Multicriticality, Scaling Operators and mKdV Flows for the Symmetric Unitary One Matrix Models

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Abstract

We present a review of the Symmetric Unitary One Matrix Models. In particular we compute the scaling operators in the double scaling limit and the corresponding mKdV flows. We briefly discuss the computation of the space of solutions to the string equation as a subspace of $Gr^{(0)} \times Gr^{(0)}$ which is invariant under the mKdV flows.

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1. Introduction

In this part of the proceedings we attempt to review some topics on the Symmetric Unitary One Matrix Models (UMM). These are statistical systems defined by partition functions of the form

$$Z_N^U = \int DU \exp\{-\frac{N}{\lambda} \operatorname{Tr} V(U+U^\dagger)\}\,, \tag{1}$$

where U is a $2N \times 2N$ or a $(2N+1)\times(2N+1)$ unitary matrix, DU is the Haar measure for the unitary group and the potential

$$V(U) = \sum_{k\geq 0} g_k\, U^k\,, \tag{2}$$

is a polynomial function in U. The interest in those models arose a long time ago when Gross and Witten [1] showed that the partition function of two dimensional $U(N)$ QCD on a lattice is given by $Z_{QCD} = (Z_U)^{\frac{V}{a^2}}$ and that the theory undergoes a third order phase transition in the large N limit (V is the volume of the two dimensional world and a is the lattice cutoff). The discovery of the double scaling limit [2–5] for the Hermitian Matrix Models (HMM) and its relation to two dimensional theories of gravity coupled to (possibly non-unitary) conformal matter raised the question of whether UMM describe a similar continuum limit for some statistical model coupled to two dimensional gravity that is relevant to string theory. The model was solved in the double scaling limit $N\to\infty$ and $\lambda\to\lambda_c$ with $t=(1-\frac{n}{N})N^{\frac{2k}{2k+1}}$ and $y=(1-\frac{\lambda}{\lambda_c})N^{\frac{2k}{2k+1}}$ held fixed in [6,7]. The scaling function v, with $v^2 = -\partial^2 \log Z$, satisfies a $2k^{\text{th}}$ order differential equation in the variable $x = t + y$, known as the string equation. It has solutions which in the weak coupling limit $x\to\infty$ are asymptotic to series that one would like to identify with the genus expansion of a string theory. The identifications of those solutions with conformal field theories coupled to two dimensional gravity or other interesting systems is still, however, an interesting open problem [8,9]. Quite recently a world sheet interpretation of the UMM as an open-closed string theory has been proposed in [10]. For another interesting suggestion see [11].

Several authors have pointed out in the past (see [8] and references therein) that one obtains the same continuum theory from the double scaling limit of the double-cut HMM as from the UMM. The reason is that the scaling behaviour of the density of eigenvalues near the multicritical points is identical for the two models. In [8] a series of multicritical points labeled by a positive integer k is found and the continuum limit of the scaling operators is computed in the spirit of [12]. The dependence of the scaling function v on the sources of the scaling operators, which are treated as perturbations, gives the NLS hierarchy. The multicritical points of the symmetric UMM correspond to even k and the corresponding flows are the mKdV hierarchy. In section 3 we prove this result directly from the UMM. The calculation has never been presented written before. We have obtained similar results for the odd order multicritical points but these will be presented elsewhere.

In section 4 we discuss aspects of the integrability of the UMM as related to the Sato Grassmannian [13]. This was the main part of the talk delivered at this meeting. Due to lack of space we summarize the results obtained in this work and refer the interested reader to [13] for the details (see also [14] for a review). In [13] we used the result of [15] that the string equation for the UMM can be written in the form $[\mathcal{P},\mathcal{Q}]=1$, where $\mathcal{P}$ and $\mathcal{Q}$ are 2×2 matrices of differential operators of specific order, in order to compute the points in the Universal Grassmannian that solve the string equation [16]. The operators $\mathcal{P}$ and $\mathcal{Q}$ correspond to the continuum limits of operators acting on the space of orthonormal functions used to solve the model. The solutions are found to correspond to a pair of points V_1 and V_2 in the (big cell of the) Sato Grassmannian satisfying certain invariance conditions. It is very important that the mKdV evolution of V_1 and V_2 gives new solutions to the string equation. The τ-functions that correspond to V_1 and V_2 are shown to satisfy the Virasoro constraints in this formalism [13], since the constraints are derived from the same invariance conditions that solutions to the string equation satisfy [17–20].

2. The Symmetric Unitary Matrix Model

The first step in solving the symmetric UMM given by (1) and (2) is to reduce the integral giving Z_N^U to an integral over the eigenvalues [1,21] $z_i=e^{\alpha_i}$ of U which lie on the unit circle in the complex z plane.

$$\begin{aligned} Z_N^U &= \int\{\prod_j \frac{dz_j}{2\pi i z_j}\}\,|\Delta(z)|^2\exp\{-\frac{N}{\lambda}\sum_i V(z_i+z_i^*)\} \\ &= \int\{\prod_j d\alpha_j\}\,|\Delta(\alpha)|^2\exp\{-\frac{N}{\lambda}\sum_i V(2\cos\alpha_i)\}\,, \end{aligned} \tag{3}$$

where $|\Delta(z)|^2=|\Delta(\alpha)|^2=\prod_{k<j}|z_k-z_j|^2=4^{2N}\prod_{k<j}\sin^2\left(\frac{\alpha_i-\alpha_j}{2}\right)$ is the Vandermonde determinant.

It is well known [1] that the critical behaviour of the model in the large N limit is governed by the stationary points of (3). The stationarity condition is given by

$$\frac{2N}{\lambda}V'(2\cos\alpha_i)\sin\alpha_i + \sum_{\substack{j=1\\ i\neq j}}^{2N}\cot\frac{\alpha_i-\alpha_j}{2} = 0\,. \tag{4}$$

The continuum version of (4) in the large N limit will be given by the replacements $\alpha_i = \alpha(\frac{i}{2N}) = \alpha(x)\, i = 1,\ldots,2N\,, x\in[0,1]$ and $\frac{1}{2N}\sum_{i\neq j} \to P\int_0^1 dx\,,\ \frac{1}{2N}\sum_i \to \int_0^1 dx$. We introduce the density of eigenvalues

$$\rho(\alpha) = \frac{dx}{d\alpha} \geq 0 \quad \text{such that} \quad \int_{\alpha_c}^{2\pi-\alpha_c} d\alpha\,\rho(\alpha) = 1. \tag{5}$$

Then condition (4) and the free energy are given by

$$\frac{1}{\lambda}V'(2\cos\alpha(x))\sin\alpha(x) = -P\int_{\alpha_c}^{2\pi-\alpha_c} d\beta\,\rho(\beta)\cot\frac{\alpha-\beta}{2} \tag{6}$$

and

$$-\mathcal{F} = -\frac{1}{\lambda}\int_{\alpha_c}^{2\pi-\alpha_c} d\alpha\,\rho(\alpha)\cos\alpha + P\int d\alpha d\beta\,\rho(\alpha)\rho(\beta)\ln|\sin\frac{\alpha-\beta}{2}| + \text{const.}\,. \tag{7}$$

Therefore the stationary solutions will be completely determined by the solutions of (6). A physical picture of the problem is obtained by realizing that (3) describes identical charged particles distributed over the unit circle subject to their mutual Coulomb repulsion and an external electric field given by $V(2\cos\alpha)$. Therefore in the limit $\lambda \to +\infty$, the particles will be distributed uniformly over the whole circle and as $\lambda \to 0^+$ they will be mostly concentrated around $\alpha = \pi$. As we will show shortly, as $\lambda \to 1^-$ the two ends of the support of the eigenvalues meet at $\alpha = 0$ and $\rho(\alpha)$ exhibits scaling behaviour at the end of its support. Then the third derivative of (7) has a discontinuity at $\lambda = 1$ obtaining a third order phase transition. Tuning the potential (2) accordingly, one can change the critical exponents of $\rho(\alpha)$ and $\mathcal{F}$ and reach a series of multicritical points labelled by an integer k.

In order to solve (6), we introduce the function [1]

$$F(z) = \int_{\alpha_c}^{2\pi-\alpha_c} d\beta\,\rho(\beta)\cot\frac{z-\beta}{2}\,. \tag{8}$$

The function $F(z)$ is periodic as $z \to z + 2n\pi$, real and analytic outside the real intervals $(2n\pi + \alpha_c, 2(n+1)\pi - \alpha_c)$ and, as a consequence of (5) and (6), when one approaches those intervals

$$F(\alpha \pm i\epsilon) = -\frac{1}{\lambda} V'(\cos\alpha)\sin\alpha \mp 2\pi i \rho(\alpha)\,. \tag{9}$$

Because of (5) and (6)

$$F(z) \to \mp i \quad \text{as} \quad z \to z_1 \pm i\infty\,. \tag{10}$$

Solutions to the above conditions are given by

$$F(z) = -\frac{1}{\lambda} V'(\cos z)\sin z \mp P(\sin^2\frac{z}{2})\sin\frac{z}{2}(\cos^2\frac{z}{2} - \cos^2\frac{\alpha_c}{2})^{\frac{1}{2}} \tag{11}$$

where $\mp$ refers to $\mathrm{Re}\, z > 0$ and $\mathrm{Re}\, z < 0$ respectively. $P(z)$ is a polynomial of degree one less than $V(z)$. The coefficients of $P(z)$ and $\cos\frac{\alpha_c}{2}$ as a function of the couplings is obtained from (10). Then (9) implies that

$$\rho(\alpha) = P(\sin^2\frac{\alpha}{2})\sin\frac{|\alpha|}{2}(\cos^2\frac{\alpha_c}{2} - \cos^2\frac{\alpha}{2})^{\frac{1}{2}}\,. \tag{12}$$

The k^{th} multicritical point is reached by tuning the couplings in the potential so that $P(z) \sim a^k z^{k-1}$ and $\cos\frac{\alpha_c}{2} \to 1$. In this case the critical density of eigenvalues is given by

$$\rho_k(\alpha) \propto \sin^{2k}\frac{z}{2}\,, \tag{13}$$

which for α close to its critical value $\alpha_c = 0$ gives

$$\rho_k(\alpha) \sim \alpha^{2k}\,. \tag{14}$$

Then we obtain a third order phase transition with $\mathcal{F} \sim (\lambda_c - \lambda)^{2+\frac{1}{k}}$. We always normalize the critical potential so that $\lambda_c = 1$. In this case the k^{th} multicritical potential is given by

$$V_k'(4Z^2 - 2) = c_k(1 - Z^2)^{k-1}(1 - \frac{1}{Z^2})_+^{\frac{1}{2}}\,, \tag{15}$$

where $Z = \cos\frac{z}{2}$ and we expand the square root around $z = \infty$ keeping only positive powers of Z. In order to solve the model in the double scaling limit we use the method of orthogonal polynomials. A convenient basis is given by [22]

$$c_n^{\pm}(z) = z^n \pm z^{-n} + \sum_{i=1}^{i_{max}} \alpha_{n,n-i}^{\pm}(\, z^{n-i} \pm\, z^{-n+i}) \tag{16}$$

where for $U(2N+1)$ n is a non-negative integer and $i_{max} = n$ and for U(2N) n is a positive half-integer and $i_{max} = n - \frac{1}{2}$. The polynomials $c_n^{\pm}(z)$ are orthogonal with respect to the inner product

$$
\begin{aligned}
\langle c_n^{\pm}, c_m^{\pm} \rangle &= \oint \frac{dz}{2\pi i z} \exp\{-\frac{N}{\lambda} V(z+z^*)\} \, c_n^{\pm}(z)^* c_m^{\pm}(z) \\
&= \mathrm{e}^{\phi_n^{\pm}} \, \delta_{n,m}^{\pm,\pm} \, .
\end{aligned}
\tag{17}
$$

Then the partition function of the model is given by the product of the norms of the orthogonal polynomials

$$
Z_N^U = \prod_n \mathrm{e}^{\phi_n^+} \mathrm{e}^{\phi_n^-} = \tau_N^{(+)} \tau_N^{(-)} \, . \tag{18}
$$

The orthogonal basis of polynomials chosen is especially useful for constructing the operator formalism of the theory. When acting on the basis of orthonormal functions $\pi_n^{\pm}(z) = \mathrm{e}^{-\phi_n^{\pm}/2} \mathrm{e}^{-\frac{N}{2\lambda} V(z_+)} c_n^{\pm}(z)$ the operators $z_{\pm} = z \pm \frac{1}{z}$ and $z\partial_z$ give finite term recursion relations

$$
\begin{aligned}
z_+ \, \pi_n^{\pm}(z) &= Q_{nm}^{(+)\pm\pm} \pi_m^{\pm}(z) = \sqrt{R_{n+1}^{\pm}} \pi_{n+1}^{\pm}(z) - r_n^{\pm} \pi_n^{\pm}(z) + \sqrt{R_n^{\pm}} \pi_{n-1}^{\pm}(z) \ , \\
z_- \, \pi_n^{\pm}(z) &= Q_{nm}^{(-)\pm\mp} \pi_m^{\mp}(z) = \sqrt{Q_{n+1}^{\mp}} \pi_{n+1}^{\mp}(z) - q_n^{\pm} \sqrt{\frac{Q_n^{\mp}}{R_n^{\pm}}} \pi_n^{\mp}(z) - \sqrt{Q_n^{\pm}} \pi_{n-1}^{\mp}(z) \, , \\
z\partial_z \pi_n^{\pm}(z) &= P_{nm}^{\pm\mp} \pi_m^{\mp}(z) = \\
&= -\frac{N}{2\lambda} \sum_{r=1}^{k} (v_z^{\pm})_{n,n+r} \pi_{n+r}^{\mp}(z) + \left\{ n \sqrt{\frac{Q_n^{\mp}}{R_n^{\pm}}} - \frac{N}{2\lambda} (v_z^{\pm})_{n,n} \right\} \pi_n^{\mp}(z) \\
&\quad + \frac{N}{2\lambda} \sum_{r=1}^{k} (v_z^{\pm})_{n,n-r} \pi_{n-r}^{\mp}(z) \ \ ,
\end{aligned}
\tag{19}
$$

where $R_n^{\pm} = \mathrm{e}^{\phi_n^{\pm} - \phi_{n-1}^{\pm}}$, $Q_n^{\pm} = \mathrm{e}^{\phi_n^{\pm} - \phi_{n-1}^{\mp}}$, $r_n^{\pm} = \frac{\partial \phi_n^{\pm}}{\partial g_1}$, $q_n^{\pm} = \frac{(Q_{n+1}^{\pm} - Q_n^{\pm}) + (R_{n+1}^{\mp} - R_n^{\pm})}{r_n^{\pm} - r_n^{\mp}}$, and $(v_z^{\pm})_{n,n-r} = \oint \frac{dz}{2\pi i z} \pi_{n-r}^{\mp}(z)^* \left\{ z\partial_z V(z_+) \right\} \pi_n^{\pm}(z)$. Then the discrete string equation is given by the relation $[z\partial_z, z_{\pm}] = z_{\mp}$.

3. The Double Scaling Limit

In the previous section we discussed the large N limit of UMM. It is possible to get non trivial contributions to the scaling part of the free energy by carefully tuning the limits

$N \to \infty$ and $\lambda \to \lambda_c$, with $t = (1-\frac{n}{N})N^{\frac{2k}{2k+1}}$, $y = (1-\frac{\lambda}{\lambda_c})N^{\frac{2k}{2k+1}}$ held fixed. It was shown in [15] that the operators $Q^{(\pm)}_{nm}$ and P_{nm} have a smooth continuum limit given by

$$\begin{aligned} &Q^{(+)}_{nm} \to 2 + N^{-\frac{2}{2k+1}}\,\mathcal{Q}_+\,, \quad Q^{(-)}_{nm} \to -2N^{-\frac{1}{2k+1}}\,\mathcal{Q}_-\,, \\ &P_{nm} \to N^{\frac{1}{2k+1}}\;\mathcal{P}_k\;, \end{aligned} \tag{20}$$

where $\mathcal{Q}_\pm$ are given by

$$\begin{aligned} \mathcal{Q}_- &= \begin{pmatrix} 0 & \partial + v \\ \partial - v & 0 \end{pmatrix}, \\ \mathcal{Q}_+ &= \begin{pmatrix} (\partial+v)(\partial-v) & 0 \\ 0 & (\partial-v)(\partial+v) \end{pmatrix} \\ &= \mathcal{Q}_-^2\,, \end{aligned} \tag{21}$$

and $\mathcal{P}_k$ by

$$\mathcal{P}_k = \begin{pmatrix} 0 & \mathbf{P}_k \\ \mathbf{P}_k^\dagger & 0 \end{pmatrix}. \tag{22}$$

Here $\partial \equiv \frac{\partial}{\partial x}$ and $x = t + y$. The scaling function v^2 is proportional to the specific heat $-\partial^2 \ln Z$ of the model. The operators $\mathcal{P}_k$ are differential operators of order $2k$.

The multicritical potentials V_m perturb the multicritical densities such that $\rho_k \to \rho_k + \tilde\rho_m$, where $\tilde\rho_m$ has the same scaling behaviour (14) and satisfies the normalization condition $\int_{\alpha_c}^{2\pi-\alpha_c} \tilde\rho_m(\alpha)\, d\alpha = 0$. Solutions for $\tilde\rho_m$ are given by $\tilde\rho_m(\alpha) \propto \frac{d}{d\alpha}\sin^{2m}\alpha(1-\cos^2\frac{\alpha}{2})_+^{\frac{1}{2}}$ and correspond to multicritical potentials

$$\tilde V_m \propto (1-Z^2)^k(1-\frac{1}{Z})_+^{\frac{1}{2}} \tag{23}$$

where $Z = \cos\frac{z}{2}$. The scaling operators of the model are defined by $<\sigma_{2k+1}>=<\mathrm{tr}\tilde V_k(U+U^\dagger)>$. Consider the expressions for the connected correlation functions [5] $<\mathrm{trF}(U)>=\mathrm{Tr}(\hat{\mathrm{F}}(U)\Pi_N)$ and $<\mathrm{trF}(U)\mathrm{trG}(U)>=\mathrm{Tr}(\hat{\mathrm{F}}(U)\Pi_N\hat{\mathrm{G}}(U)(1-\Pi_N))$, where tr is the matrix trace and Tr is the trace over the states $|n\pm>=\pi_n^\pm(z)$. Π_N is the projection operator $\Pi_N = \sum_{n=0,\pm}^N |n\pm><\pm n|$ and $\hat{\mathrm{F}}(U)$ and $\hat{\mathrm{G}}(U)$ are operators acting on the states $|n\pm>$. Then we obtain

$$<\sigma_{2k+1}>=\oint \frac{dz_+}{2\pi i z_+}\tilde V_k(z_+)\mathrm{Tr}\{\frac{1}{z_+-Q^{(+)}}\Pi_N\}\,. \tag{24}$$

Similarly the two point function $<\sigma_{2k+1}\sigma_1>=\partial<\sigma_{2k+1}>$ is given by

$$\begin{aligned} <\sigma_{2k+1}\sigma_1> &= -\oint \frac{dz_+}{2\pi i z_+}\tilde V_k(z_+)\mathrm{Tr}\{\Pi_N\frac{1}{z_+-Q^{(+)}}(1-\Pi_N)(z_+-Q^{(+)})\} \\ &\propto \oint \frac{dz_+}{2\pi i z_+}(1-Z^2)^k(1-\frac{1}{Z})^{\frac{1}{2}}\{\sqrt{R^+_{N+1}}(\frac{1}{z_+-Q^{(+)}})^{++}_{N\,N+1} \\ &\quad + \sqrt{R^-_{N+1}}(\frac{1}{z_+-Q^{(+)}})^{--}_{N\,N+1}\}\,. \end{aligned} \tag{25}$$

In the double scaling limit $z_+ = 2\cos\alpha \to 2-\alpha^2$ where $\alpha = N^{-\frac{1}{2k+1}}\nu$, $Q^{(+)} \to 2 + N^{-\frac{2}{2k+1}}\mathcal{Q}_+$, $\sqrt{R^{\pm}_{N+1}} \to 1+\frac{1}{4}N^{-\frac{2}{2k+1}}(\mp v' - v^2)$ and $|N\pm> \to N^{\frac{1}{2k+1}}|x\pm>$ and (25) becomes

$$
\begin{aligned}
<\sigma_{2k+1}\sigma_1> &\propto \oint \frac{d\nu}{2\pi i\nu}\nu^{2k+3}\Big(<+x|\frac{1}{-\nu^2-\partial^2+u_1}|x+> + <-x|\frac{1}{-\nu^2-\partial^2+u_2}|x->\Big)\\
&\propto \oint \frac{d\nu}{2\pi i\nu}\nu^{2k+3}\Big(\sum_l \frac{\mathrm{R}_l[u_1]}{\nu^{2l+1}} + \sum_l \frac{\mathrm{R}_l[u_2]}{\nu^{2l+1}}\Big)\\
&\propto \mathrm{R}_k[u_1] + \mathrm{R}_k[u_2]\,.
\end{aligned}
\tag{26}
$$

$\mathrm{R}_k[u]$ are the Gel'fand-Dikii potentials defined through the recursion relation $\partial\,\mathrm{R}_{k+1}[u] = \Big(\frac{1}{4}\partial^3 - \frac{1}{2}(\partial u + u\partial)\Big)\mathrm{R}_k[u]\,,\quad \mathrm{R}_0[u] = \frac{1}{2}$, $u_1 = v^2+v'$ and $u_2 = v^2 - v'$. Therefore $<\sigma_{2k+1}\sigma_1\sigma_1> \propto \partial\mathrm{R}_k[u_1] + \partial\mathrm{R}_k[u_2] = -v\hat{\mathcal{D}}\mathrm{R}_k[u_2]$, where $\hat{\mathcal{D}} = \partial + 2v$. Using $<\sigma_{2k+1}\sigma_1\sigma_1> = \frac{\partial}{\partial t_{2k+1}} <\sigma_1\sigma_1> = 2v\frac{\partial v}{\partial t_{2k+1}}$ we obtain

$$
\frac{\partial v}{\partial t_{2k+1}} = -\partial\hat{\mathcal{D}}\mathrm{R}_k[u]\,. \tag{27}
$$

The string equation in the presence of σ_{2k+1} is given by [13]

$$
[\mathcal{P}, \mathcal{Q}_-] = 1\,, \tag{28}
$$

where $\mathbf{P} = -\sum_{l\geq 1}(2l+1)t_{2l+1}\tilde{\mathbf{P}}_l - x$ with $\tilde{\mathbf{P}}_l = \mathbf{P}_l + x$.

4. Integrability and the Sato Grassmannian

As we already mentioned in the introduction, the analysis of the solutions of the string equation in the Sato Grassmannian Gr depends crucially on the association of the mKdV τ-functions τ_1 and τ_2 to points V_1 and V_2 in the big cell of the Sato Grassmannian $Gr^{(0)}$. The τ-functions of the mKdV hierarchy are given by $u_i = -\partial^2\log\tau_i$, $i = 1, 2$.

The Sato Grassmannian is an infinite generalization of the finite dimensional Grassmannians. The finite dimensional Grassmannian $Gr(k, N)$ consists of all k-dimensional linear subspaces of C^N. A point $V \in Gr(k, N)$ is described by a basis $\{v_i\}$ with $i = 1, \ldots, k$ and a basis of the orthogonal complement of V $\{w_i\}$ with $i = k+1, \ldots, N$. In the infinite dimensional case consider the space of formal Laurent series

$$
H = \{\sum_n a_n z^n\,,\quad a_n = 0 \quad \text{for} \quad n \gg 0\}
$$

and its decomposition

$$H = H_+ \oplus H_- \,,$$

where $H_+ = \{ \sum_{n \geq 0} a_n z^n \,, \quad a_n = 0 \quad \text{for} \quad n \gg 0 \}$. Then the big cell of the Sato Grassmannian $Gr^{(0)}$ consists of all subspaces $V \subset H$ comparable to H_+, in the sense that the natural projection $\pi_+ : V \to H_+$ is an isomorphism. Then V admits a basis of the form $\{\phi_i(z)\}_{i \geq 0}$ where $\phi_i(z) = z^i +$ lower order terms.

The spaces V_1 and V_2 are associated to τ-functions τ_1 and τ_2 via the Plücker embedding and the fermion-boson equivalence in two dimensions. They correspond to solutions of the mKdV hierarchy if and only if

$$\frac{\partial}{\partial t_{2k+1}} V_i(t) = z^{2k+1} V_i(t) \quad \text{and} \quad z^{2k} V_i(t) \subset V_i(t) \,. \tag{29}$$

Computing the space of solutions to the string equation is equivalent to determining operators $\mathcal{Q}_-$ and $\mathcal{P}$ such that (28) is true and $\mathcal{Q}_-$ has the form (21). The problem is a generalization of the Burchnall-Chaundy-Krichever (BCK) theory for non-commuting operators. One can compute this space explicitly [13]. The set of operators $\mathcal{Q}_-$ and $\mathcal{P}$ correspond to a space of pairs of points V_1 and V_2 in $Gr^{(0)}$, invariant under the mKdV flow (29), where V_1 and V_2 must satisfy the conditions

$$\begin{aligned} z\, V_1 \subset V_2 \quad z\, V_2 \subset V_1 \\ A_k\, V_1 \subset V_2 \quad A_k\, V_2 \subset V_1 \end{aligned} \tag{30}$$

for some $A_k = \frac{d}{dz} + \sum_{i=0}^{k} \alpha_i z^{2i}$.

The Virasoro constraints are a simple consequence, and in fact equivalent to, (30). The algebra of a set of operators acting on the τ-functions is simply the central extension of the algebra of the corresponding operators acting on the spaces V_1 and V_2. The operators $l_n = z^{2n+1} A$ correspond to operators L_n acting on the τ-functions, which are the Virasoro generators found long ago in [23,24]. Since operators leaving the spaces V_1 and V_2 invariant must annihilate the corresponding τ-function then as a simple consequence of $z^{2n+1} A\, V_i \subset V_i$ it is easily concluded that the L_n's annihilate τ_1 and τ_2

We conclude this presentation by mentioning that in [13] we solved for the space of solutions to (28). We found that the space of solutions to the string equation (28) is the two fold covering of the space of matrices $\left(P_{ij}(z) \right)$ with polynomial entries in z such that $P_{01}(z)$ and $P_{10}(z)$ are even polynomials having equal degree and leading terms and

$P_{00}(z)$ and $P_{11}(z)$ are odd polynomials satisfying the conditions $P_{00}(z) + P_{11}(z) = 0$ and $\deg P_{00}(z) < \deg P_{01}(z)$.

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