth dimension is replaced by two discrete points. The left- and right- chiral spinor fields live on the two separate sheets, while the gauge and Higgs fields are part of a generalized gauge operator represented by a 2×2 matrix acting on the internal (discrete) space.

The main virtue of this framework is that the scalar Higgs fields are an integral part of the gauge sector. Their gauge invariant kinetic parts in the Lagrangian as well as quartic forms of Higgs potentials arise naturally. Furthermore, the possibility of spontaneous symmetry breaking is built naturally into the Higgs potential.

While different NCG formulations of the Standard Model (and other gauge theories) have been studied extensively by several authors, we have taken in this paper a less mathematical approach and focused more on the physics of the model. The formalism allows for an easy (and transparent) construction of the Higgs sector. The choice of Higgs multiplets appearing in the theory is dictated by the choice of the underlying spinor space. The Higgs potential can be written in terms of the Yukawa couplings of the fermions and the elements of the matrix M appearing in the definition of the generalized Dirac operator

Besides the predictive power in the Higgs sector, the NCG formalism leads to sum rules for the Yukawa coupling constants of the fermions. In the case of the minimal standard model, this will lead to a prediction of the top quark mass (as well as the Higgs mass). However, such results at the tree level form of the Lagrangian cannot be taken seriously unless one knows at which scale this picture holds. If NCG-inspired theories are a description of reality, one might expect them to be valid at scales close to M_{Pl}. However, at such high scales, there are good reasons to believe that the gauge group is a larger one, corresponding to a grand unified theory of which the Standard Model is just the low energy limit.

We are led therefore to consider higher symmetries. As an example, in this paper we analyze the implications of two-sheeted space time picture for the left-right symmetric model. Our approach predicts several specific forms possible for the Higgs potential. Interestingly, these predictions allow the desired left-right symmetry breaking pattern leading to the standard model. We discuss the various scenarios for this potential depending upon the choice and the symmetries in the M matrix of the Dirac operator. A more detailed quantitative study of these features is desirable to draw concrete conclusions. But the model has several attractive features, and fewer parameters compared with the standard left-right symmetric models.

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