

12-30-2005

Finite-Dimensional Algebras with Smallest Resolutions of Simple Modules

Shashidhar Jagadeeshan
Centre for Learning Bangalore

Mark Kleiner
Syracuse University

Follow this and additional works at: <https://surface.syr.edu/mat>

 Part of the [Mathematics Commons](#)

Recommended Citation

Jagadeeshan, Shashidhar and Kleiner, Mark, "Finite-Dimensional Algebras with Smallest Resolutions of Simple Modules" (2005).
Mathematics Faculty Scholarship. 74.
<https://surface.syr.edu/mat/74>

This Article is brought to you for free and open access by the Mathematics at SURFACE. It has been accepted for inclusion in Mathematics Faculty Scholarship by an authorized administrator of SURFACE. For more information, please contact surface@syr.edu.

FINITE-DIMENSIONAL ALGEBRAS WITH SMALLEST RESOLUTIONS OF SIMPLE MODULES

SHASHIDHAR JAGADEESHAN AND MARK KLEINER

INTRODUCTION

Let Λ be an associative ring with identity and with the Jacobson radical \mathfrak{r} , let $\text{mod } \Lambda$ be the category of finitely generated left Λ -modules, and let Λ^{op} be the opposite ring of Λ . All modules are left unital modules, and if X is a module then $\text{pd } X$ is the projective dimension of X . If Λ is left artinian and $M \in \text{mod } \Lambda$, we denote by $P(M)$ a projective cover of M . Throughout the paper we fix an arbitrary field k . For terminology and notation, we refer the reader to [2, 4].

If Λ is a finite-dimensional k -algebra, any nonzero $M \in \text{mod } \Lambda$ has a minimal projective resolution

$$\cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0, \quad (0.1)$$

which is a very important but rather complicated homological invariant of the module M . An interesting numerical invariant of the above resolution is the complexity of M , $c_\Lambda(M)$, which measures the growth of the size of the n -th term, P_n , as $n \rightarrow \infty$. The notion of complexity is especially important when $\Lambda = kG$ is the group algebra over k of a finite group G , where $\text{char } k$ divides the order of G , because $c_{kG}(M) = c_G(M)$ is the dimension of the cohomological support variety $V_{kG}(M) = V_G(M)$ associated to M (see [4, Ch. 5]). We recall the definition of complexity. Let \mathbb{N} be the set of nonnegative integers, and let $\mathbf{b} = \{b_n \mid n \in \mathbb{N}\}$ be an arbitrary sequence with $b_n \in \mathbb{N}$. The growth of \mathbf{b} is the least nonnegative integer $\gamma(\mathbf{b})$ for which there is a positive real number A satisfying $b_n \leq An^{\gamma(\mathbf{b})-1}$ for sufficiently large n . If no such integer exists then $\gamma(\mathbf{b}) = \infty$. By definition, $c_\Lambda(M) = \gamma(\{\dim_k P_n\})$.

For an arbitrary finite-dimensional algebra Λ , the cohomological support variety $V_\Lambda(M)$ of $M \in \text{mod } \Lambda$ is defined in [14]. The definition works well for some classes of algebras; in particular, many properties of the support variety over a group algebra extend to self-injective algebras over an algebraically closed field that satisfy appropriate finite generation assumptions [6]. However, some of those properties do not extend to all algebras. For example, let Λ be a self-injective algebra whose Hochschild cohomology ring $\text{HH}^*(\Lambda)$ contains a commutative noetherian graded subalgebra H where (i) $H^0 = \text{HH}^0(\Lambda) = Z(\Lambda)$ and (ii) $\text{Ext}_\Lambda^*(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is a finitely generated H -module. Under these assumptions, if $M \in \text{mod } \Lambda$ then $c_\Lambda(M) < \infty$, and $V_\Lambda(M)$ is trivial if and only if M is projective. On the other hand, the authors of [14] and [6] have constructed finite-dimensional algebras Λ of infinite global dimension over an algebraically closed field for which $c_\Lambda(M) = \infty$ for some $M \in \text{mod } \Lambda$, or $V_\Lambda(M)$ is trivial for all $M \in \text{mod } \Lambda$. Some of these algebras belong to the class of algebras arising in this paper, and Example 3.5 exhibits two such algebras for which each module is eventually periodic (see the definition below). However, for one of the algebras the support variety of each module is trivial, while for the other some modules have nontrivial support

1991 *Mathematics Subject Classification.* 16P10; Secondary 16E60.

Key words and phrases. Algebra, minimal projective resolution, preorder, quiver with relations.

The first author would like to thank the Department of Mathematics at Syracuse University for having given him the opportunity to spend the academic year 2002-2003 there.

varieties. Thus, at least for the algebras in this class, there is no clear connection between the cohomological support variety of a module and the behavior of the terms of its minimal projective resolution. We also note that the notions of complexity and support variety do not distinguish between modules of finite projective dimension: all such modules have complexity zero and trivial support variety.

To study modules of arbitrary projective dimension over a class of finite-dimensional algebras containing all algebras over an algebraically closed field, we examine a numerical invariant of the resolution (0.1) that is finer than $c_\Lambda(M)$. This invariant is the sequence

$$p(M) = \{p(M)_n = \ell(P_n) \mid n \in \mathbb{N}\}$$

where $\ell(X)$ is the length of $X \in \text{mod } \Lambda$.

The obvious question is why $p(M)$ is preferable to $\{\dim_k P_n\}$. There are several answers. First of all, one can recover $c_\Lambda(M)$ from $p(M)$ because, according to [3], $\gamma(\{\dim_k P_n\}) = \gamma(p(M))$ so that $c_\Lambda(M) = \gamma(p(M))$, and the latter formula also works when Λ is a left artinian ring. Second, $p(M)$ is a categorical invariant of M , and thus is preserved under Morita equivalence, while $\{\dim_k P_n\}$ is not. Third, it may be easier to detect finite projective dimension from $p(M)$ than from $\{\dim_k P_n\}$, e.g., if $\ell(P_n) = 1$ then P_n is simple projective, so $P_{n+t} = 0$ for $t > 0$, while it is impossible to detect a simple projective module P from $\dim_k P$ when $\dim_k P > 1$. Finally, for a minimal projective resolution

$$\cdots \rightarrow Q_2 \rightarrow Q_1 \rightarrow Q_0 \rightarrow N \rightarrow 0 \tag{0.2}$$

of $N \in \text{mod } \Gamma$ where Γ is a left artinian ring, we can compare the lengths $\ell(P_m)$ and $\ell(Q_n)$, while the comparison of dimensions of P_m and Q_n over a field may be either impossible or unnatural.

Of course, certain properties of the sequence $p(M)$ have already been addressed in the literature; see, for instance, [4, Theorem 5.10.4 and Corollary 5.10.7] on the periodicity of $M \in \text{mod } \Lambda$ when $\Lambda = kG$ is the group algebra and $c_\Lambda(M) = 1$. We recall that given the minimal projective resolution (0.1), the n -th syzygy of M is $\Omega^n M = \text{Coker } d_{n+1}$, and M is said to be *eventually periodic* if there exists an integer $t > 0$ such that $\Omega^{s+t} M \simeq \Omega^s M$ for some integer $s \geq 0$; if $s = 0$ then M is *periodic*. The smallest possible t is the *period* of M .

Our approach to the set of sequences $p(X)$ is based on the introduction of a means to compare two such sequences, $p(X)$ and $p(Y)$: we define a preorder \preceq , i.e., a reflexive and transitive binary relation, on the set, and say that the minimal projective resolution of X is “smaller” than that of Y if $p(X) \preceq p(Y)$. There are several ways to define the preorder \preceq , but all preorders in this paper are such that if $p(X)_n \leq p(Y)_n$ for all $n \in \mathbb{N}$, then $p(X) \preceq p(Y)$, i.e., we respect the intuitive notion of what it means that $p(X)$ is smaller than $p(Y)$.

The following easily verifiable statement shows that among the modules of fixed length, semisimple modules have the largest resolutions.

Proposition 0.1. *Let Λ be a left artinian ring, let $L, M, N \in \text{mod } \Lambda$, and let $n \in \mathbb{N}$.*

- (a) *If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence then $p(M)_n \leq p(L)_n + p(N)_n$.*
- (b) *$p(M \oplus N)_n = p(M)_n + p(N)_n$.*
- (c) *Let S_1, \dots, S_t be a complete set of pairwise nonisomorphic simple Λ -modules and let m_i be the multiplicity of S_i in a composition series of M , $1 \leq i \leq t$. Then $p(M)_n \leq p(S_1^{m_1} \oplus \cdots \oplus S_t^{m_t})_n = m_1 p(S_1)_n + \cdots + m_t p(S_t)_n$.*

We are interested in those left artinian rings for which the largest projective resolutions are as small as possible. In view of Proposition 0.1, semisimple modules have the largest resolutions, so we wish to know when the minimal projective resolutions of their indecomposable summands, the simple modules, are as small as possible. More precisely, the problem is to describe the left artinian rings Λ with the following property.

- (*) If S is a simple Λ -module and $T \in \text{mod } \Gamma$, where Γ is any left artinian ring, then $\text{pd } S \leq \text{pd } T$ implies $p(S) \preceq p(T)$.

For a preorder \preceq described below, we solve the problem when Λ is a finite-dimensional k -algebra that is *elementary*, which means that Λ / \mathfrak{r} is isomorphic to a direct product of several copies of k . This includes all algebras over an algebraically closed field because each such algebra is Morita equivalent to an elementary algebra. Since a finite-dimensional algebra is elementary if and only if it is isomorphic to the path algebra of a finite quiver with relations (see [2, Section III.1]), we characterize the above algebras in terms of quivers and relations (Theorem 3.7). This is our main result.

We now describe the preorders \preceq considered in this paper. For a left artinian ring Λ , we do not know which sequences $\mathbf{a} = \{a_n \mid n \in \mathbb{N}\}$ of nonnegative integers satisfy $\mathbf{a} = p(M)$ for some nonzero $M \in \text{mod } \Lambda$. So we define a preorder \preceq on the collection \mathcal{A} of all sequences \mathbf{a} that satisfy some obvious necessary conditions for the existence of the indicated module M . In fact, we define not one, but infinitely many preorders that depend on the positive integer parameter r . When restricted to the sequences associated with modules, the preorder works as follows. Given an $r > 0$ and $M, N \in \text{mod } \Lambda$, we have $p(M) \preceq p(N)$ if and only if $\text{pd } M \leq \text{pd } N$ and, for all $n \in \mathbb{N}$, if $p(M)_n > p(N)_n$ then $\text{pd } M < n + r$ (Definition 1.1). Note that if $\text{pd } M < r$ and $\text{pd } N \geq r$, then $p(M) \preceq p(N)$ but $p(N) \not\preceq p(M)$, i.e., the preorder distinguishes between some modules of finite projective dimension, and if $\text{pd } M = \text{pd } N = \infty$ then $p(M) \preceq p(N)$ if and only if, for all $n \in \mathbb{N}$, $p(M)_n \leq p(N)_n$. In particular, the preorder (for any $r > 0$) is easy to understand if Λ is a selfinjective k -algebra: $p(M) \preceq p(N)$ whenever M is projective, and if M is not projective then $p(M) \preceq p(N)$ if and only if, for all $n \in \mathbb{N}$, $p(M)_n \leq p(N)_n$.

For the preorders just described, a left artinian ring Λ satisfies (*) if and only if it satisfies the following seemingly weaker condition.

- (**) If S is a simple Λ -module and T is a simple Γ -module, where Γ is any left artinian ring, then $\text{pd } S \leq \text{pd } T$ implies $p(S) \preceq p(T)$.

This equivalence of (*) and (**) follows from a careful examination of the proofs in Section 1, we have no a priori explanation for it.

It is easy to describe the elementary k -algebras satisfying (*) when $r = 1$, our main result concerns the value $r = 2$, and the case $r > 2$ is open.

For $r = 2$ and Λ a left artinian ring, condition (*) is equivalent (Corollary 1.6(iii)) to the following.

- (F) The radical of each indecomposable projective module is either projective or simple.

Condition (F) is similar to other conditions that have appeared in the literature. One such condition is the following.

- (A) Each submodule of an indecomposable projective module is either projective or simple.

The structure of left artinian rings satisfying (A) is described in [9] as part of the description of projectively stable rings, i.e., rings for which a morphism of finitely generated modules without nonzero projective direct summands must be zero if it factors through a projective module. Consider the following condition.

- (B) The injective envelope of each simple nonprojective torsionless module is projective.

Then a left artinian ring is projectively stable if and only if it satisfies (A) and (B). Another condition similar to (F) is the following.

- (D) Each indecomposable submodule of an indecomposable projective module is either projective or simple.

This condition has to do with rings that are stably equivalent to left hereditary rings. Consider the following condition.

(E) Each simple nonprojective torsionless module is cotorsionless.

Then a left artinian ring is stably equivalent to a left hereditary left artinian ring if and only if it satisfies (D) and (E); this is presented in [1, p.40] for artin algebras, but is known to be true in the indicated setting.

Clearly, (A) \Rightarrow (F) but (F) $\not\Rightarrow$ (A), (D) $\not\Rightarrow$ (F), and (E) $\not\Rightarrow$ (F). We prove that (F) \Rightarrow (D) (Proposition 2.2). However, a finite-dimensional algebra satisfying (F) need not be stably equivalent to a hereditary algebra, as follows from our main result and [5]. Hence (F) $\not\Rightarrow$ (E).

The main result of this paper says that an elementary k -algebra satisfies (F) if and only if it is isomorphic to a quadratic monomial algebra (see the definition in Section 3) determined by a (unique) triple (G, \mathcal{D}, ρ) where G is a finite quiver (directed graph) with the set of vertices $v(G)$ and without oriented cycles, \mathcal{D} is a subset of $v(G)$ consisting of sinks (vertices at which no arrow starts), and $\rho : \mathcal{D} \rightarrow v(G)$ is a function such that the sinks in $\text{Im } \rho$ belong to \mathcal{D} . A simple geometric construction produces the quadratic monomial algebra from (G, \mathcal{D}, ρ) .

Section 1 of the paper defines the set of sequences \mathcal{A} and preorders \preceq as indicated above, and we show for all of these preorders that if Λ satisfies $(*)$, then every semisimple module in $\text{mod } \Lambda$ is eventually periodic, and every $M \in \text{mod } \Lambda$ satisfies $c_\Lambda(M) = 1$. The section also contains a computation (Corollary 1.8) of the global dimension of a left artinian ring satisfying (F) in terms of sequences of morphisms between indecomposable projective modules of length two. In Section 2, for a left artinian ring Λ satisfying (F), we show that every $M \in \text{mod } \Lambda$ is eventually periodic, and develop properties of projective Λ -modules needed for the proof of the main result in Section 3. Condition (F) is not selfdual in that an algebra Λ may satisfy (F), while Λ^{op} does not (Example 3.5(c)), so we characterize (Proposition 3.9) all elementary k -algebras Λ such that both Λ and Λ^{op} satisfy (F). The point of Section 4 is that an elementary k -algebra Λ satisfying (F) not only has well-behaved projective resolutions of its modules, but possesses other good homological properties. Since such an algebra Λ is quadratic monomial, it is a Koszul algebra, and its Ext-algebra $E(\Lambda) = \bigoplus_{i=0}^{\infty} \text{Ext}_\Lambda^i(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$ is finitely generated [7, Proposition 2.2]. We show that $E(\Lambda)$ is left noetherian, which happens rarely, so considerations of [12, Section 4] apply. In particular, the Poincaré series of a finitely generated graded Λ -module is rational. We note that although the authors of [12] assume $E(\Lambda)$ noetherian, their proof only uses that $E(\Lambda)$ is left noetherian.

We are grateful to the referee for the very helpful suggestions. In particular, the referee asked the following very natural question. For which elementary algebras Λ with property (F) does the Koszul dual $E(\Lambda)$ share this property? The question is suggested by our characterization of those algebras with property (F) for which the opposite algebra shares this property. Since $E(\Lambda)$ is in general infinite-dimensional (resp. not left artinian), our results do not apply immediately because we always assume finite-dimensionality (resp. left artinianness). However, the referee was wondering to what extent the methods and constructions of this paper actually depend on these assumptions. Some few small examples the referee looked at do show that Koszul duals can very well share property (F). We intend to investigate this question in the future.

1. SEQUENCES AND PROJECTIVE RESOLUTIONS

In this section we fix a left artinian ring Λ .

Let \mathcal{A} be the set of infinite sequences $\mathbf{a} = \{a_n \mid n \in \mathbb{N}\}$ of nonnegative integers with the property that $a_0 > 0$ and, for all n , if $a_n \leq 1$ then $a_i = 0$ for $i > n$. We set $\dim \mathbf{a} = \sup \{n \mid a_n > 0\}$, $\mathcal{A}_n = \{\mathbf{a} \in \mathcal{A} \mid \dim \mathbf{a} \geq n\}$ for $0 \leq n < \infty$, and $\mathcal{A}_\infty = \bigcap_{n=0}^{\infty} \mathcal{A}_n$. We have $0 \leq \dim \mathbf{a} \leq \infty$;

$\dim \mathbf{a} = n < \infty$ if and only if $\mathbf{a} \in \mathcal{A}_n - \mathcal{A}_{n+1}$; and $\dim \mathbf{a} = \infty$ if and only if $\mathbf{a} \in \mathcal{A}_\infty$. Our motivation for studying such sequences comes from the obvious fact that if $0 \neq M \in \text{mod } \Lambda$, then $p(M) = \{\ell(P_n)\} \in \mathcal{A}$ and $\text{pd } M = \dim p(M)$.

In order to introduce appropriate preorders on \mathcal{A} , we need the following easy statement. For an arbitrary preorder (\mathcal{P}, \preceq) we say that an element $x \in \mathcal{P}$ is a *least* element if $x \preceq p$ for all $p \in \mathcal{P}$; if $x, y \in \mathcal{P}$ are least elements, then $x \preceq y$ and $y \preceq x$.

Lemma 1.1. (a) *Let $0 \leq m < \infty$ be an integer. Denote by Γ_1 the quotient of the path algebra over k of the quiver with vertices v_1, \dots, v_{m+1} and arrows $\alpha_i : v_i \rightarrow v_{i+1}$, $i = 1, \dots, m$, modulo the ideal generated by the elements $\alpha_{i+1}\alpha_i$, $i = 1, \dots, m-1$. If T_1 is the simple Γ_1 -module associated with the vertex v_1 , then $p(T_1) = \mathbf{t}_1 = \{a_n\}$ where $a_n = 2$ for $n < m$, $a_m = 1$, and $a_n = 0$ for $n > m$. In particular, $\text{pd } T_1 = m$.*
 (b) *Let $\Gamma_2 = k[X]/(X^2)$ where $k[X]$ is the polynomial algebra. If T_2 is the simple Γ_2 -module, then $p(T_2) = \mathbf{t}_2 = \{b_n\}$ where $b_n = 2$ for all n . In particular, $\text{pd } T_2 = \infty$.*
 (c) *Let \preceq be a preorder on \mathcal{A} such that $\mathbf{x} \preceq \mathbf{y}$ whenever $x_n \leq y_n$ for all n . For a fixed $0 \leq m < \infty$, \mathbf{t}_1 is a least element of \mathcal{A}_m . The element \mathbf{t}_2 is a least element of \mathcal{A}_∞ .*

Proof. (a) and (b) We leave the easy verification to the reader.

(c) If $\mathbf{c} \in \mathcal{A}_m$, then $c_m > 0$ whence $c_n > 1$ for $n < m$. Then $a_n \leq c_n$ for all n and we conclude that $\mathbf{t}_1 \preceq \mathbf{c}$. If $\mathbf{c} \in \mathcal{A}_\infty$, then $c_n > 1$ for all n , whence $b_n \leq c_n$ for all n , so $\mathbf{t}_2 \preceq \mathbf{c}$. \square

We get the most simple-minded preorder on \mathcal{A} by setting $\mathbf{a} \leq \mathbf{b}$ if and only if $a_n \leq b_n$ for all n . Then \leq is a partial order and we have the following statement.

Proposition 1.2. *For the left artinian ring Λ and partial order (\mathcal{A}, \leq) , the following are equivalent.*

- (a) *For each simple Λ -module S , $p(S)$ is the least element of $\mathcal{A}_{\text{pd } S}$.*
- (b) *Λ satisfies $(*)$.*
- (c) *For each indecomposable projective Λ -module P , $\ell(P) \leq 2$.*

Proof. (a) \Rightarrow (b) Obvious.

(b) \Rightarrow (c) Let Λ satisfy $(*)$ and let S be a simple Λ -module with $m = \text{pd } S$. If $m < \infty$ then $p(S) \leq p(T_1) = \mathbf{t}_1$ where T_1 is defined in Lemma 1.1(a), whence $\ell(P(S)) \leq \ell(P(T_1)) = 2$. If $m = \infty$ then $p(S) \leq p(T_2) = \mathbf{t}_2$ where T_2 is defined in Lemma 1.1(b), whence $\ell(P(S)) \leq \ell(P(T_2)) = 2$.

(c) \Rightarrow (a) Let S be a simple Λ -module with $m = \text{pd } S$. Since $\ell(P) \leq 2$ for each indecomposable projective Λ -module P , an easy inductive argument on the minimal projective resolution of S shows that $p(S) = p(T_1) = \mathbf{t}_1$ if $m < \infty$, and $p(S) = p(T_2) = \mathbf{t}_2$ if $m = \infty$. By Lemma 1.1(c), $p(S)$ is a least element of \mathcal{A}_m . Since (\mathcal{A}, \leq) is a partial order, $p(S)$ is the least element. \square

Proposition 1.2 shows that for the partial order (\mathcal{A}, \leq) , the left artinian rings satisfying $(*)$ form a rather small class of radical-square-zero rings. To produce more interesting classes of rings, we need coarser preorders on \mathcal{A} .

Definition 1.1. Let $r > 0$ be a fixed integer. For $\mathbf{a}, \mathbf{b} \in \mathcal{A}$ we set $\mathbf{a} \preceq \mathbf{b}$ if and only if $\dim \mathbf{a} \leq \dim \mathbf{b}$ and, for all n , $a_n > b_n$ implies $\dim \mathbf{a} < n + r$.

Applying Definition 1.1 to the minimal projective resolutions (0.1) and (0.2) of modules M and N , we see that $p(M) \preceq p(N)$ if and only if (i) $\text{pd } M \leq \text{pd } N$ and (ii) the violation of the intuitive condition $\ell(P_n) \leq \ell(Q_n)$ for some n is allowed only if $\text{pd } M < \infty$, and for at most r terms at the tail of the resolution of M . Note also that if we formally apply Definition 1.1 to $r = 0$, we will obtain the partial order (\mathcal{A}, \leq) .

Proposition 1.3. *Let $r > 0$ be a fixed integer, let (\mathcal{A}, \preceq) be given by Definition 1.1, and let $\mathbf{a}, \mathbf{b} \in \mathcal{A}$.*

- (a) The binary relation (\mathcal{A}, \preceq) is a preorder.
- (b) Both $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{b} \preceq \mathbf{a}$ hold if and only if $\dim \mathbf{a} = \dim \mathbf{b}$ and $a_i = b_i$ for $0 \leq i \leq (\dim \mathbf{a}) - r$.
- (c) For $0 \leq m < \infty$, $\mathbf{b} \in \mathcal{A}_m$ is a least element of \mathcal{A}_m if and only if $\dim \mathbf{b} = m$ and $b_i = 2$ for $0 \leq i \leq m - r$. An element $\mathbf{b} \in \mathcal{A}_\infty$ is a least element of \mathcal{A}_∞ if and only if $b_i = 2$ for $0 \leq i < \infty$.

Proof. (a) The reflexivity of \preceq is obvious. To prove the transitivity, let $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{b} \preceq \mathbf{c}$. Then $\dim \mathbf{a} \leq \dim \mathbf{b}$ and $\dim \mathbf{b} \leq \dim \mathbf{c}$ whence $\dim \mathbf{a} \leq \dim \mathbf{c}$. Let $a_n > c_n$ for some $n \geq 0$. If $b_n \leq c_n$ then $a_n > b_n$, whence $\dim \mathbf{a} < n + r$ because $\mathbf{a} \preceq \mathbf{b}$. If $b_n > c_n$, then $\dim \mathbf{b} < n + r$ because $\mathbf{b} \preceq \mathbf{c}$ so $\dim \mathbf{a} < n + r$. Since $\dim \mathbf{a} < n + r$, we get $\mathbf{a} \preceq \mathbf{c}$.

(b) This is clear.

(c) Let $m < \infty$. If $\mathbf{b} \in \mathcal{A}_m$ is a least element of \mathcal{A}_m , we must have $\mathbf{b} \preceq \mathbf{t}_1$ where \mathbf{t}_1 is defined in Lemma 1.1(a). By Lemma 1.1(c), we also have $\mathbf{t}_1 \preceq \mathbf{b}$. Now the statement is an immediate consequence of (b) and the definition of \mathbf{t}_1 . The same argument with the replacement of \mathbf{t}_1 by \mathbf{t}_2 works in the case $\mathbf{b} \in \mathcal{A}_\infty$. \square

The following statement extends Proposition 1.2.

Proposition 1.4. *Let $r > 0$ be a fixed integer. For the left artinian ring Λ and preorder (\mathcal{A}, \preceq) , the following are equivalent.*

- (a) For each simple Λ -module S , $p(S)$ is a least element of $\mathcal{A}_{\text{pd } S}$.
- (b) The ring Λ satisfies $(*)$.
- (c) For each simple Λ -module S , either $\text{pd } S < r$ or $\ell(P(S)) = 2$.

Proof. (a) \Rightarrow (b) Obvious.

(b) \Rightarrow (c) Let S be a simple module with $m = \text{pd } S \geq r$. If $m < \infty$ then $p(S) \preceq p(T_1)$ where T_1 is defined in Lemma 1.1(a). Since $m \geq r$, we cannot have $p(S)_0 > p(T_1)_0$. Hence $\ell(P(S)) = p(S)_0 \leq p(T_1)_0 = 2$. If $m = \infty$ then $p(S) \preceq p(T_2)$ where T_2 is defined in Lemma 1.1(b), whence $\ell(P(S)) \leq \ell(P(T_2)) = 2$.

(c) \Rightarrow (a) Let S be a simple Λ -module, and let $\text{pd } S = m \leq \dim \mathbf{a}$ where $\mathbf{a} \in \mathcal{A}$. If $m < r$, then $m - r < 0$ so no integer i satisfies $0 \leq i \leq m - r$. If $m \geq r$ then $p(S)_0 = \ell(P(S)) = 2$, and we obtain by induction on i that if $m < \infty$ then $p(S)_i = 2$ for $0 \leq i \leq m - r$, and if $m = \infty$ then $p(S)_i = 2$ for $0 \leq i < \infty$. By Proposition 1.3(c), $p(S)$ is a least element of $\mathcal{A}_{\text{pd } S}$. \square

Corollary 1.5. *For a fixed integer $r > 0$, denote by \preceq the preorder defined in Definition 1.1 that depends on r , and suppose that Λ satisfies $(*)$ relative to \preceq .*

- (a) Every semisimple module in $\text{mod } \Lambda$ is eventually periodic.
- (b) For all $M \in \text{mod } \Lambda$, $c_\Lambda(M) = 1$.

Proof. (a) It suffices to show that every simple Λ -module is eventually periodic, which follows from Proposition 1.4(c): if $\text{pd } S = \infty$, every projective in the minimal projective resolution of S is of length 2, so all syzygies are simple.

(b) This is a consequence of (a) and Proposition 0.1(c). \square

We write $\text{gl. dim } \Lambda$ for the global dimension of Λ .

Corollary 1.6. *Let $r > 0$ be a fixed integer.*

- (i) If $\text{gl. dim } \Lambda < r$ then Λ satisfies $(*)$.
- (ii) If $r = 1$, then Λ satisfies $(*)$ if and only if $\ell(P) \leq 2$ for each indecomposable projective Λ -module P .
- (iii) If $r = 2$, then Λ satisfies $(*)$ if and only if it satisfies (F).

Proof. The proof is a straight forward verification of condition (c) of Proposition 1.4(c). Note that for $r = 1$, the class of left artinian rings satisfying $(*)$ coincides with that described in Proposition 1.2. \square

For a fixed $r \geq 2$, we now estimate $\text{gl. dim } \Lambda$ in the case when $\text{gl. dim } \Lambda \geq r$ and Λ satisfies $(*)$. We need to recall some definitions from [8]. Let \mathcal{Q} be the full subcategory of $\text{mod } \Lambda$ determined by the indecomposable nonhereditary projective modules of length 2; recall that a projective module is hereditary if all its submodules are projective. A path of length $n \geq 0$ in \mathcal{Q} is a sequence

$$Q_n \xrightarrow{f_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_1 \xrightarrow{f_1} Q_0 \quad (1.1)$$

where, for all i , $Q_i \in \mathcal{Q}$ and f_i is a nonzero nonisomorphism. If $\text{gl. dim } \Lambda \geq r$ and Λ satisfies $(*)$, \mathcal{Q} is not empty according to Proposition 1.4. Set $l = \sup\{\text{lengths of paths in } \mathcal{Q}\}$.

Proposition 1.7. *Let $r \geq 2$ be a fixed integer. If the left artinian ring Λ satisfies $(*)$ and has the property that $\text{gl. dim } \Lambda \geq r$, then $l + 2 \leq \text{gl. dim } \Lambda \leq l + r$.*

Proof. If $\text{gl. dim } \Lambda = m$, then $\text{pd } S = m$ for some simple Λ -module S . If $m = \infty$, a minimal projective resolution of S is an infinite path in \mathcal{Q} in view of Propositions 1.4 and 1.3(c). Hence $l = \infty$ and the statement holds. If $r \leq m < \infty$, the same argument shows that the first $m - r + 1$ terms of a minimal projective resolution of S form a path in \mathcal{Q} , whence $m - r \leq l$ and $m \leq l + r$. It is clear that any path in \mathcal{Q} of the form (1.1) is an exact sequence of modules, whence $T = \text{Coker } f_1$ and $\text{Ker } f_n$ are simple nonprojective Λ -modules, so $n + 2 \leq \text{pd } T \leq m$. Since $l = \sup\{n\}$, $l + 2 \leq m$. \square

Corollary 1.8. *If Λ is nonhereditary and satisfies (F), then $\text{gl. dim } \Lambda = l + 2$.*

Proof. We set $r = 2$ and use Corollary 1.6 (iii) and Proposition 1.7. Another way to prove the statement is to use [8, Propositions 4.1 and 4.2] and replace condition (A) by (F). \square

For the rest of the paper we focus on the left artinian rings satisfying (F).

2. PROJECTIVE MODULES OVER Λ SATISFYING (F)

Throughout this section Λ is a left artinian ring. We begin by rephrasing [1, Ch. IV, Lemma 2.2].

Lemma 2.1. *Let M be an indecomposable Λ -submodule of $\bigoplus_{i=1}^n Q_i$ where each Q_i is an indecomposable projective Λ -module whose all submodules are either projective or simple. Then M is either projective or simple.*

Proposition 2.2. *If Λ satisfies (F) then it satisfies (D).*

Proof. We show by induction on $\ell(P)$ that each indecomposable projective Λ -module P satisfies the requirement of (D). This is clear when $\ell(P) = 1$ or $\ell(P) = 2$. Let $\ell(P) > 2$ and let M be an indecomposable nonprojective submodule of P . Since $\ell(P) > 2$, $\mathbf{r}P$ is not simple, hence is projective by (F), so $\mathbf{r}P = \bigoplus_{i=1}^n Q_i$ where each Q_i is indecomposable projective. Since M is not projective, $M \subset \mathbf{r}P$. Since $\ell(Q_i) < \ell(P)$, the induction hypothesis says that each indecomposable submodule of Q_i is either projective or simple. By Lemma 2.1, M is simple. \square

Corollary 2.3. *If Λ satisfies (F) then every $M \in \text{mod } \Lambda$ is eventually periodic.*

Proof. By Proposition 2.2, $\Omega^1 M = P \oplus T$ where P is projective and T is semisimple. Now the statement follows from Corollary 1.5(a). \square

Denote by \mathbf{a}_Λ the two-sided ideal of Λ equal to the sum of nonprojective submodules of $\text{Soc } \Lambda$. Recall that a module is torsionless if it is a submodule of a finitely generated projective module.

Proposition 2.4. *Suppose Λ satisfies (F).*

(a) *The ring $\Lambda/\mathfrak{a}_\Lambda$ is left hereditary.*

Let S be a simple nonprojective Λ -module with the projective cover $P(S)$.

(b) *If S is torsionless, then $S \simeq \mathbf{r}P$ for some indecomposable projective Λ -module P .*

(c) *The module S is a projective $\Lambda/\mathfrak{a}_\Lambda$ -module if and only if $\ell(P(S)) = 2$ and $\text{Soc } P(S)$ is a nonprojective Λ -module.*

Proof. (a) According to [13, Proposition 5.3], if Λ satisfies (D) then $\Lambda/\mathfrak{a}_\Lambda$ is hereditary. It remains to use Proposition 2.2.

(b) Let P be an indecomposable projective module of the smallest length with $S \simeq T$ and $T \subset P$. Since S is not projective, $T \subset \mathbf{r}P$. If $\mathbf{r}P$ is projective, $S \simeq U$ and $U \subset Q$ for some indecomposable summand Q of $\mathbf{r}P$, which is impossible because $\ell(Q) < \ell(P)$. Thus $\mathbf{r}P$ is not projective, so must be simple by (F), whence $S \simeq \mathbf{r}P$.

(c) The sufficiency is straight forward. For the necessity, we note that $P(S)/\mathfrak{a}_\Lambda P(S) \simeq S$ and $\mathfrak{a}_\Lambda P(S) \neq 0$. Therefore $\mathbf{r}P(S) = \mathfrak{a}_\Lambda P(S)$ and, by [13, Lemma 5.1], $\mathfrak{a}_\Lambda P(S)$ is not projective. Since Λ satisfies (F), $\mathfrak{a}_\Lambda P(S)$ is simple, so $\ell(P(S)) = 2$. \square

3. QUIVERS AND RELATIONS FOR ALGEBRAS SATISFYING (F)

In this section, Λ is an elementary finite-dimensional k -algebra with the Jacobson radical \mathbf{r} .

Denote by $H = (v(H), a(H))$ a finite quiver with the set of vertices $v(H)$ and the set of arrows $a(H)$. For each $\alpha \in a(H)$, $s(\alpha)$ ($t(\alpha)$) is the starting (terminal) vertex of α . A vertex $x \in v(H)$ is a source (sink) if no arrow $\alpha \in a(H)$ satisfies $t(\alpha) = x$ ($s(\alpha) = x$). If p is a path in H , $s(p)$ ($t(p)$) is the starting (terminal) vertex of p , and we write $p : s(p) \rightarrow t(p)$. For each $x \in v(H)$, e_x stands for the trivial path at x ; a nontrivial path consists of at least one arrow. The length of a path is the number of arrows in the path, and a path p is an oriented cycle if p is nontrivial and $s(p) = t(p)$.

We recall (see [2, Section III.1]) that a k -algebra Λ is finite-dimensional and elementary if and only if it is isomorphic to the algebra $k[H, \rho(H)]$ for some quiver H with a set of relations $\rho(H)$. Here $k[H, \rho(H)]$ is the quotient of the path algebra kH of H modulo the two-sided ideal $\langle \rho(H) \rangle$ generated by $\rho(H)$, where $\langle \rho(H) \rangle$ and the two-sided ideal $\langle a(H) \rangle$ generated by $a(H)$ satisfy $\langle a(H) \rangle^m \subset \langle \rho(H) \rangle \subset \langle a(H) \rangle^2$ for some positive integer m . The image of $a \in kH$ under the natural projection $\pi_\rho : kH \rightarrow k[H, \rho(H)]$ is denoted by \bar{a} .

We characterize algebras satisfying (F) in terms of the quiver H and relations $\rho(H)$. Of particular interest to us are *quadratic monomial* algebras, for which each element of $\rho(H)$ is a path of length 2.

In the following definitions ρ is a function whose domain is a set \mathcal{D} of sinks of a quiver G , and $\rho(H)$ is the set of relations on another quiver H that is constructed from the triple (G, \mathcal{D}, ρ) .

Definition 3.1. Let (G, \mathcal{D}, ρ) be a triple consisting of a finite quiver G without oriented cycles, a subset \mathcal{D} of $v(G)$ consisting of sinks, and a function $\rho : \mathcal{D} \rightarrow v(G)$ for which all sinks in $\text{Im } \rho$ belong to \mathcal{D} . A morphism $\varphi : (G, \mathcal{D}, \rho) \rightarrow (G', \mathcal{D}', \rho')$ is a morphism $\varphi : G \rightarrow G'$ of quivers satisfying $\varphi(\mathcal{D}) \subset \mathcal{D}'$ and $\rho'\varphi|_{\mathcal{D}} = \varphi\rho$.

Definition 3.2. The algebra $k(G, \mathcal{D}, \rho)$ of the triple (G, \mathcal{D}, ρ) is the algebra $k[H, \rho(H)]$, where the quiver H and the set of relations $\rho(H)$ are obtained as follows. For each $d \in \mathcal{D}$ we add a new single arrow $\alpha_d : d \rightarrow \rho(d)$, and then set $v(H) = v(G)$, $a(H) = a(G) \cup \{\alpha_d \mid d \in \mathcal{D}\}$, and $\rho(H) = \{\beta\alpha_d \mid \beta \in a(H), d \in \mathcal{D}\}$. Note that if $\mathcal{D} = \emptyset$ then $k(G, \mathcal{D}, \rho) = kG$.

Proposition 3.1. *Let $\Lambda = k(G, \mathcal{D}, \rho) = k[H, \rho(H)]$, $\mathbf{r} = \text{rad } \Lambda$, and $z \in v(H)$.*

- (a) *Every oriented cycle in H passes through a vertex in \mathcal{D} .*
- (b) *Λ is finite-dimensional.*

- (c) If $z \in \mathcal{D}$ then $\mathbf{r} \bar{e}_z$ is simple nonprojective.
(d) If $z \notin \mathcal{D}$ then $\mathbf{r} \bar{e}_z$ is projective.

Proof. (a) Since G has no oriented cycles, an oriented cycle in H must contain at least one of the added arrows α_d , $d \in \mathcal{D}$.

(b) Follows from (a) and the definition of $\rho(H)$.

Let W_z be the set of those nontrivial paths in H that start at z and have no subpath belonging to $\rho(H)$. Clearly, $\overline{W}_z = \{\bar{p} \mid p \in W_z\}$ is a k -basis for $\mathbf{r} \bar{e}_z = \text{rad } \Lambda \bar{e}_z$. For each $p \in W_z$ we have $p = q\gamma$, where $\gamma \in a(H)$ and q is a path in H .

(c) If $z = d \in \mathcal{D}$ and $p \in W_z$ then, by construction, $\gamma = \alpha_d$ and $q = e_{\rho(d)}$. Hence $\mathbf{r} \bar{e}_z = k\bar{\alpha}_d$ is 1-dimensional and, therefore, a simple Λ -module. It is not projective because $\rho(d)$ is not a sink in H .

(d) Suppose $z \notin \mathcal{D}$. If z is a sink in H , $\Lambda \bar{e}_z$ is simple projective so $\mathbf{r} \bar{e}_z = 0$. If z is not a sink, let $\gamma_1, \dots, \gamma_m$, $m > 0$, be the arrows in $a(H)$ (and in $a(G)$) starting at z . Then $W_z = \bigcup_{j=1}^m \{q\gamma_j \mid q \in W_{t(\gamma_j)}\} \cup \{\gamma_1, \dots, \gamma_m\}$, so $\mathbf{r} \bar{e}_z = \prod_{j=1}^m \Lambda \bar{e}_{t(\gamma_j)} \bar{\gamma}_j \simeq \prod_{j=1}^m \Lambda \bar{e}_{t(\gamma_j)}$ is projective. \square

Corollary 3.2. *If $\Lambda = k(G, \mathcal{D}, \rho)$ then Λ satisfies (F).*

Proof. This follows from parts (c) and (d) of Proposition 3.1. \square

Our goal now is to prove the converse of Corollary 3.2.

Notation 3.3. For the rest of the section, we fix a finite quiver H with a set of relations $\sigma(H)$, and assume that the algebra $\Sigma = k[H, \sigma(H)]$ is finite-dimensional and satisfies (F); the image of $a \in kH$ under the natural projection $\pi_\sigma : kH \rightarrow \Sigma$ is denoted by \hat{a} . Let $\mathbf{s} = \text{rad } \Sigma$ and, for each $z \in v(H)$, denote by S_z (respectively, P_z) the associated simple (indecomposable projective) Σ -module. We set

$$\mathcal{D} = \{d \in v(H) \mid \ell(P_d) = 2, \text{ Soc } P_d \text{ is not projective}\} \quad (3.1)$$

and define a function $\rho : \mathcal{D} \rightarrow v(H)$ by $\rho(d) = y$ if $S_y \simeq \text{Soc } P_d$.

Lemma 3.3. *In the setting of Notation 3.3, we have:*

- (a) For each $d \in \mathcal{D}$, there exists a unique arrow $\alpha_d \in a(H)$ such that $s(\alpha_d) = d$. We must have $t(\alpha_d) = \rho(d)$.
(b) If $d \in \mathcal{D}$ and $\beta \in a(H)$ satisfies $s(\beta) = \rho(d)$, then $\beta\alpha_d \in \langle \sigma(H) \rangle$.

Proof. The statement is an easy consequence of $\ell(P_d) = 2$. \square

Denote by \mathbf{a}_Σ the two-sided ideal of Σ equal to the sum of nonprojective submodules of $\text{Soc } \Sigma$, and let $\pi_\Sigma : \Sigma \rightarrow \Sigma/\mathbf{a}_\Sigma$ be the natural projection. By Proposition 2.4 (a), Σ/\mathbf{a}_Σ is hereditary.

Proposition 3.4. *Let $G = (v(G), a(G))$ be the quiver given by $v(G) = v(H)$ and $a(G) = a(H) - \{\alpha_d \mid d \in \mathcal{D}\}$, where α_d 's are described in Lemma 3.3(a). Let $\psi : kG \rightarrow \Sigma/\mathbf{a}_\Sigma$ be a unique morphism of k -algebras determined by $\psi(e_x) = \pi_\Sigma(\hat{e}_x)$ for $x \in v(G)$, and $\psi(\gamma) = \pi_\Sigma(\hat{\gamma})$ for $\gamma \in a(G)$. Then ψ is an isomorphism.*

Proof. In view of Lemma 3.3, ψ is onto. To show that ψ is an isomorphism, it suffices to show that $k[G]$ and Σ/\mathbf{a}_Σ have the same dimension as k -spaces. The latter will be established if we show that G is the quiver associated to the (hereditary) algebra Σ/\mathbf{a}_Σ (see [2, Proposition III.1.13]).

Since $\text{rad}(\Sigma/\mathbf{a}_\Sigma) = \mathbf{s}/\mathbf{a}_\Sigma$, then $(\text{rad}(\Sigma/\mathbf{a}_\Sigma))^2 = (\mathbf{s}^2 + \mathbf{a}_\Sigma)/\mathbf{a}_\Sigma$ and we have

$$\Sigma/\mathbf{a}_\Sigma \Big/ (\mathbf{s}^2 + \mathbf{a}_\Sigma)/\mathbf{a}_\Sigma \simeq \Sigma/(\mathbf{s}^2 + \mathbf{a}_\Sigma) \simeq \Sigma/\mathbf{s}^2 \Big/ (\mathbf{s}^2 + \mathbf{a}_\Sigma)/\mathbf{s}^2. \quad (3.2)$$

From Proposition 2.4(b) and Lemma 3.3 it is clear that $\Sigma/\mathfrak{s}^2 / (\mathfrak{s}^2 + \mathfrak{a}_\Sigma)/\mathfrak{s}^2 = \oplus k\tilde{\beta}$, where $\beta \in a(H) - \{\alpha_d \mid d \in \mathcal{D}\}$, and $\tilde{\beta}$ is the image of $\hat{\beta}$ under the canonical epimorphism

$$\Sigma/\mathfrak{a}_\Sigma \rightarrow \frac{\Sigma/\mathfrak{s}^2}{(\mathfrak{s}^2 + \mathfrak{a}_\Sigma)/\mathfrak{s}^2}$$

(see Eq. (3.2)). Therefore $\Sigma/\mathfrak{s}^2 / (\mathfrak{s}^2 + \mathfrak{a}_\Sigma)/\mathfrak{s}^2$ has G as its quiver. It remains to use the fact that if Γ is an elementary algebra, then Γ and $\Gamma/\text{rad}\Gamma^2$ have the same quiver. \square

Lemma 3.5. *In the setting of Notation 3.3, the triple (G, \mathcal{D}, ρ) satisfies the conditions of Definition 3.1.*

Proof. Viewing the isomorphism ψ of Proposition 3.4 as identification, we may assume that $\Sigma/\mathfrak{a}_\Sigma = kG$. Since $\Sigma/\mathfrak{a}_\Sigma$ is finite-dimensional, G has no oriented cycles. By Lemma 3.3 and Proposition 3.4, each $d \in \mathcal{D}$ is a sink in G . If $y \in \text{Im}\rho$ is a sink in G , then S_y is a projective $\Sigma/\mathfrak{a}_\Sigma$ -module and nonprojective Σ -module. In view of Proposition 2.4(c) and equation (3.1), we get $y \in \mathcal{D}$. \square

Theorem 3.6. *In the setting of Notation 3.3, $\Sigma = k[H, \sigma(H)] \simeq k(G, \mathcal{D}, \rho) = k[H, \rho(H)] = \Lambda$.*

Proof. Recall that by Lemma 3.5 and Corollary 3.2, Λ satisfies (F). Since $\langle \rho(H) \rangle \subset \mathfrak{a}_\Sigma$, Lemma 3.3(b) implies that there exists an epimorphism $\phi : \Lambda \rightarrow \Sigma$ of k -algebras satisfying $\phi\pi_\rho = \pi_\sigma$. To show ϕ is an isomorphism, it suffices to show that so is the restriction $\phi|_{\Lambda \bar{e}_i} = \phi_i : \Lambda \bar{e}_i \rightarrow \Sigma \hat{e}_i$, for each $i \in v(H)$. We proceed by induction on $\dim_k \Lambda \bar{e}_i$. If $\dim_k \Lambda \bar{e}_i = 1$, then $\dim_k \Sigma \hat{e}_i = 1$, and ϕ_i is an isomorphism. Suppose $\dim_k \Lambda \bar{e}_i = n > 1$ and $\phi_j : \Lambda \bar{e}_j \rightarrow \Sigma \hat{e}_j$ is an isomorphism whenever $\dim_k \Lambda \bar{e}_j < n$. Since $\Lambda \bar{e}_i$ is not simple, at least one arrow of H starts at i . Hence $\Sigma \hat{e}_i$ is not simple and $\mathfrak{s} \hat{e}_i \neq 0$.

Consider the exact commutative diagram of Λ -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{r} \bar{e}_i & \longrightarrow & \Lambda \bar{e}_i & \longrightarrow & S_i \longrightarrow 0 \\ & & \phi_i | \mathfrak{r} \bar{e}_i \downarrow & & \phi_i \downarrow & & \parallel \\ 0 & \longrightarrow & \mathfrak{s} \hat{e}_i & \longrightarrow & \Sigma \hat{e}_i & \longrightarrow & S_i \longrightarrow 0 \end{array} \quad (3.3)$$

Since Σ satisfies (F), $\mathfrak{s} \hat{e}_i$ is either projective or simple.

If $\mathfrak{s} \hat{e}_i$ is simple nonprojective, $\ell(\Sigma \hat{e}_i) = 2$ and $i \in \mathcal{D}$. By Proposition 3.1(c), $\dim_k \Lambda \bar{e}_i = 2$, and ϕ_i is an isomorphism.

If $\mathfrak{s} \hat{e}_i$ is projective, $i \notin \mathcal{D}$ and, by Proposition 3.1(d), $\mathfrak{r} \bar{e}_i$ is projective. Let $\gamma_1, \dots, \gamma_m$ be the arrows in $a(H)$ starting at i (remember, $\mathfrak{s} \hat{e}_i \neq 0$). The maps $\bigoplus_{u=1}^m \Lambda \bar{e}_{t(\gamma_u)} \rightarrow \mathfrak{r} \bar{e}_i = \sum_{u=1}^m \Lambda \bar{\gamma}_u$ and $\bigoplus_{u=1}^m \Sigma \hat{e}_{t(\gamma_u)} \rightarrow \mathfrak{s} \hat{e}_i = \sum_{u=1}^m \Sigma \hat{\gamma}_u$ induced by the right multiplication by the γ_u 's are projective covers, hence, isomorphisms, because $\mathfrak{r} \bar{e}_i$ and $\mathfrak{s} \hat{e}_i$ are projective. We obtain a commutative diagram

$$\begin{array}{ccc} \bigoplus_{u=1}^m \Lambda \bar{e}_{t(\gamma_u)} & \longrightarrow & \mathfrak{r} \bar{e}_i \\ \bigoplus_{u=1}^m \phi_{t(\gamma_u)} \downarrow & & \downarrow \phi_i | \mathfrak{r} \bar{e}_i \\ \bigoplus_{u=1}^m \Sigma \hat{e}_{t(\gamma_u)} & \longrightarrow & \mathfrak{s} \hat{e}_i \end{array}$$

By the induction hypothesis, each $\phi_{t(\gamma_u)}$ is an isomorphism, hence so are $\bigoplus_{u=1}^m \phi_{t(\gamma_u)}$ and $\phi_i | \mathfrak{r} \bar{e}_i$. Using commutative diagram (3.3), we see that ϕ_i is an isomorphism. \square

Theorem 3.7. (a) *An elementary finite-dimensional k -algebra satisfies (F) if and only if it is isomorphic to the algebra $k(G, \mathcal{D}, \rho)$ for some triple (G, \mathcal{D}, ρ) .*
 (b) *For a given algebra satisfying (F), the triple (G, \mathcal{D}, ρ) is unique up to isomorphism.*

Proof. (a) This follows from Corollary 3.2 and Theorem 3.6.

(b) Let $\Lambda = k[H, \rho(H)]$ and $\Lambda' = k[H', \rho'(H')]$ be the algebras associated with triples (G, \mathcal{D}, ρ) and $(G', \mathcal{D}', \rho')$, respectively. We show that if there exists an isomorphism of k -algebras $\psi : \Lambda \rightarrow \Lambda'$, then there exists an isomorphism $\phi : G \rightarrow G'$ such that $\phi(\mathcal{D}) = \mathcal{D}'$ and $\rho'\phi = \phi\rho$.

Since $\{\bar{e}_x | x \in v(H)\}$ is a complete set of primitive orthogonal idempotents in the elementary algebra Λ , so is $\{\psi(\bar{e}_x) | x \in v(H)\}$ in Λ' . It follows that $\psi(\bar{e}_x) = \hat{e}_{x'} + r'$, for a unique $x' \in v(H')$ and some $r' \in \text{rad } \Lambda'$, whence the map $\phi : v(H) \rightarrow v(H')$ given by $\phi(x) = x'$ is a bijection. In view of [2, Proposition 1.14], which is also valid for an elementary k -algebra, ϕ can be extended to an isomorphism $\phi : H \rightarrow H'$. Since $v(G) = v(H)$ and $a(H) = a(G) \cup \{\alpha_d | d \in \mathcal{D}\}$, where α_d is the unique arrow in H starting at d , and since $v(G'), v(H'), a(G')$, and $a(H')$ are similarly related, it suffices to show that $\phi|_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}'$ is a bijection. Indeed, assuming the map is a bijection, note that since ϕ is an isomorphism of quivers, $\phi(\alpha_d)$ is the unique arrow in $a(H')$ starting at $\phi(d)$, so we must have $\phi(\alpha_d) = \alpha_{\phi(d)}$. Hence $\phi|_{a(G)} : a(G) \rightarrow a(G')$ is a bijection, and $\rho'\phi = \phi\rho$, whence $\phi : (G, \mathcal{D}, \rho) \rightarrow (G', \mathcal{D}', \rho')$ is an isomorphism.

To show that $\phi|_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}'$ is a bijection, we prove that $\phi(\mathcal{D}) \subset \mathcal{D}'$; the proof of $\phi^{-1}(\mathcal{D}') \subset \mathcal{D}$ is similar. If $d \in \mathcal{D}$, Proposition 3.1 (c) implies that $\Lambda \bar{e}_d$ is an indecomposable projective module of length 2 with a nonprojective simple socle, and so is $\psi(\Lambda \bar{e}_d) = \Lambda' \psi(\bar{e}_d) = \Lambda'(\hat{e}_{\phi(d)} + r') = \Lambda' \hat{e}_{\phi(d)}$, where $r' \in \text{rad } \Lambda'$. Using parts (c) and (d) of Proposition 3.1, we get $\phi(d) \in \mathcal{D}'$. \square

Having shown that a finite-dimensional elementary k -algebra satisfies (F) if and only if it is isomorphic to a quadratic monomial algebra $k(G, \mathcal{D}, \rho)$, we explain how to verify whether a given quadratic monomial algebra satisfies (F). Our description is similar to that in [5] for the algebras stably equivalent to hereditary.

Notation 3.4. Let F be a finite quiver with a set of quadratic monomial relations $\sigma(F)$, and denote by \hat{p} the image of a path p in F under the natural projection $kF \rightarrow k[F, \sigma(F)]$. We set $\mathcal{X} = \{x \in v(F) | \beta\alpha \in \sigma(F) \text{ for some } \alpha, \beta \in a(F) \text{ with } s(\alpha) = x\}$.

Proposition 3.8. *In the setting of Notation 3.4, $\Sigma = k[F, \sigma(F)] \simeq k(G, \mathcal{D}, \rho)$ for some triple (G, \mathcal{D}, ρ) if and only if the following conditions hold.*

- (a) *For all $x \in \mathcal{X}$, there is precisely one $\alpha \in a(F)$ satisfying $s(\alpha) = x$.*
- (b) *If $\alpha \in a(F)$ satisfies $s(\alpha) \in \mathcal{X}$, then for all $\beta \in a(F)$ satisfying $s(\beta) = t(\alpha)$, we have $\beta\alpha \in \sigma(F)$.*
- (c) *Each oriented cycle passes through a vertex in \mathcal{X} .*

Proof. Suppose $\Sigma = k[F, \sigma(F)] \simeq k(G, \mathcal{D}, \rho)$. By Theorem 3.7(a), Σ satisfies (F). Since Σ is finite-dimensional, \mathcal{X} must satisfy (c). If $x \in \mathcal{X}$, there exist $\alpha, \beta \in a(F)$ with $s(\alpha) = x$ and $\beta\alpha \in \sigma(F)$. Since Σ is a monomial algebra, $\Sigma \hat{\alpha}$ is a direct summand of $\mathfrak{s} \hat{e}_x$ where $\mathfrak{s} = \text{rad } \Sigma$, and $\Sigma \hat{\alpha}$ is not projective because $\beta \hat{\alpha} = 0$. Since Σ satisfies (F), $\Sigma \hat{\alpha} \simeq \mathfrak{s} \hat{e}_x$ is simple, so (a) and (b) follow.

Suppose F and $\sigma(F)$ satisfy (a), (b), and (c). Denote by α_x the unique arrow with $s(\alpha_x) = x \in \mathcal{X}$. Let G be the quiver defined by $v(G) = v(F)$ and $a(G) = a(F) - \{\alpha_x | x \in \mathcal{X}\}$. By (c), G has no oriented cycles. Define $\rho : \mathcal{X} \rightarrow v(G)$ by $\rho(x) = t(\alpha_x)$. By (a), \mathcal{X} consists of sinks in G . Suppose $y \in \text{Im } \rho$ is a sink in G . By (b), there is $\beta \in a(F)$ satisfying $s(\beta) = y$ and, by assumption, $\beta \notin a(G)$. Therefore $y \in \mathcal{X}$. We have just proved that (G, \mathcal{D}, ρ) satisfies the conditions of Definition 3.1. It remains to use (b) to conclude that $k(G, \mathcal{D}, \rho) = k[F, \sigma(F)]$. \square

We now determine when both Λ and Λ^{op} satisfy (F). In the sequel, H^{op} stands for the opposite quiver of H , and if $\gamma : x \rightarrow y$ is an arrow of H , we denote by $\gamma^{op} : y \rightarrow x$ the corresponding arrow of H^{op} .

Proposition 3.9. *An elementary finite-dimensional k -algebra Λ and its opposite satisfy (F) if and only if $\Lambda \simeq k(G, \mathcal{D}, \rho)$ and the following conditions hold.*

- (a) *Each $y \in \text{Im } \rho$ is a source in G .*
- (b) *If $\beta \in a(G)$ with $s(\beta) \in \text{Im } \rho$ then β is the only arrow of G that ends at $t(\beta)$.*
- (c) *If $\rho(d_1) = \rho(d_2)$ and $d_1 \neq d_2$, then $d_i \notin \text{Im } \rho$, $i = 1, 2$.*

Proof. By Theorem 3.7(a), it suffices to show that if $\Lambda \simeq k(G, \mathcal{D}, \rho) = k[H, \rho(H)]$, then Λ^{op} satisfies (F) if and only if (a), (b) and (c) hold. In view of Definition 3.2, $\Lambda^{op} \simeq k[F, \sigma(F)]$ where $F = H^{op}$ and $\sigma(F) = \{\alpha^{op}\beta^{op} \mid \beta\alpha \in \rho(H)\}$. Using Notation 3.4, we have $\mathcal{X} = \{t(\beta) \mid \beta\alpha \in \rho(H)\}$.

Suppose Λ^{op} satisfies (F). If $\beta, \beta' \in a(G)$ with $t(\beta) = t(\beta')$ and $s(\beta) \in \text{Im } \rho$, then $\beta\alpha \in \rho(H)$ for some $\alpha \in a(H)$ so that $s(\beta^{op}) = s(\beta'^{op}) \in \mathcal{X}$. Applying Proposition 3.8(a) to Λ^{op} , we get $\beta^{op} = \beta'^{op}$ whence $\beta = \beta'$. Thus (b) holds. Let $\gamma \in a(G)$ satisfy $t(\gamma) \in \text{Im } \rho$. Then $t(\gamma) = t(\alpha)$ where $\alpha \in a(H)$ so $\beta\alpha \in \rho(H)$ for some $\beta \in a(H)$. Hence $s(\beta^{op}) = t(\beta) \in \mathcal{X}$ and $s(\gamma^{op}) = s(\alpha^{op}) = t(\beta^{op})$. Applying Proposition 3.8(b) to Λ^{op} , we get $\gamma^{op}\beta^{op} \in \sigma(F)$ whence $\beta\gamma \in \rho(H)$. The latter contradicts Theorem 3.7(b) because $\gamma \in a(G)$, so (a) holds. Let $d_1, d_2 \in \mathcal{D}$ be distinct vertices satisfying $\rho(d_1) = \rho(d_2) = x$. If, say, $d_1 = \rho(d)$ for some $d \in \mathcal{D}$, then by Definition 3.2 there exist arrows $\alpha_{d_1} : d_1 \rightarrow x$, $\alpha_{d_2} : d_2 \rightarrow x$, and $\alpha_d : d \rightarrow d_1$ in $a(H)$, and we have $\alpha_{d_1}\alpha_d \in \rho(H)$ whence $\alpha_d^{op}\alpha_{d_1}^{op} \in \sigma(F)$ and $x \in \mathcal{X}$. This contradicts Proposition 3.8(a): $\alpha_{d_1}^{op}, \alpha_{d_2}^{op} \in a(F)$ are distinct and $s(\alpha_{d_1}^{op}) = s(\alpha_{d_2}^{op}) = x$. We conclude that (c) holds.

Suppose (a), (b) and (c) hold. To show Λ^{op} satisfies (F), we verify the conditions of Proposition 3.8. Since Λ satisfies (F), Proposition 3.1(a) implies that each oriented cycle in H passes through a vertex in \mathcal{D} . By Lemma 3.3(a), for each $d \in \mathcal{D}$, there is a unique arrow $\alpha_d \in a(H)$ with $s(\alpha_d) = d$ and $t(\alpha_d) \in \text{Im } \rho$. Hence each oriented cycle contains an arrow β with $s(\beta) = t(\alpha_d) \in \text{Im } \rho$. By Definition 3.2, $\beta\alpha_d \in \rho(H)$ whence $t(\beta) \in \mathcal{X}$. Thus each oriented cycle in H passes through a vertex in \mathcal{X} , and so does each cycle in H^{op} . Hence condition (c) of Proposition 3.8 holds.

Let $x \in \mathcal{X}$, then $x = t(\beta)$ where $\beta\alpha \in \rho(H)$ for some $\alpha, \beta \in a(H)$ and $s(\beta) = t(\alpha) \in \text{Im } \rho$. Let $\gamma^{op} \in a(H^{op})$ with $s(\gamma^{op}) = t(\gamma) = x$. To show $\gamma = \beta$ and, hence, $\gamma^{op} = \beta^{op}$, assume, to the contrary, that $\gamma \neq \beta$. If $\beta \in a(G)$ then, by (b), $\gamma \notin a(G)$ whence $x = \rho(s(\gamma))$. By (a), x is a source in G , a contradiction. If $\beta \notin a(G)$ then $x = \rho(s(\beta))$ is a source in G , so $\gamma \notin a(G)$. By Definition 3.2, $s(\beta) \neq s(\gamma)$ which, together with $x = \rho(s(\beta)) = \rho(s(\gamma))$ and $s(\beta) \in \text{Im } \rho$, contradicts (c). Thus we must have $\gamma = \beta$, so condition (a) of Proposition 3.8 holds.

Suppose $\beta^{op}, \gamma^{op} \in a(H^{op})$ satisfy $s(\beta^{op}) \in \mathcal{X}$ and $s(\gamma^{op}) = t(\beta^{op})$. Then $t(\beta) \in \mathcal{X}$ and there exists $\alpha \in a(H)$ satisfying $\beta\alpha \in \rho(H)$. It follows that $t(\gamma) = s(\beta) \in \text{Im } \rho$ so that, by (a), $\gamma \notin a(G)$. By Definition 3.2, $\beta\gamma \in \rho(H)$ whence $\gamma^{op}\beta^{op} \in \sigma(F)$. Thus condition (b) of Proposition 3.8 holds. \square

We finish this section with examples of elementary k -algebras satisfying (F). In view of Theorem 3.7, it suffices to indicate the triple (G, \mathcal{D}, ρ) that determines such an algebra.

Example 3.5. (a) Let (G, \mathcal{D}, ρ) be the triple given by $v(G) = \{1\}$, $a(G) = \emptyset$, $\mathcal{D} = v(G)$, and $\rho(1) = 1$. Then $\Lambda_1 = k(G, \mathcal{D}, \rho) = k[H, \rho(H)]$ is the path algebra of the quiver H with $v(H) = v(G)$ and $a(H) = \{\alpha_1 : 1 \rightarrow 1\}$ modulo the ideal generated by $\rho(H) = \{\alpha_1^2\}$. Since Λ_1 is commutative, its opposite algebra satisfies (F).

(b) Let $v(G) = \{1, 2\}$, $a(G) = \emptyset$, $\mathcal{D} = v(G)$, and $\rho(1) = \rho(2) = 2$. Then $v(H) = v(G)$, $a(H) = \{\alpha_1 : 1 \rightarrow 2, \alpha_2 : 2 \rightarrow 2\}$, and $\rho(H) = \{\alpha_2\alpha_1, \alpha_2^2\}$. Since the triple (G, \mathcal{D}, ρ) does not

satisfy condition (c) of Proposition 3.9, the opposite algebra of $\Lambda_2 = k(G, \mathcal{D}, \rho)$ does not satisfy (F).

(c) Let $v(G) = \{1, 2\}$, $a(G) = \{\beta : 1 \rightarrow 2\}$, $\mathcal{D} = \{2\}$, and $\rho(2) = 2$. Then $v(H) = v(G)$, $a(H) = \{\beta : 1 \rightarrow 2, \alpha_2 : 2 \rightarrow 2\}$, and $\rho(H) = \{\alpha_2^2\}$. Since the triple (G, \mathcal{D}, ρ) does not satisfy condition (a) of Proposition 3.9, the opposite algebra of $\Lambda_3 = k(G, \mathcal{D}, \rho)$ does not satisfy (F).

(d) Suppose that G and \mathcal{D} are the same as in (c) but $\rho(2) = 1$. Then $v(H) = v(G)$, $a(H) = \{\beta : 1 \rightarrow 2, \alpha_2 : 2 \rightarrow 1\}$, and $\rho(H) = \{\beta\alpha_2\}$. By Proposition 3.9, the opposite algebra of $\Lambda_4 = k(G, \mathcal{D}, \rho)$ satisfies (F).

Since $\text{gl. dim } \Lambda_4 < \infty$, the support variety of each module in $\text{mod } \Lambda_4$ is trivial. The algebras Λ_1 , Λ_2 , and Λ_3 have a lot in common: all three are of infinite global dimension, and each module in $\text{mod } \Lambda_i$, $i = 1, 2, 3$, is eventually periodic by Corollary 2.3. All three are quadratic monomial, hence, Koszul algebras so the cohomology ring $\text{HH}^*(\Lambda_i)$ maps onto the graded center of the Ext-algebra $E(\Lambda_i)$. Since the graded center of $E(\Lambda_2)$ is the ground field k , the support variety of each module in $\text{mod } \Lambda_2$ is trivial. However, when i equals 1 or 3, the graded center is the polynomial algebra in one variable with a suitable grading, and $E(\Lambda_i)$ is a finitely generated module over the graded center. In this setting the methods of [14] apply, and using these methods, it is possible to show that the unique simple Λ_1 -module and the simple Λ_3 -module associated to the vertex 2 have nontrivial support varieties [15].

4. THE EXT-ALGEBRA OF Λ SATISFYING (F)

In this section, Λ is an elementary finite-dimensional k -algebra satisfying (F). We study the Ext-algebra $E(\Lambda) = \bigoplus_{i=0}^{\infty} \text{Ext}_{\Lambda}^i(\Lambda / \mathfrak{r}, \Lambda / \mathfrak{r})$.

Theorem 4.1. *Let $\Lambda \simeq k(G, \mathcal{D}, \rho) = k[H, \rho(H)]$ as described in Definitions 3.1 and 3.2.*

- (a) $E(\Lambda) \simeq k[H, \eta(H)]$ where $\eta(H) = \{\beta\gamma \mid \beta \in a(H), \gamma \in a(G)\}$.
- (b) $E(\Lambda)$ is left noetherian.

Proof. (a) The underlying quiver and relations of the Ext-algebra of a quadratic monomial algebra are described in [7, Proposition 2.2]. Applying this description to Λ , we obtain the statement as an immediate consequence of Theorem 3.7.

(b) Let Q be the quiver with $v(Q) = v(H)$ and $a(Q) = \{\alpha_d \mid d \in \mathcal{D}\}$. Let $\psi : kH \rightarrow kQ$ be a unique epimorphism of k -algebras determined by $\psi(e_i) = e_i$ for all $i \in v(H)$; $\psi(\beta) = 0$ for all $\beta \in a(G)$; and $\psi(\alpha_d) = \alpha_d$ for all $d \in \mathcal{D}$. Since $\langle \eta(H) \rangle \subset \text{Ker } \psi$, there exists a unique epimorphism of algebras $\phi : k[H, \eta(H)] \rightarrow kQ$ satisfying $\psi = \phi\pi$ where $\pi : kH \rightarrow k[H, \eta(H)]$ is the natural projection. To prove that $E(\Lambda)$ is left noetherian, we show that kQ is left noetherian, and $\text{Ker } \phi$ is finite-dimensional over k .

For kQ , we use [11, Theorem 8], which says that the path algebra $k\Gamma$ of a finite quiver Γ is left noetherian if and only if Γ satisfies the following condition. If there is an oriented cycle passing through a vertex i of Γ , then only one arrow starts at i . By Definition 3.2, Q satisfies this condition.

It is easy to see that, first, $\text{Ker } \phi$ is spanned over k by the elements $\pi(p)$ where p is a path in $\text{Ker } \psi$, and, second, $p \in \text{Ker } \psi$ if and only if p contains an arrow of G . Suppose $p \in \text{Ker } \psi$ is an oriented cycle in H . Since G has no oriented cycles, p must contain an arrow $\alpha_d, d \in \mathcal{D}$, and, hence, a subpath of the form $\alpha_d\gamma$ with $\gamma \in a(G)$, which implies $\pi(p) = 0$. Therefore $\text{Ker } \phi$ is spanned by the elements $\pi(p)$ where $p \in \text{Ker } \psi$ and no subpath of p is an oriented cycle. Since there are only finitely many such paths p , $\text{Ker } \phi$ is finite-dimensional. \square

We note that the Ext-algebra $E(\Lambda)$ need not be right noetherian. For example, for the algebra Λ_2 of Example 3.5(b), $E(\Lambda_2)$ is the path algebra of the quiver with the set of vertices $\{1, 2\}$ and

the set of arrows $\{\alpha_1 : 1 \rightarrow 2, \alpha_2 : 2 \rightarrow 2\}$. It is not right noetherian because the loop α_2 passes through the vertex 2, and more than one arrow ends at 2.

REFERENCES

- [1] M. Auslander and I. Reiten, Stable equivalence of artin algebras, in "Proceedings of the Conference on Orders, Group Rings and Related Topics," Ohio State University, 1972, Lecture Notes in Mathematics, Vol. 353, Springer-Verlag, Berlin-Heidelberg-New York, 1973, pp. 8-64.
- [2] M. Auslander, I. Reiten and S. O. Smalø, "Representation Theory of Artin Algebras," Cambridge Studies in Advanced Mathematics, Vol. 36, Cambridge University Press, New York, 1994.
- [3] L. L. Avramov, Homological asymptotics of modules over local rings, in "Commutative algebra (Berkeley, CA, 1987)," Math. Sci. Res. Inst. Publ., **15**, Springer, New York, 1989, pp. 33-62.
- [4] D. J. Benson, "Representations and Cohomology II," Cambridge Studies in Advanced Mathematics, Vol. 31, Cambridge University Press, Cambridge, 1991.
- [5] K. Bongartz and C. Riedtmann, Algèbres stablement héréditaires. (French) *C.R. Acad. Sci. Paris Sér. A-B* **288** (1979), no. 15, 703–706.
- [6] K. Erdmann, M. Holloway, N. Snashall, Ø. Solberg, R. Taillefer, Support varieties for selfinjective algebras, *K-Theory* **33** (2004), 67-87.
- [7] E.L. Green and D. Zacharia, The Cohomology Ring of a Monomial Algebra, *manuscripta math.* 85, 11-23 (1994).
- [8] S. Jagadeeshan and M. Kleiner, Stable artin algebras: the transpose and the global dimension, in "Seventh Conference on Representations of Algebras (ICRA VII)," Canadian Mathematical Society Conference Proceedings, Vol. **18**, 1996, pp. 343-351.
- [9] S. Jagadeeshan and M. Kleiner, Structure of projectively stable artinian rings, *J. Algebra* **197** (1997), 92-126.
- [10] S. Jagadeeshan and M. Kleiner, Projectively stable artin algebras: representation theory and quivers with relations, *Comm. Algebra* **27(5)** (1999), 2277-2316.
- [11] S. Liu, Isomorphism problem for tensor algebras over valued graphs, *Sci.China Ser.A*, **34**(1991), no. 3, pp. 267-272.
- [12] R. Martínez-Villa and D. Zacharia, Approximations with modules having linear resolutions. *J. Algebra* **266** (2003), no. 2, 671–697.
- [13] M. I. Platzeck, Representation theory of algebras stably equivalent to an hereditary artin algebra, *Trans. Amer. Math. Soc* **238** (1978), 89-128.
- [14] N. Snashall and Ø. Solberg, Support varieties and Hochschild cohomology rings, *Proc. London Math. Soc. (3)* **88** (2004), no. 3, 705–732.
- [15] Ø. Solberg, private communication.

CENTRE FOR LEARNING, 462, 9TH CROSS, JAYANAGAR 1ST BLOCK, BANGALORE 560 012, INDIA
E-mail address: jshashidhar@gmail.com

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13244-1150
E-mail address: mkleiner@sound.syr.edu