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HOPF DIFFERENTIALS AND SMOOTHING SOBOLEV HOMEOMORPHISMS

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Abstract. We prove that planar homeomorphisms can be approximated by diffeomorphisms in the Sobolev space $\mathscr{W}^{1,2}$ and in the Royden algebra. As an application, we show that every discrete and open planar mapping with a holomorphic Hopf differential is harmonic.

1. Introduction

It is a fundamental property of Sobolev spaces $\mathscr{W}^{1,p}$, $1 \leqslant p < \infty$, that any element can be approximated strongly (i.e., in the norm) by $\mathscr{C}^\infty$ smooth functions, or by piecewise affine ones. In the context of vector-valued Sobolev functions, that is, mappings in $\mathscr{W}^{1,p}(\Omega, \mathbb{R}^n)$, invertibility comes into play. Indeed, the studies of invertible Sobolev mappings are of great importance in nonlinear elasticity [2, 14, 26, 35]. The following natural question was put forward by John M. Ball.

Question 1.1. [4] If $u \in \mathscr{W}^{1,p}(\Omega, \mathbb{R}^n)$ is invertible, can u be approximated in $\mathscr{W}^{1,p}$ by piecewise affine invertible mappings?

J. Ball attributes this question to L.C. Evans, who was led to it through his investigation of the partial regularity of minimizers [13] of neohookean energy functionals [3, 5, 7, 33]. We provide an affirmative solution of the Ball-Evans problem in the case $p = n = 2$. The most general formulation of our result administers Royden algebras $\mathscr{A}(\Omega)$ and $\mathscr{A}_\circ(\Omega)$, see Section 2. We write

$$\mathcal{E}[h] = \mathcal{E}_\Omega[h] := \|Dh\|^2_{\mathscr{L}^2(\Omega)} = \int_\Omega |Dh(z)|^2 \, \mathrm{d}z$$

where $|Dh|$ is the Hilbert-Schmidt norm of the differential.

Theorem 1.2 (Approximation by diffeomorphisms)**.** *Let $h \colon \Omega \xrightarrow{\text{onto}} \Omega^*$ be a homeomorphism of Sobolev class $\mathscr{W}^{1,2}_{\text{loc}}(\Omega, \Omega^*)$. Then for every $\epsilon > 0$ there exist a diffeomorphism $H \colon \Omega \xrightarrow{\text{onto}} \Omega^*$ such that*

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(i) $H - h \in \mathscr{A}_\circ(\Omega)$
(ii) $\|H - h\|_{\mathscr{A}(\Omega)} \leqslant \epsilon$
(iii) $\mathcal{E}[H] \leqslant \mathcal{E}[h]$.

Part (iii) is nontirivial only in the finite energy case, $\mathcal{E}_\Omega[h] < \infty$. Let us note that the existence of smooth approximation implies the existence of piecewise-affine approximation, since a diffeomorphism can be triangulated. (In the converse direction, a piecewise-affine mapping can be smoothed in dimensions less than four [25], but not in general.) Partial results toward the Ball-Evans problem were obtained in [24] (for planar bi-Sobolev mappings that are smooth outside of a finite set) and in [6] (for planar bi-Hölder mappings, with approximation in the Hölder norm). The articles [4, 32] illustrate the difficulty of preserving invertibility in the process of smoothing a Sobolev homeomorphism.

We also give an application of Theorem 1.2 to a problem that originated in a series of papers by Eells, Lemaire and Sealey [11, 12, 31]. It concerns the nonlinear differential equation

$$\frac{\partial}{\partial \bar{z}}\left(h_z \overline{h_{\bar{z}}}\right) = 0 \tag{1.1}$$

for mappings defined in a domain in the complex plane $\mathbb{C}$. Naturally, the Sobolev space $\mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ should be considered as the domain of definition of equation (1.1). This places $h_z \overline{h_{\bar{z}}}$ in $\mathscr{L}^1_{\mathrm{loc}}(\Omega)$, so the complex Cauchy-Riemann derivative $\frac{\partial}{\partial \bar{z}}$ applies in the sense of distribution. By Weyl's lemma $h_z \overline{h_{\bar{z}}}$ is a holomorphic function.

The expression $Q_h := h_z \overline{h_{\bar{z}}}\, \mathrm{d}z \otimes \mathrm{d}z$ is known as the Hopf differential of h (named after H. Hopf, who employed a similar device, e.g., in [19, Chapter VI]). It is clear that Q_h is a holomorphic quadratic differential whenever h is harmonic, which is a general fact about energy-stationary mappings between Riemannian manifolds [10, (10.5)], [21] and [34]. Eells and Lemaire inquired about the possibility of a converse result, e.g., for mappings with finite energy and almost-everywhere positive Jacobian [11, (2.6)]. In this setting a counterexample was provided by Jost [20], who also proved the existence of $\mathscr{W}^{1,2}$-solutions of (1.1) in every homotopy class of mappings between compact Riemann surfaces. A more restricted form of the Eells-Lemaire problem, [12, (5.11)] and [31], imposed the additional assumption that h is a quasiconformal homeomorphism, and was settled by Hélein [17] in the affirmative. Here we dispose with the quasiconformality condition and treat general planar homeomorphisms of finite energy. Since the inverse of such a homeomorphism need not be in any Sobolev class [18], some difficulties are to be expected. They shall be overcome with the aid of our approximation theorem 1.2.

Theorem 1.3. *Every continuous, discrete and open mapping h of Sobolev class $\mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ that satisfies equation* (1.1) *is harmonic.*

The failure of Theorem 1.3 for uniform limits of homeomorphisms should be mentioned. This is illustrated by Example 4.1.

2. Background

Let Ω be a bounded domain in $\mathbb{R}^2 \simeq \mathbb{C}$, nonempty open connected set. We consider a class $\mathscr{A}(\Omega)$ of uniformly continuous functions $h\colon \Omega \to \mathbb{C}$ having finite Dirichlet energy, and furnish it with the norm

$$\|h\|_{\mathscr{A}(\Omega)} = \|h\|_{\mathscr{C}(\Omega)} + \|Dh\|_{\mathscr{L}^2(\Omega)} < \infty$$

$\mathscr{A}(\Omega)$ is a commutative Banach algebra with the usual multiplication of functions in which $\|h_1 h_2\|_{\mathscr{A}(\Omega)} \leqslant \|h_1\|_{\mathscr{A}(\Omega)} \|h_2\|_{\mathscr{A}(\Omega)}$. The closure of $\mathscr{C}_\circ^\infty(\Omega)$ in $\mathscr{A}(\Omega)$ will be denoted by $\mathscr{A}_\circ(\Omega)$. Suppose, to look at more specific situation, that $\Omega = \mathbb{U}$ is a Jordan domain; that is, a simply connected open set whose boundary $\Gamma = \partial \mathbb{U}$ is a closed Jordan curve. By goodness of the Carathéodory extension theorem [27, p. 18], there is a homeomorphism $\varphi\colon \overline{\mathbb{D}} \xrightarrow{\text{onto}} \overline{\mathbb{U}}$ of the closed unit disk $\overline{\mathbb{D}} = \{\xi\colon |\xi| \leqslant 1\}$ that is conformal in $\mathbb{D}$. After the change of variable, $z = \varphi(\xi)$, we obtain a function $H(\xi) = h\big(\varphi(\xi)\big)$ in $\mathscr{A}(\mathbb{D})$. The operation

$$\mathbf{T}_\varphi\colon \mathscr{A}(\mathbb{U}) \to \mathscr{A}(\mathbb{D})$$

so defined is an isometry; $\|\mathbf{T}_\varphi h\|_{\mathscr{A}(\mathbb{D})} = \|h\|_{\mathscr{A}(\mathbb{U})}$. Furthermore,

$$\mathbf{T}_\varphi\colon \mathscr{A}_\circ(\mathbb{U}) \to \mathscr{A}_\circ(\mathbb{D})$$

Proposition 2.1 (A generalization of Poisson's formula)**.** *Let $\mathbb{U}$ be a Jordan domain. There is (unique) bounded linear operator*

$$\mathbf{P}_\mathbb{U}\colon \mathscr{A}(\mathbb{U}) \to \mathscr{A}(\mathbb{U})$$

such that

$$\begin{cases} \mathbf{P}_\mathbb{U} - \mathbf{I}d\colon \mathscr{A}(\mathbb{U}) \to \mathscr{A}_\circ(\mathbb{U}) \\ \Delta \circ \mathbf{P}_\mathbb{U} = 0 \end{cases}$$

We name $\mathbf{P}_\mathbb{U}$ the *Poisson operator*. The energy of $\mathbf{P}_\mathbb{U} h$ does not exceed that of h. This fact is known as *Dirichlet's principle*

$$\int_\mathbb{U} |D\mathbf{P}_\mathbb{U} h|^2 \leqslant \int_\mathbb{U} |Dh|^2$$

The proof of this proposition reduces to the case when $\mathbb{U} = \mathbb{D}$, by conformal change of variables. A routine verification of this case is left to the reader. We only indicate that the less familiar property $\mathbf{P}_\mathbb{D} h - h \in \mathscr{A}_\circ(\mathbb{D})$, for $h \in \mathscr{A}(\mathbb{D})$, needs to be justified.

Corollary 2.2 (Harmonic replacement)**.** *Let Ω be a domain in $\mathbb{C}$ and $\mathbb{U} \subset \overline{\mathbb{U}} \subset \Omega$ a Jordan domain. There exists (unique) bounded linear operator*

$$\mathbf{R}_\mathbb{U}\colon \mathscr{A}(\Omega) \to \mathscr{A}(\Omega)$$

such that, for every $h \in \mathscr{A}(\Omega)$

$$\begin{cases} \mathbf{R}_{\mathbb{U}} h = h & \textit{on} \quad \Omega \setminus \mathbb{U} \\ \Delta \mathbf{R}_{\mathbb{U}} h = 0 & \textit{in} \quad \mathbb{U} \end{cases}$$

The Laplace equation yields $\mathcal{E}_\Omega[\mathbf{R}_{\mathbb{U}} h] \leqslant \mathcal{E}_\Omega[h]$. Equality occurs if and only if h is harmonic in $\mathbb{U}$.

A short proof of this corollary runs somewhat as follows. The unique harmonic extension of $h \colon \partial\mathbb{U} \to \mathbb{C}$ inside $\mathbb{U}$ given by $\mathbf{P}_{\mathbb{U}} h$ has the property that $\mathbf{P}_{\mathbb{U}} h - h \in \mathscr{A}_\circ(\mathbb{U})$. Therefore, the zero extension of $\mathbf{P}_{\mathbb{U}} h - h$ outside $\mathbb{U}$, denoted by $[\mathbf{P}_{\mathbb{U}} h - h]_\circ$, belongs to $\mathscr{A}(\Omega)$. We define

$$\mathbf{R}_{\mathbb{U}} h := [\mathbf{P}_{\mathbb{U}} h - h]_\circ + h \in \mathscr{A}(\Omega)$$

The desired properties of the operator $\mathbf{R}_{\mathbb{U}}$ so defined are automatically fulfilled in view of Proposition 2.1.

Proposition 2.3. *Let Ω be a domain in $\mathbb{C}$ and $\mathbb{U} \subset \overline{\mathbb{U}} \subset \Omega$ a Jordan domain. Suppose that $h \in \mathscr{A}(\Omega)$ is a homeomorphism of Ω onto $h(\Omega)$ and $h(\mathbb{U})$ is convex. Then $\mathbf{R}_{\mathbb{U}} h$ is homeomorphism in Ω and is a harmonic diffeomorphism in $\mathbb{U}$.*

The injectivity of $\mathbf{R}_{\mathbb{U}} h$ is the content of the Radó-Kneser-Choquet Theorem [9, p. 29]. Furthermore, planar harmonic homeomorphisms are $\mathscr{C}^\infty$-smooth diffeomorphisms according to Lewy's theorem [9, p. 20].

3. Smoothing Sobolev homeomorphisms, Theorem 1.2

Proof of Theorem 1.2. We may and do assume that h is not harmonic, since otherwise $H = h$ satisfies the desired properties by Lewy's theorem (mentioned above). Let $z_\circ \in \Omega$ be a point such that h fails to be harmonic in any neighborhood of $z_\circ$. By choosing the origin of the coordinate system we ensure that $h(z_\circ)$ does not lie on the boundary of any dyadic squares associated with the coordinate system.

Let us choose and fix any $\epsilon > 0$. The construction of H proceeds in 5 steps. We construct homeomorphisms $h_k \colon \Omega \xrightarrow{\text{onto}} \Omega^*$, $k = 0, \dots, 5$ such that $h_0 = h$, $h_k \in h_{k-1} + \mathscr{A}_\circ(\Omega)$, h_5 is a diffeomorphism, and $\|h_k - h_{k-1}\|$ is bounded by a multiple of ϵ for each k. In each step we modify the previous construction to gain better regularity. In steps 1, 2 and 4 we use harmonic replacement according to Proposition 2.3. In steps 3 and 5 we smoothen the mapping near the boundaries of the domains in which harmonic replacement was performed. The result of each step is denoted by $h_1, \dots, h_5$. The finite energy case $h \in \mathscr{A}(\Omega)$ requires a few additional details, which are provided at the end of each step.

We begin with a decomposition of the target domain

$$\Omega^* = \bigcup_{\nu=1}^{\infty} \overline{\mathbb{Q}_\nu} \tag{3.1}$$

into closed nonoverlapping dyadic squares $\overline{\mathbb{Q}_\nu} \subset \Omega^*$. This decomposition is made by selecting the maximal dyadic squares that lie in Ω^*. Thus the cover of Ω^* by such squares is locally finite. The preimage of $\mathbb{Q}_\nu$ under h, denoted by $\mathbb{U}_\nu$, is a Jordan domain in Ω. Hereafter $\mathbb{U}_\nu$ will be referred to as the curved-square. In fact to every partion of Ω^* into closed squares there will correspond a partition of Ω into closed curved-squares via the mapping $h\colon \Omega \xrightarrow{\text{onto}} \Omega^*$, for example:

$$\Omega = \bigcup_{\nu=1}^{\infty} \overline{\mathbb{U}_\nu}$$

Step 1. For each $\mathbb{U}_\nu$ we replace $h\colon \overline{\mathbb{U}_\nu} \xrightarrow{\text{onto}} \overline{\mathbb{Q}_\nu}$ with a piecewise harmonic homeomorphism $h_1\colon \overline{\mathbb{U}_\nu} \xrightarrow{\text{onto}} \overline{\mathbb{Q}_\nu}$ that coincides with h on $\partial\overline{\mathbb{U}_\nu}$. To this effect we partition the square $\overline{\mathbb{Q}_\nu}$,

$$\overline{\mathbb{Q}_\nu} = \overline{\mathbb{Q}_\nu^1} \cup \overline{\mathbb{Q}_\nu^2} \cup \cdots \cup \overline{\mathbb{Q}_\nu^n}, \qquad (n = n_\nu = 4^{k_\nu}) \tag{3.2}$$

into congruent dyadic squares $\overline{\mathbb{Q}_\nu^i}$, $i = 1, \dots, n$. The number n, depending on ν, will be determined later. For the moment fix ν and look at the homeomorphisms

$$h\colon \overline{\mathbb{U}_\nu^i} = h^{-1}(\overline{\mathbb{Q}_\nu^i}) \xrightarrow{\text{onto}} \overline{\mathbb{Q}_\nu^i}$$

These mappings belong to the Royden algebra $\mathscr{A}(\overline{\mathbb{U}_\nu^i})$. With the aid of Propositions 2.1 and 2.3 we replace each $h\colon \overline{\mathbb{U}_\nu^i} \to \overline{\mathbb{Q}_\nu^i}$ with a harmonic homeomorphism $h_\nu^i\colon \overline{\mathbb{U}_\nu^i} \xrightarrow{\text{onto}} \overline{\mathbb{Q}_\nu^i}$ that coincides with h on $\partial\overline{\mathbb{U}_\nu^i}$, $i = 1, 2, \dots, n$. Such mappings are $\mathscr{C}^\infty$-smooth diffeomorphisms $h_\nu^i\colon \mathbb{U}_\nu^i \xrightarrow{\text{onto}} \mathbb{Q}_\nu^i$. Moreover, $h_\nu^i - h \in \mathscr{A}_\circ(\mathbb{U}_\nu^i)$ and

$$\begin{cases} \mathcal{E}_{\mathbb{U}_\nu^i}[h_\nu^i] \leqslant \mathcal{E}_{\mathbb{U}_\nu^i}[h] & \text{for } 1, 2, \dots, n \\ \mathcal{E}_{\partial\mathbb{U}_\nu^i}[h_\nu^i] = \mathcal{E}_{\partial\mathbb{U}_\nu^i}[h], & \text{because } h_\nu^i = h \text{ on } \partial\mathbb{U}_\nu^i \end{cases} \tag{3.3}$$

We obtain a piecewise harmonic homeomorphism by gluing h_ν^i together along the common boundaries of $\mathbb{U}_\nu^i$. Denote it by

$$h_\nu^n\colon \overline{\mathbb{U}_\nu} \xrightarrow{\text{onto}} \overline{\mathbb{Q}_\nu}$$

$$h_\nu^n \in h + \mathscr{A}_\circ(\mathbb{U}_\nu)$$

Precisely we define

$$h_\nu^n = h + \sum_{i=1}^{n} [h_\nu^i - h]_\circ$$

Here and in the sequel the notation $[\varphi]_\circ$ for $\varphi \in \mathscr{A}_\circ(\mathbb{U})$ stands for zero extension of φ to the entire domain Ω. Obviously $[\varphi]_\circ \in \mathscr{A}_\circ(\Omega)$. The above construction depends on the number n. For ν fixed we actually have a sequence $\{h_\nu^n\}_{n=1,2,\dots}$ that is bounded in $\mathscr{A}(\mathbb{U}_\nu)$. However, we have uniform bounds independent of n,

$$\|h_\nu^n\|_{\mathscr{C}(\overline{\mathbb{U}_\nu})} \leqslant \operatorname{diam} \mathbb{Q}_\nu$$

and

$$\mathcal{E}_{\mathbb{U}_\nu}[h^n_\nu] \leqslant \mathcal{E}_{\mathbb{U}_\nu}[h]$$

The key observation is that

$$\begin{cases} h^n_\nu - h \in \mathscr{A}_\circ(\mathbb{U}_\nu) \\ \lim\limits_{n\to\infty} \|h^n_\nu - h\|_{\mathscr{A}(\mathbb{U}_\nu)} = 0 \end{cases} \tag{3.4}$$

Indeed, for $z \in \overline{\mathbb{U}^i_\nu}$ we have

$$|h^n_\nu(z) - h(z)| \leqslant \operatorname{diam} \mathbb{Q}^i_\nu = \frac{1}{\sqrt{n}} \operatorname{diam} \mathbb{Q}_\nu$$

Thus $h^n_\nu \rightrightarrows h$ uniformly on $\overline{\mathbb{U}_\nu}$ as $n \to \infty$. On the other hand the differential matrices Dh^n_ν are bounded in $\mathscr{L}^2(\mathbb{U}_\nu, \mathbb{R}^{2\times 2})$. Their weak limit exits and is exactly equal to Dh, because the mappings converge uniformly to h. Therefore,

$$\begin{aligned} \int_{\mathbb{U}_\nu} |Dh^n_\nu - Dh|^2 &= \int_{\mathbb{U}_\nu} \Big(|Dh^n_\nu|^2 + |Dh|^2 - 2\langle Dh^n_\nu, Dh\rangle \Big) \\ &\leqslant 2 \int_{\mathbb{U}_\nu} \Big(|Dh|^2 - \langle Dh^n_\nu, Dh\rangle \Big) \\ &= 2 \int_{\mathbb{U}_\nu} \langle Dh, Dh - Dh^n_\nu\rangle \longrightarrow 0 \end{aligned}$$

We can now determine the number $n = n_\nu$ of congruent dyadic squares in $\mathbb{Q}_\nu$, simply requiring that

$$\begin{cases} \operatorname{diam} \mathbb{Q}^i_\nu \leqslant \epsilon & \text{for every } i = 1, 2, \dots, n_\nu \\ \|Dh^n_\nu - Dh\|_{\mathscr{L}^2(\overline{\mathbb{U}_\nu})} \leqslant \epsilon \cdot 2^{-\nu} \end{cases}$$

Fix such $n = n_\nu$ and abbreviate the notation for $h^{n_\nu}_\nu$ to h^ν. We obtain a homeomorphism

$$h_1 := h + \sum_{\nu=1}^{\infty} [h^\nu - h]_\circ \in h + \mathscr{A}_\circ(\Omega)$$

where we recall that $[h^\nu - h]_\circ$ stands for the zero extension of $h^\nu - h$ to the entire domain Ω. Clearly, h_1 is harmonic in each $\mathbb{U}^i_\nu$, $\nu = 1, 2, \dots$, $i = 1, 2, \dots, n_\nu$ and we have

$$\|h_1 - h\|_{\mathscr{C}(\Omega)} \leqslant \sup\{\operatorname{diam} \mathbb{Q}^i_\nu \colon \nu = 1, 2, \dots, \ \ i = 1, \dots, n_\nu\} < \epsilon$$

$$\|h_1 - h\|_{\mathscr{A}(\Omega)} \leqslant \epsilon + \sum_{\nu=1}^{\infty} \|Dh^\nu - Dh\|_{\mathscr{L}^2(\mathbb{U}_\nu)} \leqslant \epsilon + \sum_{\nu=1}^{\infty} \epsilon \cdot 2^{-\nu} = 2\epsilon \tag{3.5}$$

For further considerations it will be convenient to number the squares $\mathbb{Q}^i_\nu$ and their preimages $\mathbb{U}^i_\nu$ using only one index. These sets will be respectively denoted by $\mathbb{Q}^\alpha$ and $\mathbb{U}^\alpha$, $\alpha = 1, 2, \dots$. For the record,

$$\operatorname{diam} \mathbb{Q}^\alpha \leqslant \epsilon, \qquad \alpha = 1, 2, \dots \tag{3.6}$$

Finite energy case. Summing up the energy inequalities for the mappings $h_\nu^i \colon \mathbb{U}_\nu^i \to \mathbb{Q}_\nu^i$ we see that the total energy of h_1 does not exceed the energy of h. Even more, since h was assumed to be not harmonic, there is at least one region $\mathbb{U}_\nu^i$ for which $h \colon \mathbb{U}_\nu^i \to \mathbb{Q}_\nu^i$ was not harmonic. Consequently, its harmonic replacement results in strictly smaller energy. Hence

$$\mathcal{E}_\Omega[h_1] < \mathcal{E}_\Omega[h], \qquad \text{so let } \delta = \|Dh\|_{\mathscr{L}^2(\Omega)} - \|Dh_1\|_{\mathscr{L}^2(\Omega)} > 0 \tag{3.7}$$

Step 2. Denote by $\mathcal{F} = \{\mathbb{Q}^\alpha \colon \alpha = 1, 2, \dots\}$ the family of all open squares $\mathbb{Q}^\alpha \subset \overline{\mathbb{Q}^\alpha} \subset \Omega^*$ that are build in Step 1 for the construction of the mapping $h_1 \colon \Omega \to \Omega^*$. Let $\mathcal{V}$ be the set of vertices of these squares. Whenever two squares $\mathbb{Q}^\alpha, \mathbb{Q}^\beta \in \mathcal{F}$, $\alpha \neq \beta$, meet along their boundaries the intersection $I^{\alpha,\beta} = \partial\mathbb{Q}^\alpha \cap \partial\mathbb{Q}^\beta$ is either a point in $\mathcal{V}$ or a closed interval with endpoints in $\mathcal{V}$. Denote by $\mathcal{J} \subset \{I^{\alpha,\beta} \colon \alpha \neq \beta,\ \alpha, \beta = 1, 2, \dots\}$ the subfamily of all such intersections, excluding empty set and vertices. For each interval $I^{\alpha,\beta} \in \mathcal{J}$ we shall construct a doubly convex lens-shaped region $\mathbb{L}^{\alpha,\beta}$ with $I^{\alpha,\beta}$ as its axis of symmetry in the following way. Let R be a number greater than the length of $I^{\alpha,\beta}$ to be chosen later. There exist exactly two open disks of radius R for which $I^{\alpha,\beta}$ is a chord. Let $\mathbb{L}_R^{\alpha,\beta}$ be their intersection. This is a symmetric doubly convex lens of curvature $\frac{1}{R}$. Thus $\mathbb{L}_R^{\alpha,\beta}$ is bounded by two circular arcs $\gamma^{\alpha,\beta} = \mathbb{Q}^\alpha \cap \partial\mathbb{L}_R^{\alpha,\beta}$ and $\gamma^{\beta,\alpha} = \mathbb{Q}^\beta \cap \partial\mathbb{L}_R^{\alpha,\beta}$. As the curvature of the lens approaches zero the area of $\mathbb{L}_R^{\alpha,\beta}$ tends to 0. This allows us to choose R depending on α and β so that the lenses $\mathbb{L}^{\alpha,\beta} = \mathbb{L}_R^{\alpha,\beta}$ have the following property.

$$\int_{\mathbb{K}^{\alpha,\beta}} |Dh_1|^2 < \frac{\epsilon^2}{2^{\alpha+\beta}}, \qquad \text{where } \mathbb{K}^{\alpha,\beta} = h_1^{-1}(\mathbb{L}_R^{\alpha,\beta}) \tag{3.8}$$

The lenses $\mathbb{L}^{\alpha,\beta}$ are disjoint because the opening angle of each lens is at most $\pi/3$ and their axes are either parallel or orthogonal. However, the closures of the lenses considered here may have a common point that lies in $\mathcal{V}$. On each $\mathbb{K}^{\alpha,\beta}$ we replace h_1 by the harmonic extension of its restriction to $\partial\mathbb{K}^{\alpha,\beta}$. Thus we obtain a homeomorphism $h_2^{\alpha,\beta} \colon \overline{\mathbb{K}^{\alpha,\beta}} \xrightarrow{\text{onto}} L^{\alpha,\beta}$ of class $h_1 + \mathscr{A}_\circ(\mathbb{K}^{\alpha,\beta})$. By Proposition 2.3 the mappings $h_2^{\alpha,\beta} \colon \mathbb{K}^{\alpha,\beta} \xrightarrow{\text{onto}} \mathbb{L}^{\alpha,\beta}$ are diffeomorphisms. Finally, we define

$$h_2 = h_1 + \sum_{\alpha,\beta} [h_2^{\alpha,\beta} - h_1]_\circ \in h_1 + \mathscr{A}_\circ(\Omega) = h + \mathscr{A}_\circ(\Omega)$$

and observe that, from (3.6),

$$\|h_2 - h_1\|_{\mathscr{C}(\Omega)} \leqslant \sup_{\alpha,\beta} \operatorname{diam}\left(\mathbb{L}^{\alpha,\beta}\right) \leqslant \epsilon.$$

Also, (3.8) and Dirichlet's principle imply

$$\int_\Omega |Dh_2 - Dh_1|^2 \leqslant \sum_{\alpha,\beta} \int_{\mathbb{K}^{\alpha,\beta}} 2\left(|Dh_2|^2 + |Dh_1|^2\right) \leqslant 4 \sum_{\alpha,\beta=1}^{\infty} \int_{\mathbb{K}^{\alpha,\beta}} |Dh_1|^2$$
$$\leqslant 4 \sum_{\alpha,\beta=1}^{\infty} \frac{\epsilon^2}{2^{\alpha+\beta}} = 4\epsilon^2$$

Thus

$$\|h_2 - h_1\|_{\mathscr{A}(\Omega)} \leqslant \epsilon + 2\epsilon = 3\epsilon$$

The boundary of $\mathbb{K}^{\alpha,\beta}$ consists of two $\mathscr{C}^\infty$-smooth arcs $\Gamma^{\alpha,\beta}$ and $\Gamma^{\beta,\alpha}$ which share common endpoints, called the apices of $\mathbb{K}^{\alpha,\beta}$. These are preimages of $\gamma^{\alpha,\beta}$ and $\gamma^{\beta,\alpha}$ under the mapping h_1, respectively. Outside of the apices, the homeomorphism $h_2 \colon \overline{\mathbb{K}^{\alpha,\beta}} \xrightarrow{\text{onto}} \overline{\mathbb{L}^{\alpha,\beta}}$ is C^∞ smooth with positive Jacobian. The smoothness is a classical result of Kellogg; *a harmonic function with $\mathscr{C}^\infty$-smooth values on a smooth part of the boundary is $\mathscr{C}^\infty$-smooth up to this part of the boundary* [16, Theorem 6.19]. The positivity of the Jacobian on such part of the boundary follows from the convexity of its image, see [9, p. 116].

In conclusion, h_2 is locally bi-Lipschitz in $\Omega \setminus h^{-1}(\mathcal{V})$. The exceptional set $h^{-1}(\mathcal{V})$ is discrete because $\mathcal{V}$ is.

Finite energy case. By (3.7) we have

$$\|Dh_2\|_{\mathscr{L}^2(\Omega)} \leqslant \|Dh_1\|_{\mathscr{L}^2(\Omega)} \leqslant \|Dh\|_{\mathscr{L}^2(\Omega)} - \delta \tag{3.9}$$

Step 3. First we cover the set of vertices $\mathcal{V}$ by disks $\{\mathbb{D}_v \colon v \in \mathcal{V}\}$ centered at v with radii small enough so that

$$\operatorname{diam} \mathbb{D}_v \leqslant \epsilon, \tag{3.10}$$

and $\{3\mathbb{D}_v \colon v \in \mathcal{V}\}$ is a disjoint collection of disks in Ω^*. Moreover, their preimages under h_2 must satisfy

$$\sum_{v\in\mathcal{V}} \int_{h_2^{-1}(3\mathbb{D}_\nu)} |Dh_2|^2 < \epsilon^2 \tag{3.11}$$

Denote by $\tilde{\Omega}^* = \Omega^* \setminus \bigcup_{v\in\mathcal{V}} \overline{\mathbb{D}}_v$ and $\tilde{\Omega} = \Omega \setminus \bigcup_{v\in\mathcal{V}} h_2^{-1}(\overline{\mathbb{D}}_v)$. Our focus for a while will be on one of the circular sides of a lens $\mathbb{L}^{\alpha,\beta}$, say

$$\gamma^{\alpha,\beta} = \mathbb{Q}^\alpha \cap \partial \mathbb{L}^{\alpha,\beta} \subset \mathbb{Q}^\alpha$$

We truncate it near the endpoints by setting $\tilde{\gamma}^{\alpha,\beta} = \tilde{\Omega} \cap \gamma^{\alpha,\beta}$. Such truncated open arcs are mutually disjoint; even more, their closures are isolated continua in Ω^*. This means that there are disjoint neighborhoods of them. We are actually interested in a neighborhood of $\tilde{\gamma}^{\alpha,\beta}$ of the shape of a thin *concavo-convex lens* that we shall denote by $\tilde{\mathbb{L}}^{\alpha,\beta}$. By definition, $\tilde{\gamma}^{\alpha,\beta} \subset \tilde{\mathbb{L}}^{\alpha,\beta} \subset \mathbb{Q}^\alpha$. The construction of such lens goes as follows. Let a and b denote the endpoints of $\tilde{\gamma}^{\alpha,\beta}$, we assemble two circular arcs $\tilde{\gamma}_+^{\alpha,\beta}$ and $\tilde{\gamma}_-^{\alpha,\beta}$ with endpoints at a and b to form together with their endpoints a concavo-convex Jordan curve. This Jordan curve constitutes the boundary of a circular lens $\tilde{\mathbb{L}}^{\alpha,\beta}$. The term concavo-convex lens refers to the configuration in which $\tilde{\mathbb{L}}^{\alpha,\beta}$ lies in the concave side of the arc $\tilde{\gamma}_-^{\alpha,\beta}$ and convex side of $\tilde{\gamma}_+^{\alpha,\beta}$. It is clear that such lenses can be made arbitrarily thin so that $\tilde{\mathbb{L}}^{\alpha,\beta} \subset \tilde{\Omega}^*$ and the closures of $\tilde{\mathbb{L}}^{\alpha,\beta}$ will still be isolated continua in Ω^*. From now on we fix the family $\{\tilde{\mathbb{L}}^{\alpha,\beta} \colon \alpha \neq \beta\}$ of such concavo-convex lenses associated with the arcs $\tilde{\gamma}^{\alpha,\beta}$. We then look at their preimages $\mathbb{U}^{\alpha,\beta} = h_2^{-1}(\tilde{\mathbb{L}}^{\alpha,\beta})$ and the $\mathscr{C}^\infty$-smooth arcs $\Upsilon^{\alpha,\beta} = h_2^{-1}(\tilde{\gamma}^{\alpha,\beta})$. The endpoints of $\Upsilon^{\alpha,\beta}$ lie in $\partial\mathbb{U}^{\alpha,\beta}$. Moreover, $\Upsilon^{\alpha,\beta}$ splits $\mathbb{U}^{\alpha,\beta}$ into two disjoint subdomains $\mathbb{U}_+^{\alpha,\beta}$ and $\mathbb{U}_-^{\alpha,\beta}$ such that $\mathbb{U}^{\alpha,\beta} \setminus \Upsilon^{\alpha,\beta} = \mathbb{U}_+^{\alpha,\beta} \cup \mathbb{U}_-^{\alpha,\beta}$. Here we have a homeomorphism $h_2 \colon \mathbb{U}^{\alpha,\beta} \xrightarrow{\text{onto}} \tilde{\mathbb{L}}^{\alpha,\beta}$ which is $\mathscr{C}^\infty$-diffeomorphism on $\overline{\mathbb{U}}_+^{\alpha,\beta}$ and $\mathscr{C}^\infty$-diffeomorphism on $\overline{\mathbb{U}}_-^{\alpha,\beta}$. Therefore, for some positive number $M_{\alpha,\beta}$, we have pointwise inequlities $|Dh_2| \leqslant M_{\alpha,\beta}$ and $\det Dh_2 \geqslant \frac{1}{M_{\alpha,\beta}}$ in both $\overline{\mathbb{U}}_+^{\alpha,\beta}$ and $\overline{\mathbb{U}}_-^{\alpha,\beta}$. Having established such a deformation of lenses and their preimages under h_2, we apply Corollary 5.4. We infer that there is also a constant $M'_{\alpha,\beta} > 0$ with the following property: to every neighborhood of $\Upsilon^{\alpha,\beta}$, say an open connected set $\mathbb{U}_\circ^{\alpha,\beta} \subset \mathbb{U}^{\alpha,\beta}$ that contains $\Upsilon^{\alpha,\beta}$, there corresponds a $\mathscr{C}^\infty$-diffeomorphism, denoted by $h_3 \colon \mathbb{U}^{\alpha,\beta} \xrightarrow{\text{onto}} \tilde{\mathbb{L}}^{\alpha,\beta}$, such that

$$\begin{cases} h_3(z) = h_2(z) & \text{for } z \in \mathbb{U}^{\alpha,\beta} \setminus \mathbb{U}_\circ^{\alpha,\beta} \\ |Dh_3| \leqslant M'_{\alpha,\beta} & \text{and} \quad \det Dh_3 \geqslant \frac{1}{M'_{\alpha,\beta}} \quad \text{in } \mathbb{U}^{\alpha,\beta} \end{cases} \tag{3.12}$$

We emphasize that $M'_{\alpha,\beta}$ is independent of the neighborhood $\mathbb{U}_\circ^{\alpha,\beta}$. We choose and fix $\mathbb{U}_\circ^{\alpha,\beta}$ thin enough to satisfy

- $\overline{\mathbb{U}}_\circ^{\alpha,\beta} \subset \mathbb{U}^{\alpha,\beta} \cup \overline{\Upsilon}^{\alpha,\beta}$
- $|\mathbb{U}_\circ^{\alpha,\beta}| \leqslant [M_{\alpha,\beta} + M'_{\alpha,\beta}]^{-2} \epsilon^2 \, 2^{-\alpha-\beta}$
- $\sup_{\mathbb{U}_\circ^{\alpha,\beta}} |Dh_2| \leqslant M_{\alpha,\beta}$
- in the finite energy case, we also assume that $|\mathbb{U}_\circ^{\alpha,\beta}| \leqslant [M'_{\alpha,\beta}]^{-2} \delta^2 \, 4^{-\alpha-\beta-1}$

Recall that δ was defined by (3.7) and later appeared in (3.9). This is certainly possible; for instance, take $\mathbb{U}_\circ^{\alpha,\beta}$ to be the preimage under h_2 of a sufficiently thin concavo-convex lens containing $\tilde{\gamma}^{\alpha,\beta}$. We call $h_3 \colon \mathbb{U}^{\alpha,\beta} \xrightarrow{\text{onto}} \tilde{\mathbb{L}}^{\alpha,\beta}$ a smoothing of $h_2 \colon \mathbb{U}^{\alpha,\beta} \xrightarrow{\text{onto}} \tilde{\mathbb{L}}^{\alpha,\beta}$ associated with a given arc $\Upsilon^{\alpha,\beta} = h_2^{-1}(\tilde{\gamma}^{\alpha,\beta})$. We now define a homeomorphism $h_3 \colon \Omega \xrightarrow{\text{onto}} \Omega^*$ by the rule

$$h_3 = \begin{cases} \text{smoothing of } h_2 & \text{in } \mathbb{U}^{\alpha,\beta} \\ h_2 & \text{in } \Omega \setminus \bigcup_{\alpha,\beta} \mathbb{U}^{\alpha,\beta} \end{cases}$$

It belongs to $h_2 + \mathscr{A}_\circ(\Omega)$. Obviously h_3 is a $\mathscr{C}^\infty$-diffeomorphism in $\widetilde{\Omega}$. We have for every $z \in \Omega$

$$
\begin{aligned}
|h_3(z) - h_2(z)| &\leqslant \begin{cases} \operatorname{diam} \widetilde{\mathbb{L}}^{\alpha,\beta} & \text{for } z \in \mathbb{U}^{\alpha,\beta} \\ 0 & \text{otherwise} \end{cases} \\
&\leqslant \operatorname{diam} \mathbb{Q}^\alpha \leqslant \epsilon
\end{aligned}
$$

see (3.6). Hence $\|h_3 - h_2\|_{\mathscr{C}(\Omega)} \leqslant \epsilon$. As regards the energy of $h_3 - h_2$ we find that

$$
\begin{aligned}
\mathcal{E}_\Omega[h_3 - h_2] &= \sum_{\alpha,\beta} \int_{\mathbb{U}_\circ^{\alpha,\beta}} |Dh_3 - Dh_2|^2 \\
&\leqslant \sum_{\alpha,\beta} |\mathbb{U}_\circ^{\alpha,\beta}| \sup_{\mathbb{U}_\circ^{\alpha,\beta}} \left(|Dh_3| + |Dh_2|\right)^2 \\
&\leqslant \sum_{\alpha,\beta} |\mathbb{U}_\circ^{\alpha,\beta}| [M'_{\alpha,\beta} + M_{\alpha,\beta}]^2 \leqslant \sum_{\alpha,\beta=1}^{\infty} \frac{\epsilon^2}{2^{\alpha+\beta}} \leqslant \epsilon^2
\end{aligned}
\tag{3.13}
$$

These estimates sum up to

$$
\|h_3 - h_2\|_{\mathscr{A}(\Omega)} \leqslant \epsilon + \epsilon = 2\epsilon
$$

Let us record for subsequent use the following estimate, obtained from (3.11) and (3.13).

$$
\begin{aligned}
\sum_{v \in \mathcal{V}} \int_{h_3^{-1}(3\mathbb{D}_\nu)} |Dh_3|^2 &\leqslant \sum_{v \in \mathcal{V}} \left(\int_{h_3^{-1}(3\mathbb{D}_\nu) \setminus h_2^{-1}(3\mathbb{D}_\nu)} |Dh_3|^2 + \int_{h_2^{-1}(3\mathbb{D}_\nu)} |Dh_3|^2 \right) \\
\leqslant \int_{\{h_3 \neq h_2\}} |Dh_3|^2 &+ 2 \sum_{v \in \mathcal{V}} \left(\int_{h_2^{-1}(3\mathbb{D}_v)} |Dh_3 - Dh_2|^2 + \int_{h_2^{-1}(3\mathbb{D}_\nu)} |Dh_2|^2 \right) \\
&\leqslant \sum_{\alpha,\beta} \left(M'_{\alpha,\beta}\right)^2 |\mathbb{U}_\circ^{\alpha,\beta}| + 2\epsilon^2 + 2\epsilon^2 \leqslant 5\epsilon^2
\end{aligned}
\tag{3.14}
$$

Finite energy case. For the energy of h_3, we observe that

$$
\begin{aligned}
\|Dh_3\|_{\mathscr{L}^2(\Omega)} &\leqslant \|Dh_3\|_{\mathscr{L}^2(\Omega \setminus \cup \mathbb{U}_\circ^{\alpha,\beta})} + \sum_{\alpha,\beta} \|Dh_3\|_{\mathscr{L}^2(\mathbb{U}_\circ^{\alpha,\beta})} \\
&= \|Dh_2\|_{\mathscr{L}^2(\Omega \setminus \cup \mathbb{U}_\circ^{\alpha,\beta})} + \sum_{\alpha,\beta} \|Dh_3\|_{\mathscr{L}^2(\mathbb{U}_\circ^{\alpha,\beta})} \\
&\leqslant \|Dh_2\|_{\mathscr{L}^2(\Omega)} + \sum_{\alpha,\beta} |\mathbb{U}_\circ^{\alpha,\beta}|^{1/2} \sup_{\mathbb{U}_\circ^{\alpha,\beta}} |Dh_3| \\
&\leqslant \|Dh\|_{\mathscr{L}^2(\Omega)} - \delta + \frac{\delta}{2}
\end{aligned}
$$

Thus

$$
\|Dh_3\|_{\mathscr{L}^2(\Omega)} \leqslant \|Dh\|_{\mathscr{L}^2(\Omega)} - \frac{\delta}{2}
\tag{3.15}
$$

Step 4. We have already upgraded the mapping h to a homeomorphism $h_3\colon \Omega \to \Omega^*$ such that $h_3 \in h + \mathscr{A}_\circ(\Omega)$ and

$$\begin{aligned} \|h_3 - h\|_{\mathscr{A}(\Omega)} &\leqslant \|h_3 - h_2\|_{\mathscr{A}(\Omega)} + \|h_2 - h_1\|_{\mathscr{A}(\Omega)} + \|h_1 - h\|_{\mathscr{A}(\Omega)} \\ &\leqslant 2\epsilon + 3\epsilon + 2\epsilon = 7\epsilon \end{aligned} \tag{3.16}$$

Moreover, h_3 is a $\mathscr{C}^\infty$-diffeomorphism on $\Omega \setminus \bigcup_{v\in\mathcal{V}} h_2^{-1}(\overline{\mathbb{D}}_v)$. We now define a homeomorphism $h_4\colon \Omega \xrightarrow{\text{onto}} \Omega^*$ by performing harmonic replacement of h_3 on each set $h_3^{-1}(2\mathbb{D}_v)$. This gives us a $\mathscr{C}^\infty$-diffeomorphism $h_4\colon h_3^{-1}(2\overline{\mathbb{D}}_v) \to 2\overline{\mathbb{D}}_v$, see Step 2 for details. For each $z \in \Omega$

$$|h_4(z) - h_3(z)| \leqslant \begin{cases} 2\operatorname{diam}\mathbb{D}_v & \text{if } z \in h_3^{-1}(2\mathbb{D}_v) \\ 0 & \text{otherwise} \end{cases} \leqslant 2\epsilon$$

Hence $\|h_4 - h_3\|_{\mathscr{C}(\Omega)} \leqslant 2\epsilon$. Using (3.14) we estimate the energy as follows.

$$\begin{aligned} \mathcal{E}_\Omega[h_4 - h_3] &= \sum_{v\in\mathcal{V}} \mathcal{E}_{h_3^{-1}(2\mathbb{D}_v)}[h_4 - h_3] \\ &\leqslant 2\sum_{v\in\mathcal{V}} \Big(\mathcal{E}_{h_3^{-1}(2\mathbb{D}_v)}[h_4] + \mathcal{E}_{h_3^{-1}(2\mathbb{D}_v)}[h_3]\Big) \\ &\leqslant 4\sum_{v\in\mathcal{V}} \mathcal{E}_{h_3^{-1}(2\mathbb{D}_v)}[h_3] \leqslant 20\epsilon^2 \end{aligned}$$

Thus, by (3.14)

$$\|h_4 - h_3\|_{\mathscr{A}(\Omega)} \leqslant \epsilon + \sqrt{20}\epsilon \leqslant 6\epsilon \tag{3.17}$$

Finite energy case. By virtue of Dirichlet's principle and (3.15) we have

$$\|Dh_4\|_{\mathscr{L}^2(\Omega)} \leqslant \|Dh_3\|_{\mathscr{L}^2(\Omega)} \leqslant \|Dh\|_{\mathscr{L}^2(\Omega)} - \frac{\delta}{2} \tag{3.18}$$

Step 5. The final step consists of smoothing h_4 in a neighborhood of each smooth Jordan curves $C_v = \partial h_3^{-1}(2\mathbb{D}_v)$. We proceed in much the same way as in Step 3, but we appeal to Corollary 5.5 instead of Corollary 5.4. By smoothing h_4 in a sufficiently thin neighborhood of each C_v we obtain a $\mathscr{C}^\infty$-diffeomorphism $h_5\colon \Omega \xrightarrow{\text{onto}} \Omega^*$, $h_5 \in h_4 + \mathscr{A}_\circ(\Omega)$ such that

$$\|h_5 - h_4\|_{\mathscr{A}(\Omega)} \leqslant \epsilon \tag{3.19}$$

We now recapitulate the estimates (3.16), (3.17) and (3.19) to obtain a $\mathscr{C}^\infty$-diffeomorphism in Ω

$$H := h_5 \in h + \mathscr{A}(\Omega)$$

such that

$$\begin{aligned} \|H - h\|_{\mathscr{A}(\Omega)} &\leqslant \|h_5 - h_4\|_{\mathscr{A}(\Omega)} + \|h_4 - h_3\|_{\mathscr{A}(\Omega)} + \|h_3 - h\|_{\mathscr{A}(\Omega)} \\ &\leqslant \epsilon + 6\epsilon + 7\epsilon = 14\epsilon \end{aligned}$$

which is as strong as (ii) in Theorem 1.2.

Finite energy case. To obtain the desired energy estimate $\mathcal{E}_\Omega[h_5] \leqslant \mathcal{E}_\Omega[h]$, we need to sharpen the energy part in (3.19). By narrowing further the neighborhoods of C_v we can be make the energy $\mathcal{E}_\Omega[h_5 - h_4]$ as small as we wish; for example to obtain

$$\|Dh_5 - Dh_4\|_{\mathscr{L}^2(\Omega)} < \frac{\delta}{2}$$

This is enough to conclude that

$$\|DH\|_{\mathscr{L}^2(\Omega)} \leqslant \|Dh\|_{\mathscr{L}^2(\Omega)}$$

because of (3.18). □

4. Hopf differentials, Theorem 1.3

A quadratic differential on a domain Ω in the complex plane $\mathbb{C}$ takes the form $Q = F(z)\,\mathrm{d}z \otimes \mathrm{d}z$, where F is a complex function on Ω. Given a conformal change of the variable z, $z = \varphi(\xi)$, where $\varphi\colon \Omega' \to \Omega$, the pull back

$$\varphi^\sharp(Q) = F\big(\varphi(\xi)\big)\,\mathrm{d}\varphi \otimes \mathrm{d}\varphi = F\big(\varphi(\xi)\big)\dot{\varphi}^2(\xi)\,\mathrm{d}\xi \otimes \mathrm{d}\xi$$

defines a quadratic differential on Ω'. It is plain that for a complex harmonic function $h\colon \Omega \to \mathbb{C}$ the associated Hopf differential

$$Q_h = h_z \overline{h_{\bar z}}\,\mathrm{d}z \otimes \mathrm{d}z$$

is holomorphic, meaning that

$$\frac{\partial}{\partial \bar z}\left(h_z \overline{h_{\bar z}}\right) = 0 \tag{4.1}$$

Conversely, if a Hopf differential $Q_h = h_z \overline{h_{\bar z}}\,\mathrm{d}z \otimes \mathrm{d}z$ is holomorphic for some $\mathscr{C}^1$-mapping h, then h is harmonic at the points where the Jacobian determinant $J(z,h) := \det Dh = |h_z|^2 - |h_{\bar z}|^2 \neq 0$, see [10, 10.5] and our Remark 4.3. Here the assumption that $J(z,h) \neq 0$ is critical. Let us illustrate it by the following.

Example 4.1. Consider a mapping $h \in \mathscr{C}^{1,1}(\mathbb{C}_\circ)$ defined on the punctured plane $\mathbb{C}_\circ = \mathbb{C} \setminus \{0\}$ by the rule

$$h(z) = \begin{cases} \frac{z}{|z|} & \text{for } 0 < |z| \leqslant 1 \\ \frac{1}{2}\left(z + \frac{1}{\bar z}\right) & \text{for } 1 \leqslant |z| < \infty \end{cases} \tag{4.2}$$

Direct computation shows that

$$h_z(z) = \begin{cases} \frac{1}{2}|z|^{-1} & \text{for } 0 < |z| \leqslant 1 \\ \frac{1}{2} & \text{for } 1 \leqslant |z| < \infty \end{cases}$$

and

$$h_{\bar z}(z) = \begin{cases} -\frac{1}{2}|z|\bar z^{-2} & \text{for } 0 < |z| \leqslant 1 \\ -\frac{1}{2}\bar z^{-2} & \text{for } 1 \leqslant |z| < \infty \end{cases}$$

Thus

$$Q_h = -\frac{\mathrm{d}z \otimes \mathrm{d}z}{4z^2} \qquad \text{in } \mathbb{C}_\circ \tag{4.3}$$

It may be worth mentioning that the mapping h in (4.2) is the unique (up to rotation of z) minimizer of the Dirichlet energy

$$\mathcal{E}[H] = \int_{\mathbb{A}} |DH|^2$$

over the annulus $\mathbb{A} = A(r,R) = \{z \colon r < |z| < R\}$, $0 < r < 1 < R$, subject to all weak limits of homeomorphisms $H \colon \mathbb{A} \xrightarrow{\text{onto}} A(1, R_*)$, where $R_* = \frac{1}{2}\left(R + \frac{1}{R}\right)$, see [1]. Note that the Hopf differential of (4.3) is real along the boundary circles of $\mathbb{A}$. The concentric circles are horizontal trajectories of Q_h. In fact this is a general property of minimizers [21, Lemma 1.2.5]. The general pattern is that with the loss of injectivity comes the loss of the Lagrange-Euler equation for the extremal mapping.

Properties of the function h with holomorphic Hopf differential $Q = h_z\overline{h_{\bar z}}\, \mathrm{d}z \otimes \mathrm{d}z$ are of interest in the studies of harmonic mappings [11, 12, 21, 30, 31], minimal surfaces [8, 34] and Teichmüller theory [15]. In this section we prove Theorem 1.3 which imposes fairly minimal assumptions that imply harmonicity of $\mathscr{W}^{1,2}$-solution to the equation (4.1). Some elements of the proof go back to [28, 29].

Proof of Theorem 1.3. As a consequence of the Stoilow factorization theorem [1, p. 56] the branch set of h is discrete, hence removable for continuous harmonic functions. Thus we assume that $h \colon \Omega \xrightarrow{\text{onto}} \Omega^*$ is a homeomorphism of Sobolev class $\mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega, \Omega^*)$ such that

$$h_z\overline{h_{\bar z}} = F(z) \quad \text{is holomorphic in } \Omega \tag{4.4}$$

By virtue of Theorem 1.2, there exists a sequence of diffeomorphisms $h^j : \Omega \xrightarrow{\text{onto}} \Omega^*$ converging c-uniformly and strongly in $\mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega, \Omega^*)$ to h. Denote by

$$h^j_z h^j_{\bar z} =: F^j \in \mathscr{L}^1_{\mathrm{loc}}(\Omega) \tag{4.5}$$

Thus $F^j \to F$ strongly in $\mathscr{L}^1_{\mathrm{loc}}(\Omega)$. Let us first dispose of an easy case.
Case 0. *The homogeneous equation* $F \equiv 0$. Since h^j are diffeomorphisms the Jacobian determinant $J(z, h^j) = |h^j_z|^2 - |h^j_{\bar z}|^2$ is either positive everywhere in Ω or negative everywhere in Ω. Let us settle the case when $J(z, h^j) > 0$ for infinitely many indices $j = 1, 2, \dots$. For such j we have $|h^j_z| > |h^j_{\bar z}|$, which yields $|h^j_{\bar z}|^2 \leqslant |h^j_z h^j_{\bar z}|$. Passing to the $\mathscr{L}^1$-limit we obtain

$$|h_{\bar z}|^2 \leqslant |h_z h_{\bar z}| = |F(z)| \equiv 0.$$

Thus h is holomorphic, by Weyl's lemma. Similarly, in case $J(z, h^j) < 0$ for infinitely many indices $j = 1, 2, \dots$, we find that h is antiholomorphic.

Remark 4.2. We observe, based on the above arguments, that for this homogeneous equation $h_z\overline{h_{\bar z}} \equiv 0$ every solution $h \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\Omega)$ obtained as the weak $\mathscr{W}^{1,2}$-limit of homeomorphisms is either holomorphic or antiholomorphic. The situation is dramatically different if $h_z\overline{h_{\bar z}} \not\equiv 0$; some topological assumption on h are necessary, as illustrated in Example 4.1.

Case 1. *Nonhomogeneous equation* $F \not\equiv 0$. The function F, being holomorphic, may vanish only at isolated points. Since isolated points are removable for bounded harmonic functions, it suffices to consider the set where $F \neq 0$. Proceeding further in this direction, we may and do assume that $F(z) \equiv 1$ (by a conformal change of the z-variable) and h is a $\mathscr{W}^{1,2}$-homeomorphism in the closure of the unit square $\mathbb{Q} = \{x + iy \colon 0 < x < 1, 0 < y < 1\}$. The problem now reduces to establishing that the equation

$$h_z \overline{h_{\bar z}} \equiv 1 \tag{4.6}$$

implies $\Delta h = 0$. This will be proved indirectly by means of the energy-minimizing property

$$\mathcal{E}_{\mathbb{Q}}[h] \leqslant \mathcal{E}_{\mathbb{Q}}[H] \tag{4.7}$$

where $H \colon \mathbb{Q} \to h(\mathbb{Q})$ is any homeomorphism in $h + \mathscr{A}_\circ(\mathbb{Q})$; in particular, $H = h$ on $\partial\mathbb{Q}$. Indeed, if h were not harmonic, we would be able to decrease its energy by harmonic replacement (Propositions 2.1 and 2.3), contradicting (4.7).

4.1. **Proof of the inequality** (4.7)**.** With the aid of the approximation theorem we need only prove (4.7) for mappings $H \in h + \mathscr{A}_\circ(\mathbb{Q})$ that are diffeomorphisms on $\mathbb{Q}$. From now on we assume that this is the case. Denote $\mathbb{Q}^* = h(\mathbb{Q}) = H(\mathbb{Q})$. We consider a sequence $h^j \in h + \mathscr{A}_\circ(\mathbb{Q})$ of diffeomorphisms $h^j \colon \mathbb{Q} \xrightarrow{\text{onto}} \mathbb{Q}^*$ converging in $\mathscr{A}(\mathbb{Q})$ to h. Moreover we may also assume that $Dh^j \to Dh$ almost everywhere in $\mathbb{Q}$ by passing to a subsequence if necessary. Now the sequence $\chi^j \colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ of self-homeomorphisms of the closed unit disk given by $\chi^j = H^{-1} \circ h^j$, where $\chi^j = \mathrm{id}$ on $\partial\mathbb{Q}$, is converging uniformly on $\overline{\mathbb{Q}}$ to $\chi = H^{-1} \circ h$. It is important to observe that $\chi \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{Q})$ and χ^j converges to χ in $\mathscr{W}^{1,2}(\mathbb{Q}')$ on any compactly contained subdomain $\mathbb{Q}' \Subset \mathbb{Q}$. Since h^j and $(\chi^j)^{-1}$ are diffeomorphisms on $\mathbb{Q}'$ and $\chi^j(\mathbb{Q}')$, respectively, the chain rule can be applied to the composition $H = h^j \circ (\chi^j)^{-1}$. For $w \in \chi^j(\mathbb{Q}')$ we have

$$\frac{\partial H(w)}{\partial w} = h^j_z(z) \frac{\partial (\chi^j)^{-1}}{\partial w} + h^j_{\bar z}(z) \frac{\partial (\chi^j)^{-1}}{\partial \bar w}$$
$$\frac{\partial H(w)}{\partial \bar w} = h^j_z(z) \frac{\partial (\chi^j)^{-1}}{\partial \bar w} + h^j_{\bar z}(z) \overline{\frac{\partial (\chi^j)^{-1}}{\partial w}}$$

where $z = (\chi^j)^{-1}(w)$.

The partial derivatives of $(\chi^j)^{-1}$ at w can be expressed in terms of $\chi^j_z(z)$ and $\chi^j_{\bar z}(z)$ by the rules

$$\frac{\partial (\chi^j)^{-1}}{\partial w} = \frac{\overline{\chi^j_z(z)}}{J(z, \chi^j)}$$
$$\frac{\partial (\chi^j)^{-1}}{\partial \bar w} = -\frac{\chi^j_{\bar z}(z)}{J(z, \chi^j)}$$

where the Jacobian determinant $J(z,\chi^j)$ is strictly positive. This yields

$$\frac{\partial H}{\partial w} = \frac{h^j_z \overline{\chi^j_z} - h^j_{\bar z}\overline{\chi^j_{\bar z}}}{J(z,\chi^j)}$$
$$\frac{\partial H}{\partial \bar w} = \frac{h^j_{\bar z} \chi^j_z - h^j_{z}\chi^j_{\bar z}}{J(z,\chi^j)}$$

We compute the energy integral of H over the set $\chi^j(\mathbb{Q}')$ by substitution $w=\chi^j(z)$,

$$\begin{aligned}
\mathcal{E}_{\chi^j(\mathbb{Q}')}[H] &= 2\int_{\chi^j(\mathbb{Q}')} \left(|H_w|^2+|H_{\bar w}|^2\right)\,\mathrm{d}w \\
&= 2\int_{\mathbb{Q}'} \frac{|h^j_z\overline{\chi^j_z} - h^j_{\bar z}\overline{\chi^j_{\bar z}}|^2 + |h^j_{\bar z}\chi^j_z - h^j_z\chi^j_{\bar z}|^2}{|\chi^j_z|^2-|\chi^j_{\bar z}|^2}\,\mathrm{d}z
\end{aligned}$$

On the other hand, the energy of h^j over the set $\mathbb{Q}'$ is

$$\mathcal{E}_{\mathbb{Q}'}[h^j] = 2\int_{\mathbb{Q}'}\Big(|h^j_z|^2+|h^j_{\bar z}|^2\Big)\,\mathrm{d}z$$

Subtract these two integrals to obtain

$$\begin{aligned}
\mathcal{E}_{\mathbb{Q}}[H]-\mathcal{E}_{\mathbb{Q}'}[h^j] &\geqslant \mathcal{E}_{\chi^j(\mathbb{Q}')}[H]-\mathcal{E}_{\mathbb{Q}'}[h^j] \\
&= 4\int_{\mathbb{Q}'} \frac{\Big(|h^j_z|^2+|h^j_{\bar z}|^2\Big)\cdot|\chi^j_{\bar z}|^2 - 2\,\mathrm{Re}\Big[h^j_z\overline{h^j_{\bar z}\chi^j_z}\chi^j_{\bar z}\Big]}{|\chi^j_z|^2-|\chi^j_{\bar z}|^2}\,\mathrm{d}z \\
&\geqslant 4\int_{\mathbb{Q}'} \frac{2|h^j_z h^j_{\bar z}||\chi^j_{\bar z}|^2 - 2\,\mathrm{Re}\Big[h^j_z\overline{h^j_{\bar z}\chi^j_z}\chi^j_{\bar z}\Big]}{|\chi^j_z|^2-|\chi^j_{\bar z}|^2}\,\mathrm{d}z \qquad (4.8)\\
&= 4\int_{\mathbb{Q}'}\left[\frac{|\chi^j_z-\sigma^j(z)\chi^j_{\bar z}|^2}{|\chi^j_z|^2-|\chi^j_{\bar z}|^2}-1\right]|h^j_z h^j_{\bar z}|\,\mathrm{d}z
\end{aligned}$$

where we have introduced the notation

$$\sigma^j=\sigma^j(z)=\begin{cases} h^j_z\overline{h^j_{\bar z}}\,|h^j_z h^j_{\bar z}|^{-1} & \text{if } h^j_z h^j_{\bar z}\neq 0\\ 1 & \text{otherwise.}\end{cases}$$

Note that $|\sigma^j|=1$ and $\sigma^j\to 1$ almost everywhere.

Upon using Hölder's inequality we continue the chain (4.8) as follows.

$$\geqslant\ 4\frac{\left[\int_{\mathbb{Q}'}\left|\chi^j_z-\sigma^j\chi^j_{\bar z}\right|\,\sqrt{|h^j_z h^j_{\bar z}|}\,\mathrm{d}z\right]^2}{\int_{\mathbb{Q}'}J(z,h^j)\,\mathrm{d}z} - 4\int_{\mathbb{Q}'}|h^j_z h^j_{\bar z}|. \qquad (4.9)$$

The denominator in (4.9) is at most 1 because

$$\int_{\mathbb{Q}'} J(z,h^j)\,\mathrm{d}z = |\chi^j(\mathbb{Q}')|\leqslant|\mathbb{Q}|=1.$$

Therefore,

$$\mathscr{E}_{\mathbb{Q}}[H] - \mathscr{E}_{\mathbb{Q}'}[h^j] \geqslant 4\left[\int_{\mathbb{Q}'} \left|\chi_z^j - \sigma^j \chi_{\bar z}^j\right| \sqrt{|h_z^j h_{\bar z}^j|}\,\mathrm{d}z\right]^2 - 4\int_{\mathbb{Q}'} |h_z^j h_{\bar z}^j|\,\mathrm{d}z.$$

It is at this point that we can pass to the limit as $j \to \infty$, to obtain

$$\mathscr{E}_{\mathbb{Q}}[H] - \mathscr{E}_{\mathbb{Q}'}[h] \geqslant 4\left[\int_{\mathbb{Q}'} |\chi_z - \chi_{\bar z}|\,\mathrm{d}z\right]^2 - 4|\mathbb{Q}'|. \tag{4.10}$$

Since $\mathbb{Q}'$ was an arbitrary compactly contained subdomain of $\mathbb{Q}$, the estimate (4.10) remains valid with $\mathbb{Q}'$ replaced by $\mathbb{Q}$.

$$\begin{aligned}
\mathscr{E}_{\mathbb{Q}}[H] - \mathscr{E}_{\mathbb{Q}}[h] &\geqslant 4\left[\int_{\mathbb{Q}} \left|\frac{\partial \chi}{\partial y}\right| \mathrm{d}x\,\mathrm{d}y\right]^2 - 4 \\
&\geqslant 4\int_0^1 \left|\int_0^1 \frac{\partial \chi(x,y)}{\partial y}\,\mathrm{d}y\right| \mathrm{d}x - 4 \\
&= 4\int_0^1 |\chi(x,1) - \chi(x,0)|\,\mathrm{d}x - 4 = 4 - 4 = 0
\end{aligned} \tag{4.11}$$

as desired. □

Remark 4.3. When specialized to the case $h \in \mathscr{C}^1$, Theorem 1.3 shows that h is harmonic outside of the zero set of its Jacobian.

5. Auxiliary smoothing results

Here we present some results concerning smoothing of piecewise differentiable planar homeomorphisms. They can be found in [25] in greater generality, but since we require quantitative control of derivatives, a self-contained proof is in order. Here it is more convenient to use the operator norm of a matrix, denoted by $\|\cdot\|$. Note that $\|A\| \leqslant |A| \leqslant 2\|A\|$ for 2×2-matrices.

Proposition 5.1. *Let $\mathbb{U} \subset \mathbb{R}^2$ be a domain containing an open segment I with endpoints on the boundary $\partial\mathbb{U}$ which splits $\mathbb{U}$ into two subdomains $\mathbb{U}_1$ and $\mathbb{U}_2$ such that $\mathbb{U} \setminus I = \mathbb{U}_1 \cup \mathbb{U}_2$. Suppose that $f\colon \overline{\mathbb{U}} \xrightarrow{\text{onto}} \overline{\mathbb{U}^*} \subset \mathbb{R}^2$ is a homeomorphism with the following properties:*

- *(i) For $j = 1, 2, \dots$ the restriction of f to $\overline{\mathbb{U}_j}$ is $\mathscr{C}^\infty$-smooth, equals the identity on I;*
- *(ii) There is a constant $M > 0$ such that for $j = 1, 2$ the restriction of f to $\overline{\mathbb{U}_j}$ satisfies $\|Df\| \leqslant M$ and $\det Df \geqslant M^{-1}$.*

Then for any open set $\mathbb{U}_\circ$ with $I \subset \mathbb{U}_\circ \subset \mathbb{U}$ there is a $\mathscr{C}^\infty$-diffeomorphism $g\colon \mathbb{U} \to \mathbb{U}^*$ such that

- *g agrees with f on $\mathbb{U} \setminus \mathbb{U}_\circ$ (and also on I);*
- *$\|Dg\| \leqslant 20M$ and $\det Dg \geqslant (20M)^{-1}$ on $\mathbb{U}$.*

Proof. Without loss of generality $I \subset \mathbb{R} = \{(x, y) : y = 0\}$. We write f in components as (u, v) where u and v are functions of x and y. Let us introduce a notation; given any $\mathscr{C}^\infty$-smooth function $\beta\colon \mathbb{R} \to [0, \infty)$, denote

$V(\beta) = \{(x, y) \in \mathbb{R}^2 \colon |y| < \beta(x)\}$. We can and do choose β so that $I \subset V(\beta) \subset \mathbb{U}_\circ$, and further scale it down until the following holds.

$$
\begin{aligned}
&|\beta'(x)| \leqslant \frac{1}{40M} && \text{for all } x \in \mathbb{R}; && \\
&|v_x| \leqslant \frac{1}{50M^2} && \text{in } V(\beta) \setminus I, && \text{because } v(x,0) = 0; \\
&|u_x - 1| \leqslant \frac{1}{10} && \text{in } V(\beta) \setminus I, && \text{because } u(x,0) = x.
\end{aligned}
\tag{5.1}
$$

As a consequence of (ii) and (5.1),

$$
v_y \geqslant \frac{M^{-1} - |u_y v_x|}{u_x} \geqslant \frac{1}{2M}. \tag{5.2}
$$

Since v is also M-Lipschitz by (ii), the following double inequality holds in $V(\beta) \setminus I$.

$$
\frac{1}{2M} \leqslant \frac{v}{y} \leqslant M. \tag{5.3}
$$

Let us fix be a nondecreasing $\mathscr{C}^\infty$ function $\alpha \colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) = 0$ for $t \leqslant 1/3$. Let $\alpha(t) = 1$ for $t \geqslant 2/3$. Moreover, $\alpha'(t) \leqslant 4$ for all $t \in \mathbb{R}$ and $\alpha(\infty) = 1$, by convention. Now we introduce a modification of u on $\mathbb{U}$ by setting

$$
\tilde{u} := \alpha(t) u + (1 - \alpha(t)) x \qquad \text{where } t = \begin{cases} \frac{|y|}{\beta(x)} & \text{if } \beta(x) \neq 0 \\ \infty & \text{otherwise} \end{cases}.
$$

Note that $\tilde{u} = u$ outside of $V(\beta)$. In $V(\beta) \setminus I$ we compute the derivatives as follows.

$$
\begin{aligned}
\tilde{u}_x &= -t^2 \alpha'(t) \beta'(x) \frac{u - x}{|y|} + \alpha(t) u_x + 1 - \alpha(t) \\
\tilde{u}_y &= t \alpha'(t) \frac{u - x}{y} + \alpha(t) u_y
\end{aligned}
\tag{5.4}
$$

Since u is M-Lipschitz by (ii), we have $|u - x| \leqslant M|y|$. From this, (5.1) and (5.4) we obtain

$$
\frac{8}{10} \leqslant \tilde{u}_x \leqslant \frac{12}{10}, \quad \text{and} \quad |\tilde{u}_y| \leqslant 5M, \tag{5.5}
$$

which combined with (5.2) yields

$$
\tilde{u}_x v_y - \tilde{u}_y v_x \geqslant \frac{8}{10} \frac{1}{2M} - \frac{5M}{50M^2} = \frac{3}{10M}. \tag{5.6}
$$

Next we modify v on $\mathbb{U}$. Specifically,

$$
\tilde{v} := \alpha(s) v + (1 - \alpha(s)) \frac{y}{2M} \qquad \text{where } s = \begin{cases} \frac{3|y|}{\beta(x)} & \text{if } \beta(x) \neq 0 \\ \infty & \text{otherwise} \end{cases}
$$

Note that $\tilde{v} = v$ outside of $V(\beta/3)$, and on the set $V(\beta/3)$ we already have $\tilde{u} \equiv x$.

Computations similar to (5.4) yield (on the set $V(\beta/3) \setminus I$)

$$\begin{aligned} \tilde{v}_x &= -\frac{1}{3}\alpha'(s)s^2\frac{v-y}{|y|} + \alpha(s)v_x; \\ \tilde{v}_y &= \frac{s\alpha'(s)}{y}\left(v - \frac{y}{2M}\right) + \alpha(s)v_y + \frac{1-\alpha(s)}{2M}. \end{aligned} \tag{5.7}$$

Straightforward estimates based on (5.1), (5.2) and (5.3) comply

$$\begin{aligned} |\tilde{v}_x| &\leqslant \frac{4M}{3} + \frac{1}{50M^2} < \frac{3M}{2}, \\ \frac{1}{2M} &\leqslant \tilde{v}_y \leqslant 5M. \end{aligned} \tag{5.8}$$

It remains to check that the mapping $g := (\tilde{u}, \tilde{v})$, which agrees with f outside of $V(\beta)$, satisfies all the requirements. As regards $\mathscr{C}^\infty$-smoothness we need only check it on $V(\beta/9)$. But in this neighborhood of I we have a linear mapping, $g(x,y) = \left(x, \frac{y}{2M}\right)$, so $\mathscr{C}^\infty$-smooth. By virtue of (5.5) and (5.8) we have $\|Dg\| \leqslant 20M$. The desired lower bound for $\det Dg$ follows from (5.6) and (5.8). Consequently, g is a local diffeomorphism, and since it agrees with f on $\partial V(\beta)$, it is in fact a diffeomorphism, by a topological result: *a local homeomorphism which shares boundary values with a homeomorphism is injective* [25, Lemma 8.2]. □

We also need a polar version of Proposition 5.1.

Corollary 5.2. *Let $\mathbb{U} \subset \mathbb{R}^2$ be a domain containing a circle $\mathbb{T}$. Suppose that $f \colon \mathbb{U} \xrightarrow{\text{onto}} \mathbb{U}^* \subset \mathbb{R}^2$ is a homeomorphism with the following properties:*

(i) The restriction of f to $\mathbb{T}$ is the identity mapping;
(ii) There is a constant $M > 0$ such that the restriction of f to either component of $\mathbb{U} \setminus \mathbb{T}$ is $\mathscr{C}^\infty$-smooth with $\|Df\| \leqslant M$ and $\det Df \geqslant M^{-1}$.

Then for any open set W with $\mathbb{T} \subset W \subset \mathbb{U}$ there is a $\mathscr{C}^\infty$-diffeomorphism $g \colon \mathbb{U} \to \mathbb{U}^*$ such that

- *g agrees with f on $\mathbb{U} \setminus W$ and on $\mathbb{T}$;*
- *$\|Dg\| \leqslant 80M$ and $\det Dg \geqslant (80M)^{-1}$ on $\mathbb{U}$.*

Proof. It is convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$. Without loss of generality $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$. Let $\psi(\zeta) = \exp(i\zeta)$. The mapping $F = \psi^{-1} \circ f \circ \psi$ is well-defined in some open horizontal strip $S_h = \{z \in \mathbb{C} \colon |\operatorname{Im} z| < \epsilon\}$ which we choose thin enough so that $\psi(S_\epsilon) \subset W$ and $|\psi'|^2 < e^{2\epsilon} \leqslant 2$. Note that F is 2π-periodic and satisfies

$$\|DF\| \leqslant 2M \quad \text{and} \quad \det DF \geqslant (2M)^{-1}.$$

The proof of Proposition 5.1 applies to F with no changes other than one simplification: $\beta > 0$ is now a small positive constant rather than a function. Thus we obtain a diffeomorphism G which agrees with F on $\mathbb{R} \cup (S \setminus V(\beta))$ and satisfies $\|DG\| \leqslant 40$ and $\det DG \geqslant (40M)^{-1}$. Since F was 2π-periodic, so is G. Thus, $g := \psi \circ G \circ \psi^{-1}$ is the desired diffeomorphism. □

Our applications require slightly more general versions of Proposition 5.1 and Corollary 5.2, where the separating curve is allowed to have other shapes and f is not required to agree with the identity on the curve.

Definition 5.3. A parametric curve $\Gamma\colon (0,1) \to \mathbb{R}^2$ is *regular* if Γ extends to a bigger interval $(a,b) \supset [0,1]$ so that the extended mapping is a $\mathscr{C}^\infty$-diffeomorphism onto its image.

Note that a regular curve Γ has well-defined endpoints $\Gamma(0)$ and $\Gamma(1)$. Also, Γ extends to an injective $\mathscr{C}^\infty$-mapping $\Phi\colon (0,1) \times (-1,1) \to \mathbb{R}^2$ such that $\|D\Phi\|$ and $\|(D\Phi)^{-1}\|$ are bounded. This follows from the existence of a tubular neighborhood of the image of Γ [23, Theorem 4.26].

Corollaries 5.4 and 5.5, given below, generalize Proposition 5.1 and Corollary 5.2 respectively.

Corollary 5.4. *Let $\mathbb{U} \subset \mathbb{R}^2$ be a domain containing the image of a regular arc Γ with endpoints on the boundary $\partial\mathbb{U}$ which divides $\mathbb{U}$ into two subdomains $\mathbb{U}_1$ and $\mathbb{U}_2$ such that $\mathbb{U} \setminus \Gamma = \mathbb{U}_1 \cup \mathbb{U}_2$. Suppose $f\colon \overline{\mathbb{U}} \to \overline{\mathbb{U}^*} \subset \mathbb{R}^2$ is a homeomorphism such that $f \circ \Gamma$ is also regular and the restriction of f to each $\overline{\mathbb{U}_i}$ is $\mathscr{C}^\infty$-smooth and satisfies*

$$|Df(z)| \leqslant M, \quad \det Df(z) \geqslant \frac{1}{M} \quad \textit{for } z \in \mathbb{U}_i$$

where M is a positive constant. Then there is a constant $M' > 0$ such that to every open set $\mathbb{U}' \subset \mathbb{U}$ with $\Gamma \subset \mathbb{U}'$ there corresponds a $\mathscr{C}^\infty$-diffeomorphism $g\colon \mathbb{U} \xrightarrow{\text{onto}} \mathbb{U}^*$ with the following properties

- *$g(z) = f(z)$ for $z \in \mathbb{U} \setminus \mathbb{U}'$ (and also on Γ)*
- *$|Dg(z)| \leqslant M'$ and $\det Dg(z) \geqslant \frac{1}{M'}$ on $\mathbb{U}$.*

Proof. Let $\mathbb{Q} = (0,1)\times(-1,1)$. Let Φ and Ψ be the extensions of Γ and $f\circ\Gamma$ to $\mathbb{Q}$ as in Definition 5.3. There is a domain $\widetilde{\mathbb{U}}$ such that $(0,1)\times\{0\} \subset \widetilde{\mathbb{U}} \subset \mathbb{Q}$, $\Phi(\widetilde{\mathbb{U}}) \Subset \mathbb{U}'$, and the composition $F := \Psi^{-1} \circ f \circ \Phi$ is defined in $\widetilde{\mathbb{U}}$. Note that $F = \mathrm{id}$ on $(0,1) \times \{0\}$. We apply Proposition 5.1 (with $\widetilde{\mathbb{U}}$ in place of $\mathbb{U}$ and with F in place of f) and obtain a $\mathscr{C}^\infty$-diffeomorphism $G\colon \widetilde{\mathbb{U}} \to F(\widetilde{\mathbb{U}})$. Finally, replace F within $\widetilde{\mathbb{U}}$ with the diffeomorphism $g = \Psi \circ G \circ \Phi^{-1}$. □

Corollary 5.5. *Let $\mathbb{U} \subset \mathbb{R}^2$ be a domain containing the image of a $\mathscr{C}^\infty$-smooth Jordan curve Γ which divides $\mathbb{U}$ into two subdomains $\mathbb{U}_1$ and $\mathbb{U}_2$ such that $\mathbb{U} \setminus \Gamma = \mathbb{U}_1 \cup \mathbb{U}_2$. Suppose $f\colon \overline{\mathbb{U}} \to \overline{\mathbb{U}^*} \subset \mathbb{R}^2$ is a homeomorphism such that the restriction of f to each $\overline{\mathbb{U}_i}$ is $\mathscr{C}^\infty$-smooth and satisfies*

$$|Df(z)| \leqslant M, \quad \det Df(z) \geqslant \frac{1}{M} \quad \textit{for } z \in \mathbb{U}_i$$

where M is a positive constant. Then there is a constant $M' > 0$ such that to every open set $\mathbb{U}' \subset \mathbb{U}$ with $\Gamma \subset \mathbb{U}'$ there corresponds a $\mathscr{C}^\infty$-diffeomorphism $g\colon \mathbb{U} \xrightarrow{\text{onto}} \mathbb{U}^*$ with the following properties

- *$g(z) = f(z)$ for $z \in \mathbb{U} \setminus \mathbb{U}'$ (and also on Γ)*

- $|Dg(z)| \leqslant M'$ *and* $\det Dg(z) \geqslant \frac{1}{M'}$ *on* $\mathbb{U}$.

Proof. The proof of Corollary 5.4 is easily adapted to this case. □

6. Concluding remarks

One may wonder whether the proof of Theorem 1.2 can be extended to the spaces $\mathscr{W}^{1,p}$, $1 < p < \infty$, by means of the p-harmonic replacement in place of Proposition 2.3. Indeed, p-harmonic mappings are $\mathscr{C}^{1,\alpha}$-smooth [36]. However, the injectivity of p-harmonic replacement of a homeomorphism is unclear.

Question 6.1. Is there a version of the Radó-Kneser-Choquet theorem for p-harmonic mappings? That is, does the p-harmonic extension of a homeomorphism onto a convex Jordan curve enjoy the injectivity property?

An attempt to extend Theorem 1.2 to higher dimensions faces another obstacle: the Radó-Kneser-Choquet theorem fails in dimensions $n \geqslant 3$ as was proved by Laugesen [22].

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