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HOPF DIFFERENTIALS AND SMOOTHING SOBOLEV HOMEOMORPHISMS

TADEUSZ IWANIEC, LEONID V. KOVALEV, AND JANI ONNINEN

ABSTRACT. We prove that planar homeomorphisms can be approximated by diffeomorphisms in the Sobolev space $\mathcal{W}^{1,2}$ and in the Royden algebra. As an application, we show that every discrete and open planar mapping with a holomorphic Hopf differential is harmonic.

1. INTRODUCTION

It is a fundamental property of Sobolev spaces $\mathcal{W}^{1,p}$, $1 \leq p < \infty$, that any element can be approximated strongly (i.e., in the norm) by \mathcal{C}^∞ smooth functions, or by piecewise affine ones. In the context of vector-valued Sobolev functions, that is, mappings in $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^n)$, invertibility comes into play. Indeed, the studies of invertible Sobolev mappings are of great importance in nonlinear elasticity [2, 14, 26, 35]. The following natural question was put forward by John M. Ball.

Question 1.1. [4] If $u \in \mathcal{W}^{1,p}(\Omega, \mathbb{R}^n)$ is invertible, can u be approximated in $\mathcal{W}^{1,p}$ by piecewise affine invertible mappings?

J. Ball attributes this question to L.C. Evans, who was led to it through his investigation of the partial regularity of minimizers [13] of neo-hookean energy functionals [3, 5, 7, 33]. We provide an affirmative solution of the Ball-Evans problem in the case $p = n = 2$. The most general formulation of our result administers Royden algebras $\mathcal{A}(\Omega)$ and $\mathcal{A}_\circ(\Omega)$, see Section 2. We write

$$\mathcal{E}[h] = \mathcal{E}_\Omega[h] := \|Dh\|_{\mathcal{L}^2(\Omega)}^2 = \int_\Omega |Dh(z)|^2 dz$$

where $|Dh|$ is the Hilbert-Schmidt norm of the differential.

Theorem 1.2 (Approximation by diffeomorphisms). *Let $h: \Omega \xrightarrow{\text{onto}} \Omega^*$ be a homeomorphism of Sobolev class $\mathcal{W}_{\text{loc}}^{1,2}(\Omega, \Omega^*)$. Then for every $\epsilon > 0$ there exist a diffeomorphism $H: \Omega \xrightarrow{\text{onto}} \Omega^*$ such that*

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- (i) $H - h \in \mathcal{A}_o(\Omega)$
- (ii) $\|H - h\|_{\mathcal{A}(\Omega)} \leq \epsilon$
- (iii) $\mathcal{E}[H] \leq \mathcal{E}[h]$.

Part (iii) is nontrivial only in the finite energy case, $\mathcal{E}_\Omega[h] < \infty$. Let us note that the existence of smooth approximation implies the existence of piecewise-affine approximation, since a diffeomorphism can be triangulated. (In the converse direction, a piecewise-affine mapping can be smoothed in dimensions less than four [25], but not in general.) Partial results toward the Ball-Evans problem were obtained in [24] (for planar bi-Sobolev mappings that are smooth outside of a finite set) and in [6] (for planar bi-Hölder mappings, with approximation in the Hölder norm). The articles [4, 32] illustrate the difficulty of preserving invertibility in the process of smoothing a Sobolev homeomorphism.

We also give an application of Theorem 1.2 to a problem that originated in a series of papers by Eells, Lemaire and Sealey [11, 12, 31]. It concerns the nonlinear differential equation

$$(1.1) \quad \frac{\partial}{\partial \bar{z}} (h_z \bar{h}_{\bar{z}}) = 0$$

for mappings defined in a domain in the complex plane \mathbb{C} . Naturally, the Sobolev space $\mathcal{W}_{\text{loc}}^{1,2}(\Omega, \mathbb{C})$ should be considered as the domain of definition of equation (1.1). This places $h_z \bar{h}_{\bar{z}}$ in $\mathcal{L}_{\text{loc}}^1(\Omega)$, so the complex Cauchy-Riemann derivative $\frac{\partial}{\partial \bar{z}}$ applies in the sense of distribution. By Weyl's lemma $h_z \bar{h}_{\bar{z}}$ is a holomorphic function.

The expression $Q_h := h_z \bar{h}_{\bar{z}} dz \otimes dz$ is known as the Hopf differential of h (named after H. Hopf, who employed a similar device, e.g., in [19, Chapter VI]). It is clear that Q_h is a holomorphic quadratic differential whenever h is harmonic, which is a general fact about energy-stationary mappings between Riemannian manifolds [10, (10.5)], [21] and [34]. Eells and Lemaire inquired about the possibility of a converse result, e.g., for mappings with finite energy and almost-everywhere positive Jacobian [11, (2.6)]. In this setting a counterexample was provided by Jost [20], who also proved the existence of $\mathcal{W}^{1,2}$ -solutions of (1.1) in every homotopy class of mappings between compact Riemann surfaces. A more restricted form of the Eells-Lemaire problem, [12, (5.11)] and [31], imposed the additional assumption that h is a quasiconformal homeomorphism, and was settled by Hélein [17] in the affirmative. Here we dispose with the quasiconformality condition and treat general planar homeomorphisms of finite energy. Since the inverse of such a homeomorphism need not be in any Sobolev class [18], some difficulties are to be expected. They shall be overcome with the aid of our approximation theorem 1.2.

Theorem 1.3. *Every continuous, discrete and open mapping h of Sobolev class $\mathcal{W}_{\text{loc}}^{1,2}(\Omega, \mathbb{C})$ that satisfies equation (1.1) is harmonic.*

The failure of Theorem 1.3 for uniform limits of homeomorphisms should be mentioned. This is illustrated by Example 4.1.

2. BACKGROUND

Let Ω be a bounded domain in $\mathbb{R}^2 \simeq \mathbb{C}$, nonempty open connected set. We consider a class $\mathcal{A}(\Omega)$ of uniformly continuous functions $h: \Omega \rightarrow \mathbb{C}$ having finite Dirichlet energy, and furnish it with the norm

$$\|h\|_{\mathcal{A}(\Omega)} = \|h\|_{\mathcal{C}(\Omega)} + \|Dh\|_{\mathcal{L}^2(\Omega)} < \infty$$

$\mathcal{A}(\Omega)$ is a commutative Banach algebra with the usual multiplication of functions in which $\|h_1 h_2\|_{\mathcal{A}(\Omega)} \leq \|h_1\|_{\mathcal{A}(\Omega)} \|h_2\|_{\mathcal{A}(\Omega)}$. The closure of $\mathcal{C}_o^\infty(\Omega)$ in $\mathcal{A}(\Omega)$ will be denoted by $\mathcal{A}_o(\Omega)$. Suppose, to look at more specific situation, that $\Omega = \mathbb{U}$ is a Jordan domain; that is, a simply connected open set whose boundary $\Gamma = \partial\mathbb{U}$ is a closed Jordan curve. By goodness of the Carathéodory extension theorem [27, p. 18], there is a homeomorphism $\varphi: \overline{\mathbb{D}} \xrightarrow{\text{onto}} \overline{\mathbb{U}}$ of the closed unit disk $\overline{\mathbb{D}} = \{\xi: |\xi| \leq 1\}$ that is conformal in \mathbb{D} . After the change of variable, $z = \varphi(\xi)$, we obtain a function $H(\xi) = h(\varphi(\xi))$ in $\mathcal{A}(\mathbb{D})$. The operation

$$\mathbf{T}_\varphi: \mathcal{A}(\mathbb{U}) \rightarrow \mathcal{A}(\mathbb{D})$$

so defined is an isometry; $\|\mathbf{T}_\varphi h\|_{\mathcal{A}(\mathbb{D})} = \|h\|_{\mathcal{A}(\mathbb{U})}$. Furthermore,

$$\mathbf{T}_\varphi: \mathcal{A}_o(\mathbb{U}) \rightarrow \mathcal{A}_o(\mathbb{D})$$

Proposition 2.1 (A generalization of Poisson's formula). *Let \mathbb{U} be a Jordan domain. There is (unique) bounded linear operator*

$$\mathbf{P}_\mathbb{U}: \mathcal{A}(\mathbb{U}) \rightarrow \mathcal{A}(\mathbb{U})$$

such that

$$\begin{cases} \mathbf{P}_\mathbb{U} - \text{Id}: \mathcal{A}(\mathbb{U}) \rightarrow \mathcal{A}_o(\mathbb{U}) \\ \Delta \circ \mathbf{P}_\mathbb{U} = 0 \end{cases}$$

We name $\mathbf{P}_\mathbb{U}$ the *Poisson operator*. The energy of $\mathbf{P}_\mathbb{U}h$ does not exceed that of h . This fact is known as *Dirichlet's principle*

$$\int_{\mathbb{U}} |D\mathbf{P}_\mathbb{U}h|^2 \leq \int_{\mathbb{U}} |Dh|^2$$

The proof of this proposition reduces to the case when $\mathbb{U} = \mathbb{D}$, by conformal change of variables. A routine verification of this case is left to the reader. We only indicate that the less familiar property $\mathbf{P}_\mathbb{D}h - h \in \mathcal{A}_o(\mathbb{D})$, for $h \in \mathcal{A}(\mathbb{D})$, needs to be justified.

Corollary 2.2 (Harmonic replacement). *Let Ω be a domain in \mathbb{C} and $\mathbb{U} \subset \overline{\mathbb{U}} \subset \Omega$ a Jordan domain. There exists (unique) bounded linear operator*

$$\mathbf{R}_\mathbb{U}: \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$$

such that, for every $h \in \mathcal{A}(\Omega)$

$$\begin{cases} \mathbf{R}_{\mathbb{U}}h = h & \text{on } \Omega \setminus \mathbb{U} \\ \Delta \mathbf{R}_{\mathbb{U}}h = 0 & \text{in } \mathbb{U} \end{cases}$$

The Laplace equation yields $\mathcal{E}_{\Omega}[\mathbf{R}_{\mathbb{U}}h] \leq \mathcal{E}_{\Omega}[h]$. Equality occurs if and only if h is harmonic in \mathbb{U} .

A short proof of this corollary runs somewhat as follows. The unique harmonic extension of $h: \partial\mathbb{U} \rightarrow \mathbb{C}$ inside \mathbb{U} given by $\mathbf{P}_{\mathbb{U}}h$ has the property that $\mathbf{P}_{\mathbb{U}}h - h \in \mathcal{A}_{\circ}(\mathbb{U})$. Therefore, the zero extension of $\mathbf{P}_{\mathbb{U}}h - h$ outside \mathbb{U} , denoted by $[\mathbf{P}_{\mathbb{U}}h - h]_{\circ}$, belongs to $\mathcal{A}(\Omega)$. We define

$$\mathbf{R}_{\mathbb{U}}h := [\mathbf{P}_{\mathbb{U}}h - h]_{\circ} + h \in \mathcal{A}(\Omega)$$

The desired properties of the operator $\mathbf{R}_{\mathbb{U}}$ so defined are automatically fulfilled in view of Proposition 2.1.

Proposition 2.3. *Let Ω be a domain in \mathbb{C} and $\mathbb{U} \subset \bar{\mathbb{U}} \subset \Omega$ a Jordan domain. Suppose that $h \in \mathcal{A}(\Omega)$ is a homeomorphism of Ω onto $h(\Omega)$ and $h(\mathbb{U})$ is convex. Then $\mathbf{R}_{\mathbb{U}}h$ is homeomorphism in Ω and is a harmonic diffeomorphism in \mathbb{U} .*

The injectivity of $\mathbf{R}_{\mathbb{U}}h$ is the content of the Radó-Kneser-Choquet Theorem [9, p. 29]. Furthermore, planar harmonic homeomorphisms are \mathcal{C}^{∞} -smooth diffeomorphisms according to Lewy's theorem [9, p. 20].

3. SMOOTHING SOBOLEV HOMEOMORPHISMS, THEOREM 1.2

Proof of Theorem 1.2. We may and do assume that h is not harmonic, since otherwise $H = h$ satisfies the desired properties by Lewy's theorem (mentioned above). Let $z_{\circ} \in \Omega$ be a point such that h fails to be harmonic in any neighborhood of z_{\circ} . By choosing the origin of the coordinate system we ensure that $h(z_{\circ})$ does not lie on the boundary of any dyadic squares associated with the coordinate system.

Let us choose and fix any $\epsilon > 0$. The construction of H proceeds in 5 steps. We construct homeomorphisms $h_k: \Omega \xrightarrow{\text{onto}} \Omega^*$, $k = 0, \dots, 5$ such that $h_0 = h$, $h_k \in h_{k-1} + \mathcal{A}_{\circ}(\Omega)$, h_5 is a diffeomorphism, and $\|h_k - h_{k-1}\|$ is bounded by a multiple of ϵ for each k . In each step we modify the previous construction to gain better regularity. In steps 1, 2 and 4 we use harmonic replacement according to Proposition 2.3. In steps 3 and 5 we smoothen the mapping near the boundaries of the domains in which harmonic replacement was performed. The result of each step is denoted by h_1, \dots, h_5 . The finite energy case $h \in \mathcal{A}(\Omega)$ requires a few additional details, which are provided at the end of each step.

We begin with a decomposition of the target domain

$$(3.1) \quad \Omega^* = \bigcup_{\nu=1}^{\infty} \bar{\mathbb{Q}}_{\nu}$$

into closed nonoverlapping dyadic squares $\overline{Q}_\nu \subset \Omega^*$. This decomposition is made by selecting the maximal dyadic squares that lie in Ω^* . Thus the cover of Ω^* by such squares is locally finite. The preimage of Q_ν under h , denoted by U_ν , is a Jordan domain in Ω . Hereafter U_ν will be referred to as the curved-square. In fact to every partition of Ω^* into closed squares there will correspond a partition of Ω into closed curved-squares via the mapping $h: \Omega \xrightarrow{\text{onto}} \Omega^*$, for example:

$$\Omega = \bigcup_{\nu=1}^{\infty} \overline{U}_\nu$$

Step 1. For each U_ν we replace $h: \overline{U}_\nu \xrightarrow{\text{onto}} \overline{Q}_\nu$ with a piecewise harmonic homeomorphism $h_1: \overline{U}_\nu \xrightarrow{\text{onto}} \overline{Q}_\nu$ that coincides with h on $\partial\overline{U}_\nu$. To this effect we partition the square \overline{Q}_ν ,

$$(3.2) \quad \overline{Q}_\nu = \overline{Q}_\nu^1 \cup \overline{Q}_\nu^2 \cup \dots \cup \overline{Q}_\nu^n, \quad (n = n_\nu = 4^{k_\nu})$$

into congruent dyadic squares \overline{Q}_ν^i , $i = 1, \dots, n$. The number n , depending on ν , will be determined later. For the moment fix ν and look at the homeomorphisms

$$h: \overline{U}_\nu^i = h^{-1}(\overline{Q}_\nu^i) \xrightarrow{\text{onto}} \overline{Q}_\nu^i$$

These mappings belong to the Royden algebra $\mathcal{A}(\overline{U}_\nu^i)$. With the aid of Propositions 2.1 and 2.3 we replace each $h: \overline{U}_\nu^i \rightarrow \overline{Q}_\nu^i$ with a harmonic homeomorphism $h_\nu^i: \overline{U}_\nu^i \xrightarrow{\text{onto}} \overline{Q}_\nu^i$ that coincides with h on $\partial\overline{U}_\nu^i$, $i = 1, 2, \dots, n$. Such mappings are \mathcal{C}^∞ -smooth diffeomorphisms $h_\nu^i: U_\nu^i \xrightarrow{\text{onto}} Q_\nu^i$. Moreover, $h_\nu^i - h \in \mathcal{A}_o(U_\nu^i)$ and

$$(3.3) \quad \begin{cases} \mathcal{E}_{U_\nu^i}[h_\nu^i] \leq \mathcal{E}_{U_\nu^i}[h] & \text{for } 1, 2, \dots, n \\ \mathcal{E}_{\partial U_\nu^i}[h_\nu^i] = \mathcal{E}_{\partial U_\nu^i}[h], & \text{because } h_\nu^i = h \text{ on } \partial U_\nu^i \end{cases}$$

We obtain a piecewise harmonic homeomorphism by gluing h_ν^i together along the common boundaries of U_ν^i . Denote it by

$$h_\nu^n: \overline{U}_\nu \xrightarrow{\text{onto}} \overline{Q}_\nu$$

$$h_\nu^n \in h + \mathcal{A}_o(U_\nu)$$

Precisely we define

$$h_\nu^n = h + \sum_{i=1}^n [h_\nu^i - h]_o$$

Here and in the sequel the notation $[\varphi]_o$ for $\varphi \in \mathcal{A}_o(U)$ stands for zero extension of φ to the entire domain Ω . Obviously $[\varphi]_o \in \mathcal{A}_o(\Omega)$. The above construction depends on the number n . For ν fixed we actually have a sequence $\{h_\nu^n\}_{n=1,2,\dots}$ that is bounded in $\mathcal{A}(U_\nu)$. However, we have uniform bounds independent of n ,

$$\|h_\nu^n\|_{\mathcal{C}(\overline{U}_\nu)} \leq \text{diam } Q_\nu$$

and

$$\mathcal{E}_{\mathbb{U}_\nu}[h_\nu^n] \leq \mathcal{E}_{\mathbb{U}_\nu}[h]$$

The key observation is that

$$(3.4) \quad \begin{cases} h_\nu^n - h \in \mathcal{A}_\circ(\mathbb{U}_\nu) \\ \lim_{n \rightarrow \infty} \|h_\nu^n - h\|_{\mathcal{A}(\mathbb{U}_\nu)} = 0 \end{cases}$$

Indeed, for $z \in \overline{\mathbb{U}_\nu^i}$ we have

$$|h_\nu^n(z) - h(z)| \leq \text{diam } \mathbb{Q}_\nu^i = \frac{1}{\sqrt{n}} \text{diam } \mathbb{Q}_\nu$$

Thus $h_\nu^n \rightrightarrows h$ uniformly on $\overline{\mathbb{U}_\nu}$ as $n \rightarrow \infty$. On the other hand the differential matrices Dh_ν^n are bounded in $\mathcal{L}^2(\mathbb{U}_\nu, \mathbb{R}^{2 \times 2})$. Their weak limit exists and is exactly equal to Dh , because the mappings converge uniformly to h . Therefore,

$$\begin{aligned} \int_{\mathbb{U}_\nu} |Dh_\nu^n - Dh|^2 &= \int_{\mathbb{U}_\nu} (|Dh_\nu^n|^2 + |Dh|^2 - 2\langle Dh_\nu^n, Dh \rangle) \\ &\leq 2 \int_{\mathbb{U}_\nu} (|Dh|^2 - \langle Dh_\nu^n, Dh \rangle) \\ &= 2 \int_{\mathbb{U}_\nu} \langle Dh, Dh - Dh_\nu^n \rangle \rightarrow 0 \end{aligned}$$

We can now determine the number $n = n_\nu$ of congruent dyadic squares in \mathbb{Q}_ν , simply requiring that

$$\begin{cases} \text{diam } \mathbb{Q}_\nu^i \leq \epsilon & \text{for every } i = 1, 2, \dots, n_\nu \\ \|Dh_\nu^n - Dh\|_{\mathcal{L}^2(\overline{\mathbb{U}_\nu})} \leq \epsilon \cdot 2^{-\nu} \end{cases}$$

Fix such $n = n_\nu$ and abbreviate the notation for $h_\nu^{n_\nu}$ to h^ν . We obtain a homeomorphism

$$h_1 := h + \sum_{\nu=1}^{\infty} [h^\nu - h]_\circ \in h + \mathcal{A}_\circ(\Omega)$$

where we recall that $[h^\nu - h]_\circ$ stands for the zero extension of $h^\nu - h$ to the entire domain Ω . Clearly, h_1 is harmonic in each \mathbb{U}_ν^i , $\nu = 1, 2, \dots$, $i = 1, 2, \dots, n_\nu$ and we have

$$\|h_1 - h\|_{\mathcal{C}(\Omega)} \leq \sup\{\text{diam } \mathbb{Q}_\nu^i : \nu = 1, 2, \dots, i = 1, \dots, n_\nu\} < \epsilon$$

$$(3.5) \quad \|h_1 - h\|_{\mathcal{A}(\Omega)} \leq \epsilon + \sum_{\nu=1}^{\infty} \|Dh^\nu - Dh\|_{\mathcal{L}^2(\mathbb{U}_\nu)} \leq \epsilon + \sum_{\nu=1}^{\infty} \epsilon \cdot 2^{-\nu} = 2\epsilon$$

For further considerations it will be convenient to number the squares \mathbb{Q}_ν^i and their preimages \mathbb{U}_ν^i using only one index. These sets will be respectively denoted by \mathbb{Q}^α and \mathbb{U}^α , $\alpha = 1, 2, \dots$. For the record,

$$(3.6) \quad \text{diam } \mathbb{Q}^\alpha \leq \epsilon, \quad \alpha = 1, 2, \dots$$

Finite energy case. Summing up the energy inequalities for the mappings $h_\nu^i: \mathbb{U}_\nu^i \rightarrow \mathbb{Q}_\nu^i$ we see that the total energy of h_1 does not exceed the energy of h . Even more, since h was assumed to be not harmonic, there is at least one region \mathbb{U}_ν^i for which $h: \mathbb{U}_\nu^i \rightarrow \mathbb{Q}_\nu^i$ was not harmonic. Consequently, its harmonic replacement results in strictly smaller energy. Hence

$$(3.7) \quad \mathcal{E}_\Omega[h_1] < \mathcal{E}_\Omega[h], \quad \text{so let } \delta = \|Dh\|_{\mathcal{L}^2(\Omega)} - \|Dh_1\|_{\mathcal{L}^2(\Omega)} > 0$$

Step 2. Denote by $\mathcal{F} = \{\mathbb{Q}^\alpha: \alpha = 1, 2, \dots\}$ the family of all open squares $\mathbb{Q}^\alpha \subset \overline{\mathbb{Q}^\alpha} \subset \Omega^*$ that are build in Step 1 for the construction of the mapping $h_1: \Omega \rightarrow \Omega^*$. Let \mathcal{V} be the set of vertices of these squares. Whenever two squares $\mathbb{Q}^\alpha, \mathbb{Q}^\beta \in \mathcal{F}$, $\alpha \neq \beta$, meet along their boundaries the intersection $I^{\alpha,\beta} = \partial\mathbb{Q}^\alpha \cap \partial\mathbb{Q}^\beta$ is either a point in \mathcal{V} or a closed interval with endpoints in \mathcal{V} . Denote by $\mathcal{J} \subset \{I^{\alpha,\beta}: \alpha \neq \beta, \alpha, \beta = 1, 2, \dots\}$ the subfamily of all such intersections, excluding empty set and vertices. For each interval $I^{\alpha,\beta} \in \mathcal{J}$ we shall construct a doubly convex lens-shaped region $\mathbb{L}^{\alpha,\beta}$ with $I^{\alpha,\beta}$ as its axis of symmetry in the following way. Let R be a number greater than the length of $I^{\alpha,\beta}$ to be chosen later. There exist exactly two open disks of radius R for which $I^{\alpha,\beta}$ is a chord. Let $\mathbb{L}_R^{\alpha,\beta}$ be their intersection. This is a symmetric doubly convex lens of curvature $\frac{1}{R}$. Thus $\mathbb{L}_R^{\alpha,\beta}$ is bounded by two circular arcs $\gamma^{\alpha,\beta} = \mathbb{Q}^\alpha \cap \partial\mathbb{L}_R^{\alpha,\beta}$ and $\gamma^{\beta,\alpha} = \mathbb{Q}^\beta \cap \partial\mathbb{L}_R^{\alpha,\beta}$. As the curvature of the lens approaches zero the area of $\mathbb{L}_R^{\alpha,\beta}$ tends to 0. This allows us to choose R depending on α and β so that the lenses $\mathbb{L}^{\alpha,\beta} = \mathbb{L}_R^{\alpha,\beta}$ have the following property.

$$(3.8) \quad \int_{\mathbb{K}^{\alpha,\beta}} |Dh_1|^2 < \frac{\epsilon^2}{2^{\alpha+\beta}}, \quad \text{where } \mathbb{K}^{\alpha,\beta} = h_1^{-1}(\mathbb{L}_R^{\alpha,\beta})$$

The lenses $\mathbb{L}^{\alpha,\beta}$ are disjoint because the opening angle of each lens is at most $\pi/3$ and their axes are either parallel or orthogonal. However, the closures of the lenses considered here may have a common point that lies in \mathcal{V} . On each $\mathbb{K}^{\alpha,\beta}$ we replace h_1 by the harmonic extension of its restriction to $\partial\mathbb{K}^{\alpha,\beta}$. Thus we obtain a homeomorphism $h_2^{\alpha,\beta}: \overline{\mathbb{K}^{\alpha,\beta}} \xrightarrow{\text{ont}\mathcal{Q}} \mathbb{L}^{\alpha,\beta}$ of class $h_1 + \mathcal{A}_\circ(\mathbb{K}^{\alpha,\beta})$. By Proposition 2.3 the mappings $h_2^{\alpha,\beta}: \mathbb{K}^{\alpha,\beta} \xrightarrow{\text{ont}\mathcal{Q}} \mathbb{L}^{\alpha,\beta}$ are diffeomorphisms. Finally, we define

$$h_2 = h_1 + \sum_{\alpha,\beta} [h_2^{\alpha,\beta} - h_1]_\circ \in h_1 + \mathcal{A}_\circ(\Omega) = h + \mathcal{A}_\circ(\Omega)$$

and observe that, from (3.6),

$$\|h_2 - h_1\|_{\mathcal{C}(\Omega)} \leq \sup_{\alpha,\beta} \text{diam} \left(\mathbb{L}^{\alpha,\beta} \right) \leq \epsilon.$$

Also, (3.8) and Dirichlet's principle imply

$$\begin{aligned} \int_{\Omega} |Dh_2 - Dh_1|^2 &\leq \sum_{\alpha,\beta} \int_{\mathbb{K}^{\alpha,\beta}} 2(|Dh_2|^2 + |Dh_1|^2) \leq 4 \sum_{\alpha,\beta=1}^{\infty} \int_{\mathbb{K}^{\alpha,\beta}} |Dh_1|^2 \\ &\leq 4 \sum_{\alpha,\beta=1}^{\infty} \frac{\epsilon^2}{2^{\alpha+\beta}} = 4\epsilon^2 \end{aligned}$$

Thus

$$\|h_2 - h_1\|_{\mathcal{A}(\Omega)} \leq \epsilon + 2\epsilon = 3\epsilon$$

The boundary of $\mathbb{K}^{\alpha,\beta}$ consists of two \mathcal{C}^∞ -smooth arcs $\Gamma^{\alpha,\beta}$ and $\Gamma^{\beta,\alpha}$ which share common endpoints, called the apices of $\mathbb{K}^{\alpha,\beta}$. These are preimages of $\gamma^{\alpha,\beta}$ and $\gamma^{\beta,\alpha}$ under the mapping h_1 , respectively. Outside of the apices, the homeomorphism $h_2: \overline{\mathbb{K}^{\alpha,\beta}} \xrightarrow{\text{onto}} \overline{\mathbb{L}^{\alpha,\beta}}$ is C^∞ smooth with positive Jacobian. The smoothness is a classical result of Kellogg; *a harmonic function with \mathcal{C}^∞ -smooth values on a smooth part of the boundary is \mathcal{C}^∞ -smooth up to this part of the boundary* [16, Theorem 6.19]. The positivity of the Jacobian on such part of the boundary follows from the convexity of its image, see [9, p. 116].

In conclusion, h_2 is locally bi-Lipschitz in $\Omega \setminus h^{-1}(\mathcal{V})$. The exceptional set $h^{-1}(\mathcal{V})$ is discrete because \mathcal{V} is.

Finite energy case. By (3.7) we have

$$(3.9) \quad \|Dh_2\|_{\mathcal{L}^2(\Omega)} \leq \|Dh_1\|_{\mathcal{L}^2(\Omega)} \leq \|Dh\|_{\mathcal{L}^2(\Omega)} - \delta$$

Step 3. First we cover the set of vertices \mathcal{V} by disks $\{\mathbb{D}_v: v \in \mathcal{V}\}$ centered at v with radii small enough so that

$$(3.10) \quad \text{diam } \mathbb{D}_v \leq \epsilon,$$

and $\{3\mathbb{D}_v: v \in \mathcal{V}\}$ is a disjoint collection of disks in Ω^* . Moreover, their preimages under h_2 must satisfy

$$(3.11) \quad \sum_{v \in \mathcal{V}} \int_{h_2^{-1}(3\mathbb{D}_v)} |Dh_2|^2 < \epsilon^2$$

Denote by $\tilde{\Omega}^* = \Omega^* \setminus \bigcup_{v \in \mathcal{V}} \overline{\mathbb{D}_v}$ and $\tilde{\Omega} = \Omega \setminus \bigcup_{v \in \mathcal{V}} h_2^{-1}(\overline{\mathbb{D}_v})$. Our focus for a while will be on one of the circular sides of a lens $\mathbb{L}^{\alpha,\beta}$, say

$$\gamma^{\alpha,\beta} = \mathbb{Q}^\alpha \cap \partial \mathbb{L}^{\alpha,\beta} \subset \mathbb{Q}^\alpha$$

We truncate it near the endpoints by setting $\tilde{\gamma}^{\alpha,\beta} = \tilde{\Omega} \cap \gamma^{\alpha,\beta}$. Such truncated open arcs are mutually disjoint; even more, their closures are isolated continua in Ω^* . This means that there are disjoint neighborhoods of them. We are actually interested in a neighborhood of $\tilde{\gamma}^{\alpha,\beta}$ of the shape of a thin *concavo-convex lens* that we shall denote by $\tilde{\mathbb{L}}^{\alpha,\beta}$. By definition, $\tilde{\gamma}^{\alpha,\beta} \subset \tilde{\mathbb{L}}^{\alpha,\beta} \subset \mathbb{Q}^\alpha$. The construction of such lens goes as follows. Let a and b denote the endpoints of $\tilde{\gamma}^{\alpha,\beta}$, we assemble two circular arcs $\tilde{\gamma}_+^{\alpha,\beta}$ and $\tilde{\gamma}_-^{\alpha,\beta}$ with endpoints at a and b to form together with their endpoints a

concavo-convex Jordan curve. This Jordan curve constitutes the boundary of a circular lens $\tilde{\mathbb{L}}^{\alpha,\beta}$. The term concavo-convex lens refers to the configuration in which $\tilde{\mathbb{L}}^{\alpha,\beta}$ lies in the concave side of the arc $\tilde{\gamma}_-^{\alpha,\beta}$ and convex side of $\tilde{\gamma}_+^{\alpha,\beta}$. It is clear that such lenses can be made arbitrarily thin so that $\tilde{\mathbb{L}}^{\alpha,\beta} \subset \tilde{\Omega}^*$ and the closures of $\tilde{\mathbb{L}}^{\alpha,\beta}$ will still be isolated continua in Ω^* . From now on we fix the family $\{\tilde{\mathbb{L}}^{\alpha,\beta} : \alpha \neq \beta\}$ of such concavo-convex lenses associated with the arcs $\tilde{\gamma}^{\alpha,\beta}$. We then look at their preimages $\mathbb{U}^{\alpha,\beta} = h_2^{-1}(\tilde{\mathbb{L}}^{\alpha,\beta})$ and the \mathcal{C}^∞ -smooth arcs $\Upsilon^{\alpha,\beta} = h_2^{-1}(\tilde{\gamma}^{\alpha,\beta})$. The endpoints of $\Upsilon^{\alpha,\beta}$ lie in $\partial\mathbb{U}^{\alpha,\beta}$. Moreover, $\Upsilon^{\alpha,\beta}$ splits $\mathbb{U}^{\alpha,\beta}$ into two disjoint subdomains $\mathbb{U}_+^{\alpha,\beta}$ and $\mathbb{U}_-^{\alpha,\beta}$ such that $\mathbb{U}^{\alpha,\beta} \setminus \Upsilon^{\alpha,\beta} = \mathbb{U}_+^{\alpha,\beta} \cup \mathbb{U}_-^{\alpha,\beta}$. Here we have a homeomorphism $h_2: \mathbb{U}^{\alpha,\beta} \xrightarrow{\text{onto}} \tilde{\mathbb{L}}^{\alpha,\beta}$ which is \mathcal{C}^∞ -diffeomorphism on $\overline{\mathbb{U}_+^{\alpha,\beta}}$ and \mathcal{C}^∞ -diffeomorphism on $\overline{\mathbb{U}_-^{\alpha,\beta}}$. Therefore, for some positive number $M_{\alpha,\beta}$, we have pointwise inequalities $|Dh_2| \leq M_{\alpha,\beta}$ and $\det Dh_2 \geq \frac{1}{M_{\alpha,\beta}}$ in both $\overline{\mathbb{U}_+^{\alpha,\beta}}$ and $\overline{\mathbb{U}_-^{\alpha,\beta}}$. Having established such a deformation of lenses and their preimages under h_2 , we apply Corollary 5.4. We infer that there is also a constant $M'_{\alpha,\beta} > 0$ with the following property: to every neighborhood of $\Upsilon^{\alpha,\beta}$, say an open connected set $\mathbb{U}_\circ^{\alpha,\beta} \subset \mathbb{U}^{\alpha,\beta}$ that contains $\Upsilon^{\alpha,\beta}$, there corresponds a \mathcal{C}^∞ -diffeomorphism, denoted by $h_3: \mathbb{U}^{\alpha,\beta} \xrightarrow{\text{onto}} \tilde{\mathbb{L}}^{\alpha,\beta}$, such that

$$(3.12) \quad \begin{cases} h_3(z) = h_2(z) & \text{for } z \in \mathbb{U}^{\alpha,\beta} \setminus \mathbb{U}_\circ^{\alpha,\beta} \\ |Dh_3| \leq M'_{\alpha,\beta} & \text{and } \det Dh_3 \geq \frac{1}{M'_{\alpha,\beta}} \text{ in } \mathbb{U}^{\alpha,\beta} \end{cases}$$

We emphasize that $M'_{\alpha,\beta}$ is independent of the neighborhood $\mathbb{U}_\circ^{\alpha,\beta}$. We choose and fix $\mathbb{U}_\circ^{\alpha,\beta}$ thin enough to satisfy

- $\overline{\mathbb{U}_\circ^{\alpha,\beta}} \subset \mathbb{U}^{\alpha,\beta} \cup \overline{\Upsilon}^{\alpha,\beta}$
- $|\mathbb{U}_\circ^{\alpha,\beta}| \leq [M_{\alpha,\beta} + M'_{\alpha,\beta}]^{-2} \epsilon^2 2^{-\alpha-\beta}$
- $\sup_{\mathbb{U}_\circ^{\alpha,\beta}} |Dh_2| \leq M_{\alpha,\beta}$
- in the finite energy case, we also assume that $|\mathbb{U}_\circ^{\alpha,\beta}| \leq [M'_{\alpha,\beta}]^{-2} \delta^2 4^{-\alpha-\beta-1}$

Recall that δ was defined by (3.7) and later appeared in (3.9). This is certainly possible; for instance, take $\mathbb{U}_\circ^{\alpha,\beta}$ to be the preimage under h_2 of a sufficiently thin concavo-convex lens containing $\tilde{\gamma}^{\alpha,\beta}$. We call $h_3: \mathbb{U}^{\alpha,\beta} \xrightarrow{\text{onto}} \tilde{\mathbb{L}}^{\alpha,\beta}$ a smoothing of $h_2: \mathbb{U}^{\alpha,\beta} \xrightarrow{\text{onto}} \tilde{\mathbb{L}}^{\alpha,\beta}$ associated with a given arc $\Upsilon^{\alpha,\beta} = h_2^{-1}(\tilde{\gamma}^{\alpha,\beta})$. We now define a homeomorphism $h_3: \Omega \xrightarrow{\text{onto}} \Omega^*$ by the rule

$$h_3 = \begin{cases} \text{smoothing of } h_2 & \text{in } \mathbb{U}^{\alpha,\beta} \\ h_2 & \text{in } \Omega \setminus \bigcup_{\alpha,\beta} \mathbb{U}^{\alpha,\beta} \end{cases}$$

It belongs to $h_2 + \mathcal{A}_\circ(\Omega)$. Obviously h_3 is a \mathcal{C}^∞ -diffeomorphism in $\tilde{\Omega}$. We have for every $z \in \Omega$

$$\begin{aligned} |h_3(z) - h_2(z)| &\leq \begin{cases} \text{diam } \tilde{\mathbb{L}}^{\alpha,\beta} & \text{for } z \in \mathbb{U}^{\alpha,\beta} \\ 0 & \text{otherwise} \end{cases} \\ &\leq \text{diam } \mathbb{Q}^\alpha \leq \epsilon \end{aligned}$$

see (3.6). Hence $\|h_3 - h_2\|_{\mathcal{C}(\Omega)} \leq \epsilon$. As regards the energy of $h_3 - h_2$ we find that

$$\begin{aligned} \mathcal{E}_\Omega[h_3 - h_2] &= \sum_{\alpha,\beta} \int_{\mathbb{U}_\circ^{\alpha,\beta}} |Dh_3 - Dh_2|^2 \\ (3.13) \quad &\leq \sum_{\alpha,\beta} |\mathbb{U}_\circ^{\alpha,\beta}| \sup_{\mathbb{U}_\circ^{\alpha,\beta}} (|Dh_3| + |Dh_2|)^2 \\ &\leq \sum_{\alpha,\beta} |\mathbb{U}_\circ^{\alpha,\beta}| [M'_{\alpha,\beta} + M_{\alpha,\beta}]^2 \leq \sum_{\alpha,\beta=1}^{\infty} \frac{\epsilon^2}{2^{\alpha+\beta}} \leq \epsilon^2 \end{aligned}$$

These estimates sum up to

$$\|h_3 - h_2\|_{\mathcal{A}(\Omega)} \leq \epsilon + \epsilon = 2\epsilon$$

Let us record for subsequent use the following estimate, obtained from (3.11) and (3.13).

$$\begin{aligned} \sum_{v \in \mathcal{V}} \int_{h_3^{-1}(3\mathbb{D}_v)} |Dh_3|^2 &\leq \sum_{v \in \mathcal{V}} \left(\int_{h_3^{-1}(3\mathbb{D}_v) \setminus h_2^{-1}(3\mathbb{D}_v)} |Dh_3|^2 + \int_{h_2^{-1}(3\mathbb{D}_v)} |Dh_3|^2 \right) \\ &\leq \int_{\{h_3 \neq h_2\}} |Dh_3|^2 + 2 \sum_{v \in \mathcal{V}} \left(\int_{h_2^{-1}(3\mathbb{D}_v)} |Dh_3 - Dh_2|^2 + \int_{h_2^{-1}(3\mathbb{D}_v)} |Dh_2|^2 \right) \\ (3.14) \quad &\leq \sum_{\alpha,\beta} (M'_{\alpha,\beta})^2 |\mathbb{U}_\circ^{\alpha,\beta}| + 2\epsilon^2 + 2\epsilon^2 \leq 5\epsilon^2 \end{aligned}$$

Finite energy case. For the energy of h_3 , we observe that

$$\begin{aligned} \|Dh_3\|_{\mathcal{L}^2(\Omega)} &\leq \|Dh_3\|_{\mathcal{L}^2(\Omega \setminus \cup \mathbb{U}_\circ^{\alpha,\beta})} + \sum_{\alpha,\beta} \|Dh_3\|_{\mathcal{L}^2(\mathbb{U}_\circ^{\alpha,\beta})} \\ &= \|Dh_2\|_{\mathcal{L}^2(\Omega \setminus \cup \mathbb{U}_\circ^{\alpha,\beta})} + \sum_{\alpha,\beta} \|Dh_3\|_{\mathcal{L}^2(\mathbb{U}_\circ^{\alpha,\beta})} \\ &\leq \|Dh_2\|_{\mathcal{L}^2(\Omega)} + \sum_{\alpha,\beta} |\mathbb{U}_\circ^{\alpha,\beta}|^{1/2} \sup_{\mathbb{U}_\circ^{\alpha,\beta}} |Dh_3| \\ &\leq \|Dh\|_{\mathcal{L}^2(\Omega)} - \delta + \frac{\delta}{2} \end{aligned}$$

Thus

$$(3.15) \quad \|Dh_3\|_{\mathcal{L}^2(\Omega)} \leq \|Dh\|_{\mathcal{L}^2(\Omega)} - \frac{\delta}{2}$$

Step 4. We have already upgraded the mapping h to a homeomorphism $h_3: \Omega \rightarrow \Omega^*$ such that $h_3 \in h + \mathcal{A}_o(\Omega)$ and

$$(3.16) \quad \begin{aligned} \|h_3 - h\|_{\mathcal{A}(\Omega)} &\leq \|h_3 - h_2\|_{\mathcal{A}(\Omega)} + \|h_2 - h_1\|_{\mathcal{A}(\Omega)} + \|h_1 - h\|_{\mathcal{A}(\Omega)} \\ &\leq 2\epsilon + 3\epsilon + 2\epsilon = 7\epsilon \end{aligned}$$

Moreover, h_3 is a \mathcal{C}^∞ -diffeomorphism on $\Omega \setminus \bigcup_{v \in \mathcal{V}} h_2^{-1}(\overline{\mathbb{D}}_v)$. We now define a homeomorphism $h_4: \Omega \xrightarrow{\text{onto}} \Omega^*$ by performing harmonic replacement of h_3 on each set $h_3^{-1}(2\mathbb{D}_v)$. This gives us a \mathcal{C}^∞ -diffeomorphism $h_4: h_3^{-1}(2\overline{\mathbb{D}}_v) \rightarrow 2\overline{\mathbb{D}}_v$, see Step 2 for details. For each $z \in \Omega$

$$|h_4(z) - h_3(z)| \leq \begin{cases} 2 \operatorname{diam} \mathbb{D}_v & \text{if } z \in h_3^{-1}(2\mathbb{D}_v) \\ 0 & \text{otherwise} \end{cases} \leq 2\epsilon$$

Hence $\|h_4 - h_3\|_{\mathcal{A}(\Omega)} \leq 2\epsilon$. Using (3.14) we estimate the energy as follows.

$$\begin{aligned} \mathcal{E}_\Omega[h_4 - h_3] &= \sum_{v \in \mathcal{V}} \mathcal{E}_{h_3^{-1}(2\mathbb{D}_v)}[h_4 - h_3] \\ &\leq 2 \sum_{v \in \mathcal{V}} \left(\mathcal{E}_{h_3^{-1}(2\mathbb{D}_v)}[h_4] + \mathcal{E}_{h_3^{-1}(2\mathbb{D}_v)}[h_3] \right) \\ &\leq 4 \sum_{v \in \mathcal{V}} \mathcal{E}_{h_3^{-1}(2\mathbb{D}_v)}[h_3] \leq 20\epsilon^2 \end{aligned}$$

Thus, by (3.14)

$$(3.17) \quad \|h_4 - h_3\|_{\mathcal{A}(\Omega)} \leq \epsilon + \sqrt{20}\epsilon \leq 6\epsilon$$

Finite energy case. By virtue of Dirichlet's principle and (3.15) we have

$$(3.18) \quad \|Dh_4\|_{\mathcal{L}^2(\Omega)} \leq \|Dh_3\|_{\mathcal{L}^2(\Omega)} \leq \|Dh\|_{\mathcal{L}^2(\Omega)} - \frac{\delta}{2}$$

Step 5. The final step consists of smoothing h_4 in a neighborhood of each smooth Jordan curves $C_v = \partial h_3^{-1}(2\mathbb{D}_v)$. We proceed in much the same way as in Step 3, but we appeal to Corollary 5.5 instead of Corollary 5.4. By smoothing h_4 in a sufficiently thin neighborhood of each C_v we obtain a \mathcal{C}^∞ -diffeomorphism $h_5: \Omega \xrightarrow{\text{onto}} \Omega^*$, $h_5 \in h_4 + \mathcal{A}_o(\Omega)$ such that

$$(3.19) \quad \|h_5 - h_4\|_{\mathcal{A}(\Omega)} \leq \epsilon$$

We now recapitulate the estimates (3.16), (3.17) and (3.19) to obtain a \mathcal{C}^∞ -diffeomorphism in Ω

$$H := h_5 \in h + \mathcal{A}(\Omega)$$

such that

$$\begin{aligned} \|H - h\|_{\mathcal{A}(\Omega)} &\leq \|h_5 - h_4\|_{\mathcal{A}(\Omega)} + \|h_4 - h_3\|_{\mathcal{A}(\Omega)} + \|h_3 - h\|_{\mathcal{A}(\Omega)} \\ &\leq \epsilon + 6\epsilon + 7\epsilon = 14\epsilon \end{aligned}$$

which is as strong as (ii) in Theorem 1.2.

Finite energy case. To obtain the desired energy estimate $\mathcal{E}_\Omega[h_5] \leq \mathcal{E}_\Omega[h]$, we need to sharpen the energy part in (3.19). By narrowing further the

neighborhoods of C_v we can make the energy $\mathcal{E}_\Omega[h_5 - h_4]$ as small as we wish; for example to obtain

$$\|Dh_5 - Dh_4\|_{\mathcal{L}^2(\Omega)} < \frac{\delta}{2}$$

This is enough to conclude that

$$\|DH\|_{\mathcal{L}^2(\Omega)} \leq \|Dh\|_{\mathcal{L}^2(\Omega)}$$

because of (3.18). \square

4. HOPF DIFFERENTIALS, THEOREM 1.3

A quadratic differential on a domain Ω in the complex plane \mathbb{C} takes the form $Q = F(z) dz \otimes dz$, where F is a complex function on Ω . Given a conformal change of the variable z , $z = \varphi(\xi)$, where $\varphi: \Omega' \rightarrow \Omega$, the pull back

$$\varphi^\#(Q) = F(\varphi(\xi)) d\varphi \otimes d\varphi = F(\varphi(\xi)) \dot{\varphi}^2(\xi) d\xi \otimes d\xi$$

defines a quadratic differential on Ω' . It is plain that for a complex harmonic function $h: \Omega \rightarrow \mathbb{C}$ the associated Hopf differential

$$Q_h = h_z \overline{h_{\bar{z}}} dz \otimes dz$$

is holomorphic, meaning that

$$(4.1) \quad \frac{\partial}{\partial \bar{z}} (h_z \overline{h_{\bar{z}}}) = 0$$

Conversely, if a Hopf differential $Q_h = h_z \overline{h_{\bar{z}}} dz \otimes dz$ is holomorphic for some \mathcal{C}^1 -mapping h , then h is harmonic at the points where the Jacobian determinant $J(z, h) := \det Dh = |h_z|^2 - |h_{\bar{z}}|^2 \neq 0$, see [10, 10.5] and our Remark 4.3. Here the assumption that $J(z, h) \neq 0$ is critical. Let us illustrate it by the following.

Example 4.1. Consider a mapping $h \in \mathcal{C}^{1,1}(\mathbb{C}_\circ)$ defined on the punctured plane $\mathbb{C}_\circ = \mathbb{C} \setminus \{0\}$ by the rule

$$(4.2) \quad h(z) = \begin{cases} \frac{z}{|z|} & \text{for } 0 < |z| \leq 1 \\ \frac{1}{2} \left(z + \frac{1}{\bar{z}} \right) & \text{for } 1 \leq |z| < \infty \end{cases}$$

Direct computation shows that

$$h_z(z) = \begin{cases} \frac{1}{2} |z|^{-1} & \text{for } 0 < |z| \leq 1 \\ \frac{1}{2} & \text{for } 1 \leq |z| < \infty \end{cases}$$

and

$$h_{\bar{z}}(z) = \begin{cases} -\frac{1}{2} |z| \bar{z}^{-2} & \text{for } 0 < |z| \leq 1 \\ -\frac{1}{2} \bar{z}^{-2} & \text{for } 1 \leq |z| < \infty \end{cases}$$

Thus

$$(4.3) \quad Q_h = -\frac{dz \otimes dz}{4z^2} \quad \text{in } \mathbb{C}_\circ$$

It may be worth mentioning that the mapping h in (4.2) is the unique (up to rotation of z) minimizer of the Dirichlet energy

$$\mathcal{E}[H] = \int_{\mathbb{A}} |DH|^2$$

over the annulus $\mathbb{A} = A(r, R) = \{z: r < |z| < R\}$, $0 < r < 1 < R$, subject to all weak limits of homeomorphisms $H: \mathbb{A} \xrightarrow{\text{ontq}} A(1, R_*)$, where $R_* = \frac{1}{2}(R + \frac{1}{R})$, see [1]. Note that the Hopf differential of (4.3) is real along the boundary circles of \mathbb{A} . The concentric circles are horizontal trajectories of Q_h . In fact this is a general property of minimizers [21, Lemma 1.2.5]. The general pattern is that with the loss of injectivity comes the loss of the Lagrange-Euler equation for the extremal mapping.

Properties of the function h with holomorphic Hopf differential $Q = h_z \overline{h_{\bar{z}}} dz \otimes dz$ are of interest in the studies of harmonic mappings [11, 12, 21, 30, 31], minimal surfaces [8, 34] and Teichmüller theory [15]. In this section we prove Theorem 1.3 which imposes fairly minimal assumptions that imply harmonicity of $\mathscr{W}^{1,2}$ -solution to the equation (4.1). Some elements of the proof go back to [28, 29].

Proof of Theorem 1.3. As a consequence of the Stoilow factorization theorem [1, p. 56] the branch set of h is discrete, hence removable for continuous harmonic functions. Thus we assume that $h: \Omega \xrightarrow{\text{ontq}} \Omega^*$ is a homeomorphism of Sobolev class $\mathscr{W}_{\text{loc}}^{1,2}(\Omega, \Omega^*)$ such that

$$(4.4) \quad h_z \overline{h_{\bar{z}}} = F(z) \quad \text{is holomorphic in } \Omega$$

By virtue of Theorem 1.2, there exists a sequence of diffeomorphisms $h^j: \Omega \xrightarrow{\text{ontq}} \Omega^*$ converging c -uniformly and strongly in $\mathscr{W}_{\text{loc}}^{1,2}(\Omega, \Omega^*)$ to h . Denote by

$$(4.5) \quad h_z^j \overline{h_{\bar{z}}^j} =: F^j \in \mathcal{L}_{\text{loc}}^1(\Omega)$$

Thus $F^j \rightarrow F$ strongly in $\mathcal{L}_{\text{loc}}^1(\Omega)$. Let us first dispose of an easy case.

Case 0. *The homogeneous equation $F \equiv 0$.* Since h^j are diffeomorphisms the Jacobian determinant $J(z, h^j) = |h_z^j|^2 - |h_{\bar{z}}^j|^2$ is either positive everywhere in Ω or negative everywhere in Ω . Let us settle the case when $J(z, h^j) > 0$ for infinitely many indices $j = 1, 2, \dots$. For such j we have $|h_z^j| > |h_{\bar{z}}^j|$, which yields $|h_z^j|^2 \leq |h_z^j h_{\bar{z}}^j|$. Passing to the \mathcal{L}^1 -limit we obtain

$$|h_{\bar{z}}|^2 \leq |h_z h_{\bar{z}}| = |F(z)| \equiv 0.$$

Thus h is holomorphic, by Weyl's lemma. Similarly, in case $J(z, h^j) < 0$ for infinitely many indices $j = 1, 2, \dots$, we find that h is antiholomorphic.

Remark 4.2. We observe, based on the above arguments, that for this homogeneous equation $h_z \overline{h_{\bar{z}}} \equiv 0$ every solution $h \in \mathscr{W}_{\text{loc}}^{1,2}(\Omega)$ obtained as the weak $\mathscr{W}^{1,2}$ -limit of homeomorphisms is either holomorphic or antiholomorphic. The situation is dramatically different if $h_z \overline{h_{\bar{z}}} \not\equiv 0$; some topological assumption on h are necessary, as illustrated in Example 4.1.

Case 1. *Nonhomogeneous equation $F \neq 0$.* The function F , being holomorphic, may vanish only at isolated points. Since isolated points are removable for bounded harmonic functions, it suffices to consider the set where $F \neq 0$. Proceeding further in this direction, we may and do assume that $F(z) \equiv 1$ (by a conformal change of the z -variable) and h is a $\mathscr{W}^{1,2}$ -homeomorphism in the closure of the unit square $\mathbb{Q} = \{x + iy : 0 < x < 1, 0 < y < 1\}$. The problem now reduces to establishing that the equation

$$(4.6) \quad h_z \overline{h_{\bar{z}}} \equiv 1$$

implies $\Delta h = 0$. This will be proved indirectly by means of the energy-minimizing property

$$(4.7) \quad \mathcal{E}_{\mathbb{Q}}[h] \leq \mathcal{E}_{\mathbb{Q}}[H]$$

where $H: \mathbb{Q} \rightarrow h(\mathbb{Q})$ is any homeomorphism in $h + \mathcal{A}_{\circ}(\mathbb{Q})$; in particular, $H = h$ on $\partial\mathbb{Q}$. Indeed, if h were not harmonic, we would be able to decrease its energy by harmonic replacement (Propositions 2.1 and 2.3), contradicting (4.7).

4.1. Proof of the inequality (4.7). With the aid of the approximation theorem we need only prove (4.7) for mappings $H \in h + \mathcal{A}_{\circ}(\mathbb{Q})$ that are diffeomorphisms on \mathbb{Q} . From now on we assume that this is the case. Denote $\mathbb{Q}^* = h(\mathbb{Q}) = H(\mathbb{Q})$. We consider a sequence $h^j \in h + \mathcal{A}_{\circ}(\mathbb{Q})$ of diffeomorphisms $h^j: \mathbb{Q} \xrightarrow{\text{onto}} \mathbb{Q}^*$ converging in $\mathcal{A}(\mathbb{Q})$ to h . Moreover we may also assume that $Dh^j \rightarrow Dh$ almost everywhere in \mathbb{Q} by passing to a subsequence if necessary. Now the sequence $\chi^j: \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}$ of self-homeomorphisms of the closed unit disk given by $\chi^j = H^{-1} \circ h^j$, where $\chi^j = \text{id}$ on $\partial\mathbb{Q}$, is converging uniformly on $\overline{\mathbb{Q}}$ to $\chi = H^{-1} \circ h$. It is important to observe that $\chi \in \mathscr{W}_{\text{loc}}^{1,2}(\mathbb{Q})$ and χ^j converges to χ in $\mathscr{W}^{1,2}(\mathbb{Q}')$ on any compactly contained subdomain $\mathbb{Q}' \Subset \mathbb{Q}$. Since h^j and $(\chi^j)^{-1}$ are diffeomorphisms on \mathbb{Q}' and $\chi^j(\mathbb{Q}')$, respectively, the chain rule can be applied to the composition $H = h^j \circ (\chi^j)^{-1}$. For $w \in \chi^j(\mathbb{Q}')$ we have

$$\begin{aligned} \frac{\partial H(w)}{\partial w} &= h_z^j(z) \frac{\partial(\chi^j)^{-1}}{\partial w} + h_{\bar{z}}^j(z) \frac{\partial(\chi^j)^{-1}}{\partial \bar{w}} \\ \frac{\partial H(w)}{\partial \bar{w}} &= h_z^j(z) \frac{\partial(\chi^j)^{-1}}{\partial \bar{w}} + h_{\bar{z}}^j(z) \frac{\partial(\chi^j)^{-1}}{\partial w} \end{aligned}$$

where $z = (\chi^j)^{-1}(w)$.

The partial derivatives of $(\chi^j)^{-1}$ at w can be expressed in terms of $\chi_z^j(z)$ and $\chi_{\bar{z}}^j(z)$ by the rules

$$\begin{aligned} \frac{\partial(\chi^j)^{-1}}{\partial w} &= \frac{\chi_z^j(z)}{J(z, \chi^j)} \\ \frac{\partial(\chi^j)^{-1}}{\partial \bar{w}} &= -\frac{\chi_{\bar{z}}^j(z)}{J(z, \chi^j)} \end{aligned}$$

where the Jacobian determinant $J(z, \chi^j)$ is strictly positive. This yields

$$\begin{aligned}\frac{\partial H}{\partial w} &= \frac{h_z^j \overline{\chi_z^j} - h_{\bar{z}}^j \overline{\chi_{\bar{z}}^j}}{J(z, \chi^j)} \\ \frac{\partial H}{\partial \bar{w}} &= \frac{h_{\bar{z}}^j \chi_z^j - h_z^j \chi_{\bar{z}}^j}{J(z, \chi^j)}\end{aligned}$$

We compute the energy integral of H over the set $\chi^j(\mathbb{Q}')$ by substitution $w = \chi^j(z)$,

$$\begin{aligned}\mathcal{E}_{\chi^j(\mathbb{Q}')}[H] &= 2 \int_{\chi^j(\mathbb{Q}')} (|H_w|^2 + |H_{\bar{w}}|^2) dw \\ &= 2 \int_{\mathbb{Q}'} \frac{|h_z^j \overline{\chi_z^j} - h_{\bar{z}}^j \overline{\chi_{\bar{z}}^j}|^2 + |h_{\bar{z}}^j \chi_z^j - h_z^j \chi_{\bar{z}}^j|^2}{|\chi_z^j|^2 - |\chi_{\bar{z}}^j|^2} dz\end{aligned}$$

On the other hand, the energy of h^j over the set \mathbb{Q}' is

$$\mathcal{E}_{\mathbb{Q}'}[h^j] = 2 \int_{\mathbb{Q}'} (|h_z^j|^2 + |h_{\bar{z}}^j|^2) dz$$

Subtract these two integrals to obtain

$$\begin{aligned}\mathcal{E}_{\mathbb{Q}}[H] - \mathcal{E}_{\mathbb{Q}'}[h^j] &\geq \mathcal{E}_{\chi^j(\mathbb{Q}')}[H] - \mathcal{E}_{\mathbb{Q}'}[h^j] \\ &= 4 \int_{\mathbb{Q}'} \frac{(|h_z^j|^2 + |h_{\bar{z}}^j|^2) \cdot |\chi_z^j|^2 - 2 \operatorname{Re} [h_z^j \overline{h_{\bar{z}}^j} \chi_z^j \overline{\chi_{\bar{z}}^j}]}{|\chi_z^j|^2 - |\chi_{\bar{z}}^j|^2} dz \\ (4.8) \quad &\geq 4 \int_{\mathbb{Q}'} \frac{2|h_z^j h_{\bar{z}}^j| |\chi_z^j|^2 - 2 \operatorname{Re} [h_z^j \overline{h_{\bar{z}}^j} \chi_z^j \overline{\chi_{\bar{z}}^j}]}{|\chi_z^j|^2 - |\chi_{\bar{z}}^j|^2} dz \\ &= 4 \int_{\mathbb{Q}'} \left[\frac{|\chi_z^j - \sigma^j(z) \chi_{\bar{z}}^j|^2}{|\chi_z^j|^2 - |\chi_{\bar{z}}^j|^2} - 1 \right] |h_z^j h_{\bar{z}}^j| dz\end{aligned}$$

where we have introduced the notation

$$\sigma^j = \sigma^j(z) = \begin{cases} h_z^j \overline{h_{\bar{z}}^j} |h_z^j h_{\bar{z}}^j|^{-1} & \text{if } h_z^j h_{\bar{z}}^j \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Note that $|\sigma^j| = 1$ and $\sigma^j \rightarrow 1$ almost everywhere.

Upon using Hölder's inequality we continue the chain (4.8) as follows.

$$(4.9) \quad \geq 4 \frac{\left[\int_{\mathbb{Q}'} |\chi_z^j - \sigma^j \chi_{\bar{z}}^j| \sqrt{|h_z^j h_{\bar{z}}^j|} dz \right]^2}{\int_{\mathbb{Q}'} J(z, h^j) dz} - 4 \int_{\mathbb{Q}'} |h_z^j h_{\bar{z}}^j|.$$

The denominator in (4.9) is at most 1 because

$$\int_{\mathbb{Q}'} J(z, h^j) dz = |\chi^j(\mathbb{Q}')| \leq |\mathbb{Q}| = 1.$$

Therefore,

$$\mathcal{E}_{\mathbb{Q}}[H] - \mathcal{E}_{\mathbb{Q}'}[h^j] \geq 4 \left[\int_{\mathbb{Q}'} |\chi_z^j - \sigma^j \chi_{\bar{z}}^j| \sqrt{|h_z^j h_{\bar{z}}^j|} dz \right]^2 - 4 \int_{\mathbb{Q}'} |h_z^j h_{\bar{z}}^j| dz.$$

It is at this point that we can pass to the limit as $j \rightarrow \infty$, to obtain

$$(4.10) \quad \mathcal{E}_{\mathbb{Q}}[H] - \mathcal{E}_{\mathbb{Q}'}[h] \geq 4 \left[\int_{\mathbb{Q}'} |\chi_z - \chi_{\bar{z}}| dz \right]^2 - 4|\mathbb{Q}'|.$$

Since \mathbb{Q}' was an arbitrary compactly contained subdomain of \mathbb{Q} , the estimate (4.10) remains valid with \mathbb{Q}' replaced by \mathbb{Q} .

$$(4.11) \quad \begin{aligned} \mathcal{E}_{\mathbb{Q}}[H] - \mathcal{E}_{\mathbb{Q}}[h] &\geq 4 \left[\int_{\mathbb{Q}} \left| \frac{\partial \chi}{\partial y} \right| dx dy \right]^2 - 4 \\ &\geq 4 \int_0^1 \left| \int_0^1 \frac{\partial \chi(x, y)}{\partial y} dy \right| dx - 4 \\ &= 4 \int_0^1 |\chi(x, 1) - \chi(x, 0)| dx - 4 = 4 - 4 = 0 \end{aligned}$$

as desired. \square

Remark 4.3. When specialized to the case $h \in \mathcal{C}^1$, Theorem 1.3 shows that h is harmonic outside of the zero set of its Jacobian.

5. AUXILIARY SMOOTHING RESULTS

Here we present some results concerning smoothing of piecewise differentiable planar homeomorphisms. They can be found in [25] in greater generality, but since we require quantitative control of derivatives, a self-contained proof is in order. Here it is more convenient to use the operator norm of a matrix, denoted by $\|\cdot\|$. Note that $\|A\| \leq |A| \leq 2\|A\|$ for 2×2 -matrices.

Proposition 5.1. *Let $\mathbb{U} \subset \mathbb{R}^2$ be a domain containing an open segment I with endpoints on the boundary $\partial\mathbb{U}$ which splits \mathbb{U} into two subdomains \mathbb{U}_1 and \mathbb{U}_2 such that $\mathbb{U} \setminus I = \mathbb{U}_1 \cup \mathbb{U}_2$. Suppose that $f: \overline{\mathbb{U}} \xrightarrow{\text{onto}} \overline{\mathbb{U}^*} \subset \mathbb{R}^2$ is a homeomorphism with the following properties:*

- (i) *For $j = 1, 2, \dots$ the restriction of f to $\overline{\mathbb{U}_j}$ is \mathcal{C}^∞ -smooth, equals the identity on I ;*
- (ii) *There is a constant $M > 0$ such that for $j = 1, 2$ the restriction of f to $\overline{\mathbb{U}_j}$ satisfies $\|Df\| \leq M$ and $\det Df \geq M^{-1}$.*

Then for any open set \mathbb{U}_\circ with $I \subset \mathbb{U}_\circ \subset \mathbb{U}$ there is a \mathcal{C}^∞ -diffeomorphism $g: \mathbb{U} \rightarrow \mathbb{U}^$ such that*

- *g agrees with f on $\mathbb{U} \setminus \mathbb{U}_\circ$ (and also on I);*
- *$\|Dg\| \leq 20M$ and $\det Dg \geq (20M)^{-1}$ on \mathbb{U} .*

Proof. Without loss of generality $I \subset \mathbb{R} = \{(x, y): y = 0\}$. We write f in components as (u, v) where u and v are functions of x and y . Let us introduce a notation; given any \mathcal{C}^∞ -smooth function $\beta: \mathbb{R} \rightarrow [0, \infty)$, denote

$V(\beta) = \{(x, y) \in \mathbb{R}^2: |y| < \beta(x)\}$. We can and do choose β so that $I \subset V(\beta) \subset \mathbb{U}_\circ$, and further scale it down until the following holds.

$$(5.1) \quad \begin{aligned} |\beta'(x)| &\leq \frac{1}{40M} && \text{for all } x \in \mathbb{R}; \\ |v_x| &\leq \frac{1}{50M^2} && \text{in } V(\beta) \setminus I, \quad \text{because } v(x, 0) = 0; \\ |u_x - 1| &\leq \frac{1}{10} && \text{in } V(\beta) \setminus I, \quad \text{because } u(x, 0) = x. \end{aligned}$$

As a consequence of (ii) and (5.1),

$$(5.2) \quad v_y \geq \frac{M^{-1} - |u_y v_x|}{u_x} \geq \frac{1}{2M}.$$

Since v is also M -Lipschitz by (ii), the following double inequality holds in $V(\beta) \setminus I$.

$$(5.3) \quad \frac{1}{2M} \leq \frac{v}{y} \leq M.$$

Let us fix be a nondecreasing \mathcal{C}^∞ function $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ such that $\alpha(t) = 0$ for $t \leq 1/3$. Let $\alpha(t) = 1$ for $t \geq 2/3$. Moreover, $\alpha'(t) \leq 4$ for all $t \in \mathbb{R}$ and $\alpha(\infty) = 1$, by convention. Now we introduce a modification of u on \mathbb{U} by setting

$$\tilde{u} := \alpha(t)u + (1 - \alpha(t))x \quad \text{where } t = \begin{cases} \frac{|y|}{\beta(x)} & \text{if } \beta(x) \neq 0 \\ \infty & \text{otherwise} \end{cases}.$$

Note that $\tilde{u} = u$ outside of $V(\beta)$. In $V(\beta) \setminus I$ we compute the derivatives as follows.

$$(5.4) \quad \begin{aligned} \tilde{u}_x &= -t^2 \alpha'(t) \beta'(x) \frac{u-x}{|y|} + \alpha(t)u_x + 1 - \alpha(t) \\ \tilde{u}_y &= t \alpha'(t) \frac{u-x}{y} + \alpha(t)u_y \end{aligned}$$

Since u is M -Lipschitz by (ii), we have $|u-x| \leq M|y|$. From this, (5.1) and (5.4) we obtain

$$(5.5) \quad \frac{8}{10} \leq \tilde{u}_x \leq \frac{12}{10}, \quad \text{and} \quad |\tilde{u}_y| \leq 5M,$$

which combined with (5.2) yields

$$(5.6) \quad \tilde{u}_x v_y - \tilde{u}_y v_x \geq \frac{8}{10} \frac{1}{2M} - \frac{5M}{50M^2} = \frac{3}{10M}.$$

Next we modify v on \mathbb{U} . Specifically,

$$\tilde{v} := \alpha(s)v + (1 - \alpha(s))\frac{y}{2M} \quad \text{where } s = \begin{cases} \frac{3|y|}{\beta(x)} & \text{if } \beta(x) \neq 0 \\ \infty & \text{otherwise} \end{cases}$$

Note that $\tilde{v} = v$ outside of $V(\beta/3)$, and on the set $V(\beta/3)$ we already have $\tilde{u} \equiv x$.

Computations similar to (5.4) yield (on the set $V(\beta/3) \setminus I$)

$$(5.7) \quad \begin{aligned} \tilde{v}_x &= -\frac{1}{3}\alpha'(s)s^2\frac{v-y}{|y|} + \alpha(s)v_x; \\ \tilde{v}_y &= \frac{s\alpha'(s)}{y}\left(v - \frac{y}{2M}\right) + \alpha(s)v_y + \frac{1-\alpha(s)}{2M}. \end{aligned}$$

Straightforward estimates based on (5.1), (5.2) and (5.3) comply

$$(5.8) \quad \begin{aligned} |\tilde{v}_x| &\leq \frac{4M}{3} + \frac{1}{50M^2} < \frac{3M}{2}, \\ \frac{1}{2M} &\leq \tilde{v}_y \leq 5M. \end{aligned}$$

It remains to check that the mapping $g := (\tilde{u}, \tilde{v})$, which agrees with f outside of $V(\beta)$, satisfies all the requirements. As regards \mathcal{C}^∞ -smoothness we need only check it on $V(\beta/9)$. But in this neighborhood of I we have a linear mapping, $g(x, y) = (x, \frac{y}{2M})$, so \mathcal{C}^∞ -smooth. By virtue of (5.5) and (5.8) we have $\|Dg\| \leq 20M$. The desired lower bound for $\det Dg$ follows from (5.6) and (5.8). Consequently, g is a local diffeomorphism, and since it agrees with f on $\partial V(\beta)$, it is in fact a diffeomorphism, by a topological result: *a local homeomorphism which shares boundary values with a homeomorphism is injective* [25, Lemma 8.2]. \square

We also need a polar version of Proposition 5.1.

Corollary 5.2. *Let $\mathbb{U} \subset \mathbb{R}^2$ be a domain containing a circle \mathbb{T} . Suppose that $f: \mathbb{U} \xrightarrow{\text{onto}} \mathbb{U}^* \subset \mathbb{R}^2$ is a homeomorphism with the following properties:*

- (i) *The restriction of f to \mathbb{T} is the identity mapping;*
- (ii) *There is a constant $M > 0$ such that the restriction of f to either component of $\mathbb{U} \setminus \mathbb{T}$ is \mathcal{C}^∞ -smooth with $\|Df\| \leq M$ and $\det Df \geq M^{-1}$.*

Then for any open set W with $\mathbb{T} \subset W \subset \mathbb{U}$ there is a \mathcal{C}^∞ -diffeomorphism $g: \mathbb{U} \rightarrow \mathbb{U}^$ such that*

- *g agrees with f on $\mathbb{U} \setminus W$ and on \mathbb{T} ;*
- *$\|Dg\| \leq 80M$ and $\det Dg \geq (80M)^{-1}$ on \mathbb{U} .*

Proof. It is convenient to identify \mathbb{R}^2 with \mathbb{C} . Without loss of generality $\mathbb{T} = \{z \in \mathbb{C}: |z| = 1\}$. Let $\psi(\zeta) = \exp(i\zeta)$. The mapping $F = \psi^{-1} \circ f \circ \psi$ is well-defined in some open horizontal strip $S_\epsilon = \{z \in \mathbb{C}: |\text{Im } z| < \epsilon\}$ which we choose thin enough so that $\psi(S_\epsilon) \subset W$ and $|\psi'|^2 < e^{2\epsilon} \leq 2$. Note that F is 2π -periodic and satisfies

$$\|DF\| \leq 2M \quad \text{and} \quad \det DF \geq (2M)^{-1}.$$

The proof of Proposition 5.1 applies to F with no changes other than one simplification: $\beta > 0$ is now a small positive constant rather than a function. Thus we obtain a diffeomorphism G which agrees with F on $\mathbb{R} \cup (S \setminus V(\beta))$ and satisfies $\|DG\| \leq 40$ and $\det DG \geq (40M)^{-1}$. Since F was 2π -periodic, so is G . Thus, $g := \psi \circ G \circ \psi^{-1}$ is the desired diffeomorphism. \square

Our applications require slightly more general versions of Proposition 5.1 and Corollary 5.2, where the separating curve is allowed to have other shapes and f is not required to agree with the identity on the curve.

Definition 5.3. A parametric curve $\Gamma: (0, 1) \rightarrow \mathbb{R}^2$ is *regular* if Γ extends to a bigger interval $(a, b) \supset [0, 1]$ so that the extended mapping is a \mathcal{C}^∞ -diffeomorphism onto its image.

Note that a regular curve Γ has well-defined endpoints $\Gamma(0)$ and $\Gamma(1)$. Also, Γ extends to an injective \mathcal{C}^∞ -mapping $\Phi: (0, 1) \times (-1, 1) \rightarrow \mathbb{R}^2$ such that $\|D\Phi\|$ and $\|(D\Phi)^{-1}\|$ are bounded. This follows from the existence of a tubular neighborhood of the image of Γ [23, Theorem 4.26].

Corollaries 5.4 and 5.5, given below, generalize Proposition 5.1 and Corollary 5.2 respectively.

Corollary 5.4. *Let $\mathbb{U} \subset \mathbb{R}^2$ be a domain containing the image of a regular arc Γ with endpoints on the boundary $\partial\mathbb{U}$ which divides \mathbb{U} into two subdomains \mathbb{U}_1 and \mathbb{U}_2 such that $\mathbb{U} \setminus \Gamma = \mathbb{U}_1 \cup \mathbb{U}_2$. Suppose $f: \overline{\mathbb{U}} \rightarrow \overline{\mathbb{U}^*} \subset \mathbb{R}^2$ is a homeomorphism such that $f \circ \Gamma$ is also regular and the restriction of f to each $\overline{\mathbb{U}_i}$ is \mathcal{C}^∞ -smooth and satisfies*

$$|Df(z)| \leq M, \quad \det Df(z) \geq \frac{1}{M} \quad \text{for } z \in \mathbb{U}_i$$

where M is a positive constant. Then there is a constant $M' > 0$ such that to every open set $\mathbb{U}' \subset \mathbb{U}$ with $\Gamma \subset \mathbb{U}'$ there corresponds a \mathcal{C}^∞ -diffeomorphism $g: \mathbb{U} \xrightarrow{\text{onto}} \mathbb{U}^*$ with the following properties

- $g(z) = f(z)$ for $z \in \mathbb{U} \setminus \mathbb{U}'$ (and also on Γ)
- $|Dg(z)| \leq M'$ and $\det Dg(z) \geq \frac{1}{M'}$ on \mathbb{U} .

Proof. Let $\mathbb{Q} = (0, 1) \times (-1, 1)$. Let Φ and Ψ be the extensions of Γ and $f \circ \Gamma$ to \mathbb{Q} as in Definition 5.3. There is a domain $\tilde{\mathbb{U}}$ such that $(0, 1) \times \{0\} \subset \tilde{\mathbb{U}} \subset \mathbb{Q}$, $\Phi(\tilde{\mathbb{U}}) \Subset \mathbb{U}'$, and the composition $F := \Psi^{-1} \circ f \circ \Phi$ is defined in $\tilde{\mathbb{U}}$. Note that $F = \text{id}$ on $(0, 1) \times \{0\}$. We apply Proposition 5.1 (with $\tilde{\mathbb{U}}$ in place of \mathbb{U} and with F in place of f) and obtain a \mathcal{C}^∞ -diffeomorphism $G: \tilde{\mathbb{U}} \rightarrow F(\tilde{\mathbb{U}})$. Finally, replace F within $\tilde{\mathbb{U}}$ with the diffeomorphism $g = \Psi \circ G \circ \Phi^{-1}$. \square

Corollary 5.5. *Let $\mathbb{U} \subset \mathbb{R}^2$ be a domain containing the image of a \mathcal{C}^∞ -smooth Jordan curve Γ which divides \mathbb{U} into two subdomains \mathbb{U}_1 and \mathbb{U}_2 such that $\mathbb{U} \setminus \Gamma = \mathbb{U}_1 \cup \mathbb{U}_2$. Suppose $f: \overline{\mathbb{U}} \rightarrow \overline{\mathbb{U}^*} \subset \mathbb{R}^2$ is a homeomorphism such that the restriction of f to each $\overline{\mathbb{U}_i}$ is \mathcal{C}^∞ -smooth and satisfies*

$$|Df(z)| \leq M, \quad \det Df(z) \geq \frac{1}{M} \quad \text{for } z \in \mathbb{U}_i$$

where M is a positive constant. Then there is a constant $M' > 0$ such that to every open set $\mathbb{U}' \subset \mathbb{U}$ with $\Gamma \subset \mathbb{U}'$ there corresponds a \mathcal{C}^∞ -diffeomorphism $g: \mathbb{U} \xrightarrow{\text{onto}} \mathbb{U}^*$ with the following properties

- $g(z) = f(z)$ for $z \in \mathbb{U} \setminus \mathbb{U}'$ (and also on Γ)

- $|Dg(z)| \leq M'$ and $\det Dg(z) \geq \frac{1}{M'}$ on U .

Proof. The proof of Corollary 5.4 is easily adapted to this case. \square

6. CONCLUDING REMARKS

One may wonder whether the proof of Theorem 1.2 can be extended to the spaces $\mathcal{W}^{1,p}$, $1 < p < \infty$, by means of the p -harmonic replacement in place of Proposition 2.3. Indeed, p -harmonic mappings are $\mathcal{C}^{1,\alpha}$ -smooth [36]. However, the injectivity of p -harmonic replacement of a homeomorphism is unclear.

Question 6.1. Is there a version of the Radó-Kneser-Choquet theorem for p -harmonic mappings? That is, does the p -harmonic extension of a homeomorphism onto a convex Jordan curve enjoy the injectivity property?

An attempt to extend Theorem 1.2 to higher dimensions faces another obstacle: the Radó-Kneser-Choquet theorem fails in dimensions $n \geq 3$ as was proved by Laugesen [22].

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