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**TESTING FOR HETEROSKEDASTICITY AND  
SERIAL CORRELATION IN A RANDOM  
EFFECTS PANEL DATA MODEL**

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## Abstract

This paper considers a panel data regression model with heteroskedastic as well as serially correlated disturbances, and derives a joint LM test for homoskedasticity and no first order serial correlation. The restricted model is the standard random individual error component model. It also derives a conditional LM test for homoskedasticity given serial correlation, as well as, a conditional LM test for no first order serial correlation given heteroskedasticity, all in the context of a random effects panel data model. Monte Carlo results show that these tests along with their likelihood ratio alternatives have good size and power under various forms of heteroskedasticity including exponential and quadratic functional forms.

Keywords: Panel data; Heteroskedasticity; Serial Correlation; Lagrange Multiplier tests; Likelihood Ratio, Random Effects.

JEL classification: C23.

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# 1 Introduction

The standard error component panel data model assumes that the disturbances have homoskedastic variances and constant serial correlation through the random individual effects, see Hsiao (2003) and Baltagi (2005). These may be restrictive assumptions for a lot of panel data applications. For example, the cross-sectional units may be varying in size and as a result may exhibit heteroskedasticity. Also, for investment behavior of firms, for example, an unobserved shock this period may affect the behavioral relationship for at least the next few periods. In fact, the standard error components model has been extended to take into account serial correlation by Lillard and Willis(1978), Baltagi and Li (1995), Galbraith and Zinde-Walsh (1995), Bera, Sosa-Escudero and Yoon (2001) and Hong and Kao (2004) to mention a few. This model has also been generalized to take into account hetetoskedasticity by Mazodier and Trognon (1978), Baltagi and Griffin (1988), Li and Stengos (1994), Lejeune (1996), Holly and Gardiol (2000), Roy (2002) and Baltagi, Bresson and Pirotte (2006) to mention a few. For a review of these papers, see Baltagi (2005). However, these strands of literature are almost separate in the panel data error components literature. When one deals with heteroskedasticity, serial correlation is ignored, and when one deals with serial correlation, heteroskedasticity is ignored. Exceptions are robust estimation of the variance-covariance matrix of the reported estimates.

Baltagi and Li (1995) for example, derived a Lagrange Multiplier (LM) test which jointly tests for the presence of serial correlation as well as random individual effects assuming homoskedasticity of the disturbances. While, Holly and Gardiol (2000), for example, derived an LM statistic which tests for homoskedasticity of the disturbances in the context of a one-way random effects panel data model. The latter LM test assumes no serial correlation in the remainder disturbances. This paper extends the Holly and Gardiol (2000) model to allow for first order serial correlation in the remainder disturbances as described in Baltagi and Li (1995). It derives a joint LM test for homoskedasticity and no first order serial correlation.

The restricted model is the standard random effects error component model. It also derives a conditional LM test for homoskedasticity given serial correlation, as well as, a conditional LM test for no first order serial correlation given heteroskedasticity. Monte Carlo results show that these tests along with their likelihood ratio alternatives have good size and power under various forms of heteroskedasticity including exponential and quadratic functional forms.

## 2 The Model

Consider the following panel data regression model :

$$y_{it} = x'_{it}\beta + u_{it}, \quad i = 1, \dots, N, \text{ and } t = 1, \dots, T, \quad (1)$$

where  $y_{it}$  is the observation on the dependent variable for the  $i$ th individual at the  $t$ th time period,  $x_{it}$  denotes the  $k \times 1$  vector of observations on the nonstochastic regressors. The regression disturbances of (1) are assumed to follow a one-way error component model

$$u_{it} = \mu_i + \nu_{it}, \quad (2)$$

where  $\mu_i$  denote the random individual effects which are assumed to be normally and independently distributed with mean 0 and variance

$$Var(\mu_i) = h(z'_i\alpha), \quad (3)$$

the function  $h(\cdot)$  is an arbitrary non-indexed (strictly) positive twice continuously differentiable function, see Breusch and Pagan (1979).  $\alpha$  is a  $p \times 1$  vector of unrestricted parameters and  $z_i$  is a  $p \times 1$  vector of strictly exogenous regressors which determine the heteroskedasticity of the individual specific effects. The first element of  $z_i$  is one, and without loss of generality,  $h(\alpha_1) = \sigma_\mu^2$ . Therefore, when the model is homoskedastic with  $\alpha_2 = \alpha_3 = \dots = \alpha_p = 0$ , this model reduces to the standard random effects model, as in Holly and Gardiol (2000). In

addition, we allow the remainder disturbances to follow an  $AR(1)$  process:  $\nu_{it} = \rho\nu_{i,t-1} + \epsilon_{it}$ , with  $|\rho| < 1$  and  $\epsilon_{it} \sim IIN(0, \sigma_\epsilon^2)$ , as described in Baltagi and Li (1995). The  $\mu_i$ 's are independent of the  $\nu_{it}$ 's, and  $\nu_{i,0} \sim N(0, \sigma_\epsilon^2/(1 - \rho^2))$ .

The model considered generalizes the one-way error component model to allow for heteroskedastic individual effects a la Holly and Gardiol (2000) *and* for first order serially correlated remainder disturbances a la Baltagi and Li (1995). The model (1) can be rewritten in matrix notation as

$$y = X\beta + u, \quad (4)$$

where  $y$  is of dimension  $NT \times 1$ ,  $X$  is  $NT \times k$ ,  $\beta$  is  $k \times 1$  and  $u$  is a  $NT \times 1$ .  $X$  is assumed to be of full column rank. The disturbance in equation (2) can be written in vector form as:

$$u = (I_N \otimes \iota_T)\mu + \nu, \quad (5)$$

where  $\iota_T$  is a vector of ones of dimension  $T$ ,  $I_N$  is an identity matrix of dimension  $N$ ,  $\mu' = (\mu_1, \dots, \mu_N)$  and  $\nu' = (\nu_{11}, \dots, \nu_{1T}, \dots, \nu_{N1}, \dots, \nu_{NT})$ . Under these assumptions, the variance-covariance matrix of  $u$  can be written as

$$\begin{aligned} \Omega &= E(uu') = (I_N \otimes \iota_T)(diag[h(z'_i\alpha)](I_N \otimes \iota_T)' + I_N \otimes V \\ &= diag[h(z'_i\alpha)] \otimes J_T + I_N \otimes V, \end{aligned} \quad (6)$$

where  $J_T$  is a matrix of ones of dimension  $T$ , and  $diag[h(z'_i\alpha)]$  is a diagonal matrix of dimension  $N \times N$  and  $V$  is the familiar  $AR(1)$  covariance matrix. It is well established that the matrix

$$C = \begin{bmatrix} (1 - \rho^2)^{1/2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\rho & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix}$$

transforms the usual  $AR(1)$  model into serially uncorrelated disturbances. For panel data, this has to be applied for  $N$  individuals, see Baltagi and Li (1995). The transformed regres-

sion disturbances are given by :

$$\begin{aligned} u^* &= (I_N \otimes C)u = (I_N \otimes C\iota_T)\mu + (I_N \otimes C)\nu \\ &= (1 - \rho)(I_N \otimes \iota_T^\delta)\mu + (I_N \otimes C)\nu, \end{aligned} \quad (7)$$

where  $C\iota_T = (1 - \rho)\iota_T^\delta$  with  $\iota_T^\delta = (\delta, \iota_{T-1}')$ , and  $\delta = \sqrt{\frac{1+\rho}{1-\rho}}$ .

Therefore, the variance-covariance matrix of transformed model is given by

$$\begin{aligned} \Omega^* &= E(u^*u^{*'}) \\ &= \text{diag}[h(z_i'\alpha)(1 - \rho)^2] \otimes i_T^\delta i_T^{\delta'} + \text{diag}[\sigma_\epsilon^2] \otimes I_T, \end{aligned} \quad (8)$$

since  $(I_N \otimes C)E(\nu\nu')(I_N \otimes C') = \text{diag}[\sigma_\epsilon^2] \otimes I_T$ . Replace  $J_T^\delta = i_T^\delta i_T^{\delta'}$  by its idempotent counterpart  $d^2 \bar{J}_T^\delta$ , where  $d^2 = i_T^{\delta'} i_T^\delta = \delta^2 + T - 1$ , and  $\bar{J}_T^\delta$  is by definition  $J_T^\delta / d^2$ . Also, replace  $I_T$  by  $E_T^\delta + \bar{J}_T^\delta$ , where  $E_T^\delta$  is by definition  $I_T - \bar{J}_T^\delta$ , and collect like terms, we get

$$\Omega^* = \text{diag}[\lambda_i^2] \otimes \bar{J}_T^\delta + \text{diag}[\sigma_\epsilon^2] \otimes E_T^\delta, \quad (9)$$

where  $\lambda_i^2 = d^2(1 - \rho)^2 h(z_i'\alpha) + \sigma_\epsilon^2$ , from which it is easy to infer, see Wansbeek and Kapteyn (1982) and Baltagi and Li (1995) that

$$\Omega^{*r} = \text{diag}[(\lambda_i^2)^r] \otimes \bar{J}_T^\delta + \text{diag}[(\sigma_\epsilon^2)^r] \otimes E_T^\delta, \quad (10)$$

where  $r$  is an arbitrary scalar.  $r = -1$  obtains the inverse, while  $r = -\frac{1}{2}$  obtains  $\Omega^{*-1/2}$ . In addition, one gets,  $|\Omega^*| = \prod_{i=1}^N (\lambda_i^2)(\sigma_\epsilon^2)^{T-1}$ , see also Magnus (1982).

## 3 LM Tests

### 3.1 Joint LM Test

In this subsection, we derive the joint LM test for testing for no heteroskedasticity and no serial correlation of the first order in a random effects panel data model. The null hypothesis



is given by

$$H_0^a : \alpha_2 = \dots = \alpha_p = 0 \quad \text{and} \quad \rho = 0. \quad (11)$$

The log-likelihood function under normality of the disturbances is given by

$$L(\beta, \sigma_\epsilon^2, \rho, \alpha) = \text{const.} - \frac{1}{2} \sum_{i=1}^N \log(\lambda_i^2) - \frac{1}{2} N(T-1) \log(\sigma_\epsilon^2) - \frac{1}{2} u^{*'} \Omega^{*-1} u^*, \quad (12)$$

where (12) uses the fact that  $\Omega = E(uu')$  is related to  $\Omega^*$  by  $\Omega^* = (I_N \otimes C)\Omega(I_N \otimes C')$  with  $|C| = \sqrt{1-\rho^2}$ ,  $|I_N \otimes C| = |C|^N$  and  $|\Omega^*| = \Pi_{i=1}^N (\lambda_i^2) (\sigma_\epsilon^2)^{T-1}$ . Let  $\theta' = (\sigma_\epsilon^2, \rho, \alpha')$ . Since, the information matrix is block diagonal between the  $\theta$  and  $\beta$  parameters, the part of the information matrix corresponding to  $\beta$  will be ignored in computing the LM statistic, see Breusch and Pagan (1980).

Under the null hypothesis  $H_0^a$ , the variance-covariance matrix reduces to  $\Omega_a = \sigma_\mu^2 I_N \otimes J_T + \sigma_\epsilon^2 I_{NT}$ . This is the familiar one-way random effects error component model, see Baltagi (2005), with  $\Omega_a^{-1} = (\sigma_1^2)^{-1} (I_N \otimes \bar{J}_T) + (\sigma_\epsilon^2)^{-1} (I_N \otimes E_T)$ , where  $\sigma_1^2 = T\sigma_\mu^2 + \sigma_\epsilon^2$ . Using general formulas on log-likelihood differentiation, see Hemmerle and Hartley(1973) and Harville(1977), Appendix 1 derives the scores of the likelihood evaluated at the restricted MLE under  $H_0^a$ :

$$\begin{aligned} \frac{\partial L}{\partial \sigma_\epsilon^2} \Big|_{H_0^a} &= D(\hat{\sigma}_\epsilon^2) = -\frac{1}{2} \left( \frac{N}{\tilde{\sigma}_1^2} + \frac{N(T-1)}{\tilde{\sigma}_\epsilon^2} \right) + \frac{1}{2} \tilde{u}' \left( \frac{1}{\tilde{\sigma}_1^4} I_N \otimes \bar{J}_T + \frac{1}{\tilde{\sigma}_\epsilon^4} I_N \otimes E_T \right) \tilde{u} = 0 \\ \frac{\partial L}{\partial \alpha_1} \Big|_{H_0^a} &= D(\tilde{\alpha}_1) = \frac{Th'(\tilde{\alpha}_1)}{2\tilde{\sigma}_1^2} \sum_{i=1}^N f_i = 0 \\ \frac{\partial L}{\partial \alpha_k} \Big|_{H_0^a} &= D(\tilde{\alpha}_k) = \frac{Th'(\tilde{\alpha}_1)}{2\tilde{\sigma}_1^2} \sum_{i=1}^N z_{ik} f_i, \quad k = 2, \dots, p \\ \frac{\partial L}{\partial \rho} \Big|_{H_0^a} &= D(\tilde{\rho}) = \frac{N(T-1)}{T} \frac{\tilde{\sigma}_1^2 - \tilde{\sigma}_\epsilon^2}{\tilde{\sigma}_1^2} \\ &\quad + \frac{\tilde{\sigma}_\epsilon^2}{2} \tilde{u}' [I_N \otimes (\bar{J}_T/\tilde{\sigma}_1^2 + E_T/\tilde{\sigma}_\epsilon^2) G(\bar{J}_T/\tilde{\sigma}_1^2 + E_T/\tilde{\sigma}_\epsilon^2)] \tilde{u}. \end{aligned} \quad (13)$$

where  $\tilde{u} = y - X\tilde{\beta}_{MLE}$  denote the restricted maximum likelihood residuals under the null hypothesis  $H_0^a$ , i.e., under a random effects panel data model.  $\tilde{\sigma}_\epsilon^2$  is the solution of  $D(\hat{\sigma}_\epsilon^2) = 0$ ,

while  $\tilde{\alpha}_1$  is the solution of  $D(\tilde{\alpha}_1) = 0$ . The latter gives the result that  $\sum_{i=1}^N f_i = 0$ , where  $f_i = [(\sum_{t=1}^T \tilde{u}_{it})^2 / T\tilde{\sigma}_1^2] - 1$ . Thus, the score vectors under  $H_0^a$  are given by

$$D(\tilde{\theta}) = \begin{pmatrix} D(\tilde{\sigma}_\epsilon^2) \\ D(\tilde{\rho}) \\ D(\tilde{\alpha}) \end{pmatrix} = \begin{pmatrix} 0 \\ D(\tilde{\rho}) \\ \frac{Th'(\tilde{\alpha}_1)}{2\tilde{\sigma}_1^2} Z' f \end{pmatrix}, \quad (14)$$

where  $D(\tilde{\alpha}) = (0, D(\tilde{\alpha}_2), \dots, D(\tilde{\alpha}_p))'$ ,  $h'(\tilde{\alpha}_1)$  is  $\partial h(z'_i \alpha) / \partial \alpha$  when  $\alpha_2 = \dots = \alpha_p = 0$ .

$$Z = (z_1, \dots, z_N)' = \begin{bmatrix} z_{11} & z_{21} & \cdots & z_{N1} \\ z_{12} & z_{22} & \cdots & z_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ z_{1p} & z_{2p} & \cdots & z_{Np} \end{bmatrix}'$$

is an  $N \times p$  matrix of observations on the  $p$  variables  $z_k$ ,  $k = 1, 2, \dots, p$ , each of dimension  $N \times 1$ , and  $f = (f_1, \dots, f_N)'$ . Note that  $\tilde{\sigma}_1^2 = \tilde{u}'(I_N \otimes \bar{J}_T)\tilde{u}/N$  and  $\tilde{\sigma}_\epsilon^2 = \tilde{u}'(I_N \otimes E_T)\tilde{u}/N(T-1)$ , are the solutions of  $\frac{\partial L}{\partial \alpha_1}|_{H_0} = 0$  and  $\frac{\partial L}{\partial \sigma_\epsilon^2}|_{H_0} = 0$ , respectively. In addition, using the results of Harville (1977), the information matrix for  $\theta$  under  $H_0^a$  is derived in Appendix 1 as:

$$\tilde{J}_a(\theta) = \begin{bmatrix} \frac{N}{2} \left( \frac{1}{\tilde{\sigma}_1^4} + \frac{T-1}{\tilde{\sigma}_\epsilon^4} \right) & \frac{N(T-1)}{T} \tilde{\sigma}_\epsilon^2 \left( \frac{1}{\tilde{\sigma}_1^4} - \frac{1}{\tilde{\sigma}_\epsilon^4} \right) & \frac{Th'(\alpha_1)}{2\tilde{\sigma}_1^4} i'_N Z \\ \frac{N(T-1)}{T} \tilde{\sigma}_\epsilon^2 \left( \frac{1}{\tilde{\sigma}_1^4} - \frac{1}{\tilde{\sigma}_\epsilon^4} \right) & \tilde{J}_{\rho\rho} & \frac{(T-1)\tilde{\sigma}_\epsilon^2 h'(\alpha_1)}{\tilde{\sigma}_1^4} i'_N Z \\ \frac{Th'(\alpha_1)}{2\tilde{\sigma}_1^4} Z' i_N & \frac{(T-1)\tilde{\sigma}_\epsilon^2 h'(\alpha_1)}{\tilde{\sigma}_1^4} Z' i_N & \frac{T^2 h'(\alpha_1)^2}{2\tilde{\sigma}_1^4} Z' Z \end{bmatrix}, \quad (15)$$

where  $\tilde{J}_{\rho\rho} = N[2a^2(T-1)^2 + 2a(2T-3) + T-1]$ ,  $a = \frac{\tilde{\sigma}_\epsilon^2 - \tilde{\sigma}_1^2}{T\tilde{\sigma}_1^2}$ . Using (14) and (15), the LM statistic for the hypothesis (11) is given by

$$\begin{aligned} LM_a &= \tilde{D}'_\theta \tilde{J}_\theta^{-1} \tilde{D}_\theta \\ &= \frac{1}{2} f' Z (Z' Z)^{-1} Z' f + \frac{T^2}{T^2 C_{\rho\rho} - 2N(T-1)} \tilde{D}(\rho)^2, \\ &= \frac{1}{2} (g' Z (Z' Z)^{-1} Z' g - N) + \frac{T^2}{T^2 C_{\rho\rho} - N(T-1)} \tilde{D}(\rho)^2, \end{aligned} \quad (16)$$

where  $C_{\rho\rho} = J_{\rho\rho} - \frac{2N(T-1)^2 \tilde{\sigma}_\epsilon^4}{T^2 \tilde{\sigma}_1^4}$ ,  $g = (g_1, \dots, g_N)'$ ,  $\tilde{J}_{\rho\rho}$  is given by (15). In (16), the second equality follows from the fact that  $f' Z (Z' Z)^{-1} Z' i_N = 0$  and the last equality uses  $f = g - i_N$  and  $g' i_N = N$ . Under the null hypothesis  $H_0^a$ , the LM statistic of (16) is asymptotically distributed as  $\chi_p^2$ .

### 3.2 Conditional LM Tests

The joint LM test derived in the previous section is useful especially when one does not reject the null hypothesis  $H_0^a$ . However, if the null hypotheses is rejected, one can not infer whether the presence of heteroskedasticity, or the presence of serial correlation, or both factors caused this rejection. In this section, we derive two conditional LM tests. The first one tests for the absence of serial correlation of the first order assuming that heteroskedasticity of the individual effects might be present. The second one tests for homoskedasticity assuming that serial correlation of the first order might be present. All in the context of a random effects panel data model.

For the first conditional LM test, the null hypothesis is given by

$$H_0^b : \rho = 0 \text{ (assuming some elements of } \alpha \text{ may not be zero)} \quad (17)$$

Under  $H_0^b$ , the variance-covariance matrix of the disturbances is given by:

$$\Omega = \text{diag}[h(z'_i\alpha)] \otimes J_T + \sigma_\epsilon^2(I_N \otimes I_T), \quad (18)$$

Replacing  $J_T$  by  $T\bar{J}_T$  and  $I_T$  by  $E_T + \bar{J}_T$ , and collecting like terms, one gets, see Wansbeek and Kapteyn (1982),

$$\Omega = \text{diag}[w_i^2] \otimes \bar{J}_T + \sigma_\epsilon^2(I_N \otimes E_T), \quad (19)$$

where  $w_i^2 = Th(z'_i\alpha) + \sigma_\epsilon^2$ . This also implies that

$$\Omega^{-1} = \text{diag}\left(\frac{1}{w_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2}(I_N \otimes E_T) \quad (20)$$

Using the general formula of Hemmerle and Hartley(1973), Appendix 2 derives the scores under  $H_0^b$  :

$$\frac{\partial \log L}{\partial \sigma_\epsilon^2} \Big|_{H_0^b} = -\frac{1}{2} \sum_{i=1}^N \left( \frac{1}{\hat{w}_i^2} + \frac{T-1}{\hat{\sigma}_\epsilon^2} \right) + \frac{1}{2} \hat{u}' \left[ \text{diag}\left(\frac{1}{\hat{w}_i^4}\right) \otimes \bar{J}_T + \frac{1}{\hat{\sigma}_\epsilon^4} I_N \otimes E_T \right] \hat{u} = 0 \quad (21)$$

$$\begin{aligned}
\frac{\partial \log L}{\partial \alpha_k} \Big|_{H_0^b} &= -\frac{T}{2} \sum_{i=1}^N \frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^2} + \frac{1}{2} \hat{u}' \left[ \text{diag} \left( \frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^4} \right) \otimes J_T \right] \hat{u} = 0, \quad k = 1, \dots, p \\
\frac{\partial \log L}{\partial \rho} \Big|_{H_0^b} &= \frac{T-1}{T} \sum_{i=1}^N \left( \frac{\hat{w}_i^2 - \hat{\sigma}_\epsilon^2}{\hat{w}_i^2} \right) + \frac{\hat{\sigma}_\epsilon^2}{2} \hat{u}' \left[ \left( \text{diag}(1/\hat{w}_i^2) \otimes \bar{J}_T + 1/\hat{\sigma}_\epsilon^2 I_N \otimes E_T \right) \right. \\
&\quad \left. (I_N \otimes G) \left( \text{diag}(1/\hat{w}_i^2) \otimes \bar{J}_T + 1/\hat{\sigma}_\epsilon^2 I_N \otimes E_T \right) \right] \hat{u} \tag{22}
\end{aligned}$$

where  $\hat{u} = y - X\hat{\beta}_{MLE}$  denotes the restricted MLE residuals under  $H_0^b$ . Also,  $\hat{w}_i^2 = Th(z'_i \hat{\alpha}) + \hat{\sigma}_\epsilon^2$ , where  $\hat{\alpha}$  and  $\hat{\sigma}_\epsilon^2$  are the restricted MLE of  $\alpha$  and  $\sigma_\epsilon^2$  under  $H_0^b$ . Therefore, the score vector under  $H_0^b$  can be written as:

$$D(\hat{\theta}) = \begin{pmatrix} D(\hat{\sigma}_\epsilon^2) \\ D(\hat{\rho}) \\ D(\hat{\alpha}) \end{pmatrix} = \begin{pmatrix} 0 \\ D(\hat{\rho}) \\ 0 \end{pmatrix}. \tag{23}$$

Appendix 2 also derives the information matrix with respect to  $\theta = (\sigma_\epsilon^2, \rho, \alpha)'$  under  $H_0^b$ .

This is given by:

$$\hat{J}_b(\theta) = \begin{bmatrix} \frac{1}{2} \sum_{i=1}^N \left( \frac{1}{\hat{w}_i^4} + \frac{T-1}{\hat{\sigma}_\epsilon^4} \right) & \frac{(T-1)\hat{\sigma}_\epsilon^2}{T} \sum_{i=1}^N \left( \frac{1}{\hat{w}_i^4} - \frac{1}{\hat{\sigma}_\epsilon^4} \right) & \frac{T}{2} i'_N H W^{-2} Z \\ \frac{(T-1)\hat{\sigma}_\epsilon^2}{T} \sum_{i=1}^N \left( \frac{1}{\hat{w}_i^4} - \frac{1}{\hat{\sigma}_\epsilon^4} \right) & \hat{\alpha}_{\rho\rho} & (T-1)\hat{\sigma}_\epsilon^2 i'_N H W^{-2} Z \\ \frac{T}{2} Z' W^{-2} H i_N & (T-1)\hat{\sigma}_\epsilon^2 Z' W^{-2} H i_N & \frac{T^2}{2} Z'^{-2} W^{-2} H^2 Z \end{bmatrix}, \tag{24}$$

where  $\hat{\alpha}_{\rho\rho} = \frac{2(T-1)^2}{T^2} \sum_{i=1}^N (\hat{\sigma}_\epsilon^2/\hat{w}_i^2 - 1)^2 + \frac{2(2T-3)}{T} \sum_{i=1}^N (\hat{\sigma}_\epsilon^2/\hat{w}_i^2 - 1) + N(T-1)$ .  $W = \text{diag}(\hat{w}_1^2, \dots, \hat{w}_N^2)$  and  $H = \text{diag}(h'(z'_1 \hat{\alpha}), \dots, h'(z'_N \hat{\alpha}))$ .

Therefore, the resulting LM test statistic for testing  $H_0^b : \rho = 0$  (assuming some elements of  $\alpha$  may not be zero) is

$$LM_b = D(\hat{\theta})' \hat{J}_b(\theta)^{-1} D(\hat{\theta}) = \hat{J}_b(\theta)^{\rho\rho} D(\hat{\rho})^2 \tag{25}$$

where  $\hat{J}_b(\theta)^{\rho\rho}$  is the element of the inverse of the information matrix corresponding to  $\rho$  evaluated under  $H_0^b$ . Under the null hypothesis,  $LM_b$  is asymptotically distributed as  $\chi_1^2$ .

The second conditional LM test the null hypothesis:

$$H_0^c : \alpha_2 = \dots = \alpha_p = 0 \text{ (given } \sigma_\mu^2 > 0 \text{ and } \rho > 0) \tag{26}$$

The variance-covariance matrix of the disturbances under  $H_0^c$  is given by

$$\Omega = \sigma_\mu^2(I_N \otimes J_T) + (I_N \otimes V) = \sigma_\mu^2(I_N \otimes J_T) + \sigma_\epsilon^2(I_N \otimes \Sigma), \quad (27)$$

where  $V = \sigma_\epsilon^2 \Sigma$ , and  $\Sigma = \frac{1}{1-\rho^2}R$ , where  $R$  is the usual AR(1) correlation matrix. Denote by  $F = \frac{\partial R}{\partial \rho}$ . Using the general formula of Hemmerle and Hartley(1973), Appendix 3 derives the scores under  $H_0^c$ . These are given by:

$$\begin{aligned} \frac{\partial L}{\partial \sigma_\epsilon^2} \Big|_{H_0^c} &= -\frac{1}{2\hat{\sigma}_\epsilon^2} \left[ NT - \left( \frac{Nd^2(1-\hat{\rho})^2\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) \right] \\ &\quad + \frac{1}{2} \hat{u}' \left[ \left\{ \frac{1}{\hat{\sigma}_\epsilon^2} (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\epsilon^2 \hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1}) \right\} (I_N \otimes \hat{\Sigma}) \right. \\ &\quad \left. \cdot \left\{ \frac{1}{\hat{\sigma}_\epsilon^2} (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\epsilon^2 \hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1}) \right\} \right] \hat{u} = 0 \\ \\ \frac{\partial L}{\partial \rho} \Big|_{H_0^c} &= -\frac{1}{2} \frac{1}{1-\hat{\rho}^2} \left[ 2\hat{\rho}NT + Ntr(\hat{\Sigma}^{-1}\hat{F}) - \left( \frac{2\hat{\rho}\hat{\sigma}_\mu^2Nd^2(1-\hat{\rho})^2}{\hat{\lambda}^2} \right) - \left( \frac{N\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) i_T' \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T \right] \\ &\quad + \frac{1}{2\hat{\sigma}_\epsilon^4} \hat{u}' \left[ \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \cdot \left\{ \frac{\hat{\sigma}_\epsilon^2}{1-\hat{\rho}^2} (2\hat{\rho} (I_N \otimes \hat{\Sigma}) + (I_N \otimes \hat{F})) \right\} \right. \\ &\quad \left. \cdot \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \right] \hat{u} = 0 \\ \\ \frac{\partial L}{\partial \alpha_k} \Big|_{H_0^c} &= D(\hat{\alpha}_k) = \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1-\hat{\rho})^2}{2\hat{\lambda}^2} \sum_{i=1}^N z_{ik} f_i = 0, \quad k = 1, 2, \dots, p \quad (28) \end{aligned}$$

where,  $\hat{u} = y - X\hat{\beta}_{MLE}$  denotes the restricted maximum likelihood residuals under the null hypothesis  $H_0^c$ . Also,  $\hat{\rho}$ ,  $\hat{\sigma}_\epsilon^2$  and  $\hat{\alpha}_1$  are the restricted ML estimates of  $\rho$ ,  $\sigma_\epsilon^2$  and  $\alpha_1$ , under  $H_0^c$ . Here

$$\begin{aligned} f_i &= \frac{\hat{\lambda}^2}{\hat{d}^2(1-\hat{\rho})^2\hat{\sigma}_\epsilon^4} \hat{u}_i' \hat{A} \hat{u}_i - 1, \text{ with} \\ \hat{A} &= \left( \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} - 2 \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} + \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4} \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right). \end{aligned}$$

Note that for  $k = 1$ ,  $z_{i1} = 1$ , and  $\left. \frac{\partial L}{\partial \alpha_1} \right|_{H_0^c}$  gives the result that  $\sum_{i=1}^N f_i = 0$ .

Therefore, the score vector under  $H_0^c$  can be written as:

$$D_c(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} D(\hat{\sigma}_\epsilon^2) \\ D(\hat{\rho}) \\ D(\hat{\boldsymbol{\alpha}}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2\hat{\lambda}^2} Z' f \end{pmatrix}, \quad (29)$$

where  $D(\hat{\boldsymbol{\alpha}}) = (0, D(\hat{\alpha}_2), \dots, D(\hat{\alpha}_p))'$ ,  $Z = (z_1, \dots, z_p)$  and  $f = (f_1, \dots, f_N)'$ .

Appendix 3 also derives the information matrix with respect to  $\boldsymbol{\theta} = (\sigma_\epsilon^2, \rho, \boldsymbol{\alpha})'$  under  $H_0^c$ .

This is given by:

$$\hat{J}_c(\boldsymbol{\theta}) = \begin{bmatrix} \frac{N}{2} \left( \frac{1}{\hat{\lambda}^4} + \frac{T-1}{\hat{\sigma}_\epsilon^4} \right) & \hat{C}(\epsilon, \rho) & a(\epsilon, \boldsymbol{\alpha}) i'_N Z \\ \hat{C}(\epsilon, \rho) & \hat{C}(\rho, \rho) & a(\rho, \boldsymbol{\alpha}) i'_N Z \\ a(\epsilon, \boldsymbol{\alpha}) Z' i_N & a(\rho, \boldsymbol{\alpha}) Z' i_N & a(\boldsymbol{\alpha}, \boldsymbol{\alpha}) Z' Z \end{bmatrix}, \quad (30)$$

where

$$\hat{C}(\epsilon, \rho) = \frac{N}{2(1 - \hat{\rho}^2) \hat{\sigma}_\epsilon^2} \left[ 2\hat{\rho} \left( \frac{\hat{\sigma}_\epsilon^4}{\hat{\lambda}^4} + T - 1 \right) - \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} (i'_T \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \left( 1 + \frac{\hat{\sigma}_\epsilon^2}{\hat{\lambda}^2} \right) + \text{tr}(\hat{\Sigma}^{-1} \hat{F}) \right]$$

$$\begin{aligned} \hat{C}(\rho, \rho) &= \frac{1}{2(1 - \hat{\rho}^2)^2} \left[ 4\hat{\rho}^2 NT + N \text{tr}(\hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} \hat{F}) + 4 \frac{N \hat{d}^4 (1 - \hat{\rho})^4 \hat{\sigma}_\mu^4 \hat{\rho}^2}{\hat{\lambda}^4} \right. \\ &\quad + \frac{N \hat{\sigma}_\mu^4}{\hat{\lambda}^4} (i'_T \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T)^2 + 4\hat{\rho} N \text{tr}(\hat{\Sigma}^{-1} \hat{F}) - 8 \frac{N \hat{d}^2 (1 - \hat{\rho})^2 \hat{\sigma}_\mu^2 \hat{\rho}^2}{\hat{\lambda}^2} \\ &\quad - 8 \frac{N \hat{\rho} \hat{\sigma}_\mu^2}{\hat{\lambda}^2} (i'_T \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) - 2 \frac{N \hat{\sigma}_\mu^2}{\hat{\lambda}^2} (i'_T \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \\ &\quad \left. + 4 \frac{N \hat{d}^2 (1 - \hat{\rho})^2 \hat{\sigma}_\mu^4 \hat{\rho}}{\hat{\lambda}^4} (i'_T \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \right] \end{aligned}$$

and

$$a(\epsilon, \boldsymbol{\alpha}) = \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2\hat{\lambda}^4}$$

$$\begin{aligned}
a(\rho, \boldsymbol{\alpha}) &= \frac{h'(\hat{\alpha}_1)\hat{\sigma}_\epsilon^2}{2(1-\hat{\rho}^2)\hat{\lambda}^4} \left( 2\hat{\rho}\hat{d}^2(1-\hat{\rho})^2 + (i_T'\hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}i_T) \right) \\
a(\boldsymbol{\alpha}, \boldsymbol{\alpha}) &= \frac{h'(\hat{\alpha}_1)^2\hat{d}^4(1-\hat{\rho})^4}{2\hat{\lambda}^4}.
\end{aligned} \tag{31}$$

Therefore, the the resulting LM test statistic for testing  $H_0^c : \alpha_2 = \dots = \alpha_p = 0$  (given  $\sigma_\mu^2 > 0$  and  $\rho > 0$ ) reduces to

$$LM_c = \frac{1}{2} f' Z (Z' Z)^{-1} Z' f \tag{32}$$

$LM_c$  is the familiar LM test used in testing the heteroskedasticity by Breusch and Pagan (1979). However, this one uses the random effects MLE residuals rather than OLS residuals. Under the null hypothesis  $H_0^c$ ,  $LM_c$  is asymptotically distributed as  $\chi_{p-1}^2$ .

### 3.3 Monte Carlo Results

The design of Monte Carlo experiments follows closely that of Baltagi et al. (2006) and Li and Stengos (1994). Consider the following simple regression model

$$y_{it} = \beta_0 + \beta_1 x_{it} + \mu_i + \nu_{it}, \quad i = 1, \dots, N, t = 1, \dots, T, \tag{33}$$

where  $\beta_0 = 5$  and  $\beta_1 = 0.5$ .  $x_{it}$  was generated using,  $x_{it} = w_{it} + 0.5w_{i,t-1}$ , where  $w_{it}$  is uniformly distributed on the interval  $[0, 2]$ . We choose  $N = 50, 100$  and  $200$  and  $T = 10$ . For each  $x_i$ , we generate  $T + 10$  observations and drop the first 10 observations in order to reduce the dependency on the initial values. In addition,  $\nu_{it}$  follows a traditional AR(1) process, namely,  $\nu_{it} = \rho\nu_{i,t-1} + \epsilon_{it}$  with  $\epsilon_{it} \sim IIN(0, \sigma_\epsilon^2)$ . The initial values  $\nu_{i0}$  were generated as  $IIN(0, \sigma_\epsilon^2/(1 - \rho^2))$ . The autocorrelation coefficient  $\rho$  varies over the set 0 to 0.5 by increments of 0.1.

For the individual heteroskedasticity, we adopt the Roy (2002) setup. More specifically, we generate  $\mu_i \sim N(0, \sigma_{\mu_i}^2)$  and  $\epsilon_{it} \sim N(0, \sigma_\epsilon^2)$  where

$$\sigma_{\mu_i}^2 = \sigma_{\mu_i}^2(\bar{x}_i) = \sigma_\mu^2(1 + \alpha\bar{x}_i)^2 \tag{34}$$

denoted as *quadratic heteroskedasticity*, or

$$\sigma_{\mu_i}^2 = \sigma_{\mu}^2(\bar{x}_i) = \sigma_{\mu}^2 \exp(\alpha \bar{x}_i), \quad (35)$$

denoted as *exponential heteroskedasticity*.  $\bar{x}_i$  is the individual mean of  $x_{it}$ . Denoting the expected variance of  $\mu_i$  by  $\bar{\sigma}_{\mu_i}^2$  and following Roy (2002) and Baltagi et al. (2006), we fix the expected total variance  $\bar{\sigma}^2 = \bar{\sigma}_{\mu_i}^2 + \sigma_{\epsilon}^2 = 8$  to make it comparable across the different data generating processes. We let  $\sigma_{\epsilon}^2$  take the values 2, 4 and 6. For each fixed value of  $\sigma_{\epsilon}^2$ ,  $\alpha$  is assigned values 0, 1, 2 and 3, with  $\alpha = 0$  denoting the homoskedastic individual specific error. For a fixed value of  $\sigma_{\epsilon}^2$ , we obtain a value of  $\bar{\sigma}_{\mu_i}^2 = (8 - \sigma_{\epsilon}^2)$  and using a specific value of  $\alpha$ , we get the corresponding value for  $\sigma_{\mu}^2$  from (34) and (35).  $\rho$  takes on values six different values (0, 0.1, 0.2, 0.3, 0.4, 0.5). In total, this amounts to 432 experiments.

For each experiment, the joint, conditional and misspecified LM and LR tests are computed and 1000 replications are performed. Not all the Monte Carlo results are presented to save space. Here we focus on the joint and conditional tests since these are new contributions to the literature.

### 3.3.1 Joint Tests for $H_0^a$ : $\alpha = \rho = 0$

Table 1 gives the empirical size of the joint *LM* and *LR* tests for  $H_0^a$  :  $\alpha = \rho = 0$  at the 5% significance level, when  $N = 50, 100$  and  $200$  and  $T = 10$ . This is done for both *quadratic* and *exponential heteroskedasticity*, and for  $\sigma_{\epsilon}^2 = 2, 4$ , and  $6$ . These correspond to cases where the percentage of the total variance due to the remainder errors are 25%, 50% and 75%, respectively. For 1000 replications, counts between 37 and 63 are not significantly different from 50 at the .05 level. Table 1 shows that at the 5% level, the size of the joint LR and LM tests are not significantly different from 5%. Figures 1 and 2 give a sample of the power of the joint LM and LR tests for  $N = 100$  and  $200$  and  $T = 10$ , for both *quadratic* and *exponential heteroskedasticity*, and for  $\sigma_{\epsilon}^2 = 4$ . This power is reasonably high as long as  $\rho$  is larger than 0.2. For  $\rho$  smaller than 0.2, the power increase with  $\alpha$ , and more so for



*exponential* rather than *quadratic heteroskedasticity*. For a fixed  $\alpha$ ,  $\rho$  and  $\sigma_\epsilon^2$ , this power increases as  $N$  increases.

### 3.3.2 Conditional Tests for $H_0^b : \rho = 0$ (given $\alpha \neq 0$ )

Table 2 gives the empirical size of the conditional *LM* and *LR* tests for the null hypothesis  $H_0^b : \rho = 0$  (given  $\alpha \neq 0$ ) at the 5% significance level, when  $N = 50, 100$  and  $200$  and  $T = 10$ . This is done for both *quadratic* and *exponential heteroskedasticity*, and for  $\sigma_\epsilon^2 = 2, 4$ , and  $6$ . The size of these conditional tests is not significantly different from 5% except in a few cases. For example, for *exponential heteroskedasticity*,  $N = 50, \alpha = 1$ , and  $\sigma_\epsilon^2 = 6$ , the size of the LM and LR tests were 7.7% and 7.4%, respectively. Figures 3 and 4 give a sample of the power of these conditional LM and LR tests for  $N = 100$  and  $200$  and  $T = 10$ , for both *quadratic* and *exponential heteroskedasticity*, and for  $\sigma_\epsilon^2 = 4$ . This power is reasonably high as long as  $\rho$  is larger than 0.2. For  $\rho$  smaller than 0.2, the power increase with  $N$ , and is about the same magnitude for both *exponential* and *quadratic heteroskedasticity*.

### 3.3.3 Conditional Tests for $H_0^c : \alpha = 0$ (given $\rho \neq 0$ )

Table 3 gives the empirical size of the conditional *LM* and *LR* tests for the null hypothesis  $H_0^c : \alpha = 0$  (given  $\rho \neq 0$ ) at the 5% significance level, when  $N = 50, 100$  and  $200$  and  $T = 10$ . This is done for both *quadratic* and *exponential heteroskedasticity*, and for  $\sigma_\epsilon^2 = 2, 4$ , and  $6$ . The size of these conditional tests is not significantly different from 5% except in a few cases. For example, for *quadratic heteroskedasticity*,  $N = 50, \rho = 0.2$ , and  $\sigma_\epsilon^2 = 2$ , the size of the LR test was 7.6% (oversized), while for *exponential heteroskedasticity*,  $N = 50, \rho = 0.5$ , and  $\sigma_\epsilon^2 = 4$ , the size of the LM test was 2.7% (undersized). Figures 5 and 6 give a sample of the power of these conditional LM and LR tests for  $H_0^c : \alpha = 0$  (given  $\rho \neq 0$ ) for  $N = 100$  and  $200$  and  $T = 10$ , for both *quadratic* and *exponential heteroskedasticity*, and for  $\sigma_\epsilon^2 = 2$  and  $4$ . This power is low for  $N = 100$  but improves for  $N = 200$  especially as  $\alpha$  increases,

and more so for *exponential* rather than *quadratic heteroskedasticity*.

## 4 Conclusion

This paper simultaneously deals with heteroskedastic as well as serially correlated disturbances in the context of a panel data regression model. This is different from the standard econometrics literature which usually deals with heteroskedasticity ignoring serial correlation or vice versa. Exceptions are robust estimation procedures which allow for a general variance-covariance matrix of the disturbances. The paper proposes a joint LM test for homoskedasticity and no first order serial correlation, as well as a conditional LM test for homoskedasticity given serial correlation, and a conditional LM test for no first order serial correlation given heteroskedasticity. Monte Carlo results show that these tests along with their likelihood ratio alternatives have good size and power under various forms of heteroskedasticity including exponential and quadratic functional forms.

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Table 1: Estimated size of joint LM and LR tests for testing  $H_0^a : \rho = 0$  and  $\alpha = 0$  when  $T = 10$ .

$\sigma_\epsilon^2$	$\alpha$	$\rho$	Quadratic heteroskedasticity						Exponential heteroskedasticity					
			N=50		N=100		N=200		N=50		N=100		N=200	
			LM	LR	LM	LR	LM	LR	LM	LR	LM	LR	LM	LR
2	0	0.0	0.040	0.060	0.041	0.057	0.041	0.041	0.049	0.057	0.043	0.049	0.055	0.053
4	0	0.0	0.043	0.054	0.046	0.053	0.049	0.059	0.043	0.051	0.047	0.051	0.041	0.039
6	0	0.0	0.051	0.058	0.049	0.060	0.044	0.046	0.045	0.044	0.045	0.050	0.047	0.047

Table 2: Estimated size of conditional LM and LR tests for testing  $H_0^b : \rho = 0$  (given  $\alpha \neq 0$ ) when  $T = 10$ .

$\sigma_\epsilon^2$	$\alpha$	$\rho$	Quadratic heteroskedasticity						Exponential heteroskedasticity					
			N=50		N=100		N=200		N=50		N=100		N=200	
			LM	LR	LM	LR	LM	LR	LM	LR	LM	LR	LM	LR
2	0	0.0	0.053	0.051	0.045	0.044	0.044	0.044	0.043	0.041	0.046	0.048	0.058	0.059
	1	0.0	0.054	0.054	0.048	0.050	0.040	0.042	0.049	0.049	0.044	0.044	0.048	0.048
	2	0.0	0.050	0.052	0.051	0.051	0.053	0.053	0.041	0.045	0.047	0.051	0.061	0.063
	3	0.0	0.044	0.045	0.047	0.051	0.060	0.059	0.046	0.044	0.047	0.046	0.056	0.058
4	0	0.0	0.052	0.050	0.043	0.042	0.044	0.045	0.050	0.048	0.048	0.049	0.051	0.051
	1	0.0	0.042	0.041	0.047	0.047	0.060	0.056	0.053	0.054	0.042	0.043	0.049	0.049
	2	0.0	0.047	0.046	0.048	0.046	0.042	0.041	0.050	0.049	0.054	0.053	0.053	0.052
	3	0.0	0.057	0.057	0.050	0.053	0.042	0.042	0.046	0.047	0.061	0.065	0.060	0.060
6	0	0.0	0.042	0.047	0.045	0.044	0.049	0.053	0.063	0.069	0.058	0.063	0.058	0.060
	1	0.0	0.043	0.045	0.049	0.050	0.047	0.049	0.077	0.074	0.058	0.062	0.054	0.054
	2	0.0	0.038	0.043	0.050	0.048	0.040	0.040	0.059	0.066	0.065	0.065	0.056	0.060
	3	0.0	0.056	0.059	0.047	0.047	0.054	0.053	0.062	0.067	0.050	0.055	0.057	0.058

Table 3: Estimated size of conditional LM and LR tests for testing  $H_0^c : \alpha = 0$  (given  $\rho \neq 0$ ) when  $T = 10$ .

$\sigma_\epsilon^2$	$\rho$	$\alpha$	Quadratic heteroskedasticity						Exponential heteroskedasticity					
			N=50		N=100		N=200		N=50		N=100		N=200	
			LM	LR	LM	LR	LM	LR	LM	LR	LM	LR	LM	LR
2	0.0	0	0.040	0.055	0.047	0.057	0.046	0.043	0.058	0.056	0.050	0.057	0.060	0.066
	0.1	0	0.049	0.059	0.050	0.051	0.056	0.055	0.051	0.060	0.046	0.050	0.040	0.047
	0.2	0	0.050	0.076	0.043	0.052	0.048	0.052	0.048	0.048	0.051	0.057	0.047	0.051
	0.3	0	0.042	0.052	0.045	0.047	0.044	0.045	0.048	0.057	0.050	0.055	0.060	0.059
	0.4	0	0.037	0.047	0.049	0.057	0.052	0.054	0.055	0.065	0.055	0.067	0.048	0.055
	0.5	0	0.048	0.063	0.045	0.042	0.045	0.046	0.050	0.054	0.043	0.051	0.038	0.048
	0.6	0	0.041	0.061	0.046	0.066	0.043	0.045	0.048	0.039	0.061	0.044	0.050	0.023
	0.7	0	0.039	0.060	0.034	0.041	0.044	0.050	0.043	0.046	0.045	0.029	0.043	0.015
	0.8	0	0.052	0.065	0.046	0.052	0.038	0.051	0.050	0.059	0.049	0.035	0.055	0.034
	0.9	0	0.048	0.084	0.036	0.054	0.054	0.076	0.035	0.046	0.047	0.049	0.053	0.032
4	0.0	0	0.040	0.054	0.045	0.050	0.060	0.050	0.045	0.061	0.053	0.063	0.039	0.043
	0.1	0	0.044	0.064	0.049	0.064	0.050	0.054	0.038	0.050	0.041	0.043	0.056	0.055
	0.2	0	0.035	0.044	0.054	0.058	0.049	0.052	0.057	0.063	0.049	0.052	0.044	0.045
	0.3	0	0.025	0.044	0.049	0.062	0.055	0.059	0.053	0.055	0.047	0.050	0.054	0.055
	0.4	0	0.040	0.050	0.050	0.067	0.040	0.045	0.058	0.068	0.041	0.045	0.045	0.046
	0.5	0	0.049	0.064	0.047	0.052	0.049	0.043	0.027	0.039	0.058	0.058	0.050	0.049
	0.6	0	0.035	0.054	0.046	0.053	0.043	0.050	0.048	0.037	0.046	0.039	0.051	0.028
	0.7	0	0.050	0.071	0.046	0.050	0.036	0.040	0.046	0.036	0.050	0.028	0.048	0.021
	0.8	0	0.048	0.065	0.058	0.074	0.041	0.051	0.053	0.055	0.043	0.037	0.048	0.028
	0.9	0	0.050	0.061	0.048	0.068	0.058	0.072	0.049	0.060	0.047	0.058	0.056	0.056
6	0.0	0	0.065	0.066	0.049	0.060	0.060	0.058	0.042	0.048	0.041	0.044	0.052	0.051
	0.1	0	0.039	0.056	0.048	0.052	0.043	0.042	0.045	0.042	0.039	0.043	0.049	0.046
	0.2	0	0.035	0.051	0.062	0.070	0.052	0.055	0.039	0.047	0.046	0.046	0.057	0.057
	0.3	0	0.049	0.059	0.047	0.047	0.055	0.057	0.044	0.054	0.041	0.047	0.051	0.052
	0.4	0	0.048	0.064	0.047	0.055	0.040	0.046	0.042	0.053	0.046	0.046	0.051	0.051
	0.5	0	0.042	0.059	0.041	0.053	0.037	0.039	0.045	0.051	0.051	0.056	0.049	0.050
	0.6	0	0.056	0.062	0.041	0.057	0.031	0.040	0.045	0.042	0.053	0.041	0.044	0.027
	0.7	0	0.039	0.052	0.057	0.069	0.052	0.059	0.060	0.037	0.054	0.035	0.050	0.029
	0.8	0	0.043	0.052	0.045	0.064	0.051	0.058	0.043	0.038	0.046	0.032	0.048	0.034
	0.9	0	0.050	0.060	0.045	0.053	0.052	0.049	0.050	0.037	0.054	0.066	0.051	0.044

Figure 1: Frequency of rejections for  $H_0^a : \rho = 0$  and  $\alpha = 0$ ,  $N=100, 200$ ,  $T=10$ , Quadratic heteroskedasticity.

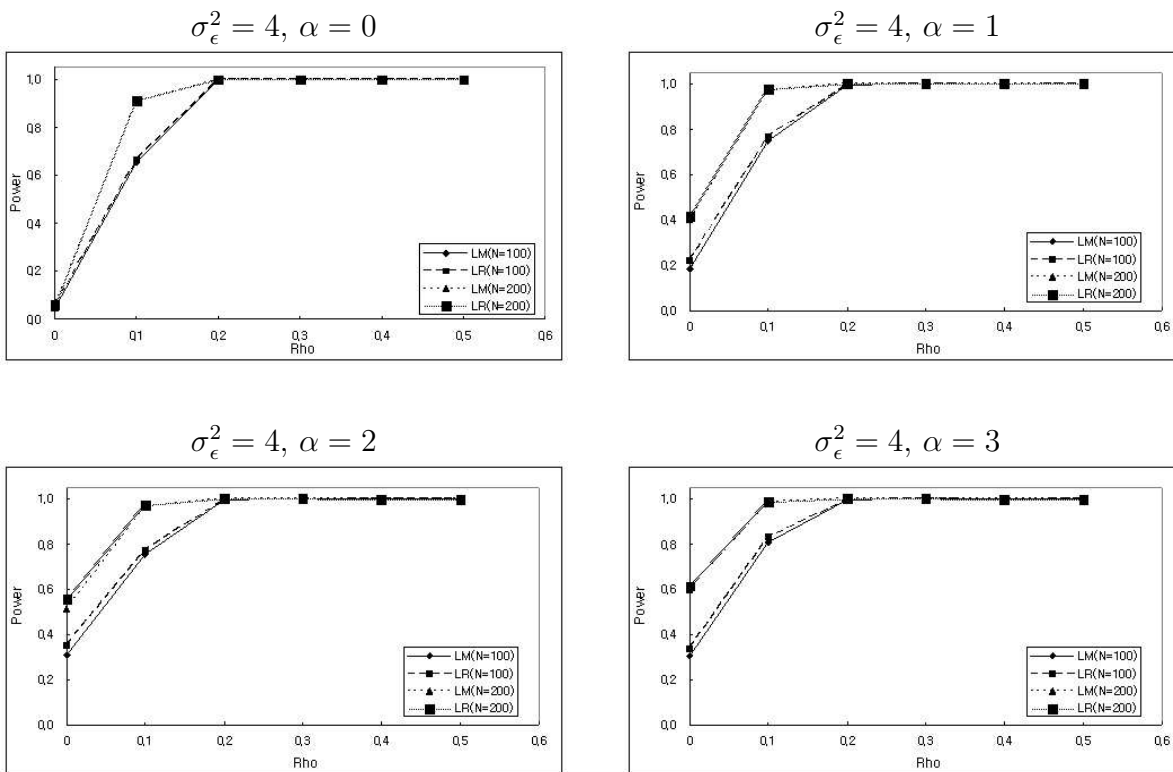


Figure 2: Frequency of rejections for  $H_0^a : \rho = 0$  and  $\alpha = 0$ ,  $N=100, 200$ ,  $T=10$ , Exponential heteroskedasticity.

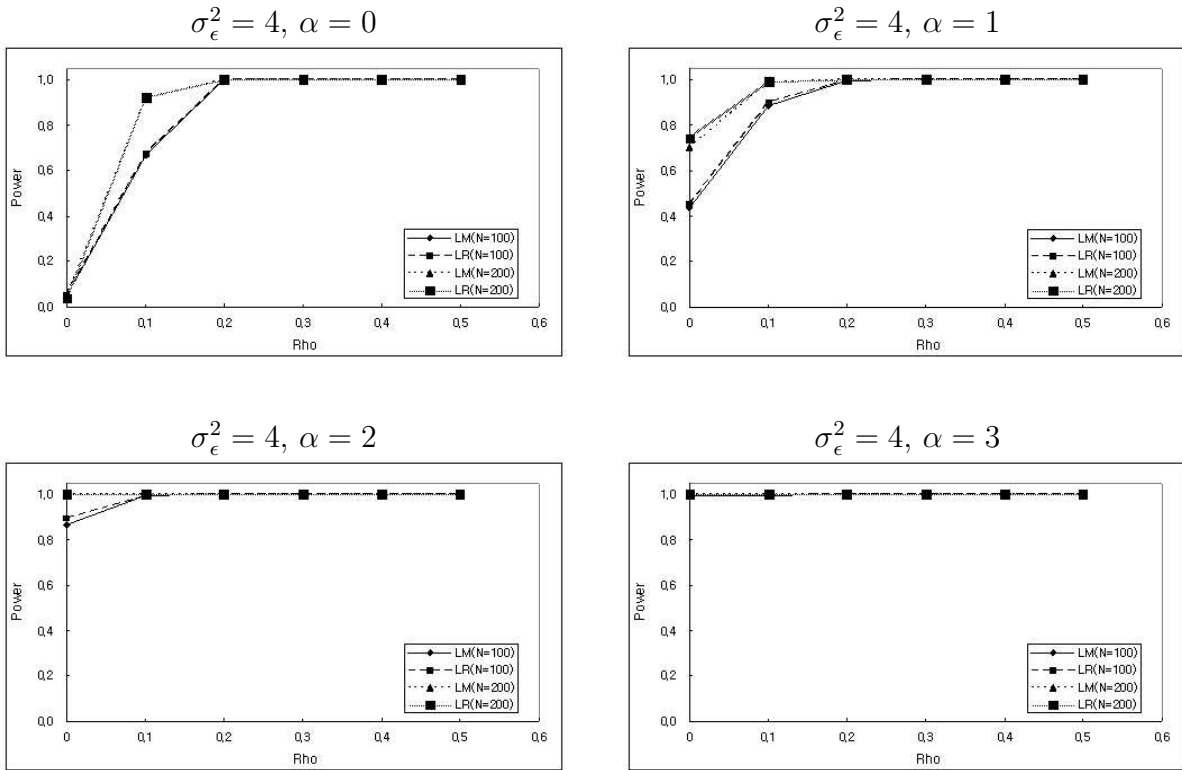




Figure 3: Frequency of rejections of the conditional tests for  $H_0^b : \rho = 0$  (given  $\alpha \neq 0$ ),  $N=100, 200, T=10$ , Quadratic heteroskedasticity.

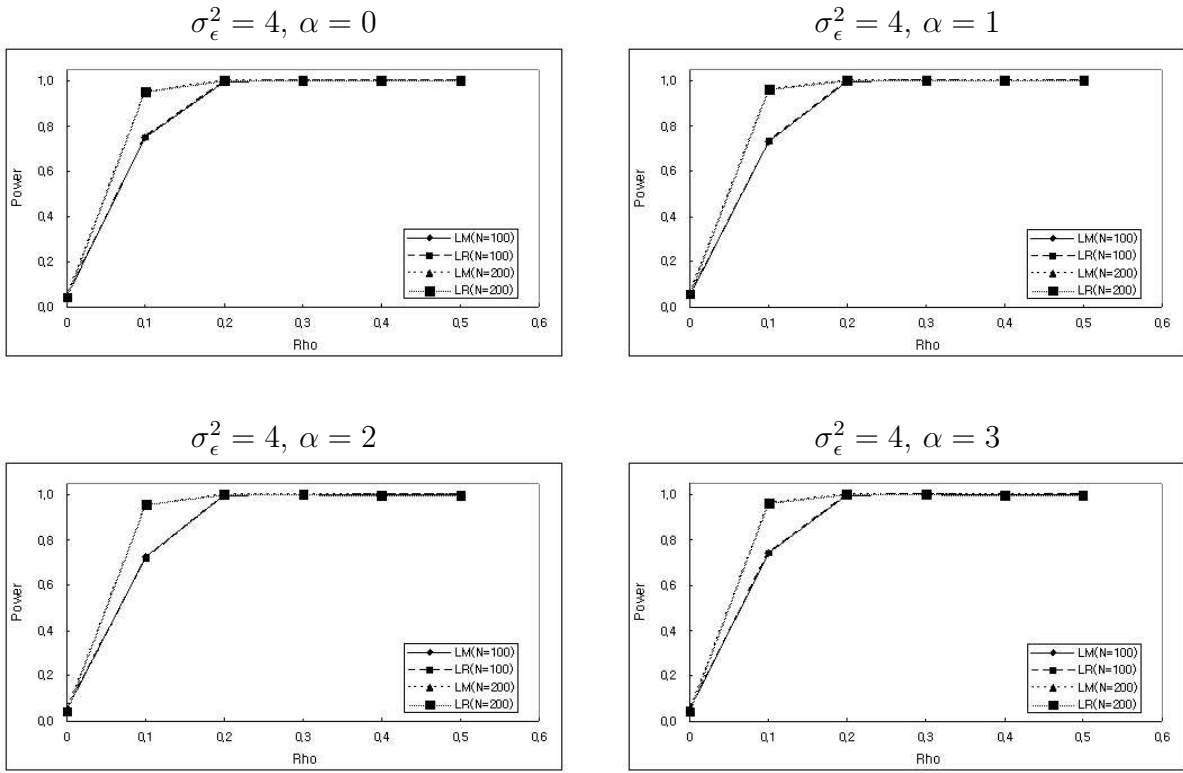


Figure 4: Frequency of rejections of the conditional tests for  $H_0^b : \rho = 0$  (given  $\alpha \neq 0$ ),  $N=100, 200, T=10$ , Exponential heteroskedasticity.

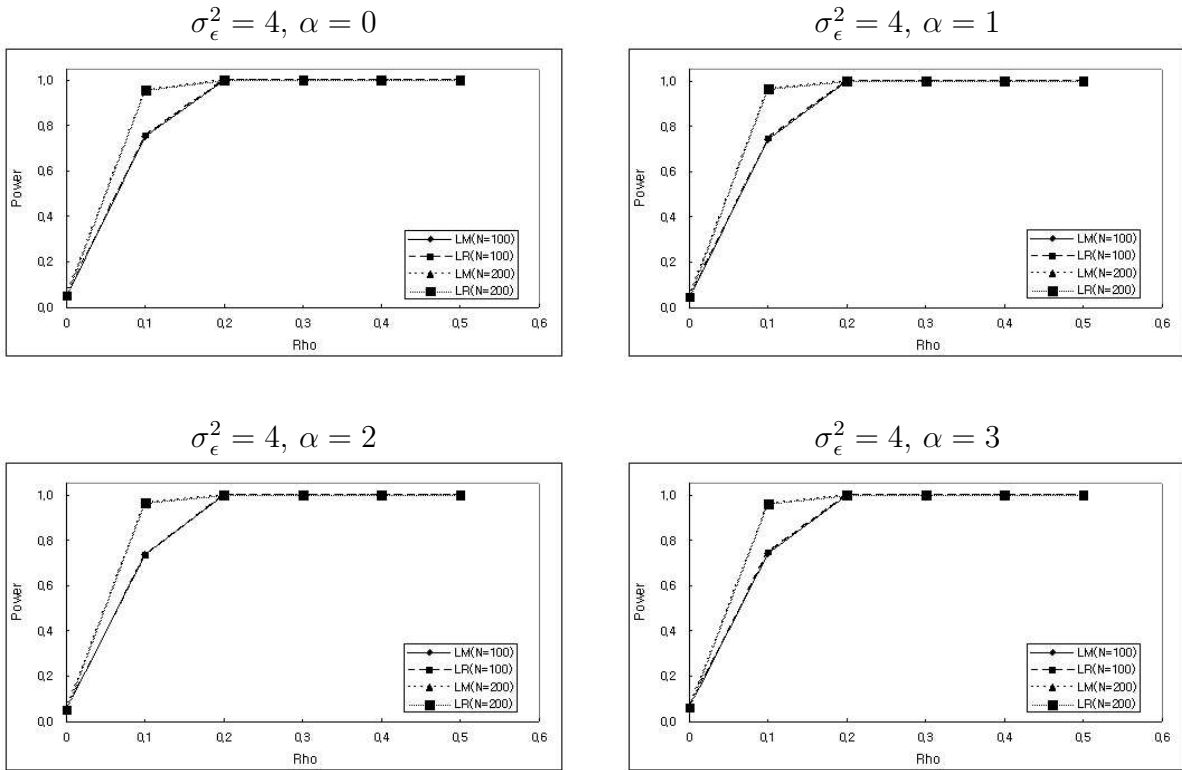


Figure 5: Frequency of rejections of the conditional tests for  $H_0^c : \alpha = 0$  (given  $\rho \neq 0$ ),  $N=100, 200, T=10$ , Quadratic heteroskedasticity.

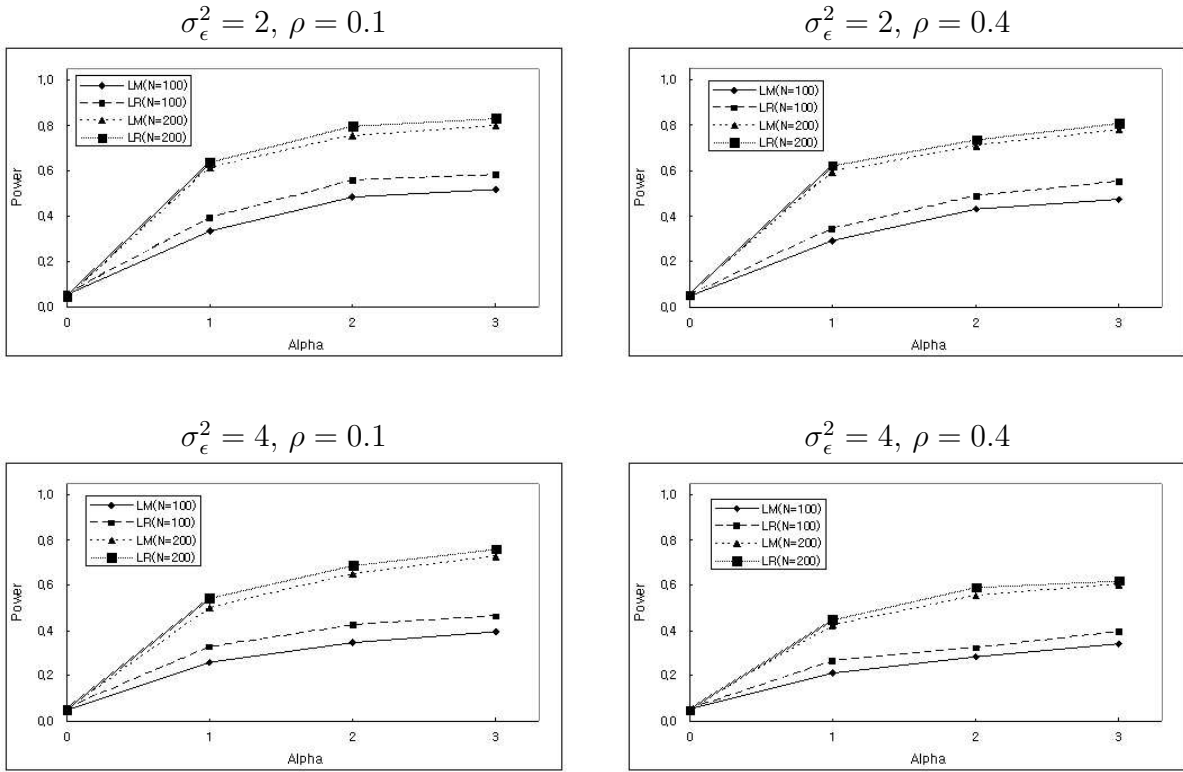
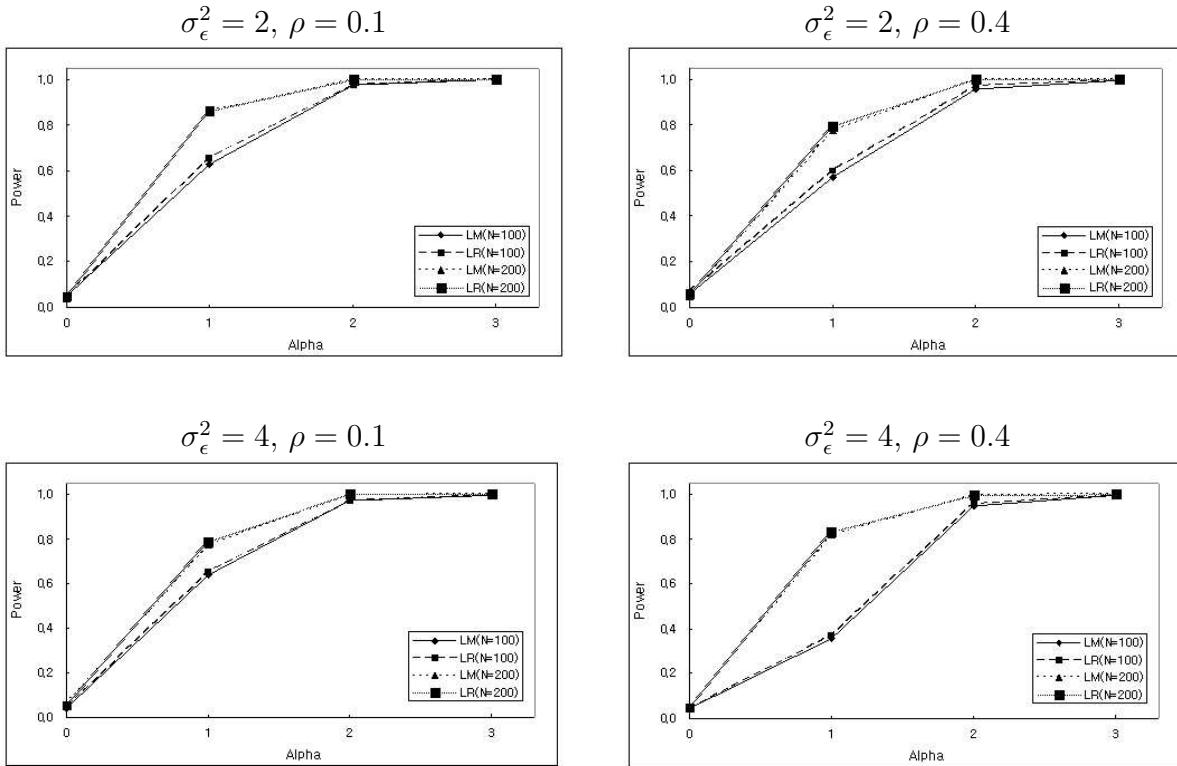


Figure 6: Frequency of rejections of the conditional tests for  $H_0^c : \alpha = 0$  (given  $\rho \neq 0$ ),  $N=100, 200, T=10$ , Exponential heteroskedasticity.



# APPENDICES

## Appendix 1

This appendix derives the joint LM test for testing  $H_0^a : \alpha_2 = \dots = \alpha_p = 0$  and  $\rho = 0$ . The variance-covariance matrix of the disturbances in (4) can be written as

$$\begin{aligned}\Omega &= (I_N \otimes \iota_T)(diag[h(z_i' \boldsymbol{\alpha})] \otimes I_T)(I_N \otimes \iota_T)' + I_N \otimes V \\ &= diag[h(z_i' \boldsymbol{\alpha})] \otimes J_T + I_N \otimes V,\end{aligned}\tag{A.1}$$

where  $J_T$  is a matrix of ones of dimension  $T$ , and  $diag[h(z_i' \boldsymbol{\alpha})]$  is a diagonal matrix of dimension  $N \times N$  and  $V$  is the familiar AR(1) covariance matrix. The log-likelihood function under normality of the disturbances is given by

$$L(\beta, \boldsymbol{\theta}) = constant - \frac{1}{2} \log |\Omega| - \frac{1}{2} u' \Omega^{-1} u,\tag{A.2}$$

where  $\boldsymbol{\theta}' = (\sigma_\epsilon^2, \rho, \boldsymbol{\alpha}')$ . The information matrix is block-diagonal between  $\beta$  and  $\boldsymbol{\theta}$ , since  $H_0^a$  involves only  $\boldsymbol{\theta}$ , the part of the information due to  $\beta$  is ignored, see Baltagi (1995). In order to obtain the joint LM statistic, we need  $D(\boldsymbol{\theta}) = (\partial L / \partial \boldsymbol{\theta})$  and the information matrix  $J(\boldsymbol{\theta}) = E[-\partial^2 L / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}']$  evaluated at the restricted ML estimator  $\tilde{\boldsymbol{\theta}}$ . Under the null hypothesis, the variance-covariance matrix reduces to  $\Omega = \sigma_\mu^2 (I_N \otimes J_T) + \sigma_\epsilon^2 (I_N \otimes I_T)$ . It is the familiar form of the one-way error component model, see Baltagi(1995). Under the null hypothesis we obtain

$$\Omega^{-1} = (\sigma_1^2)^{-1} (I_N \otimes \bar{J}_T) + (\sigma_\epsilon^2)^{-1} (I_N \otimes E_T),\tag{A.3}$$

where  $\sigma_1^2 = T\sigma_\mu^2 + \sigma_\epsilon^2$ .

Following, Hartley and Rao (1967) or Hemmerle and Hartley (1973),

$$\partial L / \partial \boldsymbol{\theta}_r = -\frac{1}{2} tr[\Omega^{-1} (\partial \Omega / \partial \boldsymbol{\theta}_r)] + \frac{1}{2} [u' \Omega^{-1} (\partial \Omega / \partial \boldsymbol{\theta}_r) \Omega^{-1} u],\tag{A.4}$$

$$E\left[-\frac{\partial^2 L}{\partial \boldsymbol{\theta}_r \partial \boldsymbol{\theta}_s}\right] = \frac{1}{2} tr\left[\Omega^{-1} \frac{\partial \Omega}{\partial \boldsymbol{\theta}_r} \Omega^{-1} \frac{\partial \Omega}{\partial \boldsymbol{\theta}_s}\right],\tag{A.5}$$

for  $r, s = 1, 2, \dots, p + 2$ , see Harville (1977). Then, we obtain the following quantities

$$\begin{aligned}
\frac{\partial \log L}{\partial \sigma_\epsilon^2} \Big|_{H_0^g} &= I_N \otimes I_T \\
\frac{\partial \log L}{\partial \alpha_k} \Big|_{H_0^g} &= \text{diag}(h'(\tilde{\alpha}_1)z_{ik}) \otimes J_T = h'(\tilde{\alpha}_1)\text{diag}(z_{ik}) \otimes J_T, \quad k = 1, \dots, p \\
\frac{\partial \log L}{\partial \rho} \Big|_{H_0^g} &= \sigma_\epsilon^2 I_N \otimes G \\
\Omega^{-1} \frac{\partial \log L}{\partial \sigma_\epsilon^2} &= \left( \frac{1}{\sigma_1^2} (I_N \otimes \bar{J}_T) + \frac{1}{\sigma_\epsilon^2} (I_N \otimes E_T) \right) (I_N \otimes I_T) \\
&= \frac{1}{\sigma_1^2} (I_N \otimes \bar{J}_T) + \frac{1}{\sigma_\epsilon^2} (I_N \otimes E_T) \\
\Omega^{-1} \frac{\partial \log L}{\partial \alpha_k} &= \left( \frac{1}{\sigma_1^2} (I_N \otimes \bar{J}_T) + \frac{1}{\sigma_\epsilon^2} (I_N \otimes E_T) \right) (h'(\alpha_1)(\text{diag}(z_{ik}) \otimes J_T)) \\
&= \frac{h'(\alpha_1)}{\sigma_1^2} (\text{diag}(z_{ik}) \otimes J_T) \\
\Omega^{-1} \frac{\partial \log L}{\partial \rho} &= \left( \frac{1}{\sigma_1^2} (I_N \otimes \bar{J}_T) + \frac{1}{\sigma_\epsilon^2} (I_N \otimes E_T) \right) (\sigma_\epsilon^2 I_N \otimes G) \\
&= \sigma_\epsilon^2 \left( \left( \frac{1}{\sigma_1^2} I_N \otimes \bar{J}_T G \right) + \left( \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T G \right) \right) = \sigma_\epsilon^2 \left[ I_N \otimes \left( \bar{J}_T G / \sigma_1^2 + E_T G / \sigma_\epsilon^2 \right) \right] \\
\Omega^{-1} \frac{\partial \log L}{\partial \sigma_\epsilon^2} \Omega^{-1} &= \frac{1}{\sigma_1^4} (I_N \otimes \bar{J}_T) + \frac{1}{\sigma_\epsilon^4} (I_N \otimes E_T) \\
\Omega^{-1} \frac{\partial \log L}{\partial \alpha_k} \Omega^{-1} &= \frac{h'(\alpha_1)}{\sigma_1^2} (\text{diag}(z_{ik}) \otimes J_T) \left( \frac{1}{\sigma_1^2} (I_N \otimes \bar{J}_T) + \frac{1}{\sigma_\epsilon^2} (I_N \otimes E_T) \right) \\
&= \frac{h'(\alpha_1)}{\sigma_1^4} (\text{diag}(z_{ik}) \otimes J_T) \\
\Omega^{-1} \frac{\partial \log L}{\partial \rho} \Omega^{-1} &= \sigma_\epsilon^2 \left[ I_N \otimes \left( \bar{J}_T G / \sigma_1^2 + E_T G / \sigma_\epsilon^2 \right) \right] \left( \frac{1}{\sigma_1^2} (I_N \otimes \bar{J}_T) + \frac{1}{\sigma_\epsilon^2} (I_N \otimes E_T) \right) \\
&= \sigma_\epsilon^2 \left[ I_N \otimes \left( (\bar{J}_T / \sigma_1^2 + E_T / \sigma_\epsilon^2) G (\bar{J}_T / \sigma_1^2 + E_T / \sigma_\epsilon^2) \right) \right]
\end{aligned}$$

Straightforward calculation of partial derivatives, evaluated at the restricted MLE, yield

$$\begin{aligned}
\frac{\partial L}{\partial \sigma_\epsilon^2} \Big|_{H_0^g} &= -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \sigma_\epsilon^2} \right] + \frac{1}{2} \tilde{u}' \left( \Omega^{-1} \frac{\partial \Omega}{\partial \sigma_\epsilon^2} \Omega^{-1} \right) \tilde{u} \\
&= -\frac{1}{2} \text{tr} \left[ \frac{1}{\sigma_1^2} (I_N \otimes \bar{J}_T) + \frac{1}{\sigma_\epsilon^2} (I_N \otimes E_T) \right] + \frac{1}{2} \tilde{u}' \left( \frac{1}{\sigma_1^4} (I_N \otimes \bar{J}_T) + \frac{1}{\sigma_\epsilon^4} (I_N \otimes E_T) \right) \tilde{u} \\
&= -\frac{1}{2} (N / \tilde{\sigma}_1^2 + N(T-1) / \tilde{\sigma}_\epsilon^2) + \frac{1}{2} \tilde{u}' \left( \frac{1}{\tilde{\sigma}_1^4} I_N \otimes \bar{J}_T + \frac{1}{\tilde{\sigma}_\epsilon^4} I_N \otimes E_T \right) \tilde{u} = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \alpha_1} \Big|_{H_0^a} &= -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \alpha_1} \right] + \frac{1}{2} \tilde{u}' \left( \Omega^{-1} \frac{\partial \Omega}{\partial \alpha_1} \Omega^{-1} \right) \tilde{u} \\
&= -\frac{1}{2} \text{tr} \left[ \frac{h'(\alpha_1)}{\sigma_1^2} (\text{diag}(z_{i1}) \otimes J_T) \right] + \frac{1}{2} \tilde{u}' \left( \frac{h'(\alpha_1)}{\sigma_1^4} (\text{diag}(z_{i1}) \otimes J_T) \right) \tilde{u} \\
&= -\frac{Th'(\alpha_1)}{2\sigma_1^2} \sum_{i=1}^N z_{i1} + \frac{h'(\alpha_1)}{2\sigma_1^4} \tilde{u}' (\text{diag}(z_{i1}) \otimes J_T) \tilde{u} \\
&= -\frac{Th'(\alpha_1)}{2\sigma_1^2} \sum_{i=1}^N z_{i1} + \frac{h'(\alpha_1)}{2\sigma_1^4} \sum_{i=1}^N z_{i1} \tilde{u}'_i J_T \tilde{u}_i \\
&= -\frac{Th'(\alpha_1)}{2\sigma_1^2} \sum_{i=1}^N z_{i1} + \frac{h'(\alpha_1)}{2\sigma_1^4} \sum_{i=1}^N z_{i1} \left( \sum_{t=1}^T \tilde{u}_{it} \right)^2 \\
&= \frac{Th'(\tilde{\alpha}_1)}{2\tilde{\sigma}_1^2} \sum_{i=1}^N z_{i1} \left( \frac{(\sum_{t=1}^T \tilde{u}_{it})^2}{T\tilde{\sigma}_1^2} - 1 \right) \\
&= \frac{Th'(\tilde{\alpha}_1)}{2\tilde{\sigma}_1^2} \sum_{i=1}^N \left( \frac{(\sum_{t=1}^T \tilde{u}_{it})^2}{T\tilde{\sigma}_1^2} - 1 \right) \quad (\text{since } z_{i1} = 1) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \alpha_k} \Big|_{H_0^a} &= D(\tilde{\alpha}_k) = -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \alpha_k} \right] + \frac{1}{2} \tilde{u}' \left( \Omega^{-1} \frac{\partial \Omega}{\partial \alpha_k} \Omega^{-1} \right) \tilde{u} \\
&= -\frac{1}{2} \text{tr} \left[ \frac{h'(\alpha_1)}{\sigma_1^2} (\text{diag}(z_{ik}) \otimes J_T) \right] + \frac{1}{2} \tilde{u}' \left( \frac{h'(\alpha_1)}{\sigma_1^4} (\text{diag}(z_{ik}) \otimes J_T) \right) \tilde{u} \\
&= -\frac{Th'(\alpha_1)}{2\sigma_1^2} \sum_{i=1}^N z_{ik} + \frac{h'(\alpha_1)}{2\sigma_1^4} \tilde{u}' (\text{diag}(z_{ik}) \otimes J_T) \tilde{u} \\
&= -\frac{Th'(\alpha_1)}{2\sigma_1^2} \sum_{i=1}^N z_{ik} + \frac{h'(\alpha_1)}{2\sigma_1^4} \sum_{i=1}^N z_{ik} \tilde{u}'_i J_T \tilde{u}_i \\
&= -\frac{Th'(\alpha_1)}{2\sigma_1^2} \sum_{i=1}^N z_{ik} + \frac{h'(\alpha_1)}{2\sigma_1^4} \sum_{i=1}^N z_{ik} \left( \sum_{t=1}^T \tilde{u}_{it} \right)^2 \\
&= \frac{Th'(\tilde{\alpha}_1)}{2\tilde{\sigma}_1^2} \sum_{i=1}^N z_{ik} \left( \frac{(\sum_{t=1}^T \tilde{u}_{it})^2}{T\tilde{\sigma}_1^2} - 1 \right), \quad k = 2, \dots, p
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \rho} \Big|_{H_0^a} &= D(\tilde{\rho}) = -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \rho} \right] + \frac{1}{2} \tilde{u}' \left( \Omega^{-1} \frac{\partial \Omega}{\partial \rho} \Omega^{-1} \right) \tilde{u} \\
&= -\frac{1}{2} \text{tr} \left[ \sigma_\epsilon^2 \left\{ I_N \otimes \left( \bar{J}_T G / \sigma_1^2 + E_T G / \sigma_\epsilon^2 \right) \right\} \right] \\
&\quad + \frac{1}{2} \tilde{u}' \left[ \sigma_\epsilon^2 \left\{ I_N \otimes \left( (\bar{J}_T / \sigma_1^2 + E_T / \sigma_\epsilon^2) G (\bar{J}_T / \sigma_1^2 + E_T / \sigma_\epsilon^2) \right) \right\} \right] \tilde{u} \\
&= -\frac{N\sigma_\epsilon^2}{2} \left( \text{tr}(\bar{J}_T G) / \sigma_1^2 + \text{tr}(E_T G) / \sigma_\epsilon^2 \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\tilde{\sigma}_\epsilon^2}{2} \tilde{u}' \left[ I_N \otimes \left( (\bar{J}_T / \tilde{\sigma}_1^2 + E_T / \tilde{\sigma}_\epsilon^2) G (\bar{J}_T / \tilde{\sigma}_1^2 + E_T / \tilde{\sigma}_\epsilon^2) \right) \right] \tilde{u} \\
= & - \frac{N \sigma_\epsilon^2}{2} \left( \frac{2(T-1)}{T \sigma_1^2} - \frac{2(T-1)}{T \sigma_\epsilon^2} \right) \quad (\text{since } \text{tr}(G) = 0, \text{tr}(\bar{J}_T G) = 2(T-1)/T) \\
& + \frac{\tilde{\sigma}_\epsilon^2}{2} \tilde{u}' \left[ I_N \otimes \left( (\bar{J}_T / \tilde{\sigma}_1^2 + E_T / \tilde{\sigma}_\epsilon^2) G (\bar{J}_T / \tilde{\sigma}_1^2 + E_T / \tilde{\sigma}_\epsilon^2) \right) \right] \tilde{u} \\
= & \frac{N(T-1)}{T} \frac{\tilde{\sigma}_1^2 - \tilde{\sigma}_\epsilon^2}{\tilde{\sigma}_1^2} \\
& + \frac{\tilde{\sigma}_\epsilon^2}{2} \tilde{u}' \left[ I_N \otimes \left( (\bar{J}_T / \tilde{\sigma}_1^2 + E_T / \tilde{\sigma}_\epsilon^2) G (\bar{J}_T / \tilde{\sigma}_1^2 + E_T / \tilde{\sigma}_\epsilon^2) \right) \right] \tilde{u}. \tag{A.6}
\end{aligned}$$

where  $\tilde{u} = y - X\tilde{\beta}_{MLE}$  is the maximum likelihood residuals under the null hypothesis  $H_0^a$ , and  $\tilde{\alpha}_1$  is the solution of  $D(\tilde{\alpha}_1) = 0$  while  $\tilde{\sigma}_\epsilon^2$  is the solution of  $D(\tilde{\sigma}_\epsilon^2) = 0$  from (A.6). Thus, the partial derivatives under  $H_0^a$  are rewritten in vector form as

$$D(\tilde{\theta}) = \begin{pmatrix} D(\tilde{\sigma}_\epsilon^2) \\ D(\tilde{\alpha}) \\ D(\tilde{\rho}) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{Th'(\tilde{\alpha}_1)}{2\tilde{\sigma}_1^2} Z' f \\ D(\tilde{\rho}) \end{pmatrix}, \tag{A.7}$$

where  $D(\tilde{\alpha}) = (0, D(\tilde{\alpha}_2), \dots, D(\tilde{\alpha}_p))'$ ,  $h'(\tilde{\alpha}_1)$  is the evaluated value of  $\partial h(z_i' \alpha) / \partial z_i' \alpha$  when  $\alpha_2 = \dots = \alpha_p = 0$ , and  $Z = (z_1, \dots, z_N)'$  and  $f = (f_1, \dots, f_N)'$ ,  $f_i = (\sum_{t=1}^T \tilde{u}_{it})^2 / T \tilde{\sigma}_1^2 - 1$ .

Also, using the the results of Harville (1977), we obtain the information matrix under the null hypothesis  $H_0^a$ :

$$\begin{aligned}
E \left[ - \frac{\partial^2 \log L}{\partial \sigma_\epsilon^4} \right] \Big|_{H_0^a} &= \frac{1}{2} \text{tr} \left[ \left\{ \frac{1}{\sigma_1^2} (I_N \otimes \bar{J}_T) + \frac{1}{\sigma_\epsilon^2} (I_N \otimes E_T) \right\}^2 \right] \\
&= \frac{1}{2} \text{tr} \left[ 1/\tilde{\sigma}_1^4 I_N \otimes \bar{J}_T + 1/\tilde{\sigma}_\epsilon^4 I_N \otimes E_T \right] \\
&= \frac{N}{2} \left( 1/\tilde{\sigma}_1^4 + (T-1)/\tilde{\sigma}_\epsilon^4 \right) \\
E \left[ - \frac{\partial^2 \log L}{\partial \sigma_\epsilon^2 \partial \alpha_k} \right] \Big|_{H_0^a} &= \frac{1}{2} \text{tr} \left[ \left( \frac{1}{\tilde{\sigma}_1^2} I_N \otimes \bar{J}_T + \frac{1}{\tilde{\sigma}_\epsilon^2} I_N \otimes E_T \right) \left( \frac{h'(\tilde{\alpha}_1)}{\tilde{\sigma}_1^2} \text{diag}(z_{ik}) \otimes J_T \right) \right] \\
&= \frac{h'(\tilde{\alpha}_1)}{2\tilde{\sigma}_1^4} \text{tr} \left[ \text{diag}(z_{ik}) \otimes J_T \right]
\end{aligned}$$



$$\begin{aligned}
&= \frac{Th'(\tilde{\alpha}_1)}{2\tilde{\sigma}_1^4} \sum_{i=1}^N z_{ik}, \quad k = 1, \dots, p \\
E\left[-\frac{\partial^2 \log L}{\partial \sigma_\epsilon^2 \partial \rho}\right] \Big|_{H_0^a} &= \frac{1}{2} \text{tr} \left[ \left( \frac{1}{\tilde{\sigma}_1^2} I_N \otimes \bar{J}_T + \frac{1}{\tilde{\sigma}_\epsilon^2} I_N \otimes E_T \right) \left( \frac{\tilde{\sigma}_\epsilon^2}{\tilde{\sigma}_1^2} I_N \otimes \bar{J}_T G + I_N \otimes E_T G \right) \right] \\
&= \frac{1}{2} \text{tr} \left[ \frac{\tilde{\sigma}_\epsilon^2}{\tilde{\sigma}_1^4} I_N \otimes \bar{J}_T G + \frac{1}{\tilde{\sigma}_\epsilon^2} I_N \otimes E_T G \right] \\
&= \frac{N(T-1)}{T} \tilde{\sigma}_\epsilon^2 \left( \frac{1}{\tilde{\sigma}_1^4} - \frac{1}{\tilde{\sigma}_\epsilon^4} \right) \\
E\left[-\frac{\partial^2 \log L}{\partial \rho^2}\right] \Big|_{H_0^a} &= \frac{1}{2} \text{tr} \left[ \left( \frac{\tilde{\sigma}_\epsilon^2}{\tilde{\sigma}_1^2} I_N \otimes \bar{J}_T G + I_N \otimes E_T G \right)^2 \right] \\
&= \frac{1}{2} \text{tr} \left[ \frac{\tilde{\sigma}_\epsilon^4}{\tilde{\sigma}_1^4} I_N \otimes \bar{J}_T G \bar{J}_T G + 2 \frac{\tilde{\sigma}_\epsilon^2}{\tilde{\sigma}_1^2} I_N \otimes \bar{J}_T G E_T G + I_N \otimes E_T G E_T G \right] \\
&= N \left( 2a^2(T-1)^2 + 2a(2T-3) + T-1 \right) \\
E\left[-\frac{\partial^2 \log L}{\partial \rho \partial \alpha_k}\right] \Big|_{H_0^a} &= \frac{1}{2} \text{tr} \left[ \left( \frac{\tilde{\sigma}_\epsilon^2}{\tilde{\sigma}_1^2} I_N \otimes \bar{J}_T G + I_N \otimes E_T G \right) \left( \frac{h'(\tilde{\alpha}_1)}{\tilde{\sigma}_1^2} \text{diag}(z_{ik}) \otimes J_T \right) \right] \\
&= \frac{h'(\tilde{\alpha}_1) \tilde{\sigma}_\epsilon^2}{2\tilde{\sigma}_1^4} \text{tr} \left[ \text{diag}(z_{ik}) \otimes J_T G \right] \\
&= \frac{h'(\tilde{\alpha}_1) \tilde{\sigma}_\epsilon^2 2(T-1)}{2\tilde{\sigma}_1^4} \sum_{i=1}^N z_{ik} = \frac{(T-1)h'(\tilde{\alpha}_1) \tilde{\sigma}_\epsilon^2}{\tilde{\sigma}_1^4} \sum_{i=1}^N z_{ik} \\
E\left[-\frac{\partial^2 \log L}{\partial \alpha_k \partial \alpha_l}\right] \Big|_{H_0^a} &= \frac{1}{2} \text{tr} \left[ \left( \frac{h'(\tilde{\alpha}_1)}{\tilde{\sigma}_1^2} \text{diag}(z_{ik}) \otimes J_T \right) \left( \frac{h'(\tilde{\alpha}_1)}{\tilde{\sigma}_1^2} \text{diag}(z_{il}) \otimes J_T \right) \right] \\
&= \frac{Th'(\tilde{\alpha}_1)^2}{2\tilde{\sigma}_1^4} \text{tr} \left[ \text{diag}(z_{ik} z_{il}) \otimes J_T \right] \\
&= \frac{T^2 h'(\tilde{\alpha}_1)^2}{2\tilde{\sigma}_1^4} \sum_{i=1}^N z_{ik} z_{il}, \quad k, l = 1, \dots, p. \tag{A.8}
\end{aligned}$$

Therefore, information matrix under the null hypothesis  $H_0^a$  can be obtained in matrix form

as

$$\tilde{J}_a(\boldsymbol{\theta}) = \begin{bmatrix} \frac{N}{2} \left( \frac{1}{\tilde{\sigma}_1^4} + \frac{T-1}{\tilde{\sigma}_\epsilon^4} \right) & \frac{N(T-1)}{T} \tilde{\sigma}_\epsilon^2 \left( \frac{1}{\tilde{\sigma}_1^4} - \frac{1}{\tilde{\sigma}_\epsilon^4} \right) & \frac{Th'(\alpha_1)}{2\tilde{\sigma}_1^4} i'_N Z \\ \frac{N(T-1)}{T} \tilde{\sigma}_\epsilon^2 \left( \frac{1}{\tilde{\sigma}_1^4} - \frac{1}{\tilde{\sigma}_\epsilon^4} \right) & \tilde{J}_{\rho\rho} & \frac{(T-1)\tilde{\sigma}_\epsilon^2 h'(\alpha_1)}{\tilde{\sigma}_1^4} i'_N Z \\ \frac{Th'(\alpha_1)}{2\tilde{\sigma}_1^4} Z' i_N & \frac{(T-1)\tilde{\sigma}_\epsilon^2 h'(\alpha_1)}{\tilde{\sigma}_1^4} Z' i_N & \frac{T^2 h'(\alpha_1)^2}{2\tilde{\sigma}_1^4} Z' Z \end{bmatrix}, \tag{A.9}$$

where  $\tilde{J}_{\rho\rho} = N[2a^2(T-1)^2 + 2a(2T-3) + T-1]$ ,  $a = \frac{\tilde{\sigma}_\epsilon^2 - \tilde{\sigma}_1^2}{T\sigma_1^2}$ .

Let

$$A = \begin{bmatrix} \frac{N}{2} \left( \frac{1}{\tilde{\sigma}_1^4} + \frac{T-1}{\tilde{\sigma}_\epsilon^4} \right) & \frac{N(T-1)}{T} \tilde{\sigma}_\epsilon^2 \left( \frac{1}{\tilde{\sigma}_1^4} - \frac{1}{\tilde{\sigma}_\epsilon^4} \right) \\ \frac{N(T-1)}{T} \tilde{\sigma}_\epsilon^2 \left( \frac{1}{\tilde{\sigma}_1^4} - \frac{1}{\tilde{\sigma}_\epsilon^4} \right) & \tilde{J}_{\rho\rho} \end{bmatrix}, B = \begin{bmatrix} \frac{Th'(\alpha_1)}{2\tilde{\sigma}_1^4} i'_N Z \\ \frac{(T-1)\tilde{\sigma}_\epsilon^2 h'(\alpha_1)}{\tilde{\sigma}_1^4} i'_N Z \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{Th'(\alpha_1)}{2\tilde{\sigma}_1^4} Z' i_N & \frac{(T-1)\tilde{\sigma}_\epsilon^2 h'(\alpha_1)}{\tilde{\sigma}_1^4} Z' i_N \end{bmatrix}, D = \begin{bmatrix} \frac{T^2 h'(\alpha_1)^2}{2\tilde{\sigma}_1^4} Z' Z \end{bmatrix},$$

then  $\tilde{J}_a(\boldsymbol{\theta})$  can be written as

$$\tilde{J}_a(\boldsymbol{\theta}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (\text{A.10})$$

Using Searle (), the inverse of partitioned matrix can be obtained as

$$\tilde{J}_a(\boldsymbol{\theta})^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D^{-1} \end{bmatrix} + \begin{bmatrix} I_2 \\ -D^{-1}C \end{bmatrix} (A - BD^{-1}C)^{-1} [I_2 \quad -BD^{-1}]. \quad (\text{A.11})$$

In (A.11), we obtain

$$BD^{-1}C = \begin{bmatrix} \frac{Th'(\alpha_1)}{2\tilde{\sigma}_1^4} i'_N Z \\ \frac{(T-1)\tilde{\sigma}_\epsilon^2 h'(\alpha_1)}{\tilde{\sigma}_1^4} i'_N Z \end{bmatrix} \left( \frac{2\tilde{\sigma}_1^4}{T^2 h'(\alpha_1)^2} (Z'Z)^{-1} \right) \begin{bmatrix} \frac{Th'(\alpha_1)}{2\tilde{\sigma}_1^4} Z' i_N & \frac{(T-1)\tilde{\sigma}_\epsilon^2 h'(\alpha_1)}{\tilde{\sigma}_1^4} Z' i_N \end{bmatrix}$$

$$= \begin{bmatrix} \frac{N}{2\tilde{\sigma}_1^4} & \frac{N(T-1)\tilde{\sigma}_\epsilon^2}{T\tilde{\sigma}_1^4} \\ \frac{N(T-1)\tilde{\sigma}_\epsilon^2}{T\tilde{\sigma}_1^4} & \frac{2N(T-1)^2\tilde{\sigma}_\epsilon^4}{T^2\tilde{\sigma}_1^4} \end{bmatrix},$$

$$A - BD^{-1}C = \begin{bmatrix} \frac{N(T-1)}{2\tilde{\sigma}_\epsilon^4} & -\frac{N(T-1)}{T\tilde{\sigma}_\epsilon^2} \\ -\frac{N(T-1)}{T\tilde{\sigma}_\epsilon^2} & \tilde{J}_{\rho\rho} - \frac{2N(T-1)^2\tilde{\sigma}_\epsilon^4}{T^2\tilde{\sigma}_1^4} \end{bmatrix} = \begin{bmatrix} \frac{N(T-1)}{2\tilde{\sigma}_\epsilon^4} & -\frac{N(T-1)}{T\tilde{\sigma}_\epsilon^2} \\ -\frac{N(T-1)}{T\tilde{\sigma}_\epsilon^2} & \tilde{C}_{\rho\rho} \end{bmatrix}$$

$$\det(A - BD^{-1}C) = \frac{N(T-1)(T^2\tilde{C}_{\rho\rho} - 2N(T-1))}{2T^2\tilde{\sigma}_\epsilon^4}$$

$$(A - BD^{-1}C)^{-1} = \begin{bmatrix} \frac{2T^2\tilde{\sigma}_\epsilon^4\tilde{C}_{\rho\rho}}{N(T-1)(T^2\tilde{C}_{\rho\rho} - 2N(T-1))} & \frac{2T\tilde{\sigma}_\epsilon^2}{T^2\tilde{C}_{\rho\rho} - 2N(T-1)} \\ \frac{2T\tilde{\sigma}_\epsilon^2}{T^2\tilde{C}_{\rho\rho} - 2N(T-1)} & \frac{T^2}{T^2\tilde{C}_{\rho\rho} - 2N(T-1)} \end{bmatrix}. \quad (\text{A.12})$$

Also we obtain

$$\tilde{D}(\boldsymbol{\theta})' \begin{bmatrix} I_2 \\ -D^{-1}C \end{bmatrix}$$

$$\begin{aligned}
&= (0 \ D(\tilde{\rho}) \ D(\tilde{\alpha})') \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{1}{Th'(\tilde{\alpha}_1)}(Z'Z)^{-1}Z'i_N & -\frac{2(T-1)\tilde{\sigma}_\epsilon^2}{T^2h'(\tilde{\alpha}_1)}(Z'Z)^{-1}Z'i_N \end{bmatrix} \\
&= \left(0 \ D(\tilde{\rho}) \ \frac{Th'(\tilde{\alpha}_1)}{2\tilde{\sigma}_1^2}f'Z\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{1}{Th'(\tilde{\alpha}_1)}(Z'Z)^{-1}Z'i_N & -\frac{2(T-1)\tilde{\sigma}_\epsilon^2}{T^2h'(\tilde{\alpha}_1)}(Z'Z)^{-1}Z'i_N \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2\tilde{\sigma}_1^2}f'Z(Z'Z)^{-1}Z'i_N & D(\tilde{\rho}) - \frac{(T-1)\tilde{\sigma}_\epsilon^2}{T\tilde{\sigma}_1^2}f'Z(Z'Z)^{-1}Z'i_N \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2\tilde{\sigma}_1^2}f'i_N & D(\tilde{\rho}) - \frac{(T-1)\tilde{\sigma}_\epsilon^2}{T\tilde{\sigma}_1^2}f'i_N \end{bmatrix} \\
&= [0 \ D(\tilde{\rho})], \tag{A.13}
\end{aligned}$$

where the fourth equality follows from the fact that the first column of  $Z$  is  $i_N$  and the last equality follows from the first-order condition in (A.6).

Therefore, the LM statistic for the hypothesis  $H_0^a$  is obtained by

$$\begin{aligned}
LM_a &= \tilde{D}(\boldsymbol{\theta})' \tilde{J}^{-1}(\boldsymbol{\theta}) \tilde{D}(\boldsymbol{\theta}) \\
&= \tilde{D}(\boldsymbol{\theta})' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D^{-1} \end{bmatrix} \tilde{D}(\boldsymbol{\theta}) \\
&+ \tilde{D}(\boldsymbol{\theta})' \begin{bmatrix} I_2 \\ -D^{-1}C \end{bmatrix} (A - BD^{-1}C)^{-1} [I_2 \ -BD^{-1}] \tilde{D}(\boldsymbol{\theta}) \\
&= D(\tilde{\alpha})' D^{-1} D(\tilde{\alpha}) + [0 \ D(\tilde{\rho})] (A - BD^{-1}C)^{-1} \begin{bmatrix} 0 \\ D(\tilde{\rho}) \end{bmatrix} \\
&= \frac{1}{2} f'Z(Z'Z)^{-1}Z'f + \frac{T^2}{T^2C_{\rho\rho} - 2N(T-1)} \tilde{D}(\rho)^2, \tag{A.14}
\end{aligned}$$

where  $C_{\rho\rho} = J_{\rho\rho} - \frac{2N(T-1)^2}{T^2} \frac{\tilde{\sigma}_\epsilon^4}{\tilde{\sigma}_1^4}$ ,  $\tilde{J}_{\rho\rho}$  is given by (A.9). The LM statistic of (A.14) is the familiar term used in testing the heteroscedasticity in Breusch and Pagan (1979). Under the null hypothesis, the LM statistic of (A.14) is asymptotically distributed as  $\chi_p^2$ .

## Appendix 2

This appendix derives the conditional LM test for testing  $H_0^b : \rho = 0$  (given  $\alpha \neq 0$ ). The variance-covariance matrix of the disturbances is given by (A.1). Under  $H_0^b$  we obtain

$$\begin{aligned}\Omega &= \text{diag}[h(z'_i \alpha)] \otimes J_T + \sigma_\epsilon^2 I_N \otimes I_T, \\ \Omega^{-1} &= \text{diag}\left(\frac{1}{w_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T\end{aligned}\quad (\text{B.1})$$

where  $w_i^2 = Th(z'_i \alpha) + \sigma_\epsilon^2$ .

Then, we obtain the following quantities

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma_\epsilon^2} \Big|_{H_0^b} &= I_N \otimes I_T \\ \frac{\partial \log L}{\partial \alpha_k} \Big|_{H_0^b} &= \text{diag}(h'(z'_i \hat{\alpha}) z_{ik}) \otimes J_T \\ \frac{\partial \log L}{\partial \rho} \Big|_{H_0^b} &= \sigma_\epsilon^2 I_N \otimes G \\ \Omega^{-1} \frac{\partial \log L}{\partial \sigma_\epsilon^2} &= \left( \text{diag}\left(\frac{1}{w_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T \right) (I_N \otimes I_T) \\ &= \text{diag}\left(\frac{1}{w_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T \\ \Omega^{-1} \frac{\partial \log L}{\partial \alpha_k} &= \left( \text{diag}\left(\frac{1}{w_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T \right) \left( \text{diag}(h'(z'_i \hat{\alpha}) z_{ik}) \otimes J_T \right) \\ &= \text{diag}\left(\frac{h'(z'_i \hat{\alpha}) z_{ik}}{w_i^2}\right) \otimes J_T \\ \Omega^{-1} \frac{\partial \log L}{\partial \rho} &= \left( \text{diag}\left(\frac{1}{w_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T \right) (\sigma_\epsilon^2 I_N \otimes G) \\ &= \left( \text{diag}\left(\frac{\sigma_\epsilon^2}{w_i^2}\right) \otimes \bar{J}_T G + I_N \otimes E_T G \right) \\ \Omega^{-1} \frac{\partial \log L}{\partial \sigma_\epsilon^2} \Omega^{-1} &= \text{diag}\left(\frac{1}{w_i^4}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^4} I_N \otimes E_T \\ \Omega^{-1} \frac{\partial \log L}{\partial \alpha_k} \Omega^{-1} &= \left( \text{diag}\left(\frac{h'(z'_i \hat{\alpha}) z_{ik}}{w_i^2}\right) \otimes J_T \right) \left( \text{diag}\left(\frac{1}{w_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T \right) \\ &= \left( \text{diag}\left(\frac{h'(z'_i \hat{\alpha}) z_{ik}}{w_i^4}\right) \otimes J_T \right) \\ \Omega^{-1} \frac{\partial \log L}{\partial \rho} \Omega^{-1} &= \left( \text{diag}\left(\frac{1}{w_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T \right) (\sigma_\epsilon^2 I_N \otimes G) \left( \text{diag}\left(\frac{1}{w_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T \right)\end{aligned}$$

$$= \sigma_\epsilon^2 \left( \text{diag}\left(\frac{1}{\hat{w}_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T \right) (I_N \otimes G) \left( \text{diag}\left(\frac{1}{\hat{w}_i^2}\right) \otimes \bar{J}_T + \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T \right)$$

Using the results of Hemmerle and Hartly(1973), we obtain under the null hypothesis  $H_0^b$  :

$$\begin{aligned} \frac{\partial \log L}{\partial \sigma_\epsilon^2} \Big|_{H_0^b} &= D(\hat{\sigma}_\epsilon^2) = -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \sigma_\epsilon^2} \right] + \frac{1}{2} \hat{u}' \left( \Omega^{-1} \frac{\partial \Omega}{\partial \sigma_\epsilon^2} \Omega^{-1} \right) \hat{u} \\ &= -\frac{1}{2} \text{tr} \left[ \text{diag}\left(\frac{1}{\hat{w}_i^2}\right) \otimes \bar{J}_T + \frac{1}{\hat{\sigma}_\epsilon^2} I_N \otimes E_T \right] + \frac{1}{2} \hat{u}' \left( \text{diag}\left(\frac{1}{\hat{w}_i^4}\right) \otimes \bar{J}_T + \frac{1}{\hat{\sigma}_\epsilon^4} I_N \otimes E_T \right) \hat{u} \\ &= -\frac{1}{2} \sum_{i=1}^N \left( \frac{1}{\hat{w}_i^2} + \frac{T-1}{\hat{\sigma}_\epsilon^2} \right) + \frac{1}{2} \hat{u}' \left[ \text{diag}\left(\frac{1}{\hat{w}_i^4}\right) \otimes \bar{J}_T + \frac{1}{\hat{\sigma}_\epsilon^4} I_N \otimes E_T \right] \hat{u} \\ &= 0 \\ \frac{\partial \log L}{\partial \alpha_k} \Big|_{H_0} &= D(\hat{\alpha}_k) = -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \alpha_k} \right] + \frac{1}{2} \hat{u}' \left( \Omega^{-1} \frac{\partial \Omega}{\partial \alpha_k} \Omega^{-1} \right) \hat{u} \\ &= -\frac{1}{2} \text{tr} \left[ \text{diag}\left(\frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^2}\right) \otimes J_T \right] + \frac{1}{2} \hat{u}' \left( \text{diag}\left(\frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^4}\right) \otimes J_T \right) \hat{u} \\ &= -\frac{T}{2} \sum_{i=1}^N \frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^2} + \frac{1}{2} \hat{u}' \left[ \text{diag}\left(\frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^4}\right) \otimes J_T \right] \hat{u} \\ &= 0, \quad k = 1, \dots, p \end{aligned}$$

$$\begin{aligned} \frac{\partial \log L}{\partial \rho} \Big|_{H_0} &= D(\hat{\rho}) \\ &= -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \rho} \right] + \frac{1}{2} \hat{u}' \left( \Omega^{-1} \frac{\partial \Omega}{\partial \rho} \Omega^{-1} \right) \hat{u} \\ &= -\frac{1}{2} \text{tr} \left[ \text{diag}\left(\frac{\hat{\sigma}_\epsilon^2}{\hat{w}_i^2}\right) \otimes \bar{J}_T G + I_N \otimes E_T G \right] \\ &+ \frac{\hat{\sigma}_\epsilon^2}{2} \hat{u}' \left[ \left( \text{diag}(1/\hat{w}_i^2) \otimes \bar{J}_T + 1/\hat{\sigma}_\epsilon^2 I_N \otimes E_T \right) (I_N \otimes G) \left( \text{diag}(1/\hat{w}_i^2) \otimes \bar{J}_T + 1/\hat{\sigma}_\epsilon^2 I_N \otimes E_T \right) \right] \hat{u} \\ &= -\frac{1}{2} \left( \frac{2(T-1)}{T} \sum_{i=1}^N \frac{\hat{\sigma}_\epsilon^2}{\hat{w}_i^2} - \frac{2N(T-1)}{T} \right) \quad (\text{since } \text{tr}(G) = 0, \text{tr}(\bar{J}_T G) = 2(T-1)/T) \\ &+ \frac{\hat{\sigma}_\epsilon^2}{2} \hat{u}' \left[ \left( \text{diag}(1/\hat{w}_i^2) \otimes \bar{J}_T + 1/\hat{\sigma}_\epsilon^2 I_N \otimes E_T \right) (I_N \otimes G) \left( \text{diag}(1/\hat{w}_i^2) \otimes \bar{J}_T + 1/\hat{\sigma}_\epsilon^2 I_N \otimes E_T \right) \right] \hat{u} \\ &= \frac{T-1}{T} \sum_{i=1}^N \left( \frac{\hat{w}_i^2 - \hat{\sigma}_\epsilon^2}{\hat{w}_i^2} \right) \\ &+ \frac{\hat{\sigma}_\epsilon^2}{2} \hat{u}' \left[ \left( \text{diag}(1/\hat{w}_i^2) \otimes \bar{J}_T + 1/\hat{\sigma}_\epsilon^2 I_N \otimes E_T \right) (I_N \otimes G) \left( \text{diag}(1/\hat{w}_i^2) \otimes \bar{J}_T + 1/\hat{\sigma}_\epsilon^2 I_N \otimes E_T \right) \right] \hat{u} \end{aligned} \tag{B.2}$$

where  $\hat{u} = y - X\hat{\beta}_{GLS}$  is the GLS residuals under  $H_0^b$ ,  $\hat{w}_i^2 = Th(z_i'\hat{\alpha}) + \hat{\sigma}_\epsilon^2$ , where  $\hat{\alpha}$  is the ML estimator of  $\alpha$  and  $\hat{\sigma}_\epsilon^2$  is the solution of  $D(\hat{\sigma}_\epsilon^2) = 0$  under  $H_0^b$ , and  $h'(z_i'\hat{\alpha})$  is the evaluated value of  $\partial h(z_i'\alpha) / \partial z_i'\alpha$ . Therefore, the partial derivatives under  $H_0^b$  can be written in vector form as

$$D(\hat{\theta}) = \begin{pmatrix} D(\hat{\sigma}_\epsilon^2) \\ D(\hat{\alpha}) \\ D(\hat{\rho}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ D(\hat{\rho}) \end{pmatrix}. \quad (\text{B.3})$$

where  $D(\hat{\alpha}) = (D(\hat{\alpha}_1), \dots, D(\hat{\alpha}_p))'$ . Also, using the the results of Harville (1977), we obtain the information matrix under the null hypothesis  $H_0^b$ :

$$\begin{aligned} E\left[-\frac{\partial^2 \log L}{\partial \sigma_\epsilon^4}\right]_{H_0^b} &= \frac{1}{2} \text{tr} \left[ \left( \text{diag} \left( \frac{1}{\hat{w}_i^2} \right) \otimes \bar{J}_T + \frac{1}{\hat{\sigma}_\epsilon^2} I_N \otimes E_T \right)^2 \right] \\ &= \frac{1}{2} \text{tr} \left[ \left( \text{diag} \left( \frac{1}{\hat{w}_i^4} \right) \otimes \bar{J}_T + \frac{1}{\hat{\sigma}_\epsilon^4} I_N \otimes E_T \right) \right] \\ &= \frac{1}{2} \sum_{i=1}^N \left( \frac{1}{\hat{w}_i^4} + \frac{T-1}{\hat{\sigma}_\epsilon^4} \right) \\ E\left[-\frac{\partial^2 \log L}{\partial \sigma_\epsilon^2 \partial \alpha_k}\right]_{H_0^b} &= \frac{1}{2} \text{tr} \left[ \left( \text{diag} \left( \frac{1}{\hat{w}_i^2} \right) \otimes \bar{J}_T + \frac{1}{\hat{\sigma}_\epsilon^2} I_N \otimes E_T \right) \left( \text{diag} \left( \frac{h'(z_i'\hat{\alpha})z_{ik}}{\hat{w}_i^2} \right) \otimes J_T \right) \right] \\ &= \frac{1}{2} \text{tr} \left[ \left( \text{diag} \left( \frac{h'(z_i'\hat{\alpha})z_{ik}}{\hat{w}_i^4} \right) \otimes J_T \right) \right] \\ &= \frac{T}{2} \sum_{i=1}^N \left( \frac{h'(z_i'\hat{\alpha})z_{ik}}{\hat{w}_i^4} \right), \quad k = 1, \dots, p \\ E\left[-\frac{\partial^2 \log L}{\partial \sigma_\epsilon^2 \partial \rho}\right]_{H_0^b} &= \frac{1}{2} \text{tr} \left[ \left( \text{diag} \left( \frac{1}{\hat{w}_i^2} \right) \otimes \bar{J}_T + \frac{1}{\hat{\sigma}_\epsilon^2} I_N \otimes E_T \right) \left( \text{diag} \left( \frac{\hat{\sigma}_\epsilon^2}{\hat{w}_i^2} \right) \otimes \bar{J}_T G + I_N \otimes E_T G \right) \right] \\ &= \frac{1}{2} \text{tr} \left[ \text{diag} \left( \frac{\hat{\sigma}_\epsilon^2}{\hat{w}_i^4} \right) \otimes \bar{J}_T G + \frac{1}{\hat{\sigma}_\epsilon^2} I_N \otimes E_T G \right] \\ &= \frac{(T-1)\hat{\sigma}_\epsilon^2}{T} \sum_{i=1}^N \left( \frac{1}{\hat{w}_i^4} - \frac{1}{\hat{\sigma}_\epsilon^4} \right) \end{aligned}$$

$$\begin{aligned} &E\left[-\frac{\partial^2 \log L}{\partial \rho^2}\right]_{H_0^b} \\ &= \frac{1}{2} \text{tr} \left[ \left( \text{diag} \left( \frac{\hat{\sigma}_\epsilon^2}{\hat{w}_i^2} \right) \otimes \bar{J}_T G + I_N \otimes E_T G \right)^2 \right] \\ &= \frac{1}{2} \text{tr} \left[ \text{diag} \left( \frac{\hat{\sigma}_\epsilon^4}{\hat{w}_i^4} \right) \otimes \bar{J}_T G \bar{J}_T G + 2 \text{diag} \left( \frac{\hat{\sigma}_\epsilon^2}{\hat{w}_i^2} \right) \otimes \bar{J}_T G E_T G + I_N \otimes E_T G E_T G \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[ \sum_{i=1}^N \hat{\sigma}_\epsilon^4 / \hat{w}_i^4 \text{tr}(\bar{J}_T G \bar{J}_T G) + 2 \sum_{i=1}^N \frac{\hat{\sigma}_\epsilon^2}{\hat{w}_i^2} \text{tr}(\bar{J}_T G E_T G) + N \text{tr}(E_T G E_T G) \right] \\
&= \frac{1}{2} \left[ \sum_{i=1}^N \hat{\sigma}_\epsilon^4 / \hat{w}_i^4 \text{tr}(\bar{J}_T G \bar{J}_T G) + 2 \sum_{i=1}^N \frac{\hat{\sigma}_\epsilon^2}{\hat{w}_i^2} \text{tr}(\bar{J}_T G^2 - \bar{J}_T G \bar{J}_T G) \right. \\
&\quad \left. + N \text{tr}(G^2 - 2\bar{J}_T G^2 + \bar{J}_T G \bar{J}_T G) \right] \\
&= \frac{1}{2} \left[ \sum_{i=1}^N \hat{\sigma}_\epsilon^4 / \hat{w}_i^4 4(T-1)^2/T^2 + 2 \sum_{i=1}^N \frac{\hat{\sigma}_\epsilon^2}{\hat{w}_i^2} (2(2T-3)/T - 4(T-1)^2/T^2) \right. \\
&\quad \left. + N (2(T-1) - 4(2T-3)/T + 4(T-1)^2/T^2) \right] \\
&\quad \left( \text{since } \text{tr}(\bar{J}_T G \bar{J}_T G) = 4(T-1)^2/T^2, \text{tr}(\bar{J}_T G^2) = 2(2T-3)/T, \text{tr}(G^2) = 2(T-1) \right) \\
&= \frac{2(T-1)^2}{T^2} \sum_{i=1}^N (\hat{\sigma}_\epsilon^2 / \hat{w}_i^2 - 1)^2 + \frac{2(2T-3)}{T} \sum_{i=1}^N (\hat{\sigma}_\epsilon^2 / \hat{w}_i^2 - 1) + (T-1) \\
&= \hat{a}_{\rho\rho}
\end{aligned}$$

$$\begin{aligned}
E \left[ - \frac{\partial^2 \log L}{\partial \rho \partial \alpha_k} \right] \Big|_{H_0^a} &= \frac{1}{2} \text{tr} \left[ \left( \text{diag} \left( \frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^2} \right) \otimes J_T \right) \left( \text{diag} \left( \frac{\hat{\sigma}_\epsilon^2}{\hat{w}_i^2} \right) \otimes \bar{J}_T G + I_N \otimes E_T G \right) \right] \\
&= \frac{\hat{\sigma}_\epsilon^2}{2} \text{tr} \left[ \left( \text{diag} \left( \frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^4} \right) \otimes J_T G \right) \right] \\
&= \frac{2(T-1) \hat{\sigma}_\epsilon^2}{2} \sum_{i=1}^N \frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^4} = (T-1) \hat{\sigma}_\epsilon^2 \sum_{i=1}^N \frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^4} \\
E \left[ - \frac{\partial^2 \log L}{\partial \alpha_k \partial \alpha_l} \right] \Big|_{H_0^a} &= \frac{1}{2} \text{tr} \left[ \left( \text{diag} \left( \frac{h'(z'_i \hat{\alpha}) z_{ik}}{\hat{w}_i^2} \right) \otimes J_T \right) \left( \text{diag} \left( \frac{h'(z'_i \hat{\alpha}) z_{il}}{\hat{w}_i^2} \right) \otimes J_T \right) \right] \\
&= \frac{T}{2} \text{tr} \left[ \left( \text{diag} \left( \frac{h'(z'_i \hat{\alpha})^2 z_{ik} z_{il}}{\hat{w}_i^4} \right) \otimes J_T \right) \right] \\
&= \frac{T^2}{2} \sum_{i=1}^N \frac{h'(z'_i \hat{\alpha})^2 z_{ik} z_{il}}{\hat{w}_i^4}, \quad k, l = 1, \dots, p. \tag{B.4}
\end{aligned}$$

Let  $W = \text{diag}(\hat{w}_1^2, \dots, \hat{w}_N^2)$  and  $H = \text{diag}(h'(z'_1 \hat{\alpha}), \dots, h'(z'_N \hat{\alpha}))$ , then, in vector form, we obtain the following quantity

$$\begin{aligned}
E \left[ - \frac{\partial^2 \log L}{\partial \sigma_\epsilon^2 \partial \alpha} \right] \Big|_{H_0^b} &= \frac{T}{2} Z' W^{-2} H i_N \\
E \left[ - \frac{\partial^2 \log L}{\partial \rho \partial \alpha} \right] \Big|_{H_0^b} &= (T-1) \hat{\sigma}_\epsilon^2 Z' W^{-2} H i_N \\
E \left[ - \frac{\partial^2 \log L}{\partial \alpha \partial \alpha'} \right] \Big|_{H_0^b} &= \frac{T^2}{2} Z' W^{-2} H^2 Z. \tag{B.5}
\end{aligned}$$

Note that from (B.5) we obtain

$$\begin{aligned}
& Z'W^{-2}Hi_N = \\
& \begin{pmatrix} z_{11} & z_{21} & \cdots & z_{N1} \\ z_{12} & z_{22} & \cdots & z_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ z_{1p} & z_{2p} & \cdots & z_{Np} \end{pmatrix} \begin{pmatrix} 1/\hat{w}_1^4 & 0 & \cdots & 0 \\ 0 & 1/\hat{w}_2^4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/\hat{w}_N^4 \end{pmatrix} \begin{pmatrix} h'(z'_1\hat{\alpha}) & 0 & \cdots & 0 \\ 0 & h'(z'_2\hat{\alpha}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & h'(z'_N\hat{\alpha}) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\
& = \begin{pmatrix} z_{11} & z_{21} & \cdots & z_{N1} \\ z_{12} & z_{22} & \cdots & z_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ z_{1p} & z_{2p} & \cdots & z_{Np} \end{pmatrix} \begin{pmatrix} h'(z'_1\hat{\alpha})/\hat{w}_1^4 \\ h'(z'_2\hat{\alpha})/\hat{w}_2^4 \\ \vdots \\ h'(z'_N\hat{\alpha})/\hat{w}_N^4 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^N \frac{h'(z'_i\hat{\alpha})z_{i1}}{\hat{w}_i^4} \\ \sum_{i=1}^N \frac{h'(z'_i\hat{\alpha})z_{i2}}{\hat{w}_i^4} \\ \vdots \\ \sum_{i=1}^N \frac{h'(z'_i\hat{\alpha})z_{ip}}{\hat{w}_i^4} \end{pmatrix} \\
& Z'W^{-2}H^2Z = \\
& \begin{pmatrix} z_{11} & z_{21} & \cdots & z_{N1} \\ z_{12} & z_{22} & \cdots & z_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ z_{1p} & z_{2p} & \cdots & z_{Np} \end{pmatrix} \begin{pmatrix} 1/\hat{w}_1^4 & 0 & \cdots & 0 \\ 0 & 1/\hat{w}_2^4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/\hat{w}_N^4 \end{pmatrix} \begin{pmatrix} h'(z'_1\hat{\alpha})^2 & 0 & \cdots & 0 \\ 0 & h'(z'_2\hat{\alpha})^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & h'(z'_N\hat{\alpha})^2 \end{pmatrix} \\
& \times \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1p} \\ z_{21} & z_{22} & \cdots & z_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ z_{N1} & z_{N2} & \cdots & z_{Np} \end{pmatrix} \\
& = \begin{pmatrix} z_{11} & z_{21} & \cdots & z_{N1} \\ z_{12} & z_{22} & \cdots & z_{N2} \\ \vdots & \vdots & \vdots & \vdots \\ z_{1p} & z_{2p} & \cdots & z_{Np} \end{pmatrix} \begin{pmatrix} \frac{h'(z'_1\hat{\alpha})^2}{\hat{w}_1^4} & 0 & \cdots & 0 \\ 0 & \frac{h'(z'_2\hat{\alpha})^2}{\hat{w}_2^4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{h'(z'_N\hat{\alpha})^2}{\hat{w}_N^4} \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1p} \\ z_{21} & z_{22} & \cdots & z_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ z_{N1} & z_{N2} & \cdots & z_{Np} \end{pmatrix} \\
& = \begin{pmatrix} \frac{h'(z'_1\hat{\alpha})^2 z_{11}}{\hat{w}_1^4} & \frac{h'(z'_2\hat{\alpha})^2 z_{21}}{\hat{w}_2^4} & \cdots & \frac{h'(z'_N\hat{\alpha})^2 z_{N1}}{\hat{w}_N^4} \\ \frac{h'(z'_1\hat{\alpha})^2 z_{12}}{\hat{w}_1^4} & \frac{h'(z'_2\hat{\alpha})^2 z_{22}}{\hat{w}_2^4} & \cdots & \frac{h'(z'_N\hat{\alpha})^2 z_{N2}}{\hat{w}_N^4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{h'(z'_1\hat{\alpha})^2 z_{1p}}{\hat{w}_1^4} & \frac{h'(z'_2\hat{\alpha})^2 z_{2p}}{\hat{w}_2^4} & \cdots & \frac{h'(z'_N\hat{\alpha})^2 z_{Np}}{\hat{w}_N^4} \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1p} \\ z_{21} & z_{22} & \cdots & z_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ z_{N1} & z_{N2} & \cdots & z_{Np} \end{pmatrix}
\end{aligned}$$



$$= \begin{pmatrix} \sum_{i=1}^N \frac{h'(z'_i \hat{\boldsymbol{\alpha}})^2 z_{i1}^2}{\hat{w}_i^4} & \sum_{i=1}^N \frac{h'(z'_i \hat{\boldsymbol{\alpha}})^2 z_{i1} z_{i2}}{\hat{w}_i^4} & \cdots & \sum_{i=1}^N \frac{h'(z'_i \hat{\boldsymbol{\alpha}})^2 z_{i1} z_{ip}}{\hat{w}_i^4} \\ \sum_{i=1}^N \frac{h'(z'_i \hat{\boldsymbol{\alpha}})^2 z_{i1} z_{i2}}{\hat{w}_i^4} & \sum_{i=1}^N \frac{h'(z'_i \hat{\boldsymbol{\alpha}})^2 z_{i2}^2}{\hat{w}_i^4} & \cdots & \sum_{i=1}^N \frac{h'(z'_i \hat{\boldsymbol{\alpha}})^2 z_{i2} z_{ip}}{\hat{w}_i^4} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=1}^N \frac{h'(z'_i \hat{\boldsymbol{\alpha}})^2 z_{i1} z_{ip}}{\hat{w}_i^4} & \sum_{i=1}^N \frac{h'(z'_i \hat{\boldsymbol{\alpha}})^2 z_{i2} z_{ip}}{\hat{w}_i^4} & \cdots & \sum_{i=1}^N \frac{h'(z'_i \hat{\boldsymbol{\alpha}})^2 z_{ip}^2}{\hat{w}_i^4} \end{pmatrix}. \quad (\text{B.6})$$

Thus, the information matrix with respect to  $\boldsymbol{\theta} = (\sigma_\epsilon^2, \rho, \boldsymbol{\alpha})'$  under the  $H_0^b$  can be written in vector form as

$$\hat{J}_b(\boldsymbol{\theta}) = \begin{bmatrix} \frac{1}{2} \sum_{i=1}^N \left( \frac{1}{\hat{w}_i^4} + \frac{T-1}{\hat{\sigma}_\epsilon^4} \right) & \frac{(T-1)\hat{\sigma}_\epsilon^2}{T} \sum_{i=1}^N \left( \frac{1}{\hat{w}_i^4} - \frac{1}{\hat{\sigma}_\epsilon^4} \right) & \frac{T}{2} Z'W^{-2}Hl_N \\ \frac{(T-1)\hat{\sigma}_\epsilon^2}{T} \sum_{i=1}^N \left( \frac{1}{\hat{w}_i^4} - \frac{1}{\hat{\sigma}_\epsilon^4} \right) & \hat{a}_{\rho\rho} & (T-1)\hat{\sigma}_\epsilon^2 Z'W^{-2}Hl_N \\ \frac{T}{2} l'_N HW^{-2}Z & (T-1)\hat{\sigma}_\epsilon^2 l'_N HW^{-2}Z & \frac{T^2}{2} Z'W^{-2}H^2Z \end{bmatrix}, \quad (\text{B.7})$$

where  $\hat{a}_{\rho\rho} = \frac{2(T-1)^2}{T^2} \sum_{i=1}^N (\hat{\sigma}_\epsilon^2/\hat{w}_i^2 - 1)^2 + \frac{2(2T-3)}{T} \sum_{i=1}^N (\hat{\sigma}_\epsilon^2/\hat{w}_i^2 - 1) + (T-1)$ .

Therefore, the resulting LM test statistic for testing  $H_0^b : \rho = 0$  (given  $\boldsymbol{\alpha} \neq \mathbf{0}$ ) is

$$LM_b = D(\hat{\boldsymbol{\theta}})' \hat{J}_b(\boldsymbol{\theta})^{-1} D(\hat{\boldsymbol{\theta}}) = \hat{J}_b(\boldsymbol{\theta})^{\rho\rho} D(\hat{\rho})^2 \quad (\text{B.8})$$

where  $\hat{J}_b(\boldsymbol{\theta})^{\rho\rho}$  is the element of the estimate of the inverse information matrix corresponding to  $\rho$  evaluated under  $H_0$ . Under the null hypothesis,  $LM_b$  in (A.21) is asymptotically distributed as  $\chi^2(1)$ .

### Appendix 3

Let us consider the LM test for  $\alpha_2 = \dots = \alpha_p = 0$  (given  $\sigma_\mu^2 > 0$  and  $\rho > 0$ ). The null hypothesis for this model is

$$H_0^c : \alpha_2 = \dots = \alpha_p = 0 \text{ (given } \sigma_\mu^2 > 0 \text{ and } \rho > 0) \quad vs \quad H_1^c : \text{not } H_0 \quad (\text{C.1})$$

The variance-covariance matrix of the disturbances is given by (A.1). Under  $H_0^c$  we obtain

$$\begin{aligned} \Omega &= \sigma_\mu^2 (I_N \otimes J_T) + (I_N \otimes V) \\ &= \sigma_\mu^2 (I_N \otimes J_T) + \sigma_\epsilon^2 (I_N \otimes \Sigma), \end{aligned} \quad (\text{C.2})$$

where  $\Sigma = \frac{1}{1-\rho^2} R$ , where  $R$  is the AR(1) correlation matrix. It is well established, see for e.g. Kadiyala(1968), that

$$C = \begin{bmatrix} (1-\rho^2)^{1/2} & 0 & 0 & \dots & 0 & 0 & 0 \\ -\rho & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\rho & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -\rho & 1 \end{bmatrix} \quad (\text{C.3})$$

transform the usual AR(1) model into a serially uncorrelated regression with independent observations. Therefore, one can obtain the transformed covariance matrix and given by

$$\begin{aligned} \Omega^* &= (I_N \otimes C) \Omega (I_N \otimes C') \\ &= \sigma_\mu^2 (I_N \otimes C J_T C') + \sigma_\epsilon^2 (I_N \otimes I_T) \\ &= \sigma_\mu^2 (1-\rho)^2 (I_N \otimes J_T^\delta) + \sigma_\epsilon^2 (I_N \otimes I_T) \\ &= (d^2 \sigma_\mu^2 (1-\rho)^2 I_N \otimes \bar{J}_T^\delta) + \sigma_\epsilon^2 (I_N \otimes I_T) \\ &= \lambda^2 (I_N \otimes \bar{J}_T^\delta) + \sigma_\epsilon^2 (I_N \otimes E_T^\delta) \end{aligned} \quad (\text{C.4})$$

where  $C i_T = (1-\rho) i_T^\delta$ ,  $i_T^\delta = (\delta, 1, \dots, 1)'$ ,  $\delta = \sqrt{\frac{1+\rho}{1-\rho}}$ ,  $d^2 = i_T^\delta' i_T^\delta = \delta^2 + T - 1$  and  $\lambda^2 = d^2 \sigma_\mu^2 (1-\rho)^2 + \sigma_\epsilon^2$ .

Therefore,  $\Omega^{*-1}$  given by

$$\Omega^{*-1} = \frac{1}{\lambda^2} (I_N \otimes \bar{J}_T^\delta) + \frac{1}{\sigma_\epsilon^2} (I_N \otimes E_T^\delta). \quad (\text{C.5})$$

Since  $\Omega$  is related to  $\Omega^*$  by  $\Omega^* = (I_N \otimes C) \Omega (I_N \otimes C')$ ,  $\Omega^{-1}$  is given by

$$\begin{aligned}
\Omega^{-1} &= (I_N \otimes C') \Omega^{*-1} (I_N \otimes C) \\
&= (I_N \otimes C') \left( \frac{1}{\lambda^2} I_N \otimes \bar{J}_T^\delta \right) (I_N \otimes C) + (I_N \otimes C') \left( \frac{1}{\sigma_\epsilon^2} I_N \otimes E_T^\delta \right) (I_N \otimes C) \\
&= \frac{1}{\lambda^2} (I_N \otimes C' \bar{J}_T^\delta C) + \frac{1}{\sigma_\epsilon^2} (I_N \otimes C' C) - \frac{1}{\sigma_\epsilon^2} (I_N \otimes C' \bar{J}_T^\delta C) \\
&= \frac{1}{\sigma_\epsilon^2} (I_N \otimes \Sigma^{-1}) - \left( \frac{1}{\sigma_\epsilon^2} - \frac{1}{\lambda^2} \right) (I_N \otimes C' \bar{J}_T^\delta C) \\
&= \frac{1}{\sigma_\epsilon^2} (I_N \otimes \Sigma^{-1}) - \left( \frac{1}{\sigma_\epsilon^2} - \frac{1}{\lambda^2} \right) \frac{1}{d^2(1-\rho)^2} (I_N \otimes \Sigma^{-1} J_T \Sigma^{-1}) \\
&= \frac{1}{\sigma_\epsilon^2} (I_N \otimes \Sigma^{-1}) - \left( \frac{\sigma_\mu^2}{\sigma_\epsilon^2 \lambda^2} \right) (I_N \otimes \Sigma^{-1} J_T \Sigma^{-1})
\end{aligned} \tag{C.6}$$

where the last equation follows from  $i_T^\delta = C i_T / (1 - \rho)$  and  $C' C = \Sigma^{-1}$ .

## 1) Partial Derivatives

Using the formula of Hemmerle and Hartly (1973), we obtain

$$\begin{aligned}
\frac{\partial \Omega}{\partial \rho} &= \frac{\partial}{\partial \rho} \left( \frac{\sigma_\epsilon^2}{1 - \rho^2} (I_N \otimes R) \right) \\
&= \sigma_\epsilon^2 \left( \frac{2\rho}{(1 - \rho^2)^2} (I_N \otimes R) + \frac{1}{1 - \rho^2} (I_N \otimes F) \right) \\
&= \frac{\sigma_\epsilon^2}{1 - \rho^2} \left( 2\rho (I_N \otimes \Sigma) + (I_N \otimes F) \right) \\
\frac{\partial \Omega}{\partial \alpha_k} &= \frac{\partial}{\partial \alpha_k} \text{diag}(h(z'_i \boldsymbol{\alpha})) \otimes J_T \\
&= h'(\alpha_1) \left( \text{diag}(z_{ik}) \otimes J_T \right), \quad k = 1, \dots, p
\end{aligned} \tag{C.7}$$

where

$$F = \frac{\partial R}{\partial \rho} = \begin{bmatrix} 0 & 1 & 2\rho & \dots & (T-1)\rho^{T-2} \\ 1 & 0 & 1 & \dots & (T-2)\rho^{T-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (T-1)\rho^{T-2} & (T-2)\rho^{T-3} & (T-3)\rho^{T-4} & \dots & 0 \end{bmatrix}$$

Also, we obtain the following quantities,

$$\Omega^{-1} \frac{\partial \Omega}{\partial \sigma_\epsilon^2} \Big|_{H_0^c} = \frac{1}{\hat{\sigma}_\epsilon^2} \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1}) \right\} (I_N \otimes \hat{\Sigma})$$

$$\begin{aligned}
&= \frac{1}{\hat{\sigma}_\epsilon^2} \left[ (I_N \otimes I_T) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T) \right] \\
\Omega^{-1} \frac{\partial \Omega}{\partial \rho} \Big|_{H_0^c} &= \frac{1}{\hat{\sigma}_\epsilon^2} \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1}) \right\} \left\{ \frac{\hat{\sigma}_\epsilon^2}{1 - \hat{\rho}^2} (2\hat{\rho}(I_N \otimes \hat{\Sigma}) + (I_N \otimes \hat{F})) \right\} \\
&= \frac{1}{1 - \hat{\rho}^2} \left\{ 2\hat{\rho}(I_N \otimes I_T) + (I_N \otimes \hat{\Sigma}^{-1} \hat{F}) - \left( \frac{2\hat{\rho}\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T) \right. \\
&\quad \left. - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \hat{F}) \right\} \\
\Omega^{-1} \frac{\partial \Omega}{\partial \alpha_k} \Big|_{H_0^c} &= \frac{1}{\hat{\sigma}_\epsilon^2} \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1}) \right\} \left\{ h'(\hat{\alpha}_1) (\text{diag}(z_{ik}) \otimes J_T) \right\} \\
&= \frac{h'(\hat{\alpha}_1)}{\hat{\sigma}_\epsilon^2} \left\{ (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right\}. \tag{C.8}
\end{aligned}$$

Therefore, we obtain the following partial derivatives with respect to  $\boldsymbol{\theta} = (\sigma_\epsilon^2, \boldsymbol{\alpha}, \rho)'$  under  $H_0^c$ :

$$\begin{aligned}
\frac{\partial L}{\partial \sigma_\epsilon^2} \Big|_{H_0^c} &= -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \sigma_\epsilon^2} \right] + \frac{1}{2} \hat{u}' \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \sigma_\epsilon^2} \Omega^{-1} \right] \hat{u} \\
&= -\frac{1}{2} \text{tr} \left[ \frac{1}{\hat{\sigma}_\epsilon^2} \left( (I_N \otimes I_T) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T) \right) \right] \\
&\quad + \frac{1}{2} \hat{u}' \left[ \left\{ \frac{1}{\hat{\sigma}_\epsilon^2} (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\epsilon^2 \hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1}) \right\} (I_N \otimes \hat{\Sigma}) \right. \\
&\quad \left. \cdot \left\{ \frac{1}{\hat{\sigma}_\epsilon^2} (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\epsilon^2 \hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1}) \right\} \right] \hat{u} \\
&= -\frac{1}{2\hat{\sigma}_\epsilon^2} \left[ NT - \left( \frac{Nd^2(1 - \hat{\rho})^2 \hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) \right] \\
&\quad + \frac{1}{2} \hat{u}' \left[ \left\{ \frac{1}{\hat{\sigma}_\epsilon^2} (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\epsilon^2 \hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1}) \right\} (I_N \otimes \hat{\Sigma}) \right. \\
&\quad \left. \cdot \left\{ \frac{1}{\hat{\sigma}_\epsilon^2} (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\epsilon^2 \hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1}) \right\} \right] \hat{u} \\
&= 0 \tag{C.9}
\end{aligned}$$

$$\frac{\partial L}{\partial \rho} \Big|_{H_0^c} = -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \rho} \right] + \frac{1}{2} \hat{u}' \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \rho} \Omega^{-1} \right] \hat{u}$$

$$\begin{aligned}
&= \frac{1}{1-\hat{\rho}^2} \text{tr} \left[ \left\{ 2\hat{\rho}(I_N \otimes I_T) + (I_N \otimes \hat{\Sigma}^{-1} \hat{F}) - \left( \frac{2\hat{\rho}\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T) \right. \right. \\
&\quad \left. \left. - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \hat{F}) \right\} \right] \\
&+ \frac{1}{2\hat{\sigma}_\epsilon^4} \hat{u}' \left[ \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \cdot \left\{ \frac{\hat{\sigma}_\epsilon^2}{1-\hat{\rho}^2} (2\hat{\rho}(I_N \otimes \hat{\Sigma}) + (I_N \otimes \hat{F})) \right\} \right. \\
&\quad \left. \cdot \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \right] \hat{u} \\
&= \frac{1}{1-\hat{\rho}^2} \left[ 2\hat{\rho}NT + N \text{tr}(\hat{\Sigma}^{-1} \hat{F}) - \left( \frac{2\hat{\rho}\hat{\sigma}_\mu^2 N d^2 (1-\hat{\rho})^2}{\hat{\lambda}^2} \right) - \left( \frac{N\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) i_T' \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T \right] \\
&+ \frac{1}{2\hat{\sigma}_\epsilon^4} \hat{u}' \left[ \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \cdot \left\{ \frac{\hat{\sigma}_\epsilon^2}{1-\hat{\rho}^2} (2\hat{\rho}(I_N \otimes \hat{\Sigma}) + (I_N \otimes \hat{F})) \right\} \right. \\
&\quad \left. \cdot \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \right] \hat{u} \\
&= 0 \tag{C.10}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \alpha_1} \Big|_{H_0^c} &= -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \alpha_1} \right] + \frac{1}{2} \hat{u}' \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \alpha_1} \Omega^{-1} \right] \hat{u} \\
&= -\frac{1}{2} \text{tr} \left[ \frac{h'(\hat{\alpha}_1)}{\hat{\sigma}_\epsilon^2} \left\{ (\text{diag}(z_{i1}) \otimes \hat{\Sigma}^{-1} J_T) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (\text{diag}(z_{i1}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right\} \right] \\
&+ \frac{1}{2\hat{\sigma}_\epsilon^4} \hat{u}' \left[ \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \left\{ h'(\hat{\alpha}_1) (\text{diag}(z_{i1}) \otimes J_T) \right\} \right. \\
&\quad \left. \cdot \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \right] \hat{u} \\
&= -\frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^2} \left[ d^2 (1-\hat{\rho})^2 \sum_{i=1}^N z_{i1} - \frac{\hat{\sigma}_\mu^2 d^4 (1-\hat{\rho})^4}{\hat{\lambda}^2} \sum_{i=1}^N z_{i1} \right] \\
&+ \frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^4} \hat{u}' \left[ \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \left\{ (\text{diag}(z_{i1}) \otimes J_T) \right\} \right. \\
&\quad \left. \cdot \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \right] \hat{u} \\
&= -\frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^2} d^2 (1-\hat{\rho})^2 \left[ 1 - \frac{\hat{\sigma}_\mu^2 d^2 (1-\hat{\rho})^2}{\hat{\lambda}^2} \right] \sum_{i=1}^N z_{i1} \\
&+ \frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^4} \hat{u}' \left[ \text{diag}(z_{i1}) \otimes \left( \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} - 2 \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4} \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \Big] \hat{u} \\
= & - \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2\hat{\lambda}^2} \sum_{i=1}^N z_{i1} + \frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^4} \sum_{i=1}^N z_{i1} \hat{u}'_i \hat{A} \hat{u}_i \\
= & \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2\hat{\lambda}^2} \sum_{i=1}^N \left( \frac{\hat{\lambda}^2}{\hat{d}^2 (1 - \hat{\rho})^2 \hat{\sigma}_\epsilon^4} \hat{u}'_i \hat{A} \hat{u}_i - 1 \right) \\
= & 0 \tag{C.11}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \alpha_k} \Big|_{H_0^c} &= D(\hat{\alpha}_k) = -\frac{1}{2} \text{tr} \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \alpha_k} \right] + \frac{1}{2} \hat{u}' \left[ \Omega^{-1} \frac{\partial \Omega}{\partial \alpha_k} \Omega^{-1} \right] \hat{u} \\
&= -\frac{1}{2} \text{tr} \left[ \frac{h'(\hat{\alpha}_1)}{\hat{\sigma}_\epsilon^2} \left\{ (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right\} \right] \\
&+ \frac{1}{2\hat{\sigma}_\epsilon^4} \hat{u}' \left[ \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \left\{ h'(\hat{\alpha}_1) (\text{diag}(z_{ik}) \otimes J_T) \right\} \right. \\
&\quad \left. \cdot \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \right] \hat{u} \\
&= -\frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^2} \left[ \hat{d}^2 (1 - \hat{\rho})^2 \sum_{i=1}^N z_{ik} - \frac{\hat{\sigma}_\mu^2 \hat{d}^4 (1 - \hat{\rho})^4}{\hat{\lambda}^2} \sum_{i=1}^N z_{ik} \right] \\
&+ \frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^4} \hat{u}' \left[ \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \left\{ (\text{diag}(z_{ik}) \otimes J_T) \right\} \right. \\
&\quad \left. \cdot \left\{ (I_N \otimes \hat{\Sigma}^{-1}) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right\} \right] \hat{u} \\
&= -\frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^2} \hat{d}^2 (1 - \hat{\rho})^2 \left[ 1 - \frac{\hat{\sigma}_\mu^2 \hat{d}^2 (1 - \hat{\rho})^2}{\hat{\lambda}^2} \right] \sum_{i=1}^N z_{ik} \\
&+ \frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^4} \hat{u}' \left[ \text{diag}(z_{ik}) \otimes \left( \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} - 2 \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right. \right. \\
&\quad \left. \left. + \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4} \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \right) \right] \hat{u} \\
&= -\frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2\hat{\lambda}^2} \sum_{i=1}^N z_{ik} + \frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^4} \sum_{i=1}^N z_{ik} \hat{u}'_i \hat{A} \hat{u}_i \\
&= \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2\hat{\lambda}^2} \sum_{i=1}^N z_{ik} \left( \frac{\hat{\lambda}^2}{\hat{d}^2 (1 - \hat{\rho})^2 \hat{\sigma}_\epsilon^4} \hat{u}'_i \hat{A} \hat{u}_i - 1 \right), \quad k = 2, \dots, p \tag{C.12}
\end{aligned}$$

where  $\hat{u} = y - X\hat{\beta}_{GLS}$  is the maximum likelihood residuals under the null hypothesis  $H_0^c$ ,  $\hat{\rho}$ ,  $\hat{\sigma}_\epsilon^2$  and  $\hat{\alpha}_1$  is the ML estimates of  $\rho$ ,  $\sigma_\epsilon^2$  and  $\alpha_1$ , respectively. Also,  $\hat{\sigma}_\mu^2$  is the value of  $h(\hat{\alpha}_1)$  and  $h'(\hat{\alpha}_1)$  is the evaluated value of  $\partial h(z'_i \boldsymbol{\alpha}) / \partial z'_i \boldsymbol{\alpha}$  when  $\alpha_2 = \dots = \alpha_p = 0$ . In addition, the second equality of  $\frac{\partial L}{\partial \alpha_1} \Big|_{H_0^c}$  and  $\frac{\partial L}{\partial \alpha_k} \Big|_{H_0^c}$  uses the fact that  $tr(\Sigma^{-1} J_T) = tr(i'_T \Sigma^{-1} i_T) = d^2(1 - \rho)^2$  and  $tr(\Sigma^{-1} J_T \Sigma^{-1} J_T) = tr(i'_T \Sigma^{-1} i_T)^2 = d^4(1 - \rho)^4$ , and  $\hat{A} = (\hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} - 2 \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} + \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4} \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1})$ ,  $\hat{u}_i = (\hat{u}_{i1}, \dots, \hat{u}_{iT})'$ .

Thus, the partial derivatives under  $H_0^c$  are rewritten in vector form as

$$D_c(\hat{\boldsymbol{\theta}}) = \begin{pmatrix} D(\hat{\sigma}_\epsilon^2) \\ D(\hat{\rho}) \\ D(\hat{\boldsymbol{\alpha}}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2 \hat{\lambda}^2} Z' f \end{pmatrix}, \quad (\text{C.13})$$

where  $D(\hat{\boldsymbol{\alpha}}) = (0, D(\hat{\alpha}_2), \dots, D(\hat{\alpha}_p))'$ , and  $Z = (z_1, \dots, z_N)'$  and  $f = (f_1, \dots, f_N)'$ , where  $f_i = \frac{\hat{\lambda}^2}{\hat{d}^2 (1 - \hat{\rho})^2 \hat{\sigma}_\epsilon^4} \hat{u}'_i \hat{A} \hat{u}_i - 1$ .

## 2) Information Matrix

Also, using the the formula of Harville (1977), we obtain

$$\begin{aligned} E \left[ - \frac{\partial^2 L}{\partial (\sigma_\epsilon^2)^2} \right]_{H_0^c} &= \frac{1}{2} tr \left[ \left\{ \frac{1}{\hat{\sigma}_\epsilon^2} (I_N \otimes I_T) - \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\epsilon^2 \hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} J_T) \right\}^2 \right] \\ &= \frac{1}{2} tr \left[ \frac{1}{\hat{\sigma}_\epsilon^4} (I_N \otimes I_T) - 2 \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\epsilon^4 \hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} J_T) + \frac{\hat{\sigma}_\mu^4}{\hat{\sigma}_\epsilon^4 \hat{\lambda}^4} (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right] \\ &= \frac{NT}{2 \hat{\sigma}_\epsilon^4} - \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_\epsilon^4 \hat{\lambda}^2} tr \left[ (I_N \otimes i'_T \hat{\Sigma}^{-1} i_T) \right] + \frac{\hat{\sigma}_\mu^4}{2 \hat{\sigma}_\epsilon^4 \hat{\lambda}^4} tr \left[ (I_N \otimes (i'_T \hat{\Sigma}^{-1} i_T)(i'_T \hat{\Sigma}^{-1} i_T)) \right] \\ &= \frac{NT}{2 \hat{\sigma}_\epsilon^4} - \frac{N \hat{\sigma}_\mu^2 \hat{d}^2 (1 - \hat{\rho})^2}{\hat{\sigma}_\epsilon^4 \hat{\lambda}^2} + \frac{N \hat{\sigma}_\mu^4 \hat{d}^4 (1 - \hat{\rho})^4}{2 \hat{\sigma}_\epsilon^4 \hat{\lambda}^4} \\ &= \frac{N}{2 \hat{\sigma}_\epsilon^4} \left( 1 - 2 \frac{\hat{\sigma}_\mu^2 \hat{d}^2 (1 - \hat{\rho})^2}{\hat{\lambda}^2} + \frac{\hat{\sigma}_\mu^4 \hat{d}^4 (1 - \hat{\rho})^4}{\hat{\lambda}^4} + (T - 1) \right) \\ &= \frac{N}{2 \hat{\sigma}_\epsilon^4} \left( 1 - \frac{\hat{\sigma}_\mu^2 \hat{d}^2 (1 - \hat{\rho})^2}{\hat{\lambda}^2} \right)^2 + \frac{N(T - 1)}{2 \hat{\sigma}_\epsilon^4} \end{aligned}$$

$$\begin{aligned}
&= \frac{N}{2} \left( \frac{1}{\hat{\lambda}^4} + \frac{T-1}{\hat{\sigma}_\epsilon^4} \right) \\
&= \hat{C}(\epsilon, \epsilon)
\end{aligned} \tag{C.14}$$

$$\begin{aligned}
E \left[ - \frac{\partial^2 L}{\partial \sigma_\epsilon^2 \partial \rho} \right]_{H_0^c} &= \frac{1}{2} \text{tr} \left[ \frac{1}{\hat{\sigma}_\epsilon^2} \left\{ (I_N \otimes I_T) - \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} J_T) \right\} \right. \\
&\quad \cdot \frac{1}{1 - \hat{\rho}^2} \left\{ 2\hat{\rho} (I_N \otimes I_T) + (I_N \otimes \hat{\Sigma}^{-1} \hat{F}) - 2 \frac{\hat{\rho} \hat{\sigma}_\mu^2}{\hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} J_T) \right. \\
&\quad \quad \left. \left. - \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \hat{F}) \right\} \right] \\
&= \frac{1}{2(1 - \hat{\rho}^2) \hat{\sigma}_\epsilon^2} \text{tr} \left[ 2\hat{\rho} (I_N \otimes I_T) + (I_N \otimes \hat{\Sigma}^{-1} \hat{F}) \right. \\
&\quad \left. - 2 \frac{2\hat{\rho} \hat{\sigma}_\mu^2}{\hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} J_T) - 2 \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \hat{F}) \right. \\
&\quad \left. + \frac{2\hat{\rho} \hat{\sigma}_\mu^4}{\hat{\lambda}^4} (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) + \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4} (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \hat{F}) \right] \\
&= \frac{1}{2(1 - \hat{\rho}^2) \hat{\sigma}_\epsilon^2} \left[ 2\hat{\rho} N T + N \text{tr}(\hat{\Sigma}^{-1} \hat{F}) - 4 \frac{N \hat{d}^2 (1 - \hat{\rho})^2 \hat{\sigma}_\mu^2 \hat{\rho}}{\hat{\lambda}^2} \right. \\
&\quad \left. - 2 \frac{N \hat{\sigma}_\mu^2}{\hat{\lambda}^2} (i_T' \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) + 2 \frac{N \hat{d}^4 (1 - \hat{\rho})^4 \hat{\sigma}_\mu^4 \hat{\rho}}{\hat{\lambda}^4} \right. \\
&\quad \left. + \frac{N \hat{d}^2 (1 - \hat{\rho})^2 \hat{\sigma}_\mu^4}{\hat{\lambda}^4} (i_T' \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \right] \\
&= \frac{1}{2(1 - \hat{\rho}^2) \hat{\sigma}_\epsilon^2} \left[ 2N\hat{\rho} \left( \frac{\hat{d}^4 (1 - \hat{\rho})^4 \hat{\sigma}_\mu^4}{\hat{\lambda}^4} - 2 \frac{\hat{d}^2 (1 - \hat{\rho})^2 \hat{\sigma}_\mu^2}{\hat{\lambda}^2} + T \right) \right. \\
&\quad \left. - \frac{N \hat{\sigma}_\mu^2}{\hat{\lambda}^2} (i_T' \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \left( 2 - \frac{\hat{d}^2 (1 - \hat{\rho})^2 \hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) + N \text{tr}(\hat{\Sigma}^{-1} \hat{F}) \right] \\
&= \frac{1}{2(1 - \hat{\rho}^2) \hat{\sigma}_\epsilon^2} \left[ 2N\hat{\rho} \left( \frac{\hat{\sigma}_\epsilon^4}{\hat{\lambda}^4} + T - 1 \right) - \frac{N \hat{\sigma}_\mu^2}{\hat{\lambda}^2} (i_T' \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \left( 1 + \frac{\hat{\sigma}_\epsilon^2}{\hat{\lambda}^2} \right) \right. \\
&\quad \left. + N \text{tr}(\hat{\Sigma}^{-1} \hat{F}) \right] \\
&= \frac{N}{2(1 - \hat{\rho}^2) \hat{\sigma}_\epsilon^2} \left[ 2\hat{\rho} \left( \frac{\hat{\sigma}_\epsilon^4}{\hat{\lambda}^4} + T - 1 \right) - \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} (i_T' \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \left( 1 + \frac{\hat{\sigma}_\epsilon^2}{\hat{\lambda}^2} \right) \right. \\
&\quad \left. + \text{tr}(\hat{\Sigma}^{-1} \hat{F}) \right]
\end{aligned}$$



$$= \hat{C}(\epsilon, \rho) \quad (\text{C.15})$$

$$\begin{aligned}
E\left[-\frac{\partial^2 L}{\partial \sigma_\epsilon^2 \partial \alpha_k}\right]_{H_0^c} &= \frac{1}{2} \text{tr} \left[ \frac{1}{\hat{\sigma}_\epsilon^2} \left\{ (I_N \otimes I_T) - \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} J_T) \right\} \right. \\
&\quad \left. \cdot \frac{h'(\hat{\alpha}_1)}{\hat{\sigma}_\epsilon^2} \left\{ (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right\} \right] \\
&= \frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^4} \text{tr} \left[ (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T) - 2 \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right. \\
&\quad \left. + \left( \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4} \right) (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right] \\
&= \frac{h'(\hat{\alpha}_1)}{2\hat{\sigma}_\epsilon^4} \left[ \hat{d}^2 (1 - \hat{\rho})^2 \sum_{i=1}^N z_{ik} - 2 \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \hat{d}^4 (1 - \hat{\rho})^4 \sum_{i=1}^N z_{ik} + \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4} \hat{d}^6 (1 - \hat{\rho})^6 \sum_{i=1}^N z_{ik} \right] \\
&= \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2\hat{\sigma}_\epsilon^4} \sum_{i=1}^N z_{ik} \left[ 1 - 2 \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \hat{d}^2 (1 - \hat{\rho})^2 + \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4} \hat{d}^4 (1 - \hat{\rho})^4 \right] \\
&= \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2\hat{\sigma}_\epsilon^4} \sum_{i=1}^N z_{ik} \left( 1 - \frac{\hat{d}^2 (1 - \hat{\rho})^2 \hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right)^2 \\
&= \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2\hat{\lambda}^4} \sum_{i=1}^N z_{ik} \\
&= a(\epsilon, \boldsymbol{\alpha}) i'_N Z \quad (\text{C.16})
\end{aligned}$$

$$\begin{aligned}
E\left[-\frac{\partial^2 L}{\partial \rho^2}\right]_{H_0^c} &= \frac{1}{2} \left[ \frac{1}{(1 - \hat{\rho}^2)^2} \left\{ 2\hat{\rho} (I_N \otimes I_T) + (I_N \otimes \hat{\Sigma}^{-1} \hat{F}) - \left( \frac{2\hat{\rho}\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T) \right. \right. \\
&\quad \left. \left. - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \hat{F}) \right\}^2 \right] \\
&= \frac{1}{2(1 - \hat{\rho}^2)^2} \text{tr} \left[ 4\hat{\rho}^2 (I_N \otimes I_T) + (I_N \otimes \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} \hat{F}) \right. \\
&\quad \left. + 4 \frac{\hat{\rho}^2 \hat{\sigma}_\mu^4}{\hat{\lambda}^4} (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) + \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4} (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \hat{F}) \right. \\
&\quad \left. + 4\hat{\rho} (I_N \otimes \hat{\Sigma}^{-1} \hat{F}) - 8 \frac{\hat{\rho}^2 \hat{\sigma}_\mu^2}{\hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} \hat{J}_T) \right. \\
&\quad \left. - 4 \frac{\hat{\rho} \hat{\sigma}_\mu^2}{\hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} \hat{F}) - 4 \frac{\hat{\rho} \hat{\sigma}_\mu^2}{\hat{\lambda}^2} (I_N \otimes \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} J_T) \right]
\end{aligned}$$

$$\begin{aligned}
& -2\frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2}(I_N \otimes \hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}\hat{F}) + 4\frac{\hat{\rho}\hat{\sigma}_\mu^4}{\hat{\lambda}^4}(I_N \otimes \hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}\hat{F})] \\
= & \frac{1}{2(1-\hat{\rho}^2)^2} \left[ 4\hat{\rho}^2 NT + Ntr(\hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}\hat{F}) + 4\frac{Nd^4(1-\hat{\rho})^4\hat{\sigma}_\mu^4\hat{\rho}^2}{\lambda^4} \right. \\
& + \frac{N\hat{\sigma}_\mu^4}{\hat{\lambda}^4}(i'_T\hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}i_T)^2 + 4\hat{\rho}Ntr(\hat{\Sigma}^{-1}\hat{F}) - 8\frac{Nd^2(1-\hat{\rho})^2\hat{\sigma}_\mu^2\hat{\rho}^2}{\hat{\lambda}^2} \\
& - 8\frac{N\hat{\rho}\hat{\sigma}_\mu^2}{\hat{\lambda}^2}(i'_T\hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}i_T) - 2\sum\frac{N\hat{\sigma}_\mu^2}{\hat{\lambda}^2}(i'_T\hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}i_T) \\
& \left. + 4\frac{Nd^2(1-\hat{\rho})^2\hat{\sigma}_\mu^4\hat{\rho}}{\hat{\lambda}^4}(i'_T\hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}i_T) \right] \\
= & \hat{C}(\rho, \rho) \tag{C.17}
\end{aligned}$$

$$\begin{aligned}
E\left[-\frac{\partial^2 L}{\partial \rho \partial \alpha_k}\right]_{H_0^\epsilon} &= \frac{1}{2}tr\left[\frac{1}{(1-\hat{\rho}^2)}\left\{2\hat{\rho}(I_N \otimes I_T) + (I_N \otimes \hat{\Sigma}^{-1}\hat{F}) - \left(\frac{2\hat{\rho}\hat{\sigma}_\mu^2}{\hat{\lambda}^2}\right)(I_N \otimes \hat{\Sigma}^{-1}J_T) \right. \right. \\
& \quad \left. \left. - \left(\frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2}\right)(I_N \otimes \hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}\hat{F})\right\} \right. \\
& \quad \left. \cdot \frac{h'(\hat{\alpha}_1)}{\hat{\sigma}_\epsilon^2}\left\{(diag(z_{ik}) \otimes \hat{\Sigma}^{-1}J_T) - \left(\frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2}\right)(diag(z_{ik}) \otimes \hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}J_T)\right\}\right] \\
= & \frac{h'(\hat{\alpha}_1)}{2(1-\hat{\rho}^2)\hat{\sigma}_\epsilon^2}tr\left[2\hat{\rho}(diag(z_{ik}) \otimes \hat{\Sigma}^{-1}J_T) + (diag(z_{ik}) \otimes \hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}J_T) \right. \\
& - 4\frac{\hat{\rho}\hat{\sigma}_\mu^2}{\hat{\lambda}^2}(diag(z_{ik}) \otimes \hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}J_T) - 2\frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2}(diag(z_{ik}) \otimes \hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}J_T) \\
& + 2\frac{\hat{\rho}\hat{\sigma}_\mu^4}{\hat{\lambda}^4}(diag(z_{ik}) \otimes \hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}J_T) \\
& \left. + \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4}(diag(z_{ik}) \otimes \hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}J_T\hat{\Sigma}^{-1}\hat{F})\right] \\
= & \frac{h'(\hat{\alpha}_1)}{2(1-\hat{\rho}^2)\hat{\sigma}_\epsilon^2}\left[2\hat{\rho}\hat{d}^2(1-\hat{\rho})^2\sum_{i=1}^N z_{ik} + \sum_{i=1}^N z_{ik}(i'_T\hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}i_T) \right. \\
& - 4\frac{\hat{\rho}\hat{d}^4(1-\hat{\rho})^4\hat{\sigma}_\mu^2}{\hat{\lambda}^2}\sum_{i=1}^N z_{ik} - 2\frac{\hat{d}^2(1-\hat{\rho})^2\hat{\sigma}_\mu^2}{\hat{\lambda}^2}\sum_{i=1}^N z_{ik}(i'_T\hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}i_T) \\
& \left. + 2\frac{\hat{\rho}\hat{d}^6(1-\hat{\rho})^6\hat{\sigma}_\mu^4}{\hat{\lambda}^4}\sum_{i=1}^N z_{ik} + \frac{\hat{d}^4(1-\hat{\rho})^4\hat{\sigma}_\mu^4}{\hat{\lambda}^4}\sum_{i=1}^N z_{ik}(i'_T\hat{\Sigma}^{-1}\hat{F}\hat{\Sigma}^{-1}i_T)\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{h'(\hat{\alpha}_1)}{2(1-\hat{\rho}^2)\hat{\sigma}_\epsilon^2} \sum_{i=1}^N z_{ik} \left[ 2\hat{\rho}\hat{d}^2(1-\hat{\rho})^2 \left( 1 - 2\frac{\hat{d}^2(1-\hat{\rho})^2\hat{\sigma}_\mu^2}{\hat{\lambda}^2} + \frac{\hat{d}^4(1-\hat{\rho})^4\hat{\sigma}_\mu^4}{\hat{\lambda}^4} \right) \right. \\
&\quad \left. + (i'_T \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \left( 1 - 2\frac{\hat{d}^2(1-\hat{\rho})^2\hat{\sigma}_\mu^2}{\hat{\lambda}^2} + \frac{\hat{d}^4(1-\hat{\rho})^4\hat{\sigma}_\mu^4}{\hat{\lambda}^4} \right) \right] \\
&= \frac{h'(\hat{\alpha}_1)}{2(1-\hat{\rho}^2)\hat{\sigma}_\epsilon^2} \sum_{i=1}^N z_{ik} \left[ \left\{ 2\hat{\rho}\hat{d}^2(1-\hat{\rho})^2 + (i'_T \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \right\} \left( \frac{\hat{\sigma}_\epsilon^4}{\hat{\lambda}^4} \right) \right] \\
&= \frac{h'(\hat{\alpha}_1)\hat{\sigma}_\epsilon^2}{2(1-\hat{\rho}^2)\hat{\lambda}^4} \left( 2\hat{\rho}\hat{d}^2(1-\hat{\rho})^2 + (i'_T \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \right) \sum_{i=1}^N z_{ik} \\
&= a(\rho, \boldsymbol{\alpha}) i'_N Z
\end{aligned} \tag{C.18}$$

$$\begin{aligned}
E \left[ -\frac{\partial^2 L}{\partial \alpha_k \partial \alpha_l} \right]_{H_0^c} &= \frac{1}{2} \text{tr} \left[ \frac{h'(\hat{\alpha}_1)}{\hat{\sigma}_\epsilon^2} \left\{ (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (\text{diag}(z_{ik}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right\} \right. \\
&\quad \cdot \frac{h'(\hat{\alpha}_1)}{\hat{\sigma}_\epsilon^2} \left\{ (\text{diag}(z_{il}) \otimes \hat{\Sigma}^{-1} J_T) - \left( \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} \right) (\text{diag}(z_{il}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right\} \left. \right] \\
&= \frac{h'(\hat{\alpha}_1)^2}{2\hat{\sigma}_\epsilon^4} \text{tr} \left[ (\text{diag}(z_{ik} z_{il}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right. \\
&\quad - 2 \frac{\hat{\sigma}_\mu^2}{\hat{\lambda}^2} (\text{diag}(z_{ik} z_{il}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \\
&\quad \left. + \frac{\hat{\sigma}_\mu^4}{\hat{\lambda}^4} (\text{diag}(z_{ik} z_{il}) \otimes \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T \hat{\Sigma}^{-1} J_T) \right] \\
&= \frac{h'(\hat{\alpha}_1)^2}{2\hat{\sigma}_\epsilon^4} \left( \hat{d}^4(1-\hat{\rho})^4 \sum_{i=1}^N z_{ik} z_{il} - 2 \frac{\hat{d}^6(1-\hat{\rho})^6 \hat{\sigma}_\mu^2}{\hat{\lambda}^2} \sum_{i=1}^N z_{ik} z_{il} \right. \\
&\quad \left. + \frac{\hat{d}^8(1-\hat{\rho})^8 \hat{\sigma}_\mu^4}{\hat{\lambda}^4} \sum_{i=1}^N z_{ik} z_{il} \right) \\
&= \frac{h'(\hat{\alpha}_1)^2 \hat{d}^4(1-\hat{\rho})^4}{2\hat{\sigma}_\epsilon^4} \sum_{i=1}^N z_{ik} z_{il} \left( 1 - 2 \frac{\hat{d}^2(1-\hat{\rho})^2 \hat{\sigma}_\mu^2}{\hat{\lambda}^2} + \frac{\hat{d}^4(1-\hat{\rho})^4 \hat{\sigma}_\mu^4}{\hat{\lambda}^4} \right) \\
&= \frac{h'(\hat{\alpha}_1)^2 \hat{d}^4(1-\hat{\rho})^4}{2\hat{\lambda}^4} \sum_{i=1}^N z_{ik} z_{il} \\
&= a(\boldsymbol{\alpha}, \boldsymbol{\alpha}) Z' Z
\end{aligned} \tag{C.19}$$

Therefore, information matrix under the null hypothesis  $H_0^c$  can be obtained in matrix form as

$$\hat{J}_c(\boldsymbol{\theta}) = \begin{bmatrix} \hat{C}(\epsilon, \epsilon) & \hat{C}(\epsilon, \rho) & a(\epsilon, \boldsymbol{\alpha})i'_N Z \\ \hat{C}(\epsilon, \rho) & \hat{C}(\rho, \rho) & a(\rho, \boldsymbol{\alpha})i'_N Z \\ a(\epsilon, \boldsymbol{\alpha})Z'i_N & a(\rho, \boldsymbol{\alpha})Z'i_N & a(\boldsymbol{\alpha}, \boldsymbol{\alpha})Z'Z \end{bmatrix}, \quad (\text{C.20})$$

where  $\hat{C}(\epsilon, \epsilon)$ ,  $\hat{C}(\epsilon, \rho)$  and  $\hat{C}(\rho, \rho)$  are given by (A.11), (A.12) and (A.14), respectively, and  $a(\epsilon, \boldsymbol{\alpha})$ ,  $a(\rho, \boldsymbol{\alpha})$ , and  $a(\boldsymbol{\alpha}, \boldsymbol{\alpha})$  are given by

$$\begin{aligned} a(\epsilon, \boldsymbol{\alpha}) &= \frac{h'(\hat{\alpha}_1)\hat{d}^2(1-\hat{\rho})^2}{2\hat{\lambda}^4} \\ a(\rho, \boldsymbol{\alpha}) &= \frac{h'(\hat{\alpha}_1)\hat{\sigma}_\epsilon^2}{2(1-\hat{\rho}^2)\hat{\lambda}^4} \left( 2\hat{\rho}\hat{d}^2(1-\hat{\rho})^2 + (i'_T \hat{\Sigma}^{-1} \hat{F} \hat{\Sigma}^{-1} i_T) \right) \\ a(\boldsymbol{\alpha}, \boldsymbol{\alpha}) &= \frac{h'(\hat{\alpha}_1)^2 \hat{d}^4 (1-\hat{\rho})^4}{2\hat{\lambda}^4}. \end{aligned} \quad (\text{C.21})$$

Let

$$\begin{aligned} A &= \begin{bmatrix} \hat{C}(\epsilon, \epsilon) & \hat{C}(\epsilon, \rho) \\ \hat{C}(\epsilon, \rho) & \hat{C}(\rho, \rho) \end{bmatrix}, B = \begin{bmatrix} a(\epsilon, \boldsymbol{\alpha})i'_N Z \\ a(\rho, \boldsymbol{\alpha})i'_N Z \end{bmatrix}, \\ C &= [ a(\epsilon, \boldsymbol{\alpha})Z'i_N \quad a(\rho, \boldsymbol{\alpha})Z'i_N ], D = [ a(\boldsymbol{\alpha}, \boldsymbol{\alpha})Z'Z ], \end{aligned}$$

then  $\hat{J}_c(\boldsymbol{\theta})$  can be written as

$$\hat{J}_c(\boldsymbol{\theta}) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \quad (\text{C.22})$$

Using Searle (), the inverse of partitioned matrix can be obtained as

$$\hat{J}_c(\boldsymbol{\theta})^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D^{-1} \end{bmatrix} + \begin{bmatrix} I_2 \\ -D^{-1}C \end{bmatrix} (A - BD^{-1}C)^{-1} [I_2 \quad -BD^{-1}]. \quad (\text{C.23})$$

In (A.20), we obtain

$$D^{-1} = \frac{1}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})} (Z'Z)^{-1}$$

$$\begin{aligned}
D^{-1}C &= \begin{bmatrix} \frac{a(\epsilon, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}(Z'Z)^{-1}Z'i_N & \frac{a(\rho, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}(Z'Z)^{-1}Z'i_N \\ 1 & 0 \\ 0 & 1 \\ \frac{a(\epsilon, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}(Z'Z)^{-1}Z'i_N & \frac{a(\rho, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}(Z'Z)^{-1}Z'i_N \end{bmatrix} \\
\begin{bmatrix} I_2 \\ -D^{-1}C \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{a(\epsilon, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}(Z'Z)^{-1}Z'i_N & \frac{a(\rho, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}(Z'Z)^{-1}Z'i_N \end{bmatrix} \tag{C.24}
\end{aligned}$$

Thus,

$$\begin{aligned}
&D_c(\hat{\boldsymbol{\theta}})' \begin{bmatrix} I_2 \\ -D^{-1}C \end{bmatrix} \\
&= \begin{pmatrix} 0 & 0 & \frac{h'(\hat{\alpha}_1)\hat{d}^2(1-\hat{\rho})^2}{2\hat{\lambda}^2}f'Z \end{pmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{a(\epsilon, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}(Z'Z)^{-1}Z'i_N & \frac{a(\rho, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}(Z'Z)^{-1}Z'i_N \end{bmatrix} \\
&= \frac{h'(\hat{\alpha}_1)\hat{d}^2(1-\hat{\rho})^2}{2\hat{\lambda}^2} \begin{pmatrix} \frac{a(\epsilon, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}f'Z(Z'Z)^{-1}Z'i_N & \frac{a(\rho, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}f'Z(Z'Z)^{-1}Z'i_N \end{pmatrix} \\
&= \frac{h'(\hat{\alpha}_1)\hat{d}^2(1-\hat{\rho})^2}{2\hat{\lambda}^2} \begin{pmatrix} \frac{a(\epsilon, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}f'i_N & \frac{a(\rho, \boldsymbol{\alpha})}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})}f'i_N \end{pmatrix} \\
&= (0 \ 0), \tag{C.25}
\end{aligned}$$

where the third equality uses the fact that the first column of  $Z$  is  $i_N$  and the last equality follows from the first-order condition in (A.9).

Therefore, the LM statistic for the hypothesis  $H_0^c$  is obtained by

$$\begin{aligned}
LM_c &= \hat{D}_c(\boldsymbol{\theta})' \hat{J}_c^{-1}(\boldsymbol{\theta}) \hat{D}_c(\boldsymbol{\theta}) \\
&= \hat{D}_c(\boldsymbol{\theta})' \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D^{-1} \end{bmatrix} \hat{D}_c(\boldsymbol{\theta}) \\
&+ \hat{D}_c(\boldsymbol{\theta})' \begin{bmatrix} I_2 \\ -D^{-1}C \end{bmatrix} (A - BD^{-1}C)^{-1} [I_2 \ -BD^{-1}] \hat{D}_c(\boldsymbol{\theta}) \\
&= D(\hat{\boldsymbol{\alpha}})' D^{-1} D(\hat{\boldsymbol{\alpha}}) \\
&= \left( \frac{h'(\hat{\alpha}_1)\hat{d}^2(1-\hat{\rho})^2}{2\hat{\lambda}^2} \right)^2 \left( \frac{1}{a(\boldsymbol{\alpha}, \boldsymbol{\alpha})} \right) f'Z(Z'Z)^{-1}Z'f
\end{aligned}$$

$$\begin{aligned}
&= \left( \frac{h'(\hat{\alpha}_1) \hat{d}^2 (1 - \hat{\rho})^2}{2 \hat{\lambda}^2} \right)^2 \left( \frac{2 \hat{\lambda}^4}{h'(\hat{\alpha}_1)^2 \hat{d}^4 (1 - \hat{\rho})^4} \right) f' Z (Z' Z)^{-1} Z' f \\
&= \frac{1}{2} f' Z (Z' Z)^{-1} Z' f \tag{C.26}
\end{aligned}$$

The LM statistic in (A.23) is the familiar term used in testing the heteroscedasticity in Breusch and Pagan (1979). Under the null hypothesis  $H_0^c$ , the LM statistic is asymptotically distributed as  $\chi_{p-1}^2$ .