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Pruitt's Estimates in Banach Space

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Abstract. Pruitt's estimates on the expectation and the distribution of the time taken by a random walk to exit a ball of radius r are extended to the infinite dimensional setting. It is shown that they separate into two pairs of estimates depending on whether the space is type 2 or cotype 2. It is further shown that these estimates characterize type 2 and cotype 2 spaces.

1. Introduction

Let $X, X_1, X_2, \dots$ be a sequence of non-degenerate, independent and identically distributed random variables taking values in a separable Banach space $(B, \| \ \|)$. Set $S_n = \sum_{j=1}^{n} X_j$ and let $T_r = \min\{n : \|S_n\| > r\}$ be the first time the random walk leaves the ball of radius r. For $r > 0$ define

$$\begin{aligned} G(r) &= P(\|X\| > r),\ K(r) = r^{-2}E(\|X\|^2; \|X\| \le r), \\ M(r) &= r^{-1}E(X; \|X\| \le r),\ h(r) = G(r) + K(r) + \|M(r)\|. \end{aligned} \tag{1.1}$$

In the case that $B = \mathbb{R}^d$ with the usual Euclidean norm, Pruitt [15] obtained two fundamental estimates on the size of T_r in terms of $h(r)$. The first gives bounds on the distribution of T_r; there exist constants $0 < c < C < \infty$ such that for all $r > 0$ and $n \ge 1$

$$P(T_r > n) \le \frac{c}{nh(r)}, \quad P(T_r \le n) \le Cnh(r). \tag{1.2}$$

The second gives bounds on the expectation of T_r; there exist constants $0 < c < C < \infty$ such that for all $r > 0$

$$\frac{c}{h(r)} \le ET_r \le \frac{C}{h(r)}. \tag{1.3}$$

In both cases the constants are independent of the underlying distribution and of r and n, but they do depend on the dimension d. These estimates underly one approach to recent work on moments of the overshoot and related topics in renewal theory; see [5], [6], [7] and [8] for examples of this work.

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Apart from the problem's intrinsic interest, the motivation for this paper came from trying to extend to infinite dimensions some of the work cited above. To take advantage of the techniques already developed, infinite dimensional versions of Pruitt's estimates are needed. We will show that when B is a Hilbert space this can be achieved, but (1.2) and (1.3) do not hold in a general Banach space. They split into two pairs of estimates depending on whether B is type 2 or cotype 2. Recall B is type 2 if there exists a constant $c \in (0, \infty)$ such that for every $n \geq 1$ and every sequence of independent random variables $X_1, \ldots, X_n$ with $EX_i = 0$ and $E\|X_i\|^2 < \infty$, $1 \leq i \leq n$, the following inequality holds;

$$E\|\sum_{i=1}^{n} X_i\|^2 \leq c \sum_{i=1}^{n} E\|X_i\|^2. \tag{1.4}$$

We refer to any c for which (1.4) holds as a type 2 constant for B. B is cotype 2 if (1.4) holds with the inequality reversed. For example, the ℓ^p spaces are type 2 for $2 \leq p < \infty$ and cotype 2 for $1 \leq p \leq 2$. The main results of this paper are summarized in the following two Theorems which may be viewed as giving alternative characterizations of type 2 and cotype 2 spaces;

Theorem 1.1. *The following are equivalent:*

$$B \text{ is type } 2; \tag{1.5}$$

There is a constant c such that for all distributions X and all $r > 0$

$$ET_r \geq \frac{c}{h(r)}; \tag{1.6}$$

There is a constant c such that for all distributions X, all $r > 0$ and all $n \geq 1$

$$P(T_r \leq n) \leq cnh(r). \tag{1.7}$$

The constants appearing in the three equivalent statements may be taken to depend only on each other.

Theorem 1.2. *The following are equivalent:*

$$B \text{ is cotype } 2; \tag{1.8}$$

There is a constant c such that for all distributions X and all $r > 0$

$$ET_r \leq \frac{c}{h(r)}; \tag{1.9}$$

There is a constant c such that for all distributions X, all $r > 0$ and all $n \geq 1$

$$P(T_r > n) \leq \frac{c}{nh(r)}. \tag{1.10}$$

The constants appearing in the three equivalent statements may be taken to depend only on each other.

As a consequence it follows that Pruitt's estimates hold precisely when B is both type 2 and cotype 2, that is by Kwapien's theorem [11], when B is isomorphic to a Hilbert space. We observe in passing that since $\mathbb{R}^d$ with the usual norm can be isometrically embedded in the Hilbert space ℓ^2, this implies the constants in (1.2) and (1.3) may in fact be taken independent of dimension.

The method of proof of these results, while similar in places to Pruitt's original proofs, necessarily differs in important ways and provides some new insights to the estimates. For example it is shown that it suffices to establish the estimates asymptotically and just for distributions with mean zero and finite second moment; see the statements of Theorems 3.1 and 3.2.

It would be interesting to know if there is a different function which plays the role of h in a general Banach space. That is, is there a function g for which (1.2) and (1.3) hold with g replacing h? In this direction it is interesting to note that by Lemma 2.1 below, if it is the case that ET_{2r} and ET_r are comparable, then such a function exists, namely $g(r) = (ET_r)^{-1}$. The problem then of course becomes how to calculate g from the underlying distribution in a manner similar to h. This is discussed further in Section 4.

2. **Preliminaries and General Results**

The interplay between the functions $G(r), K(r)$, and $M(r)$ plays an important role in our analysis. Each of these functions is right continuous with left limits and approaches 0 as $r \to \infty$. Their behavior near $r = 0$ will not be of much importance, but we point out that $G(r) \to P(\|X\| > 0) > 0$ as $r \to 0$, hence $h(r)$ is bounded away from 0 as $r \to 0$. Further h is strictly positive for all $r > 0$ since an integration by parts shows that

$$G(r) + K(r) = r^{-2} \int_0^r 2uG(u)\, du. \tag{2.1}$$

As $\|M(r)\| \le 1$, this also shows that $h(r) \le 2$ for all r. In dealing with h, it is useful to think of h as decreasing and $r^2 h(r)$ as increasing. While this is not quite true the following inequalities provide suitable substitutes (see (2.3) of [15]); for any $0 < r \le s < \infty$

$$\frac{r^2}{2s^2} \le \frac{h(s)}{h(r)} \le 2. \tag{2.2}$$

We denote by L the collection of random variables taking values in B. The following simple result shows why the estimates in (1.2) take the form they do, and how they relate to (1.3).

Lemma 2.1. *For any $X \in L$, any $r > 0$ and any $n \ge 1$*

$$P(T_{2r} \le n) \le \frac{n}{ET_r}, \tag{2.3}$$

$$P(T_r > n) \le \frac{ET_r}{n}. \tag{2.4}$$

Proof. The second inequality is an immediate consequence of Markov's inequality. For the first we just observe that by the Markov property, for any $k \geq 0$

$$P(T_r > kn) \leq P(T_{2r} > n)^k.$$

Hence

$$\frac{ET_r}{n} \leq \sum_{k=0}^{\infty} P(T_r > kn) \leq \frac{1}{1 - P(T_{2r} > n)}$$

from which (2.3) immediately follows. □

As mentioned in the introduction, this result raises the possibility of finding a function which plays the role of h in the general setting. It also allows us to show how (1.2) and (1.3) split into two separate pairs of estimates.

Lemma 2.2. *The following are equivalent: There exists a constant c such that for any $X \in L$ and all $r > 0$*

$$ET_r \geq \frac{c}{h(r)}; \tag{2.5}$$

There exists a constant C such that for all $X \in L$, $r > 0$ and $n \geq 1$

$$P(T_r \leq n) \leq Cnh(r). \tag{2.6}$$

The constants appearing in these statements may be taken to depend only on each other.

Proof. First assume (2.5), then by (2.3), for all $X \in L$, $r > 0$ and $n \geq 1$

$$\begin{aligned} P(T_r \leq n) &\leq \frac{n}{ET_{r/2}} \\ &\leq \frac{nh(r/2)}{c} \\ &\leq \frac{8nh(r)}{c} \end{aligned}$$

by (2.2). Thus we may take $C = 8c^{-1}$.

Now assume (2.6), then for any $X \in L$ and all $r > 0$

$$\begin{aligned} ET_r &\geq E(T_r; T_r > \frac{1}{2Ch(r)}) \\ &\geq \frac{1}{2Ch(r)} P(T_r > \frac{1}{2Ch(r)}) \\ &\geq \frac{1}{4Ch(r)}. \end{aligned}$$

Thus we may take $c = (4C)^{-1}$. □

Lemma 2.3. *The following are equivalent: There exists a constant c such that for any $X \in L$ and all $r > 0$*

$$ET_r \leq \frac{c}{h(r)}; \tag{2.7}$$

There exists a constant C such that for all $X \in L$, $r > 0$ and $n \geq 1$

$$P(T_r > n) \leq \frac{C}{nh(r)}. \tag{2.8}$$

The constants appearing in these statements may be taken to depend only on each other.

Proof. First assume (2.7), then (2.8) follows immediately from Markov's inequality with $C = c$.

Now assume (2.8), then by the Markov property, for any $X \in L$, $r > 0$ and $n \geq 3$

$$\begin{aligned} P(T_r > n) &\leq P(T_r > \lfloor n/2 \rfloor) P(T_{2r} > \lfloor n/2 \rfloor) \\ &\leq \frac{9C^2}{n^2 h(r) h(2r)} \\ &\leq \frac{72C^2}{n^2 h^2(r)} \end{aligned}$$

by (2.2). Hence recalling that $h(r) \leq 2$ for all r we have

$$\begin{aligned} ET_r &= \sum_{n=0}^{\infty} P(T_r > n) \\ &\leq \sum_{n \leq \frac{2}{h(r)}+1} 1 + \sum_{n > \frac{2}{h(r)}+1} \frac{72C^2}{n^2 h^2(r)} \\ &\leq \frac{2}{h(r)} + 2 + \frac{36C^2}{h(r)} \\ &\leq \frac{6}{h(r)} + \frac{36C^2}{h(r)} = \frac{c}{h(r)} \end{aligned}$$

where $c = 6(1 + 6C^2)$. □

Recall that L denotes the class of all random variables taking values in B. We will also have need to consider the following two subclasses;

$$\begin{aligned} L^2 &= \{X \in L : E\|X\|^2 < \infty\}, \\ L_0^2 &= \{X \in L^2 : EX = 0\}. \end{aligned}$$

The definitions of type 2 and cotype 2 involve sums of independent random variables in L_0^2. An apparently weaker condition frequently encountered in the literature is; B is

Gaussian type 2 if there exists a constant c such that for every finite sequence $\{x_j\}_{j=1}^N$ in B

$$E\|\sum_{j=1}^N g_j x_j\|^2 \le c \sum_{j=1}^N \|x_j\|^2 \tag{2.9}$$

where $g_1, \ldots, g_N$ are real valued IID random variables with standard normal distribution. B is Gaussian cotype 2 if (2.9) holds with the inequality reversed.

It is well known that these notions of type 2 (respectively cotype 2) are equivalent and furthermore the constants in each formulation may be chosen to depend only on each other; see for example pp 53 and 130 of [13] combined with Proposition 9.11 of [12].

It will be convenient to introduce one more equivalent formulation dealing with sums of IID random variables in L_0^2. By Propositions 9.19 and 9.20 of [12], B is type 2 if there exists a constant c such that for every $n \ge 1$ and every sequence of IID random variables $X, X_1, \ldots, X_n$ with $X \in L_0^2$, the following inequality holds;

$$E\|S_n\|^2 \le cnE\|X\|^2. \tag{2.10}$$

B is cotype 2 if the inequality is reversed. We will need the following generalization;

Proposition 2.1. *B is type 2 if (and only if) there is a constant c such that for all $X \in L_0^2$*

$$\limsup_{n\to\infty} \frac{E\|S_n\|^2}{n} \le cE\|X\|^2. \tag{2.11}$$

The type 2 constant may be chosen to depend only on the constant c in (2.11).

B is cotype 2 if (and only if) there is a constant c such that for all $X \in L_0^2$

$$\liminf_{n\to\infty} \frac{E\|S_n\|^2}{n} \ge cE\|X\|^2. \tag{2.12}$$

The cotype 2 constant may be chosen to depend only on the constant c in (2.12).

Proof. Let $x_j \in B$ for $1 \le j \le N$. On some sufficiently rich probability space consider a family A_j of disjoint sets with $P(A_j) = N^{-1}$ for $1 \le j \le N$ and a Rademacher variable ε independent of $A_1, \ldots, A_N$. Let $(\varepsilon_i, I_{A_{i,1}}, \ldots, I_{A_{i,N}})$ be independent copies of $(\varepsilon, I_{A_1}, \ldots, I_{A_N})$ and set

$$X_i = \varepsilon_i \sqrt{N} \sum_{j=1}^N I_{A_{i,j}} x_j.$$

Then

$$E\|X_1\|^2 = \sum_{j=1}^N \|x_j\|^2. \tag{2.13}$$

Observe that

$$\frac{S_n}{\sqrt{n}} = \sum_{j=1}^N Z_{n,j} x_j$$

where

$$Z_{n,j} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \sqrt{N} I_{A_{i,j}}.$$

By the central limit theorem in $\mathbb{R}^N$,

$$(Z_{n,1}, \ldots, Z_{n,N}) \xrightarrow{d} (g_1, \ldots, g_N)$$

where $g_1, \ldots, g_N$ are IID random variables with standard normal distribution. Thus

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} \sum_{j=1}^{N} g_j x_j.$$

Since $\|X_1\|$ is bounded, it then follows from Theorem 5.1 of [3] that

$$\frac{E\|S_n\|^2}{n} \to E\| \sum_{j=1}^{N} g_j x_j \|^2. \tag{2.14}$$

Both results now follow from (2.13), (2.14) and the discussion preceeding the proposition. □

3. **Main Results**

The following inequality related to (2.10), and which is valid in any Banach space, will prove useful below; for any $X \in L^2$

$$Var(\|S_n\|) \le 4nE\|X\|^2. \tag{3.1}$$

This was first observed by de Acosta [2]. The proof is based on elementary properties of conditional expectations and martingales. Alternatively (3.1) may be viewed as a simple consequence of the Efron-Stein inequality, see [16].

Before stating and proving the main results, we note that if $X \in L_0^2$ then

$$\begin{aligned} r^2\|M(r)\| &= r\|E(X : \|X\| \le r)\| \\ &= r\|E(X : \|X\| > r)\| \\ &\le rE(\|X\| : \|X\| > r) \\ &\le E(\|X\|^2 : \|X\| > r) \to 0 \end{aligned}$$

as $r \to \infty$, hence by (2.1)

$$\lim_{r\to\infty} r^2 h(r) = E\|X\|^2 \quad \text{as } r \to \infty. \tag{3.2}$$

Theorem 3.1. *The following are equivalent:*

$$B \textit{ is type } 2; \tag{3.3}$$

There is a constant c such that for all $X \in L$ and all $r > 0$

$$ET_r \ge \frac{c}{h(r)}; \tag{3.4}$$

There is a constant c such that for all $X \in L$, all $r > 0$ and all $n \geq 1$

$$P(T_r \leq n) \leq cnh(r); \tag{3.5}$$

There is a constant c such that for all $X \in L_0^2$

$$\liminf_{r\to\infty} \frac{ET_r}{r^2} \geq \frac{c}{E\|X\|^2}; \tag{3.6}$$

There is a constant c such that for all $X \in L_0^2$

$$\limsup_{r\to\infty} \sup_{n\geq 1} \frac{r^2 P(T_r \leq n)}{n} \leq cE\|X\|^2. \tag{3.7}$$

The constants appearing in these statements may be taken to depend only on each other.

Proof. The dependence of the constants on each other follows from the proof. To avoid repitition this will not be pointed out explicitly during the proof, but the reader is asked to keep this in mind.

It follows from Lemma 2.2 that (3.4) is equivalent to (3.5). Clearly (3.4) implies (3.6) by (3.2), while (3.6) implies (3.7) by (2.3). Thus it suffices to prove (3.3) implies (3.5) and (3.7) implies (3.3).

Assume (3.3). Let $Y_k = X_k I(\|X_k\| \leq r)$ and

$$U_n = \sum_{k=1}^{n} Y_k.$$

Observe that

$$\|EU_n\| = n\|EXI(\|X\| \leq r)\| = nr\|M(r)\| \tag{3.8}$$

while by (3.3)

$$\begin{aligned} E\|U_n - EU_n\|^2 &\leq cnE\|Y_1 - EY_1\|^2 \\ &\leq cnE(\|Y_1\| + E\|Y_1\|)^2 \\ &= cn(E\|Y_1\|^2 + 3(E\|Y_1\|)^2) \\ &\leq 4cnE\|Y_1\|^2 \\ &= 4cnr^2K(r). \end{aligned}$$

Now if $\|EU_n\| \le r/2$, then

$$\begin{aligned}
P(T_r \le n) &= P(\max_{1\le k\le n} \|S_k\| > r) \\
&\le P(\max_{1\le k\le n} \|X_k\| > r) + P(\max_{1\le k\le n} \|U_k\| > r) \\
&\le nG(r) + P(\max_{1\le k\le n} \|U_k - EU_k\| > r/2) \\
&\le nG(r) + \frac{4E\|U_n - EU_n\|^2}{r^2} \qquad \text{by Doob's inequality} \\
&\le nG(r) + 16cnK(r) \\
&\le (16c+1)nh(r).
\end{aligned}$$

On the other hand if $\|EU_n\| \ge r/2$, then by (3.8)

$$P(T_r \le n) \le 1 \le \frac{2\|EU_n\|}{r} = 2n\|M(r)\| \le 2nh(r).$$

Thus in either case (3.5) holds.

Now assume (3.7) and let $X \in L_0^2$. Then for n sufficiently large (depending on X)

$$\begin{aligned}
E\|S_n\| &= \int_0^\infty P(\|S_n\| > r)\ dr \\
&\le \int_0^\infty P(T_r \le n)\ dr \\
&\le \int_0^{\sqrt{cnE\|X\|^2}} dr + \int_{\sqrt{cnE\|X\|^2}}^\infty \frac{2cnE\|X\|^2}{r^2}\ dr \\
&\le 3\sqrt{cnE\|X\|^2}.
\end{aligned}$$

Hence by (3.1) for such n,

$$E\|S_n\|^2 = Var(\|S_n\|) + (E\|S_n\|)^2 \le 4nE\|X\|^2 + 9cnE\|X\|^2$$

which verifies (2.11). □

Fix $r > 0$ and set

$$\hat{X}_k = X_k I(\|X_k\| \le 3r) + \frac{3rX_k}{\|X_k\|} I(\|X_k\| > 3r) \tag{3.9}$$

$$\hat{S}_n = \sum_{k=1}^n \hat{X}_k, \quad \hat{T}_r = \inf\{n : \|\hat{S}_n\| > r\}. \tag{3.10}$$

Since a jump of any size greater than or equal to $3r$ results in S_n leaving the ball of radius r, it follows that $\hat{T}_r = T_r$. It is clear from the definition of $\hat{X}$ that

$$E\|\hat{X}\|^2 = 9r^2(G(3r) + K(3r)). \tag{3.11}$$

Observe further that

$$
\begin{aligned}
E\|\hat{X} - E\hat{X}\|^2 &\geq E\big(\|\hat{X}\| - \|E\hat{X}\|\big)^2 \\
&= E\|\hat{X}\|^2 - 2E\|\hat{X}\|\|E\hat{X}\| + \|E\hat{X}\|^2 \\
&\geq E\|\hat{X}\|^2 - 6r\|E\hat{X}\|. \qquad (3.12)
\end{aligned}
$$

while

$$
\begin{aligned}
\big|\|E\hat{X}\| - \|EXI(\|X\| \leq 3r)\|\big| &\leq \|E\hat{X} - EXI(\|X\| \leq 3r)\| \\
&\leq 3rG(3r). \qquad (3.13)
\end{aligned}
$$

Theorem 3.2. *The following are equivalent:*

$$B \textit{ is cotype } 2; \qquad (3.14)$$

There is a constant c such that for all $X \in L$ and all $r > 0$

$$ET_r \leq \frac{c}{h(r)}; \qquad (3.15)$$

There is a constant c such that for all $X \in L$, all $r > 0$ and all $n \geq 1$

$$P(T_r > n) \leq \frac{c}{nh(r)}; \qquad (3.16)$$

There is a constant c such that for all $X \in L_0^2$

$$\limsup_{r\to\infty} \frac{ET_r}{r^2} \leq \frac{c}{E\|X\|^2}; \qquad (3.17)$$

There is a constant c such that for all $X \in L_0^2$

$$\limsup_{r\to\infty} \sup_{n\geq 1} \frac{nP(T_r > n)}{r^2} \leq \frac{c}{E\|X\|^2}. \qquad (3.18)$$

The constants appearing in these statements may be taken to depend only on each other.

Proof. The dependence of the constants on each other is again a consequence of the proof.

It follows from Lemma 2.3 that (3.15) is equivalent to (3.16). Clearly (3.15) implies (3.17) by (3.2), while (3.17) implies (3.18) by (2.4). Thus it suffices to prove (3.14) implies (3.16) and (3.18) implies (3.14).

Assume (3.14). Let

$$\phi(n) = E \max_{1\leq k\leq n} \big(\|\hat{S}_k - E\hat{S}_k\|\big)^2.$$

Then

$$
\begin{aligned}
\phi(n) &\geq \max_{1\leq k\leq n} E\big(\|\hat{S}_k - E\hat{S}_k\|\big)^2 \\
&\geq cnE\|\hat{X} - E\hat{X}\|^2 \qquad (3.19)
\end{aligned}
$$

by (3.14). By (1.3) of Klass [10] (alternatively see Theorem 2.3.1 of [4]), there exists an absolute constant α such that

$$\begin{aligned} E \max_{1\le k\le T_r} \left(\|\hat{S}_k - E\hat{S}_k\|\right)^2 &\ge \alpha E\phi(T_r) \\ &\ge \alpha c E T_r E\|\hat{X} - E\hat{X}\|^2 \end{aligned} \tag{3.20}$$

by (3.19). Since $\|\hat{S}_k\| \le 4r$ for all $1 \le k \le T_r$, it follows that

$$E \max_{1\le k\le T_r} \left(\|\hat{S}_k - E\hat{S}_k\|\right)^2 \le E \max_{1\le k\le T_r} \left(\|\hat{S}_k\| + \|E\hat{S}_k\|\right)^2 \le 64r^2.$$

Hence by (3.12) and (3.20)

$$\alpha c E T_r \left(E\|\hat{X}\|^2 - 6r\|E\hat{X}\|\right) \le 64r^2. \tag{3.21}$$

We now consider two cases:
Case 1; $E\|\hat{X}\|^2 > 12r\|E\hat{X}\|$.
Then by (3.21)

$$\alpha c E T_r E\|\hat{X}\|^2 \le 128r^2.$$

On the other hand by (3.11) and (3.13)

$$\begin{aligned} 9r^2 h(3r) &= E\|\hat{X}\|^2 + 3r\|EXI(\|X\| \le 3r)\| \\ &\le E\|\hat{X}\|^2 + 3r\|E\hat{X}\| + 9r^2 G(3r) \\ &\le (9/4)E\|\hat{X}\|^2. \end{aligned}$$

Hence by (2.2)

$$ET_r \le \frac{576}{\alpha c h(r)}.$$

Thus (3.16) follows by Markov's inequality.
Case 2; $E\|\hat{X}\|^2 \le 12r\|E\hat{X}\|$.
First observe that by (3.13) and (3.11)

$$\|EXI(\|X\| \le 3r)\| \le \|E\hat{X}\| + 3rG(3r) \le \|E\hat{X}\| + \frac{E\|\hat{X}\|^2}{3r} \le 5\|E\hat{X}\|,$$

hence

$$3rh(3r) = \frac{\|E\hat{X}\|^2}{3r} + \|EXI(\|X\| \le 3r)\| \le 9\|E\hat{X}\|. \tag{3.22}$$

Subcase 2a; $n\|E\hat{X}\| \le 3r$.
Then

$$P(T_r > n) \le 1 \le \frac{3r}{n\|E\hat{X}\|} \le \frac{9}{nh(3r)} \le \frac{162}{nh(r)}$$

by (2.2).

Subcase 2b; $n\|E\hat{X}\| > 3r$.

In this case we have

$$E\|\hat{S}_n\| \geq \|E\hat{S}_n\| = n\|E\hat{X}\|.$$

In particular $E\|\hat{S}_n\| > 3r$ and so

$$\begin{aligned}
P(T_r > n) &\leq P(\|\hat{S}_n\| \leq r) \\
&\leq P\big(3(E\|\hat{S}_n\| - \|\hat{S}_n\|) > 2E\|\hat{S}_n\|\big) \\
&\leq P\big(3\big|\|\hat{S}_n\| - E\|\hat{S}_n\|\big| > 2n\|E\hat{X}\|\big) \\
&\leq \frac{9Var(\|\hat{S}_n\|)}{\big(2n\|E\hat{X}\|\big)^2} \\
&\leq \frac{36nE\|\hat{X}\|^2}{\big(2n\|E\hat{X}\|\big)^2} \quad \text{by (3.1)} \\
&\leq \frac{108r}{n\|E\hat{X}\|} \quad \text{by Case 2} \\
&\leq \frac{5832}{nh(r)}
\end{aligned}$$

by (3.22) and (2.2). Thus (3.16) holds.

Now assume (3.18). Let

$$M_n = \max_{1\leq k\leq n} \|S_k\|.$$

Since $P\{M_n \leq r\} = \{T_r > n\}$ we have that for sufficiently large n (depending on X)

$$P(M_n^2 \leq \frac{nE\|X\|^2}{4c}) \leq 1/2.$$

Hence by Doob's inequality, for such n

$$\begin{aligned}
4E\|S_n\|^2 &\geq EM_n^2 \\
&\geq E(M_n^2; M_n^2 > \frac{nE\|X\|^2}{4c}) \\
&\geq \frac{nE\|X\|^2}{8c}.
\end{aligned}$$

which verifies (2.12). □

4. **Examples**

As was indicated earlier, in order to extend Pruitt's results to an arbitrary separable Banach space a function other than h is needed to measure the size of ET_r. In this section we consider the asymptotic behavior of ET_r when $B = \ell^p$ for $2 \leq p < \infty$. In particular we obtain sharp bounds in (3.6) for symmetric $X \in L^2$ showing that $1/h(r)$ underestimates the size of ET_r when $p > 2$.

Let $X, X_1, X_2, \dots$ be a sequence of IID symmetric random variables taking values in ℓ^p and let $X_{i,j}$ denote the jth coordinate of X_i. In the following $\approx_p$ means the ratio of the two quantities is bounded above and below by constants which depend only on p.

Lemma 4.1. *Let $\{X_i\}$ be a sequence of IID symmetric random variables taking values in ℓ^p for $1 \le p < \infty$. Then*

$$E\|S_n\|^p \approx_p \sum_{j=1}^{\infty} E\Big(\sum_{i=1}^{n} X_{i,j}^2\Big)^{p/2}.$$

The proof is a standard computation using the Khintchine-Kahane inequalities for Rademacher functions, so we will omit it. The upper bound is recorded in Lemma 5.2 of Pisier and Zinn [14]. Next we need Rosenthal's inequality (Theorem 1.5.9 of [4]) from which it follows that for $p \ge 2$

$$E\Big(\sum_{i=1}^{n} X_{i,j}^2\Big)^{p/2} \approx_p (nEX_{1,j}^2)^{p/2} + nE|X_{1,j}|^p.$$

Combining these two results we find that for $2 \le p < \infty$

$$E\|S_n\|^p \approx_p n^{p/2} \sum_{j=1}^{\infty} (EX_{1,j}^2)^{p/2} + n \sum_{j=1}^{\infty} E|X_{1,j}|^p.$$

Now ℓ^p is type 2 when $2 \le p < \infty$. Thus by Proposition 9.24 of [12], if $X \in L^2$ then X is pregaussian . In that case let $G(X)$ denote a Gaussian random variable with the same covariance structure as X. Then

$$E\|G(X)\|^p = E|g|^p \sum_{j=1}^{\infty} (EX_{1,j}^2)^{p/2}, \tag{4.1}$$

where g has a standard normal distribution; see p 261 of [12]. Since all L^q norms of a Gaussian random variable are comparable (Corollary 3.2 of [12]), we can conclude that

$$E\|S_n\|^p \approx_p n^{p/2}(E\|G(X)\|^2)^{p/2} + nE\|X\|^p. \tag{4.2}$$

This is only useful if $E\|X\|^p < \infty$. But since the constants in (4.2) do not depend on the random variable X we can apply this estimate to $\hat{X}$, as defined in (3.9), and obtain that for any symmetric random variable X

$$E\|\hat{S}_n\|^p \approx_p n^{p/2}(E\|G(\hat{X})\|^2)^{p/2} + nE\|\hat{X}\|^p. \tag{4.3}$$

Observe also that since $\hat{X}_1 = \xi_r X_1$ where

$$\xi_r = I(\|X_1\| \le 3r) + \frac{3r}{\|X_1\|} I(\|X_1\| > 3r),$$

it follows from (4.1) and monotone convergence that

$$E\|G(\hat{X})\|^p \to E\|G(X)\|^p. \tag{4.4}$$

It is not hard to show that $G(\hat{X}) \xrightarrow{d} G(X)$ and so in fact $E\|G(\hat{X})\|^q \to E\|G(X)\|^q$ for all $q \geq 0$; see Theorem 3.8.11 of [1]. However (4.4), together with the comparability of Gaussian norms, will suffice for our needs below.

Proposition 4.1. *Let $\{X_i\}$ be a sequence of IID symmetric random variables taking values in ℓ^p for $2 \leq p < \infty$, for which $E\|X\|^2 < \infty$. Then there is a constant c, depending only on p, such that for every $r > 0$*

$$ET_r \leq \frac{cr^2}{E\|G(\hat{X})\|^2}. \tag{4.5}$$

On the other hand there is a constant c, again depending only on p, such that

$$\liminf_{r\to\infty} \frac{ET_r}{r^2} \geq \frac{c}{E\|G(X)\|^2}. \tag{4.6}$$

Proof. Recalling the definitions of $\hat{S}_n$ and $\hat{T}_r$ in (3.10), if we let

$$\phi(n) = E \max_{1\leq k\leq n} \|\hat{S}_k\|^p,$$

then by (1.3) of [10] (alternatively see Theorem 2.3.1 of [4]) there exists a constant α_p, depending only on p, such that

$$E \max_{1\leq k\leq \hat{T}_r} \|\hat{S}_k\|^p \geq \alpha_p E\phi(\hat{T}_r).$$

Now by (4.3), there exists a constant c_p, depending only on p, such that

$$\phi(n) \geq c_p n^{p/2}(E\|G(\hat{X})\|^2)^{p/2},$$

hence

$$(4r)^p \geq \alpha_p c_p ET_r^{p/2}(E\|G(\hat{X})\|^2)^{p/2}.$$

Thus (4.5) follows by Jensen's inequality.

For the lower bound we let

$$\phi(n) = E \max_{1\leq k\leq n} \|\hat{S}_k\|^2.$$

By (1.10) of [9] (alternatively see Theorem 2.3.1 of [4]) there exists a universal constant β such that

$$E \max_{1\leq k\leq \hat{T}_r} \|\hat{S}_k\|^2 \leq \beta E\phi(\hat{T}_r).$$

By Doob's inequality, Jensen's inequality and (4.3), for some constant C_p depending only on p,

$$\phi(n) \leq 4E\|\hat{S}_n\|^2 \leq 4(E\|\hat{S}_n\|^p)^{2/p} \leq C_p nE\|G(\hat{X})\|^2 + C_p n^{2/p}(E\|\hat{X}\|^p)^{2/p}.$$

Thus

$$r^2 \le \beta C_p ET_r E\|G(\hat{X})\|^2 + \beta C_p ET_r^{2/p}(E\|\hat{X}\|^p)^{2/p}. \tag{4.7}$$

If $p = 2$ then $E\|G(\hat{X})\|^2 = E\|\hat{X}\|^2$ by (4.1), and so the following stronger (than (4.6)) condition holds; for all $r > 0$

$$\frac{ET_r}{r^2} \ge \frac{c}{E\|G(\hat{X})\|^2} \tag{4.8}$$

where $c = (2\beta C_p)^{-1}$. If $p > 2$ then $r^{2-p}E\|\hat{X}\|^p \to 0$ since $E\|X\|^2 < \infty$. In that case (4.6) follows from (4.7), Jensen's inequality, and (4.4) together with the comparability of Gaussian norms. □

When $p = 2$ the pair of inequalities (4.5) and (4.8) also follow easily from (3.4) and (3.15). The main interest in Proposition 4.1 as a source of examples is when $p > 2$.

Using the bounds in (4.5) and (4.6) we can compare $h(r)$ and ET_r for large r. For convenience we will write $h(r)ET_r \approx L$ if

$$cL \le \liminf_{r\to\infty} h(r)ET_r \le \limsup_{r\to\infty} h(r)ET_r \le CL, \tag{4.9}$$

where the constants depend only on the Banach space. Thus in the setting of Proposition 4.1, it follows that

$$h(r)ET_r \approx \frac{E\|X\|^2}{E\|G(X)\|^2}. \tag{4.10}$$

The RHS of (4.10) is always bounded below by a constant depending on the type 2 constant of B by Proposition 9.24 of [12]. However there is no corresponding upper bound (unless $p = 2$).

We conclude with some specific examples. Let

$$X_{i,j} = \alpha_j r_{i,j} I(j \in \Theta_i)$$

where $\{\Theta_i\}_{i\ge1}$ is a sequence of IID random variables taking values in the power set of the natural numbers $\mathbb{N}$, $\{r_{i,j}\}_{i\ge1,j\ge1}$ is a sequence of IID Rademacher random variables independent of $\{\Theta_i\}_{i\ge1}$ and $\{\alpha_j\}_{j\ge1}$ is a sequence of non-negative real numbers. In this case (4.10) becomes

$$h(r)ET_r \approx \frac{E\big(\sum_{j=1}^\infty \alpha_j^p I(j\in\Theta)\big)^{2/p}}{\big(\sum_{j=1}^\infty \alpha_j^p p_j^{p/2}\big)^{2/p}} \tag{4.11}$$

where $p_j = P(j \in \Theta_1)$.

Example 1. Θ *is non-random; thus* $\Theta \equiv A$ *for some fixed subset* A *of* $\mathbb{N}$.

In that case X is simply a Rademacher sum and in order that it take values in ℓ^p we require $\|X\| \equiv \sum_{j\in A} \alpha_j^p < \infty$. Thus all moments of $\|X\|$ are finite and in particular

$$E\Big(\sum_{j=1}^\infty \alpha_j^p I(j\in\Theta)\Big)^{2/p} = \Big(\sum_{j\in A}\alpha_j^p\Big)^{2/p}.$$

Since $p_j \equiv 1$ on A and 0 otherwise,

$$h(r)ET_r \approx 1.$$

Example 2. $|\Theta| = k$ *for some* $k \geq 1$ *and* $\alpha_i \equiv 1$.

In that case

$$\|X\|^p = \sum_{j=1}^{\infty} I(j \in \Theta) = k$$

and so again all moments are finite, and

$$h(r)ET_r \approx \frac{k^{2/p}}{\left(\sum_{j=1}^{\infty} p_j^{p/2}\right)^{2/p}}.$$

As a particular case assume $1 \leq k \leq d < \infty$ and Θ is uniformly distributed on subsets of $\{1, \ldots, d\}$ of size k. Then $p_j = k/d$ for $1 \leq j \leq d$ and is 0 otherwise, hence

$$h(r)ET_r \approx (d/k)^{1-\frac{2}{p}}.$$

This class of examples should be contrasted with Example 1 when A is taken to be a subset of $\{1, \ldots, d\}$ of size k. If the k directions in which S_n can move are fixed, then h can be used to approximate ET_r, but if at each step the k directions are chosen at random from $\{1, \ldots, d\}$ then this is no longer the case (if $p > 2$). These examples also illustrate that the RHS of (4.10) can not be bounded above by a constant independent of the distribution of X when $p > 2$.

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