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William C. Purdy
Syracuse University, wcpurdy@ecs.syr.edu

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School of Computer and Information Science
Suite 4-116
Center for Science and Technology
Syracuse, New York 13244-4100

(315) 443-2368

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William C. Purdy*

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*School of Computer and Information Science, Syracuse University

Abstract

This paper describes a language called $\mathcal{L}_N$ whose structure mirrors that of natural language. $\mathcal{L}_N$ is characterized by absence of variables and individual constants. Singular predicates assume the role of both individual constants and free variables. The role of bound variables is played by predicate functors called "selection operators." Like natural languages, $\mathcal{L}_N$ is implicitly many-sorted. $\mathcal{L}_N$ does not have an identity relation. Its expressive power lies between the predicate calculus without identity and the predicate calculus with identity. The loss in expressiveness relative to the predicate calculus with identity however is not significant. Deduction in $\mathcal{L}_N$ is intended to parallel reasoning in natural language, and therefore is termed "surface reasoning." In contrast to deduction in a disparate underlying logic such as clausal form, each step of a proof in $\mathcal{L}_N$ has a direct counterpart in the surface language. A sound and complete axiomatization is given. Derived rules, corresponding to monotonicity and conservativity of quantifiers and to unification and resolution in conventional logic, are presented. Several problems are worked to illustrate reasoning in $\mathcal{L}_N$.

1 **Introduction** It is a popular view that spoken or written language is a "surface" phenomenon, that its logical structure and meaning reside in an underlying base language, and that complex transformations relate these two levels. Reasoning takes place at the base level with the surface language providing only an input/output function. Put into practice, this view would require difficult transformations from surface to base language and back again. Even more difficult would be providing an intelligible account in the surface language of reasoning performed in the base language.

This paper is motivated by an alternative view [13], viz., that the surface language directly conveys logical structure and meaning, and that the base level and transformations are unnecessary. Reasoning conducted in the surface language will be termed "surface reasoning" to distinguish it from deduction performed in some base language such as clausal form of first-order logic.

The paper describes $\mathcal{L}_N$, a logic designed for surface reasoning. $\mathcal{L}_N$ is characterized by absence of variables and individual constants. Singular predicates assume the role of both individual constants and free variables. The role of bound variables is played by predicate functors called "selection operators." Like natural languages, $\mathcal{L}_N$ is implicitly many-sorted. $\mathcal{L}_N$ does not have an identity relation.

The elimination of bound variables borrows from Quine's Predicate Functor Logic [5, 9]. The elimination of the identity relation and the central role of singular predicates are inspired by Sommers' Term Calculus [6, 7, 10, 11]. But the principal influence is the recent work on generalized quantifiers in natural language [1, 2]. This work gave rise to the conviction underlying $\mathcal{L}_N$, viz., that monotonicity properties constitute a unifying principle in surface reasoning.

Two claims are made for $\mathcal{L}_N$: (i) the language is structurally similar to natural language in the sense that there exist well-translatable grammars [3] relating $\mathcal{L}_N$ and natural languages; (ii) the logic is similar to natural language reasoning in that the monotonicity principle captures an essential and important element of natural language reasoning.

The paper is organized as follows. First the syntax and semantics of $\mathcal{L}_N$ are defined. Next a complete axiomatization is given. Then several theorems establishing the monotonicity principle are presented. The monotonicity principle is shown to subsume unification and resolution. To support the claim that $\mathcal{L}_N$ is structurally similar to natural language, a fragment of English and its translation to $\mathcal{L}_N$ are defined. To support the claim that $\mathcal{L}_N$ mirrors reasoning in natural language, several example problems are solved and discussed.

2 Definition of the Language

The alphabet of $\mathcal{L}_N$ consists of the following.

1. Predicate symbols $\mathcal{P} = \mathcal{S} \cup (\bigcup_{j \in \omega} \mathcal{R}_j)$ where $\mathcal{R}_j = \{R_i^j : i \in \omega\}$, $\mathcal{S} = \{S_i : i \in \omega\}$, and $\mathcal{S}$ and the $\mathcal{R}_j$ are mutually disjoint.

2. Selection operators $\{\langle k_1, \ldots, k_n \rangle : n \in (\omega - \{0\}), k_i \in (\omega - \{0\}), 1 \leq i \leq n\}$.

3. Boolean operators $\cap$ and $^-$.

4. Parentheses (and).

$\mathcal{L}_N$ is partitioned into sets of n-ary expressions for $n \in \omega$. These sets are defined to be the smallest satisfying the following conditions.

1. Each $S_i \in \mathcal{S}$ is a unary expression.

2. For all $n \in \omega$, each $R_i^n \in \mathcal{R}_n$ is a n-ary expression.

3. For each predicate symbol $P \in \mathcal{P}$ of arity m, $\langle k_1, \ldots, k_m \rangle P$ is a n-ary expression where $n = max(k_i)_{1 \leq i \leq m}$.

4. If X is a n-ary expression then $\overline{(X)}$ is a n-ary expression.

5. If X is a m-ary expression and Y is a l-ary expression then $(X \cap Y)$ is a n-ary expression where $n = max(l, m)$.

6. If X is a unary expression and Y is a $(n+1)$-ary expression then (XY) is a n-ary expression.

In the sequel, superscripts and parentheses are dropped whenever no confusion can result. Metavariables are used as follows: S_i, S range over $\mathcal{S}$; R^n ranges over $\mathcal{R}_n$; P ranges over $\mathcal{P}$; $X, Y, Z, X_i, Y_i, Z_i, W_i$ range over $\mathcal{L}_N$; and X^n, Y^n, Z^n, W^n, V^n range over n-ary expressions of $\mathcal{L}_N$.

An *interpretation* of $\mathcal{L}_N$ is a pair $\mathcal{I} = \langle \mathcal{D}, \mathcal{F} \rangle$ where $\mathcal{D}$ is a nonempty set and $\mathcal{F}$ is a mapping defined on $\mathcal{P}$ satisfying:

1. for each $S_i \in \mathcal{S}$, $\mathcal{F}(S_i) = \{\langle d \rangle\}$ for some (not necessarily unique) $d \in \mathcal{D}$, and
2. for each $R^n \in \mathcal{R}_n$, $\mathcal{F}(R^n) \subseteq \mathcal{D}^n$.

Note that $\mathcal{D}^0 = \{\langle\rangle\}$, so $\mathcal{F}(R^0)$ must be either $\{\langle\rangle\}$ or $\emptyset$.

Let $\alpha = \langle d_1, d_2, \ldots \rangle \in \mathcal{D}^\omega$ (a sequence of individuals). Then $X \in \mathcal{L}_N$ *is satisfied by* α *in* $\mathcal{I}$ (written $\mathcal{I} \models_\alpha X$) iff one of the following holds:

1. $X \in \mathcal{P}$ with arity n and $\langle d_1, \ldots, d_n \rangle \in \mathcal{F}(X)$
2. $X = \langle k_1, \ldots, k_m \rangle P$ where $P \in \mathcal{P}$ with arity m and $\langle d_{k_1}, \ldots, d_{k_m} \rangle \models P$
3. $X = \overline{Y}$ and $\mathcal{I} \not\models_\alpha Y$
4. $X = Y \cap Z$ and $\mathcal{I} \models_\alpha Y$ and $\mathcal{I} \models_\alpha Z$
5. $X = Y^1 Z^{n+1}$ and for some $d \in \mathcal{D}$, $\langle d \rangle \models Y^1$ and $\langle d \rangle \models Z^{n+1}$

where $\mathcal{I} \not\models_\alpha X$ is an abbreviation for not$(\mathcal{I} \models_\alpha X)$ and $\langle d_{i_1}, \ldots, d_{i_n} \rangle \models X$ is an abbreviation for $\mathcal{I} \models_{\langle d_{i_1}, \ldots, d_{i_n}, d_1, d_2, \ldots \rangle} X$.

A *sentence* of $\mathcal{L}_N$ is a 0-ary expression. Let X be a sentence of $\mathcal{L}_N$. X *is true in* $\mathcal{I}$ (written $\mathcal{I} \models X$) iff $\mathcal{I} \models_\alpha X$ (i.e., $\langle\rangle \models X$) for every $\alpha \in \mathcal{D}^\omega$. X *is valid* (written $\models X$) iff X is true in every interpretation of $\mathcal{L}_N$. A set Γ of sentences *is satisfied in* $\mathcal{I}$ iff each $X \in \Gamma$ is true in $\mathcal{I}$.

It can be shown that the pure predicate calculus without identity ($\mathcal{PP}$) is equivalent to a proper subset of $\mathcal{L}_N$, which in turn is equivalent to a proper subset of the pure predicate calculus with identity ($\mathcal{PPI}$). The first inclusion is shown by defining a recursive translation function τ which, given a well-formed subexpression of $\mathcal{PP}$ and a binding environment (a string over the set of variables of $\mathcal{PP}$), computes the corresponding subexpression of $\mathcal{L}_N$. The *translation* of a closed wff $\phi \in \mathcal{PP}$ is then defined to be $\tau(\phi, \epsilon)$. That the inclusion is proper is proved by a routine application of Padoa's Principle to show that $\mathcal{PP}$ cannot express the property of being a singular predicate. The second inclusion is shown similarly.

In subsequent sections the following abbreviations are used to improve readability.

1. $\breve{R}^n := \langle n, \ldots, 1\rangle R^n$
2. $X \cup Y := \overline{(\overline{X} \cap \overline{Y})}$
3. $X \subseteq Y := \overline{X \cap \overline{Y}}$
4. $X \equiv Y := (X \subseteq Y) \cap (Y \subseteq X)$
5. $T := (S_0 \subseteq S_0)$
6. $X_n X_{n-1} \cdots X_1 Y := (X_n(X_{n-1} \cdots (X_1 Y) \cdots)$

7. $X^1Y^2_n \circ Y^2_{n-1} \circ \cdots \circ Y^2_1 := (\cdots(X^1Y^2_n)Y^2_{n-1})\cdots Y^2_1)$

8. $\wedge X^1 Y := \overline{X^1\overline{Y}}$

It is easy to see that:

1. $\mathcal{I} \models_\alpha X \cup Y$ iff ($\mathcal{I} \models_\alpha X$ or $\mathcal{I} \models_\alpha Y$)

2. $\mathcal{I} \models_\alpha X \subseteq Y$ iff ($\mathcal{I} \models_\alpha X$ implies $\mathcal{I} \models_\alpha Y$)

3. $\mathcal{I} \models_\alpha X \equiv Y$ iff ($\mathcal{I} \models_\alpha X$ iff $\mathcal{I} \models_\alpha Y$)

4. $\mathcal{I} \models_\alpha T$ for every $\mathcal{I}$ and α

5. $\mathcal{I} \models_\alpha X^1Y^2_n \circ \cdots \circ Y^2_1$ iff for some $d \in \mathcal{D}$, $\langle d \rangle \models X^1$ and $\langle d \rangle \models Y^2_n \circ \cdots \circ Y^2_1$ where $\circ$ denotes composition of relations in $\mathcal{I}$

6. $\mathcal{I} \models_\alpha \wedge X^1 Y$ iff for all $d \in \mathcal{D}$, $\langle d \rangle \models X^1$ implies $\langle d \rangle \models Y$

3 **Axiomatization of $\mathcal{L}_N$**

The axiom schemas of $\mathcal{L}_N$ are the following.

BT. Every schema that can be obtained from a tautologous Boolean wff by uniform substitution of nullary metavariables of $\mathcal{L}_N$ for sentential variables, $\cap$ for $\wedge$, and $\bar{}$ for $\neg$

C1. $S_{i_n} \cdots S_{i_1} \langle k_1, \ldots, k_m \rangle P \subseteq S_{i_{k_m}} \cdots S_{i_{k_1}} P$ where P is of arity m and $n = max(k_j)_{1 \leq j \leq m}$

C2. $S_{i_n} \cdots S_{i_1} \overline{\langle k_1, \ldots, k_m \rangle P} \subseteq S_{i_{k_m}} \cdots S_{i_{k_1}} \overline{P}$ where P is of arity m and $n = max(k_j)_{1 \leq j \leq m}$

EG. $(SX^1 \cap S_{i_n} \cdots S_{i_1} SY^{n+1}) \subseteq S_{i_n} \cdots S_{i_1} X^1 Y^{n+1}$

S1. SS

S2. $S_{i_n} \cdots S_{i_1} \overline{(SX^{n+1})} \equiv S_{i_n} \cdots S_{i_1} S\overline{X^{n+1}}$

D. $S_{i_n} \cdots S_{i_1} (X^m \cap Y^l) \equiv (S_{i_m} \cdots S_{i_1} X^m \cap S_{i_l} \cdots S_{i_1} Y^l)$ where $n = max(l, m)$

The inference rules of $\mathcal{L}_N$ are the following.

MP. From X^0 and $X^0 \subseteq Y^0$ infer Y^0

EI. From $\overline{(Z^0 \cap SX^1 \cap S_{i_n} \cdots S_{i_1} SY^{n+1})}$, where S does not occur in X^1, Y^{n+1}, or Z^0, and is distinct from $S_{i_1}, \ldots, S_{i_n}$, infer $\overline{(Z^0 \cap S_{i_n} \cdots S_{i_1} X^1 Y^{n+1})}$

The set $\mathcal{T}$ of theorems of $\mathcal{L}_N$ is the smallest set containing the axioms and closed under MP and EI.

Observe that by the definition of satisfaction, $\langle \mathcal{F}(S_{i_1}), \ldots, \mathcal{F}(S_{i_n})\rangle \models X^n$ iff $\langle \mathcal{F}(S_{i_2}), \ldots, \mathcal{F}(S_{i_n})\rangle \models S_{i_1}X^n$ iff $\cdots$ iff $\mathcal{I} \models S_{i_n} \cdots S_{i_1}X^n$. It follows easily from this observation and the definition of validity that the axioms are valid and that validity is preserved by the inference rules. Hence the axiomatization is sound.

Next completeness of the axiomatization is shown. Since the proof is a straightforward Henkin proof [8], a sketch will suffice. Let $\Gamma \subseteq \mathcal{L}_N$ be a set of sentences. Γ is *consistent* iff it does not contain $X_1, \ldots, X_n$ such that $\overline{X_1 \cap \cdots \cap X_n}$ is in $\mathcal{T}$. Γ is *complete* iff for every sentence $X \in \mathcal{L}_N$, either X or $\overline{X}$ is in Γ. Γ is *saturated* iff it is complete, consistent and contains SX^1 and $S_{i_n} \cdots S_{i_1}SY^{n+1}$ for some $S \in \mathcal{S}$ whenever it contains $S_{i_n} \cdots S_{i_1}X^1Y^{n+1}$. Γ^* is the set of sentences obtained from Γ by uniform substitution of S_{2i} for S_i in each $X \in \Gamma$. Thus only singular predicate symbols with even index occur in Γ^*, leaving a denumerably infinite number of "fresh" singular predicate symbols. Notice that the axioms do not reference any *particular* singular predicates. Therefore any uniform substitution of distinct singular predicates for distinct singular predicates preserves consistency and inconsistency.

Now given a set of sentences $\Gamma \subseteq \mathcal{L}_N$ it is shown that if Γ^* is consistent it can be extended to a saturated set of sentences $\Gamma^+ \subseteq \mathcal{L}_N$. An interpretation $\mathcal{I} = \langle \mathcal{D}, \mathcal{F}\rangle$ of $\mathcal{L}_N$ satisfying Γ^+ can be constructed with $\mathcal{D} = \mathcal{S}/{\sim}$, where $S_i \sim S_j$ iff $S_iS_j \in \Gamma^+$. $\mathcal{I}$ is also a model of Γ^*. Thus Γ^* is consistent iff it has a model. Obviously the same holds for Γ. It then follows that $\models X$ only if $X \in \mathcal{T}$.

4 **Some useful theorems** The main results of this section are two monotonicity theorems. These theorems establish the monotonicity properties of quantifiers (which include the resolution principle). Monotonicity is the foundation of surface reasoning. In addition, several other properties of quantifiers, including conservativity, are proved.

In the proofs of this section it often will be necessary to introduce singular predicates $S_{i_1}, \ldots, S_{i_n}$ $(n \geq 1)$ that are distinct and have no previous occurrences in the proof. To avoid unnecessary repetition, this circumstance will be conveyed by the phrase: *Let* $S_{i_1}, \ldots, S_{i_n}$ *be fresh.* To further reduce unnecessary repetition, axiom BT and rule MP will be used implicitly whenever that use is clear from the context. Most of the theorems of this section can be succinctly stated as schemas, i.e., using schematic letters or metavariables. The proof of such a schema is concerned with an arbitrary instance, or in the case of a refutation, with some particular instance, of the schema. To reduce proliferation of symbols, the same letters are used in the proof, but with the understanding that in the proof these letters represent fixed instances.

First five lemmas are stated. Their proofs are obvious and left to the reader. The first two facilitate application of axiom EG. The next two correspond to universal instantiation and generalization. The last combines axioms S2 and D.

LEMMA 1 *(schema)* ST. $\square$

LEMMA 2 *(schema)* $\overline{S_{i_n} \cdots S_{i_1} X^1 Y^{n+1}} \subseteq \overline{SX^1 \cap S_{i_n} \cdots S_{i_1} SY^{n+1}}$. $\square$

LEMMA 3 *(schema)* $(\wedge T)^n X^n \subseteq S_{i_n} \cdots S_{i_1} X^n$. □

LEMMA 4 *If* $S_{i_1}, \ldots, S_{i_n} \in \mathcal{S}$ *are distinct and do not occur in* X^n, *then* $S_{i_n} \cdots S_{i_1} X^n \in \mathcal{T}$ *implies* $(\wedge T)^n X^n \in \mathcal{T}$. □

LEMMA 5 *Let* ϕ *be obtained from a Boolean wff in sentential variables* $p_1, \ldots, p_k$ *by uniform substitution of* $\cap$ *for* $\wedge$ *and* $^-$ *for* $\neg$. *Let* $X_1^{n_1}, \ldots, X_k^{n_k} \in \mathcal{L}_N$. *Let* $n = max(n_1, \ldots, n_k)$ *and* $S_{i_1}, \ldots, S_{i_n} \in \mathcal{S}$. *Then* $\phi[S_{i_{n_1}} \cdots S_{i_1} X_1^{n_1}, \ldots, S_{i_{n_k}} \cdots S_{i_1} X_k^{n_k} / p_1, \ldots, p_k] \equiv S_{i_n} \cdots S_{i_1} \phi[X_1^{n_1}, \ldots, X_n^{n_k} / p_1, \ldots, p_k]$. □

The first theorem generalizes axiom BT.

THEOREM 6 *Let* X^n *be obtained from a Boolean tautology by uniform substitution of expressions of* $\mathcal{L}_N$ *for sentential variables,* $\cap$ *for* $\wedge$ *and* $^-$ *for* $\neg$. *Then* $(\wedge T)^n X^n \in \mathcal{T}$.

proof: Let $S_{i_1}, \ldots, S_{i_n}$ be fresh members of $\mathcal{S}$. Then Lemma 5 can be followed by Lemma 4 to yield the desired result. □

It follows from definitions given previously that the statements of Lemma 5 and Theorem 6 can be extended to read ... *by uniform substitution of* $\cap$ *for* $\wedge$, $^-$ *for* $\neg$, $\cup$ *for* $\vee$, $\subseteq$ *for* $\rightarrow$, *and* $\equiv$ *for* $\leftrightarrow$.

The next theorem is the first of two which establish the monotonicity properties of the image operation. These properties play a dominant role in reasoning in $\mathcal{L}_N$. In the examples of section 6, invocation of this theorem will be indicated by the abbreviation *MON*. First some definitions are needed.

An occurrence of a subexpression Y in an expression W has *positive (negative) polarity* if that occurrence of Y lies in the scope of an even (odd) number of $^-$ operations in W. An occurrence of a subexpression Y^m, where $m \geq 1$, is *governed by X in W* if W is $X\langle k_1, \ldots, k_m\rangle Y^m$, XY^m, $X\overline{Y^m}$, or $X(Y^m \cap Z^l)$, or the complement of one of these expressions. An occurrence of Y^m is *governed by $X_n \cdots X_1$ in W*, where $1 \leq n \leq m$, if V is governed by X_n in W and that occurrence of Y^m is governed by $X_{n-1} \cdots X_1$ in V.

THEOREM 7 (First Monotonicity Theorem) *Let Y^m occur in W with positive (respectively, negative) polarity. Let $({\wedge}T)^m(Y^m \subseteq Z^l)$ (respectively, $({\wedge}T)^m(Z^l \subseteq Y^m)$), where $l \leq m$. Let W' be obtained from W by (i) substituting Z^l for that occurrence of Y^m, (ii) substituting $\langle k_1, \ldots, k_l\rangle$ for selection operator $\langle k_1, \ldots, k_m\rangle$ on Y^m, if any, and (iii) eliminating all occurrences of governing subexpressions that no longer govern after the substitutions in (i) and (ii). Finally, let TX for every governing subexpression X with an occurrence of negative polarity that was eliminated in (iii). Then $({\wedge}T)^h(W \subseteq W')$, where h is the arity of W.*

proof: Proof is by induction on the depth of Y^m in W. If the depth is zero, then $W = Y^m$, $W' = Z^l$, and $({\wedge}T)^m(W \subseteq W')$. For the induction step, let V occur in W at some arbitrary depth and Y^m occur in V at depth one.

Case 1. $V = \langle k_1, \ldots, k_m\rangle Y^m$, where $r = max(k_i)_{1 \leq i \leq m}$.

(a) Suppose V occurs in W with positive polarity, and therefore Y^m has positive polarity in W. Let $S_{i_1}, \ldots, S_{i_r}$ be fresh and suppose $S_{i_r} \cdots S_{i_1} \langle k_1, \ldots, k_m \rangle Y^m \cap S_{i_q} \cdots S_{i_1} \overline{\langle k_1, \ldots, k_l \rangle Z^l}$, where $q = max(k_i)_{1 \leq i \leq l}$. By axiom C1, $S_{i_r} \cdots S_{i_1} \langle k_1, \ldots, k_m \rangle Y^m \subseteq S_{i_{k_m}} \cdots S_{i_{k_1}} Y^m$. By axiom C2, $S_{i_q} \cdots S_{i_1} \overline{\langle k_1, \ldots, k_l \rangle Z^l} \subseteq S_{i_{k_l}} \cdots S_{i_{k_1}} \overline{Z^l}$. Then by Lemma 5, $S_{i_{k_m}} \cdots S_{i_{k_1}} (Y^m \cap \overline{Z^l})$. However, $(\wedge T)^m (Y^m \subseteq Z^l)$ and Lemma 3 yield $S_{i_{k_m}} \cdots S_{i_{k_1}} \overline{(Y^m \cap \overline{Z^l})}$, leading by axiom S2 to a contradiction. Therefore by axioms D and S2, $S_{i_r} \cdots S_{i_1} \overline{\langle k_1, \ldots, k_m \rangle Y^m \cap \overline{\langle k_1, \ldots, k_l \rangle Z^l}}$, and by Lemma 4, $(\wedge T)^r (\langle k_1, \ldots, k_m \rangle Y^m \subseteq \langle k_1, \ldots, k_l \rangle Z^l)$. The theorem follows by the induction hypothesis.

(b) Suppose V occurs in W with negative polarity, and therefore Y^m has negative polarity in W. Suppose $S_{i_r} \cdots S_{i_1} \overline{\langle k_1, \ldots, k_m \rangle Y^m} \cap S_{i_q} \cdots S_{i_1} \langle k_1, \ldots, k_l \rangle Z^l$. Then reasoning similar to the above yields $S_{i_{k_m}} \cdots S_{i_{k_1}} (\overline{Y^m} \cap Z^l)$, while $(\wedge T)^m (Z^l \subseteq Y^m)$ and Lemma 3 yield $S_{i_{k_m}} \cdots S_{i_{k_1}} \overline{(Z^l \cap \overline{Y^m})}$, again leading to a contradiction. The theorem follows as above.

Case 2. $V = Y^m X$, where $m = l = 1$ and g is the arity of X.

(a) Suppose V occurs in W with positive polarity, and therefore Y^m has positive polarity in W. Let $S_{i_1}, \ldots, S_{i_g}$ be fresh and suppose $S_{i_g} \cdots S_{i_2} Y^1 X \cap \overline{S_{i_g} \cdots S_{i_2} Z^1 X}$. By Lemma 2, $\overline{S_{i_g} \cdots S_{i_2} S_{i_1} X \cap S_{i_1} Z^1}$. From $(\wedge T)(Y^1 \subseteq Z^1)$ and Lemma 3 follows $S_{i_1} \overline{(Y^1 \cap \overline{Z^1})}$, whence by Lemma 5, $\overline{S_{i_1} Y^1 \cap \overline{S_{i_1} Z^1}}$. Combining these results, $\overline{S_{i_g} \cdots S_{i_2} S_{i_1} X \cap S_{i_1} Z^1 \cup S_{i_1} Y^1 \cap \overline{S_{i_1} Z^1}}$. Using the Boolean tautology $p \wedge q \rightarrow p \wedge r \vee q \wedge \neg r$, it follows that $\overline{S_{i_g} \cdots S_{i_2} S_{i_1} X \cap S_{i_1} Y^1}$. Now rule EI yields $\overline{S_{i_g} \cdots S_{i_2} Y^1 X}$, which contradicts the assumption. Therefore, $S_{i_g} \cdots S_{i_2} \overline{(Y^1 X \cap \overline{Z^1 X})}$, and by Lemma 4, $(\wedge T)^{g-1} (Y^1 X \subseteq Z^1 X)$. The theorem follows by the induction hypothesis.

(b) Suppose V occurs in W with negative polarity, and therefore Y^m has negative polarity in W. Let $S_{i_1}, \ldots, S_{i_g}$ be fresh and suppose $\overline{S_{i_g} \cdots S_{i_2} Y^1 X} \cap S_{i_g} \cdots S_{i_2} Z^1 X$. Reasoning as above, this assumption yields $\overline{S_{i_g} \cdots S_{i_2} Z^1 X}$, a contradiction. This leads to the conclusion $(\wedge T)^{g-1}(Z^1 X \subseteq Y^1 X)$. The theorem again follows by the induction hypothesis.

Case 3. $V = XY^m$, where $m \geq 1$ and $l \geq 0$.

(a) Suppose V occurs in W with positive polarity, and therefore Y^m has positive polarity in W. Let $S_{i_1}, \ldots, S_{i_m}$ be fresh. Two subcases must be considered.

(i) Let $l \geq 1$ and suppose $S_{i_m} \cdots S_{i_2} X Y^m \cap \overline{S_{i_l} \cdots S_{i_2} X Z^l}$. By Lemma 2, $\overline{S_{i_l} \cdots S_{i_2} S_{i_1} Z^l \cap S_{i_1} X}$. From $(\wedge T)^m (Y^m \subseteq Z^l)$ and Lemma 3 follows $\overline{S_{i_m} \cdots S_{i_2} S_{i_1} (Y^m \cap \overline{Z^l})}$, whence by Boolean tautology $\neg p \to \neg(p \wedge q)$, $\overline{S_{i_m} \cdots S_{i_2} S_{i_1} (Y^m \cap \overline{Z^l}) \cap S_{i_1} X}$. Combining these results, $\overline{S_{i_m} \cdots S_{i_2} S_{i_1} (Z^l \cup Y^m \cap \overline{Z^l}) \cap S_{i_1} X}$, which by Lemma 5 and axiom BT yields $\overline{S_{i_m} \cdots S_{i_2} S_{i_1} Y^m \cap S_{i_1} X}$. By rule EI, $\overline{S_{i_m} \cdots S_{i_2} X Y^m}$, which contradicts the assumption. Therefore, $\overline{S_{i_m} \cdots S_{i_2} X Y^m \cap \overline{S_{i_l} \cdots S_{i_2} X Z^l}}$, which by Lemmas 5 and 4 leads to $(\wedge T)^{m-1}(XY^m \subseteq XZ^l)$. The theorem follows by the induction hypothesis.

(ii) Let $l = 0$ and suppose $S_{i_m} \cdots S_{i_2} X Y^m \cap \overline{Z^0}$. By Boolean tautology $\neg p \to \neg(p \wedge q)$, $\overline{Z^0 \cap S_{i_1} X}$. Reasoning as in subcase (i) again leads to $(\wedge T)^{m-1}(XY^m \subseteq Z^0)$ and the theorem follows by the induction hypothesis.

(b) Suppose V occurs in W with negative polarity, and therefore Y^m has negative polarity in W. Let $S_{i_1}, \ldots, S_{i_m}$ be fresh and again consider two subcases.

(i) Let $l \geq 1$ and suppose $\overline{S_{i_m} \cdots S_{i_2} X Y^m} \cap S_{i_l} \cdots S_{i_2} X Z^l$. Reasoning as above, this assumption yields $\overline{S_{i_m} \cdots S_{i_2} S_{i_1} (Y^m \cup Z^l \cap \overline{Y^m}) \cap S_{i_1} X}$, which by Lemma 5 and axiom BT yields $\overline{S_{i_l} \cdots S_{i_2} S_{i_1} Z^l \cap S_{i_1} X}$. Then by rule EI, $\overline{S_{i_l} \cdots S_{i_2} X Z^l}$, contradicting the assumption. As above, the theorem follows by Lemmas 5 and 4 and the induction hypothesis.

(ii) Let $l = 0$ and suppose $\overline{S_{i_m} \cdots S_{i_2} X Y^m} \cap Z^0$. Then as in subcase (i), $\overline{Z^0 \cap S_{i_1} X}$. Now by Lemma 1 and rule EI, $\overline{Z^0 \cap TX}$. Since TX is a condition of the theorem, this again contradicts the assumption and the theorem follows as above.

Case 4 ($V = \overline{Y^m}$) and Case 5 ($V = Y^m \cap X$) are straightforward. □

From previous definitions, it follows that if the expression ${}_\wedge YX$ occurs with positive (negative) polarity, then the occurrence of Y has negative (positive) polarity while the occurrence of X has positive (negative) polarity; if the expression $Y \subseteq X$ occurs with positive (negative) polarity, then the occurrence of Y has negative (positive) polarity while the occurrence of X has positive (negative) polarity; if the expression $Y \cup X$ occurs with positive (negative) polarity, then the occurrence of Y and the occurrence of X both have positive (negative) polarity; and if the expression $Y \equiv X$ occurs with either positive or negative polarity, then the occurrence of Y and the occurrence of X both have positive *and* negative polarity. With these provisions, Theorem 7 applies to expressions containing occurrences of defined operators. In this connection, singular predicates require special mention. Since ${}_\wedge SX := \overline{S\overline{X}} \equiv S\overline{\overline{X}} \equiv SX$, any occurrence of a singular predicate can be taken to have *either* positive or negative polarity.

COROLLARY 8 *(schema)* $(({}_\wedge T)^m (Y^m \subseteq Z^l) \cap ({}_\wedge T)^l (Z^l \subseteq W^k)) \subseteq ({}_\wedge T)^m (Y^m \subseteq W^k)$ *where* $k \leq l \leq m$. □

The following theorem provides a useful distributive property.

THEOREM 9 *(schema)* $(\wedge T)^{n-k}((\wedge X_h \cdots \wedge X_1 Y^m \cap \wedge X_j \cdots \wedge X_1 Z^l) \subseteq \wedge X_k \cdots \wedge X_1 (Y^m \cap Z^l))$ *where* $n = max(l, m)$*, and either (i)* $h = m \leq k \wedge j = k \leq l$ *or (ii)* $j = l \leq k \wedge h = k \leq m$ *or (iii)* $h = j = k \leq n$*.*

proof: Let $S_{i_1}, \ldots, S_{i_n}$ be fresh. Suppose $S_{i_n} \cdots S_{i_{k+1}}((\wedge X_h \cdots \wedge X_1 Y^m \cap \wedge X_j \cdots \wedge X_1 Z^l) \cap \overline{\wedge X_k \cdots \wedge X_1 (Y^m \cap Z^l)})$. Then by axiom D, $S_{i_m} \cdots S_{i_{h+1}} \overline{X_h \cdots X_1 \overline{Y^m}}$ and by axiom S2, $\overline{S_{i_m} \cdots S_{i_{h+1}} X_h \cdots X_1 \overline{Y^m}}$. Lemma 2 yields $\overline{S_{i_m} \cdots S_{i_1} \overline{Y^m} \cap S_{i_1} X_1 \cap \cdots \cap S_{i_h} X_h}$. By a similar argument, $\overline{S_{i_l} \cdots S_{i_1} \overline{Z^l} \cap S_{i_1} X_1 \cap \cdots \cap S_{i_j} X_j}$. Using axiom BT and rule MP, these two results yield $\overline{(S_{i_m} \cdots S_{i_1} \overline{Y^m} \cup S_{i_l} \cdots S_{i_1} \overline{Z^l}) \cap S_{i_1} X_1 \cap \cdots \cap S_{i_k} X_k}$. By Lemma 5, $\overline{S_{i_n} \cdots S_{i_1} (\overline{Y^m} \cup \overline{Z^l}) \cap S_{i_1} X_1 \cap \cdots \cap S_{i_k} X_k}$. By rule EI, $\overline{S_{i_n} \cdots S_{i_{k+1}} X_k \cdots X_1 (\overline{Y^m} \cup \overline{Z^l})}$, or equivalently, $S_{i_n} \cdots S_{i_{k+1}} \wedge X_k \cdots \wedge X_1 (Y^m \cap Z^l)$. Since this is a contradiction, the theorem follows by Lemma 4. □

Now the second monotonicity theorem can be presented. First a definition is needed.

A subexpression Y^m will be said to *occur disjunctively* in expression W iff (i) $W = \wedge X_n \cdots \wedge X_1 Y^m \cup Z$ where $n \leq m$; or (ii) $W = \wedge X_n \cdots \wedge X_{k+1}(Z_1 \cup Z_2)$ where $0 \leq k \leq n$ and Y^m occurs disjunctively in Z_1.

THEOREM 10 (Second Monotonicity Theorem) *Let* Y^m *occur disjunctively in* W*, governed by* $X_k \cdots X_1$*. Let* W' *be obtained from* W *by replacing that occurrence of* Y^m *with* Z^l $(l \leq m)$ *and deleting all occurrences of* $\wedge X_i$ *that no longer govern a subexpression. Let* TX_i *for every* $\wedge X_i$ *that was deleted. Then* $(\wedge T)^h((W \cap \wedge X_k \cdots \wedge X_1 (Y^m \subseteq$

$Z^l)) \subseteq W')$, *where* h *is the arity of* W.

proof: Proof is by induction on the depth of Y^m in W.

Basis (depth = 1). $W = {}_\wedge X_k \cdots X_1 Y^m \cup V$.

$({}_\wedge T)^h((W \cap {}_\wedge X_k \cdots {}_\wedge X_1(Y^m \subseteq Z^l)) \subseteq ({}_\wedge X_k \cdots {}_\wedge X_1 Y^m \cap {}_\wedge X_k \cdots {}_\wedge X_1(Y^m \subseteq Z^l) \cup V))$ by Theorem 6 and Boolean tautology $((p \vee q) \wedge r) \rightarrow (p \wedge r \vee q)$. Then $({}_\wedge T)^h(({}_\wedge X_k \cdots {}_\wedge X_1 Y^m \cap {}_\wedge X_k \cdots {}_\wedge X_1(Y^m \subseteq Z^l) \cup V) \subseteq ({}_\wedge X_k \cdots {}_\wedge X_1(Y^m \cap (Y^m \subseteq Z^l)) \cup V))$ by Theorems 9 and 7. By Theorem 6 and Boolean tautology $(p \wedge (p \rightarrow q)) \rightarrow q$, $({}_\wedge T)^{h+k}((Y^m \cap (Y^m \subseteq Z^l)) \subseteq Z^l)$, whence by Theorem 7, $({}_\wedge T)^h(({}_\wedge X_k \cdots {}_\wedge X_1(Y^m \cap (Y^m \subseteq Z^l)) \cup V) \subseteq ({}_\wedge X_j \cdots {}_\wedge X_1 Z^l \cup V))$. Finally, $({}_\wedge T)^h((W \cap {}_\wedge X_k \cdots {}_\wedge X_1(Y^m \subseteq Z^l)) \subseteq W')$, by Corollary 8.

Induction (depth > 1). $W = {}_\wedge X_n \cdots X_{q+1}(Z_1 \cup Z_2)$ where Y^m occurs disjunctively in Z_1, governed by $X_j \cdots X_1$, $0 \leq j \leq q, k$.

$({}_\wedge T)^h((W \cap {}_\wedge X_k \cdots {}_\wedge X_1(Y^m \subseteq Z^l)) \subseteq {}_\wedge X_n \cdots {}_\wedge X_{q+1}({}_\wedge X_j \cdots {}_\wedge X_1(Y^m \subseteq Z^l) \cap (Z_1 \cup Z_2)))$ by Theorem 9. Then $({}_\wedge T)^{h+n-q}(({}_\wedge X_j \cdots {}_\wedge X_1(Y^m \subseteq Z^l) \cap (Z_1 \cup Z_2)) \subseteq ({}_\wedge X_j \cdots {}_\wedge X_1(Y^m \subseteq Z^l) \cap Z_1 \cup Z_2))$ by Theorem 6. By Theorem 7, $({}_\wedge T)^h({}_\wedge X_n \cdots {}_\wedge X_{q+1}({}_\wedge X_j \cdots {}_\wedge X_1(Y^m \subseteq Z^l) \cap (Z_1 \cup Z_2)) \subseteq {}_\wedge X_n \cdots {}_\wedge X_{q+1}({}_\wedge X_j \cdots {}_\wedge X_1(Y^m \subseteq Z^l) \cap Z_1 \cup Z_2))$. Now by the induction hypothesis, $({}_\wedge T)^g(({}_\wedge X_j \cdots {}_\wedge X_1(Y^m \subseteq Z^l) \cap Z_1) \subseteq Z_1')$ where g is the arity of Z_1 and Z' is obtained from Z as W' was obtained from W. Again by Theorem 7, $({}_\wedge T)^h({}_\wedge X_n \cdots {}_\wedge X_{q+1}({}_\wedge X_j \cdots {}_\wedge X_1(Y^m \subseteq Z^l) \cap Z_1 \cup Z_2) \subseteq {}_\wedge X_{n'} \cdots {}_\wedge X_{q+1}(Z_1' \cup Z_2))$, where $n' \leq n$. Finally, $({}_\wedge T)^h((W \cap {}_\wedge X_k \cdots {}_\wedge X_1(Y^m \subseteq Z^l)) \subseteq W')$, by Corollary 8. □

It is easy to see (from the equivalence $(Y^m \subseteq Z^l) \equiv (\overline{Y^m} \cup Z^l))$ that this theorem corresponds to the resolution principle in conventional logic. A corollary provides a rule corresponding to unit resolution. It will be referred to as the *Cancellation Rule.* In section 6, its invocation will be indicated by the abbreviation *CANC.*

COROLLARY 11 *Let Y^m occur disjunctively in W, governed by $X_k \cdots X_1$. Let W' be obtained from W by deleting that occurrence of Y^m and all occurrences of $\wedge X_i$ that no longer govern a subexpression. Let TX_i for every $\wedge X_i$ that was deleted. Then $(\wedge T)^h((W \cap \wedge X_k \cdots \wedge X_1 \overline{Y^m}) \subseteq W')$, where h is the arity of W.* □

The image operation is further characterized by the next theorem. The first corollary establishes the property of conservativity. The second provides equivalent forms and gives the rules for conversion in the case of unary predicates.

THEOREM 12 *(schema) $X_m \cdots X_2 X_1 Y^m \equiv X_m \cdots X_2 T(Y^m \cap X_1)$.*

proof: Suppose $\overline{X_m \cdots X_2 X_1 Y^m} \cap X_m \cdots X_2 T(Y^m \cap X_1)$. Let $S_{i_1}, \ldots, S_{i_m}$ be fresh. Then by Lemma 2, $\overline{S_{i_m} \cdots S_{i_1} Y^m \cap S_{i_1} X_1 \cap \cdots \cap S_{i_m} X_m}$. By axiom D, $\overline{S_{i_m} \cdots S_{i_1}(Y^m \cap X_1) \cap S_{i_2} X_2 \cap \cdots \cap S_{i_m} X_m}$ and therefore also $\overline{S_{i_m} \cdots S_{i_1}(Y^m \cap X_1) \cap S_{i_1} T \cap S_{i_2} X_2 \cap \cdots \cap S_{i_m} X_m}$. Rule EI yields $\overline{X_m \cdots X_2 T(Y^m \cap X_1)}$, resulting in a contradiction. Conversely, suppose $X_m \cdots X_2 X_1 Y^m \cap \overline{X_m \cdots X_2 T(Y^m \cap X_1)}$. Let $S_{i_1}, \ldots, S_{i_m}$ be fresh. Then by Lemma 2, $\overline{S_{i_m} \cdots S_{i_1}(Y^m \cap X_1) \cap S_{i_1} T \cap S_{i_2} X_2 \cap \cdots \cap S_{i_m} X_m}$. By axiom D, $\overline{S_{i_m} \cdots S_{i_1} Y^m \cap S_{i_1} T \cap S_{i_1} X_1 \cap \cdots \cap S_{i_m} X_m}$ and therefore also $\overline{S_{i_m} \cdots S_{i_1} Y^m \cap S_{i_1} X_1 \cap \cdots \cap S_{i_m} X_m}$. Rule EI yields $\overline{X_m \cdots X_2 X_1 Y^m}$, resulting in a contradiction. □

COROLLARY 13 (Conservativity) *(schema) (i) $X_m \cdots X_2 X_1 Y^m \equiv X_m \cdots X_2 X_1 (Y^m \cap X_1)$ (ii) $\wedge X_m \cdots \wedge X_2 \wedge X_1 Y^m \equiv \wedge X_m \cdots \wedge X_2 \wedge X_1 (Y^m \cap X_1)$.* □

COROLLARY 14 *For unary expressions* X *and* Y, *(i)* $XY \equiv T(X \cap Y)$ *(ii)* $XY \equiv YX$ *(iii)* $\wedge XY \equiv \wedge T(X \subseteq Y)$ *(iv)* $\wedge XY \equiv \wedge(\overline{Y})\,\overline{X}$. □

It is now easy to prove that the image operation defines an identity relation on $\mathcal{S}$. Indeed if I is the identity relation, then it can be axiomatized by the schema $S_iS_jI \equiv S_iS_j$.

THEOREM 15 *(schema) (i)* S_iS_i *(ii)* $S_iS_j \equiv S_jS_i$ *(iii)* $(S_iS_j \cap S_jS_k) \subseteq S_iS_k$ *(iv) If* S_i *occurs in* W, W' *is obtained from* W *by substituting* S_j *for that occurrence of* S_i, *and* S_iS_j, *then* $(\wedge T)^h(W \subseteq W')$, *where* h *is the arity of* W *(v) From the schema* $S_iX^1 \equiv S_jX^1$, *infer* S_iS_j.

proof: (i) Axiom S1. (ii) Corollary 14. (iii) Corollary 14 and Theorem 7. (iv) Corollary 14 and Theorem 7. (v) If the schema holds, then $S_iS_j \equiv S_jS_j$. Therefore S_iS_j. □

5 $\mathcal{L}_N$ **and natural language structure** In this section an English fragment is offered in support of the claim that $\mathcal{L}_N$ is structurally similar to natural language. The syntax of the fragment and its translation to $\mathcal{L}_N$ are defined by an attribute grammar. To make the grammar brief, some inessential simplifications are adopted. Morphological rules necessary to achieve proper noun and verb forms are omitted. Only the conjunction **and** is shown; **or** can be dealt with similarly. The grammar is allowed to be syntactically ambiguous.

To further enhance the presentation, the following "syntactic sugar" is added to $\mathcal{L}_N$.

$$\textbf{thing} := T$$

$$\textbf{some}X^1Y := X^1Y$$

$$\textbf{all}X^1Y := {\wedge}X^1Y$$

$$\textbf{no}X^1Y := \overline{\textbf{some}X^1Y}$$

The attribute grammar follows. τ is the translation mapping.

S	$\rightarrow$ S **and** S	$\tau(\mathrm{S}_1) \leftarrow \tau(\mathrm{S}_2) \cap \tau(\mathrm{S}_3)$
	$\vert$D CN VP	$\tau(\mathrm{S}) \leftarrow \tau(\mathrm{D})\tau(\mathrm{CN})\tau(\mathrm{VP})$
	$\vert$D CN **do not** VP	$\tau(\mathrm{S}) \leftarrow \overline{\tau(\mathrm{D})\tau(\mathrm{CN})\tau(\mathrm{VP})}$
	$\vert$PN VP	$\tau(\mathrm{S}) \leftarrow \tau(\mathrm{PN})\tau(\mathrm{VP})$
	$\vert$PN **do not** VP	$\tau(\mathrm{S}) \leftarrow \overline{\tau(\mathrm{PN})\tau(\mathrm{VP})}$
	$\vert$**there be** VP	$\tau(\mathrm{S}) \leftarrow$ **some thing** $\tau(\mathrm{VP})$

CN	$\rightarrow$ CN **and** CN	$\tau(\text{CN}_1) \leftarrow \tau(\text{CN}_2) \cap \tau(\text{CN}_3)$
	|A CN	$\tau(\text{CN}_1) \leftarrow \tau(\text{A}) \cap \tau(\text{CN}_2)$
	|CN **that** VP	$\tau(\text{CN}) \leftarrow \tau(\text{CN}) \cap \tau(\text{VP})$
	|BCN	$\tau(\text{CN}) \leftarrow \tau(\text{BCN})$
PN	$\rightarrow$ PN **and** PN	$\tau(\text{PN}_1) \leftarrow \tau(\text{PN}_2) \cap \tau(\text{PN}_3)$
	|BPN	$\tau(\text{PN}) \leftarrow \tau(\text{BPN})$
VP	$\rightarrow$ VP **and** VP	$\tau(\text{VP}_1) \leftarrow \tau(\text{VP}_2) \cap \tau(\text{VP}_3)$
	|TV D CN	$\tau(\text{VP}) \leftarrow \tau(\text{D})\tau(\text{CN})\ (\tau(\breve{\text{TV}}))$
	|TV PN	$\tau(\text{VP}) \leftarrow \tau(\text{PN})\ (\tau(\breve{\text{TV}}))$
	|**be-en** TV **by** D CN	$\tau(\text{VP}) \leftarrow \tau(\text{D})\tau(\text{CN})\tau(\text{TV})$
	|**be-en** TV **by** PN	$\tau(\text{VP}) \leftarrow \tau(\text{PN})\tau(\text{TV})$
	|IV	$\tau(\text{VP}) \leftarrow \tau(\text{IV})$
TV	$\rightarrow$ **do not** BTV	$\tau(\text{TV}) \leftarrow \overline{\tau(\text{BTV})}$
	|BTV	$\tau(\text{TV}) \leftarrow \tau(\text{BTV})$
IV	$\rightarrow$ **do not** BIV	$\tau(\text{IV}) \leftarrow \overline{\tau(\text{BIV})}$
	|BIV	$\tau(\text{IV}) \leftarrow \tau(\text{BIV})$

A small lexicon is provided.

D: **some, all, no, a, every**

A: **black, spotted**

BCN: **dog, cat**

BPN: **Bert, Helen**

BIV: **run, bark**

BTV: **like, chase**

a translates to **some** and **every** translates to **all**; otherwise τ is the identity function on the lexicon.

In view of the incomplete understanding of human language, it cannot be *proved* that $\mathcal{L}_N$ has the same structure as natural language; but the above grammar demonstrates that a well-translation [3] can be defined between $\mathcal{L}_N$ and a simple English fragment.

This grammar is of further interest because of the interpretation of English it induces. It deviates from Montagovian semantics [1, 13] in several respects. Most significant is the absence of term phrases, which denote (in a purely extensional Montagovian semantics) sets of sets of individuals. In the fragment defined above, determiners are functors that combine directly with two predicates; a determiner and one predicate do not form a phrase. Determiners thus denote binary relations on subsets of the universe of individuals. The fragment has no phrases that denote sets of sets. As a dividend, proper nouns always denote individuals - or, more precisely, singleton sets rather than sometimes individuals and other times sets of sets of individuals [1].

Relative clauses are always unary predicates. Thus for example the sentence **every dog that chases a cat barks** can be given the *de dicto* reading **every (dog** $\cap$ **some cat** $\breve{\textbf{chase}}$**)bark**. In contrast to this, the *de re* reading (which incidentally lies outside the above grammar) would be S_i**cat** $\cap$ **every (dog** $\cap S_i$ $\breve{\textbf{chase}}$**)bark**.

In a sense, these deviations are in the direction of a simpler semantics. This will influence the form that reasoning takes in $\mathcal{L}_N$. The next section discusses this further.

6 $\mathcal{L}_N$ **and natural language reasoning** Theorem 15 implies that $\mathcal{L}_N$ has an expressiveness essentially equivalent to that of $\mathcal{PPI}$. For example, elementary group theory can be axiomatized and developed in $\mathcal{L}_N$ in essentially the same way as in $\mathcal{PPI}$. This however is not the principal claim made for $\mathcal{L}_N$. Rather $\mathcal{L}_N$ is claimed to mirror the structure of natural language and the process of natural language reasoning. The previous section provided some support for the first claim; this section will address the second.

The organizing principle of reasoning in $\mathcal{L}_N$ is that of monotonicity as enunciated by the first and second monotonicity theorems and their corollaries. The importance of this principle is illustrated below by several examples. In addition, the examples demonstrate the following. (i) Not only the problem statement but each step in the reasoning process is *directly* intertranslatable with English. (ii) The reasoning process is one of incrementally building a model of the world entailed by the problem statement.

In general only a partial model is needed. If a partial model entailed by the premises contains the desired conclusion, then a direct proof has been constructed. If a model entailed by the premises conjoined with the denial of the conclusion does not exist (i.e., the attempt to build such a model fails), then an indirect proof has been constructed. Each step in building a model adds another fact about the kinds of individuals in the world entailed by the problem statement, that is, about the subsets of the model universe.

6.1 **Exercises from introductory logic** These examples are taken from Sommers [11]. They are intentionally simple so that the details of each step in the reasoning process can be given. Each step consists of a $\mathcal{L}_N$ expression, its justification, and a direct English equivalent. To make the Boolean character of reasoning in $\mathcal{L}_N$ apparent, "universal closure" is implicit. For example, $\wedge T(D \subseteq M\breve{F})$ is abbreviated $D \subseteq M\breve{F}$.

example 1 Some horses are faster than some dogs. All dogs are faster than some men. Therefore, some horses are faster than some men. (Implicit assumption: **faster** and its converse are transitive relations.)
proof (direct):

1	$\mathbf{some}H\mathbf{some}D\breve{F}$	P	some horses are faster than some dogs
2	$\mathbf{all}D\mathbf{some}M\breve{F}$	P	all dogs are faster than some men
3	$D \subseteq \mathbf{some}M\breve{F}$	2,Cor14	all dogs are faster than some men
4	$\mathbf{some}H\mathbf{some}(\mathbf{some}M\breve{F})\breve{F}$	1,3,MON	some horses are faster than some things faster than some men
5	$\mathbf{some}H\mathbf{some}M(\breve{F} \circ \breve{F})$	4,Defn	some horses are faster than some things faster than some men
6	$\breve{F} \circ \breve{F} \subseteq \breve{F}$	P	for all pairs of things, the first being faster than something faster than the second implies the first being faster than the second (converse of **faster** is transitive)

7	**some**H**some**$M\breve{F}$	5,6,MON	some horses are faster than some men

This proof can also be presented graphically in the form of a Hasse diagram (see Figure 1). Each node is labelled with a $\mathcal{L}_N$ expression. Consider a pair of nodes with labels X and Y, and let n be the greater of their arities. An arc ascending from X to Y represents the assertion $(\mathbf{all}T)^n(X \subseteq Y)$, which may be abbreviated $X \subseteq Y$. A pair of arcs descending from X and Y to a common node represents the assertion $(\mathbf{some}T)^n(X \cap Y)$, which also may be written $(X \cap Y) \not\subseteq \overline{T}$. The premises are represented in the diagram by heavy arcs; the lighter arcs represent inferences. H' denotes a nonempty set. There are two inferences, both based on the monotonicity principle. The conclusion follows from the circumstance $H \cap \mathbf{some}M\breve{F} \not\subseteq \overline{T}$. The Hasse diagram of the partial model is easy to grasp intuitively and has a compelling similarity to human reasoning.

example 2: All supporters of Nixon will vote for Reagan. Avery will vote for none but a friend of Harriman. No friend of Khrushchev has Reagan for a friend. Harriman is a friend of Khrushchev. Therefore, Avery will not support Nixon.

proof (indirect):

1	**all**$(N\breve{S})R\breve{V}$	P	all supporters of Nixon will vote for Reagan
2	**all**$(AV)H\breve{F}$	P	all those for whom Avery will vote are friends of Harriman
3	**no**$(K\breve{F})RF$	P	no friend of Khrushchev has Reagan for a friend
4	$HK\breve{F}$	P	Harriman is a friend of Khrushchev

5	$AN\breve{S}$	Denial	Avery is a supporter of Nixon
6	$N\breve{S} \subseteq R\breve{V}$	1,Cor14	all supporters of Nixon will vote for Reagan
7	$AR\breve{V}$	5,6,MON	Avery will vote for Reagan
8	RAV	7,C1	Reagan is one for whom Avery will vote
9	$AV \subseteq H\breve{F}$	2,Cor14	all those for whom Avery will vote are friends of Harriman
10	$RH\breve{F}$	8,9,MON	Reagan is a friend of Harriman
11	HRF	10,C1	Harriman has Reagan for a friend
12	$H \subseteq K\breve{F}$	4,S2,Cor14	Harriman is a friend of Khrushchev
13	$\textbf{no}HRF$	3,12,MON	Harriman does not have Reagan for a friend (contradicts 11)

Again the proof can be presented graphically. Using the same conventions as before, the Hasse diagram is shown in Figure 2. In this example, inferences are based on conversion (axiom C1) as well as the monotonicity principle. That the premises and the denial of the conclusion have no model is seen from the contradictory circumstance $RF \cap \overline{RF} \not\subseteq \overline{T}$.

This example illustrates that an indirect proof can be viewed as a process of model elimination (in contrast to model building), with the result that all models are finally eliminated.

6.2 **Schubert's Steamroller** In 1978 Lenhart Schubert formulated the following problem as a challenge to automated reasoning systems.

Wolves, foxes, birds, caterpillars, and snails are animals, and there are some of each of them. Also there are some grains, and grains are plants. Every animal either likes to eat all plants or all animals much smaller than itself that like to eat some plants. Caterpillars and snails are much smaller than birds, which are much smaller than foxes, which in turn are much smaller than wolves. Wolves do not like to eat foxes or grains, while birds like to eat caterpillars but not snails. Caterpillars and snails like to eat some plants. Therefore there is an animal that likes to eat a grain-eating animal.

To save space the proof is given without English translations. The translations are easy. As an example, step 24 can be translated **all wolves either like to eat all grains or all foxes are either not much smaller than they or do not like to eat all plants or are liked to be eaten by them.** It might be remarked in passing that $\mathbf{all}W(\mathbf{all}P\breve{E} \cup \mathbf{all}(A \cap M \cap \mathbf{some}P\breve{E})\breve{E})$, which may seem more direct than 24, is not a well-formed expression of $\mathcal{L}_N$.

proof (direct): The premises 1-23 are stated first. The conclusion is given by 36-37.

1 $\mathbf{all}A(\mathbf{all}P\breve{E} \cup \mathbf{all}A(\overline{M} \cup \mathbf{all}P\overline{\breve{E}} \cup \breve{E}))$

2-7	$\mathbf{all}WA$	$\mathbf{all}FA$	$\mathbf{all}BA$	$\mathbf{all}CA$	$\mathbf{all}SA$	$\mathbf{all}GP$
8-13	TW	TF	TB	TC	TS	TG
14-17	$\mathbf{all}W\mathbf{all}FM$	$\mathbf{all}F\mathbf{all}BM$	$\mathbf{all}B\mathbf{all}CM$	$\mathbf{all}B\mathbf{all}SM$		
18-21	$\mathbf{all}W\mathbf{all}F\overline{\breve{E}}$	$\mathbf{all}W\mathbf{all}G\overline{\breve{E}}$	$\mathbf{all}B\mathbf{all}C\breve{E}$	$\mathbf{all}B\mathbf{all}S\overline{\breve{E}}$		

22-23 $\mathbf{all}C\mathbf{some}P\breve{E}$ $\mathbf{all}S\mathbf{some}P\breve{E}$

24	$\mathbf{all}W(\mathbf{all}G\breve{E} \cup \mathbf{all}F(\overline{M} \cup \mathbf{all}P\overline{\breve{E}} \cup \breve{E}))$	1,2,3,7,MON
25	$\mathbf{all}W\mathbf{all}F(\overline{M} \cup \mathbf{all}P\overline{\breve{E}} \cup \breve{E})$	19,24,CANC
26	$\mathbf{all}W\mathbf{all}F(\mathbf{all}P\overline{\breve{E}} \cup \breve{E})$	14,25,CANC
27	$\mathbf{all}F\mathbf{all}P\overline{\breve{E}}$	18,26,CANC
28	$\mathbf{all}F(\mathbf{all}P\breve{E} \cup \mathbf{all}B(\overline{M} \cup \mathbf{all}P\overline{\breve{E}} \cup \breve{E}))$	1,3,4,MON
29	$\mathbf{all}F\mathbf{all}B(\overline{M} \cup \mathbf{all}P\overline{\breve{E}} \cup \breve{E})$	27,28,CANC
30	$\mathbf{all}F\mathbf{all}B(\mathbf{all}P\overline{\breve{E}} \cup \breve{E})$	15,29,CANC
31	$\mathbf{all}B(\mathbf{all}P\breve{E} \cup \mathbf{all}S(\overline{M} \cup \mathbf{all}P\overline{\breve{E}} \cup \breve{E}))$	1,4,6,MON
32	$\mathbf{all}B(\mathbf{all}P\breve{E} \cup \mathbf{all}S(\mathbf{all}P\overline{\breve{E}} \cup \breve{E}))$	17,31,CANC
33	$\mathbf{all}\overline{S\mathbf{all}P\overline{\breve{E}}}$	23,Defn
34	$\mathbf{all}B(\mathbf{all}P\breve{E} \cup \mathbf{all}S\breve{E})$	32,33,CANC
35	$\mathbf{all}B\mathbf{all}P\breve{E}$	21,34,CANC
36	$\mathbf{all}F\mathbf{all}B\breve{E}$	30,35,CANC
37	$\mathbf{all}B\mathbf{all}G\breve{E}$	35,MON

Because of its larger size, the partial model for Schubert's Steamroller will not be presented as a Hasse diagram. The first "lemma" (steps 24-27) can be so presented however (see Figure 3). The heavy arcs represent the inferences from step 24 to step 27. For example, the inference from 25 to 26 is: if $W \subseteq \mathbf{all}F(\overline{M} \cup \mathbf{all}P\overline{\breve{E}} \cup \breve{E})$ and

$W \subseteq \mathbf{all}FM$, then $W \subseteq \mathbf{all}F(\mathbf{all}P\overline{\breve{E}} \cup \breve{E})$.

Notice that $\mathbf{all}F\mathbf{all}P\overline{\breve{E}}$ is nullary, in contrast to the other expressions, which are unary. To interpret this, observe that $\mathbf{all}T(W \subseteq \mathbf{all}F\mathbf{all}P\overline{\breve{E}}) \equiv (TW \subseteq \mathbf{all}F\mathbf{all}P\overline{\breve{E}})$. Since TW, the result $\mathbf{all}F\mathbf{all}P\overline{\breve{E}}$ follows.

Schubert's Steamroller remained a challenge to automated reasoning systems for a number of years because of its potentially enormous search space. See [12] for a good review. It finally yielded to reasoning systems employing many-sorted logic. $\mathcal{L}_N$ is implicitly a many-sorted logic. Indeed, as with all natural languages, reasoning with sorts is intrinsic to $\mathcal{L}_N$. It is remarkable that the restriction imposed by sorts and the Cancellation Rule strategy together reduce the total search space for Schubert's Steamroller to 30 expressions. Remarkable also is the use of the First Monotonicity Theorem to accomplish unification without complexities such as the "occur-check."

6.3 **Discussion** Although psychological theories of human reasoning abound, it can be said with confidence that human reasoning is not well enough understood to permit anything to be proved about it. Consequently, the claim that $\mathcal{L}_N$ mirrors natural language reasoning must be argued on intuitive grounds.

It is clear from the examples that reasoning in $\mathcal{L}_N$ is concerned with describing a world or model in terms of classes of individuals and the ways in which they are related. Specifically it is concerned with inclusion, exclusion and overlap as represented by expressions of the forms $X \subseteq Y$, $X \subseteq \overline{Y}$, and $X \cap Y \not\subseteq \overline{T}$. These are precisely the relations conveyed by the categorical statements (A, E and I, respectively) of syllogistic. Syllogistic is often proposed, by psychologists and philosophers alike, as a model of human reasoning competence. Its survival for twenty three centuries is testimony to the fundamental importance of these relations in human reasoning.

The monotonicity properties of the Boolean connectives can be viewed as basic to reasoning in propositional logic. Adding the monotonicity properties of quantifiers, syllogistic extends this mode of reasoning to monadic logic. By generalization of these monotonicity properties as enunciated by the two monotonicity theorems and their corollaries, $\mathcal{L}_N$ extends this mode of reasoning to polyadic logic.

As a consequence, reasoning in $\mathcal{L}_N$ is essentially building models of the world entailed by the set of premises. While similar to building semantic trees or model (Hintikka) sets, reasoning in $\mathcal{L}_N$ differs because of the Boolean character of the relations which constrain the classes of individuals that may exist in the world.

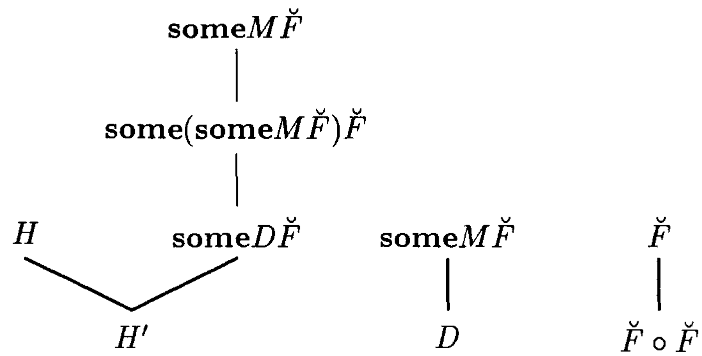

Figure 1: A partial model for the first exercise

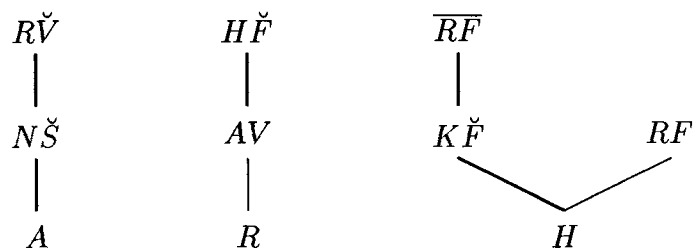

Figure 2: Model construction fails for the second exercise

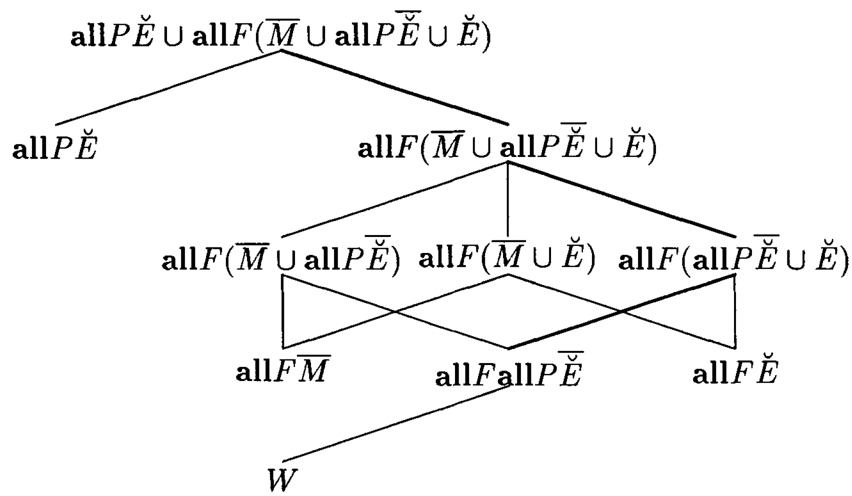

Figure 3: A fragment of the model for Schubert's Streamroller

7 Conclusion

The claims that $\mathcal{L}_N$ mirrors natural language structure and natural language reasoning have been argued on intuitive grounds using examples. The state of knowledge in cognitive science does not permit more. Additional evidence will be presented in subsequent papers on $\mathcal{L}_N$. This will take the form of extending the language to additional constructs of natural language, and further analysis of reasoning in $\mathcal{L}_N$ to establish further connections with natural language reasoning.

In the first direction, $\mathcal{L}_N$ will be extended to include generalized quantifiers of natural language. The cardinal quantifiers **at least k** can be axiomatized much the same as **some**, requiring the addition of two axiom schemas and a rule of inference. **exactly k** and **less than k** can then be introduced by definition. The second-order quantifier **most** can also be axiomatized, but here completeness requires restriction of model size to not exceed some fixed limit N. Monotonicity properties and conversion rules can then be derived. This can be accomplished by definition in first-order logic with identity; the axiomatization in $\mathcal{L}_N$ is equivalent.

In the second direction, reasoning in $\mathcal{L}_N$ will be investigated in relation to Hintikka's notion of surface information [4]. Hintikka has suggested that natural language meaning and understanding are best understood in terms of surface information, that is, the results of deduction in which depth does not exceed that of the premises. Here depth is defined as the maximum number of nested quantifiers or the maximum number of individuals simultaneously considered. When depth is allowed to increase beyond that of the premises, depth information is produced. This seems to closely match the intuitive notion of reasoning involved in natural language understanding. The reasoning in the examples of the previous section illustrate this. The distinction is not only a philosophical one. It also promises to shed light on the kinds of reasoning that characterize natural language understanding.

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