2008

Exact Lattice Supersymmetry: the Two-Dimensional N = 2 Wess-Zumino Model

Simon Catterall
Syracuse University

Sergey Karamov
Syracuse University

SU-4252-735
February 1, 2008

Exact Lattice Supersymmetry: the Two-Dimensional $N = 2$ Wess-Zumino Model

Simon Catterall and Sergey Karamov
Physics Department, Syracuse University, Syracuse, NY 13244

Abstract

We study the two-dimensional Wess-Zumino model with extended $N = 2$ supersymmetry on the lattice. The lattice prescription we choose has the merit of preserving *exactly* a single supersymmetric invariance at finite lattice spacing a. Furthermore, we construct three other transformations of the lattice fields under which the variation of the lattice action vanishes to $O(ga^2)$ where g is a typical interaction coupling. These four transformations correspond to the two Majorana supercharges of the continuum theory. We also derive lattice Ward identities corresponding to these exact and approximate symmetries. We use dynamical fermion simulations to check the equality of the massgaps in the boson and fermion sectors and to check the lattice Ward identities. At least for weak coupling we see no problems associated with a lack of reflection positivity in the lattice action and find good agreement with theory. At strong coupling we provide evidence that problems associated with a lack of reflection positivity are evaded for small enough lattice spacing.

1 Introduction

Supersymmetry is thought to be an important ingredient of many theories which attempt to unify the separate interactions contained in the standard model of particle physics. Since low energy physics is manifestly not supersymmetric it is necessary that this symmetry be broken at some energy scale. A set of non-renormalization theorems ensure that if SUSY is not

[0]Corresponding author: Simon Catterall, email: smc@physics.syr.edu

broken at tree level then it cannot be broken in any finite order of perturbation theory see eg. [1]. Thus we are led to investigate non-perturbative mechanisms for SUSY breaking. The lattice furnishes the only tool for a systematic investigation of non-perturbative effects in field theories and so significant effort has gone into formulating SUSY theories on the lattice [2].

Unfortunately, there are several barriers to such lattice formulations. Firstly, supersymmetry is a spacetime symmetry which is generically broken by the discretization procedure. In this it resembles Poincare invariance which is also not preserved in a lattice theory. However, unlike Poincare invariance there is usually no SUSY analog of the discrete translation and cubic rotation groups which are left unbroken on the lattice. In the latter case the existence of these remaining discrete symmetries is sufficient to prohibit the appearance of relevant operators in the long wavelength lattice effective action which violate the *full* symmetry group. This ensures that Poincare invariance is achieved automatically *without fine tuning* in the continuum limit. Since generic latticizations of supersymmetric theories do not have this property their effective actions typically contain relevant supersymmetry breaking interactions. To achieve a supersymmetric continuum limit then requires fine tuning the bare lattice couplings of all these SUSY violating terms - typically a very difficult proposition.

Secondly, supersymmetric theories necessarily involve fermionic fields which suffer from so-called doubling problems when we attempt to define them on the lattice. The presence of extra fermionic modes furnishes yet another source of supersymmetry breaking since typically they are not paired with corresponding bosonic states. Furthermore, most methods of eliminating the extra fermionic modes serve to break supersymmetry also.

In this paper we employ a lattice formulation of the two-dimensional Wess-Zumino model which was first written down in [3], [4]. In these earlier works the lattice formulation is found by discretizing the Nicolai map for the model [5]. In our case we rederive the formulation in a slightly different way – we start from a simple discrete model which exhibits a one parameter local, supersymmetric invariance and show how this model may be generalized to a two dimensional Euclidean lattice field theory provided certain integrability conditions are satisfied. The $N = 2$ Wess-Zumino model is then found as the essentially unique solution to these conditions. Since the continuum model contains two Majorana supercharges we would expect the lattice model to possess three further transformations which are invariances of the action in the naive continuum limit. We construct these transformations explicitly and from them derive a set of exact and broken lattice Ward identities.

To check these ideas explicitly we have simulated the simplest realization of the model for a range of masses and couplings, computing both boson and fermion massgaps and the correlation functions needed for checking the supersymmetric Ward identities. To perform the simulations we have replaced the fermionic fields by commuting pseudofermionic fields in the usual manner.

The outline of the paper is as follows; first we introduce a simple discrete model with an exact SUSY-like symmetry, showing how it can be used to describe a lattice version of supersymmetric quantum mechanics and then discussing its extension to two-dimensional field theory. The Ward identities are then introduced and we show how the expectation value of the total action (including the contribution of pseudofermion fields) can be used as an order parameter for SUSY breaking. Following from this theoretical introduction we present our numerical results both for weak and strong coupling. The final section contains our conclusions.

2 Simple SUSY Lattice Model

Consider a set of P real commuting variables x_i and two sets of P real grassmann variables ψ_i and $\overline{\psi}_i$ with $i = 1 \ldots P$ governed by an action $S(x, \psi, \overline{\psi})$ of the form

$$S = \frac{1}{2} N_i(x) N_i(x) + \overline{\psi}_i \frac{\partial N_i}{\partial x_j} \psi_j \tag{1}$$

with the field $N_i(x)$ an arbitrary function of x_i. It is easy to see that this action is invariant under the following SUSY transformation.

$$\begin{aligned} \delta_1 x_i &= \psi_i \xi \\ \delta_1 \overline{\psi}_i &= N_i \xi \\ \delta_1 \psi_i &= 0 \end{aligned}$$

$$\begin{aligned} \delta S &= N_i \frac{\partial N_i}{\partial x_j} \delta x_j + \delta \overline{\psi}_i \frac{\partial N_i}{\partial x_j} \psi_j \\ &= N_i \frac{\partial N_i}{\partial x_j} \{\psi_j, \xi\}_+ \end{aligned}$$

which vanishes on account of the grassmann nature of the infinitesimal parameter ξ. Notice that the variation of the matrix $\frac{\partial N_i}{\partial x_j}$

$$\delta \frac{\partial N_i}{\partial x_j} \psi_j = \frac{1}{2} \frac{\partial^2 N_i}{\partial x_j x_k} \{\psi_k, \psi_j\}_+ \xi$$

also vanishes for similar reasons.

Let us now choose the fields $x, \psi, \overline{\psi}$ to lie on a spatial lattice equipped with periodic boundary conditions and take the fermion matrix $M_{ij} = \frac{\partial N_i}{\partial x_j}$ to be of the form

$$M_{ij} = D^S_{ij} + P''_{ij}(x)$$

The symmetric difference operator D^S_{ij} replaces the continuum derivative and can be written in terms of the usual forward and backward difference operators.

$$D^S_{ij} = \frac{1}{2}(D^+_{ij} + D^-_{ij})$$

and $P''_{ij}(x)$ is some (local) interaction matrix polynomial in the scalar fields x. The resulting model is easily recognized as supersymmetric quantum mechanics regularized as a $0+1$ dimensional Euclidean lattice theory [6]. Furthermore it is trivial to find a field $N_i(x)$ which yields this fermion matrix under differentiation

$$N_i = D^S_{ij} x_j + P'_i(x)$$

Notice, however, that the resulting bosonic action $\frac{1}{2}N_i^2$ is not a simple discretization of its continuum counterpart

$$S^E_{\rm cont} = \int d\tau \frac{1}{2}\left((\partial_\tau x)^2 + (P'(x))^2\right)$$

as it contains a new cross term $C = P'_i(x) D^S_{ij} x_j$ which would be a total derivative (and hence zero) in the continuum but is non-vanishing on the lattice and required to ensure the transformation eqn. 2 is an exact symmetry of the theory. Notice that this extra term also vanishes on the lattice for a *free* theory where $P'(x) = mx$ because of the antisymmetry of the matrix D^S_{ij}.

Notice that if I imagine changing variables in the partition function $Z = \int Dx e^{-S(x,\overline{\psi},\psi)}$ from x to N the Jacobian resulting from this transformation cancels the fermion determinant yielding a trivial gaussian theory in the field N. This is an example of a Nicolai map and the existence of such a transformation of the bosonic degrees of freedom can be shown to imply an exact supersymmetry [5]. While most supersymmetric theories admit such a map, in the generic case it is non-local - that is the mapped Nicolai field N will be a function of arbitrarily high derivatives of the original boson field x. In the case of SUSY quantum mechanics (and as we will see later the $N=2$ Wess-Zumino model) the expression is local. It can then serve as a basis for constructing a lattice theory with an exact supersymmetry as was pointed out in [3], [4] and [8].

So far we have neglected the fact that the form of the fermion action appears to admit doubles - the symmetric difference operator D^S_{ij} behaves like $\sin ka$ in lattice momentum space yielding zeroes at both $ka = 0$ and the Brillouin zone boundary $ka = \pi$. Indeed, both the fermionic *and bosonic* actions now contain spurious modes which are not part of the continuum theory. The extra bosonic modes arise from using $D^S D^S$ as the kinetic operator rather than the usual scalar lattice Laplacian $\Box = D_+ D_-$. However, we can use our freedom in choosing the interaction matrix $P''_{ij}(x)$ to add a Wilson term to the fermion action

$$P''_{ij}(x) = -D^A_{ij} + \text{local interaction terms}$$

where the matrix $D^A_{ij} = \frac{1}{2}(D^+_{ij} - D^-_{ij}) = \Box_{ij}$. By construction this eliminates the doubles from the free fermion action completely; what is, perhaps, more surprising is that it also renders the boson spectrum double free too. This can be seen to be a consequence of the lattice supersymmetry.

One further observation is in order. Consider a second supersymmetry transformation

$$\begin{aligned}
\delta_2 x_i &= \overline{\psi}_i \xi \\
\delta_2 \psi_i &= \overline{N}_i \xi \\
\delta_2 \overline{\psi}_i &= 0
\end{aligned}$$

where

$$\frac{\partial \overline{N}_i}{\partial x_j} = -M^T_{ij} = D^S_{ij} - P''_{ij}(x)$$

The action in eqn. 1 is no longer invariant under this transformation

$$\begin{aligned}
\delta_2 S &= \frac{1}{2}\delta_2 \left(N_i^2 - \overline{N}_i^2\right) \\
&= 2\delta_2 C
\end{aligned} \tag{2}$$

but transforms into the supersymmetry variation of (twice) the cross term C. As we have argued, for a free lattice theory or in the naive continuum limit this term will vanish and the model will be invariant under this second supersymmetry. For the lattice theory in the presence of interactions ($P'(x) \sim gx^n, n > 1$), this second symmetry will be broken by terms $\mathrm{O}(ga^2)$ where the suppression by two powers of the lattice spacing reflects the fact that $D^S = \partial_{\text{cont}} + O(a^2)$. Thus the second supersymmetry is broken only by *irrelevant* operators. Since quantum mechanics is a finite theory we then expect that the continuum theory will have the two invariances that we expect of supersymmetric quantum mechanics [7]. We have verified this explicitly in [6] in which a computation of both the mass spectrum and the supersymmetric Ward identities revealed the existence of $N = 2$ supersymmetry in the continuum limit.

3 The Lattice Wess-Zumino Model

The action eqn. 1 and supersymmetry transformations eqn. 2 do not depend strongly on the existence of a background lattice of given dimensionality – indeed this physical interpretation only arises when we choose the form of the fermion operator. This allows us to use it as a basis for constructing candidate lattice field theories in higher dimensions which admit supersymmetry.

In two dimensions the fermions will be represented by two independent two-component spinors whose components we will assume to be real (this restriction will turn out to be valid for $N = 2$ theories in Euclidean space). Thus we will imagine that the indices i, j can be promoted to compound indices $i \to i, \alpha$, $j \to j, \beta$ labeling spacetime and spinor components respectively. We immediately realize that there will be two scalar fields now in the theory $x_i \to x_i^\alpha$ and the fermion matrix will take the form $D_{ij} \to D_{ij}^{\alpha\beta}$ (from now on we will use D in place of D^S). To maintain contact with the simple, discrete model we will require a Euclidean fermion operator which is also entirely real. Then the most general fermion matrix respecting this condition takes the form

$$M_{ij}^{\alpha\beta} = \gamma^\mu_{\alpha\beta} D^\mu_{ij} + A_{ij}\delta_{\alpha\beta} + B_{ij} i\gamma^3_{\alpha\beta}$$

where $A(x)$ and $B(x)$ are real matrix fields and we have chosen a Majorana basis for the Dirac matrices so that γ^1,γ^2 and $i\gamma^3$ are also real.

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad i\gamma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

To remove the doubles we again add a Wilson term to the interaction matrix $A_{ij} = -D^A_{ij} +$ interactions where

$$D^A_{ij} = \frac{1}{2}\sum_{\mu=1}^{2}(D^{\mu+}_{ij} - D^{\mu-}_{ij})$$

The resultant fermion matrix is easily recognized as a discrete version of the continuum Wess-Zumino model and is the same fermion operator appearing in [3],[4]. Having chosen this fermion matrix we can attempt to find a vector N_i^α whose derivative yields $M_{ij}^{\alpha\beta}$. Clearly N_i^α must have the form

$$N_i^\alpha = \gamma^\mu_{\alpha\beta} D^\mu_{ij} x_j^\beta + f_i^\alpha$$

where f_i^α which represent mass and interaction terms must still be determined. Ignoring for a moment the spacetime indices it is clear that strong restrictions are placed on the vector f^α. We must have

$$A = \frac{\partial f^1}{\partial x^1} = \frac{\partial f^2}{\partial x^2} \quad B = \frac{\partial f_2}{\partial x^1} = -\frac{\partial f^1}{\partial x_2}$$

Of course these are just Cauchy-Riemann conditions. In other words the integrability condition that M be a derivative of some vector N imposes a complex structure on the scalar fields in the theory. Indeed, the bosonic part of the action can now be rewritten in terms of a complex vector $\eta^{(1)}(\phi)$ whose real and imaginary parts are just the two components N^1 and N^2 respectively (we have again suppressed spacetime indices for clarity) $S_B = \frac{1}{2}\overline{\eta}^{(1)}\eta^{(1)}$ where $\mathrm{Re}\phi = x^1$ and $\mathrm{Im}\phi = x^2$ and

$$\eta^{(1)} = D_z\overline{\phi} + W'(\phi)$$

where we have introduced complex coordinates $z = (x+iy)/2$, $\overline{z} = (x-iy)/2$ so that

$$D_z = D_1 - iD_2$$

with D_1, D_2 derivative operators in the two dimensional lattice. The significance of the superscript on $\eta^{(1)}$ will be become apparent later. $W'(\phi)$ is an arbitrary analytic function of the complex field ϕ with $\overline{\phi}$ its complex conjugate. Furthermore, in this language the fields A and B are nothing but the real and imaginary parts of $W''(\phi)$. Expanding the bosonic action yields

$$S_B = \frac{1}{2}\sum_{z,\overline{z}} D_{\overline{z}}\phi D_z\overline{\phi} + W'(\phi)W'(\overline{\phi}) + D_z\overline{\phi}W'(\overline{\phi}) + D_{\overline{z}}\phi W'(\phi)$$

The first two terms go over as $a \to 0$ to the bosonic part of the continuum action for the $N = 2$ Wess-Zumino model while the last two terms are clearly total derivatives which will vanish both in the continuum and for a free lattice theory. For an interacting theory they are necessary to preserve the lattice supersymmetry transformation. However they spoil the reflection positivity of the lattice action, a point we shall return to when we present our numerical results.

So far we have shown that the lattice action

$$S = \frac{1}{2} N_i^\alpha N_i^\alpha + \overline{\psi}_i^\alpha M_{ij}^{\alpha\beta} \psi_j^\beta$$

admits the following invariance

$$\begin{aligned} \delta_1 x_i^\alpha &= \psi_i^\alpha \xi \\ \delta_1 \psi_i^\alpha &= 0 \\ \delta_1 \overline{\psi}_i^\alpha &= N_i^\alpha \xi \end{aligned} \tag{3}$$

determined by a single grassmann parameter ξ corresponding to a single supercharge. We know that the continuum $N = 2$ Wess-Zumino model possesses four such supercharges corresponding to two independent two component Majorana charges. Thus we might expect that the lattice model will admit three further transformations which become invariances as $a \to 0$. The complex form of the bosonic action immediately suggests three further bosonic actions which will differ from each other by terms which become total derivatives in the continuum limit. These are

$$\begin{aligned} \eta^{(2)} &= D_z \overline{\phi} - W'(\phi) \\ \eta^{(3)} &= D_z \phi - W'(\overline{\phi}) \\ \eta^{(4)} &= D_z \phi + W'(\overline{\phi}) \end{aligned} \tag{4}$$

Let $\overline{N}_i^\alpha$ be the (real) two component vector corresponding to the complex field $\eta^{(2)}$. Under differentiation it generates a new fermion matrix

$$(M^{(2)})_{ij}^{\alpha\beta} = \frac{\partial \overline{N}_i^\alpha}{\partial x_j^\beta}$$

Using the arguments of the previous section we can now write down a new lattice action $S^{(2)}$.

$$S^{(2)} = \frac{1}{2} \eta^{(2)} \overline{\eta}^{(2)} + \chi M^{(2)} \omega$$

where χ and ω are new anticommuting spinor fields. $S^{(2)}$ will, of course, possess a new supersymmetry invariance involving now not the vector N but $\overline{N}$. Furthermore it is easy to see that $M^{(2)} = i\gamma_3 M i\gamma_3$. Hence these two lattice theories generate (up to total derivative-like terms) the *same* continuum action. Indeed, if we make the identifications

$$\begin{aligned} \psi &= i\gamma_3 \omega \\ \overline{\psi} &= i\gamma_3 \chi \end{aligned} \tag{5}$$

we can see that the original lattice action has a second approximate supersymmetry given by

$$\begin{aligned}
\delta_2 x_i^\alpha &= i\gamma_3^{\alpha\beta}\psi_i^\beta\xi \\
\delta_2 \psi_i^\alpha &= 0 \\
\delta_2 \overline{\psi}_i^\alpha &= i\gamma_3^{\alpha\beta}\overline{N}_i^\beta\xi
\end{aligned} \tag{6}$$

The variation of the action under this second supersymmetry involves the supersymmetry variation of terms which vanish as total derivatives in the continuum limit. On the lattice these terms will be of order ga^2 with g a typical interaction coupling. Hence, at least in perturbation theory such a term would constitute an irrelevant operator and the continuum limit should exhibit this second supersymmetry. One might worry that the presence of such a SUSY-violating term in the bare lattice action might lead to relevant breaking terms in the long distance effective action. However, it is not possible to write down any such counterterms which simultaneously preserve the one exact SUSY. Thus, the existence of a subset of the full SUSY in the lattice model is indeed sufficient to protect the broken supersymmetries so that no fine tuning is required to achieve the full symmetry in the continuum limit.

Turning to $\eta^{(3)}$ we can see that it generates yet another fermion matrix of the form

$$(M^{(3)})_{ij}^{\alpha\beta} = \frac{\partial Q_i^\alpha}{\partial x_j^\beta}$$

where the vector Q^α again carries the real and imaginary parts of $\eta^{(3)}$. Again, $M^{(3)}$ may be expressed in terms of the original M

$$M^{(3)} = -M^T\gamma^1$$

which proves that an action based around $\eta^{(3)}$ will once again constitute a lattice theory of the continuum Wess-Zumino model with yet another supersymmetry. In terms of the original fermion fields this third transformation will yield another approximate invariance of the original action

$$\begin{aligned}
\delta_3 x_i^\alpha &= \gamma_1^{\alpha\beta}\overline{\psi}_i^\beta\xi \\
\delta_3 \overline{\psi}_i^\alpha &= 0 \\
\delta_3 \psi_i^\alpha &= Q_i^\alpha\xi
\end{aligned} \tag{7}$$

The final approximate invariance can be derived similarly from $\eta^{(4)}$ (or its real vector form $\overline{Q}_i^\alpha$) and yields the transformations

$$\delta_4 x_i^\alpha = \gamma_2^{\alpha\beta}\overline{\psi}_i^\beta\xi$$

$$
\begin{aligned}
\delta_4 \overline{\psi}_i^\alpha &= 0 \\
\delta_4 \psi_i^\alpha &= i\gamma_3^{\alpha\beta} \overline{Q}_i^\beta \xi
\end{aligned} \tag{8}
$$

Thus far we have again assumed that the variation of the fermion matrix under these supersymmetry transformations is zero. However, the simple proof we gave in the previous section for the absence of such a term in δS does not hold when the variation of the field x involves non-trivial gamma matrices acting on ψ or $\overline{\psi}$. If we examine the general structure of such a variation we find that it has the form (we suppress spacetime indices which play no essential role)

$$
\overline{\theta}\delta M \psi = \overline{\theta}^\alpha \frac{\partial^2 f^\alpha}{\partial x^\beta \partial x^\gamma} \Gamma^{\gamma\delta} \theta^\delta \theta^\beta
$$

where θ, $\overline{\theta}$ represent either ψ or $\overline{\psi}$. This can be seen to be the trace of a product of a symmetric matrix (the term involving derivatives of f) with the gamma matrix Γ and the antisymmetric matrix formed by the product of the θ terms. Thus, for $\Gamma = \gamma_1$ or $\Gamma = \gamma_2$ this is the trace of an antisymmetric matrix and is hence zero. For $\Gamma = i\gamma_3$ the resultant matrix is now symmetric but the trace can be shown to still vanish as a consequence of the Cauchy-Riemann conditions applying to the derivatives of f.

4 Ward Identities

4.1 Quantum Mechanics

The invariance of the quantum mechanical lattice action under the discrete supersymmetry transformation eqn. 2 leads to a set of Ward identities connecting bosonic and fermionic correlation functions. We can derive these following the usual procedure by adding a set of source terms to the action and carrying out an (infinitesimal) supersymmetry variation of the fields. Since the partition function, measure and action are all invariant under this change of variables we immediately derive the result

$$
\delta Z = 0 = \int D\overline{\psi} D\psi D x e^{-S + J.x + \theta.\overline{\psi}} \left(J.\delta_1 x + \theta.\delta_1 \overline{\psi} \right)
$$

Indeed any derivative of this expression with respect to the source terms (which are set to zero at the end) is also vanishing. Thus we are led immediately to the first non-trivial supersymmetric Ward identity

$$
\left\langle \overline{\psi}_i \psi_j \right\rangle + \left\langle N_i x_j \right\rangle = 0
$$

relating the fermion correlation function to one depending only on bosonic fields. Notice also that in the continuum limit there will be a second set of Ward identities following from the second invariance given by the variation δ_2.

$$\left\langle \psi_i \overline{\psi}_j \right\rangle + \left\langle \overline{N}_i x_j \right\rangle = 0$$

To perform a simulation of this model we will replace the integral over anticommuting fields $\overline{\psi}$, ψ by one over a (real) pseudofermion field χ whose action $S_{PF} = \chi^T (M^T M)^{-1} \chi$ yields the same fermion determinant $\det(M(x))$. Consider now the generalized partition function $Z(\alpha)$ where

$$Z(\alpha) = \int DxD\chi e^{-\alpha S(x,\chi)} \tag{9}$$

This allows us to write down a simple expression for the mean action including the pseudofermions

$$\langle S \rangle = -\frac{\partial \ln Z(\alpha)}{\partial \alpha}$$

We will from now on restrict ourselves to lattice actions which derive from a field N_i of the form

$$N_i = D_{ij} x_j + M_{ij} x_j + g x_i^Q$$

In this case a simple scaling argument allows us to rewrite eqn. 9 as

$$Z(\alpha, g) = \alpha^{-N/2} Z(1, g')$$

where $g'/g = \alpha^{\frac{(1-Q)}{2}}$ and N is just the total number of degrees of freedom we integrate over. Hence we find the following expression for the expectation value of the total action including the pseudofermions

$$< S >= \frac{N}{2\alpha} + \frac{1-Q}{2\alpha} g \frac{\partial}{\partial g} \ln Z(1, g)$$

The second term on the right vanishes by virtues of the fact that the partition function does not depend on g - as guaranteed by the existence of the Nicolai map. Thus we see that the mean action (with $\alpha = 1$) merely counts the number of degrees of freedom including the pseudofermions. Furthermore, since the existence of the Nicolai map implies a supersymmetry we can also regard the value of the mean action computed in the simulation as an order parameter for supersymmetry breaking – if we find it depends on coupling and differs from its value for the free theory we know that supersymmetry has been broken.

4.2 Wess-Zumino Model

The analysis of the previous section carries over to the Wess-Zumino model with the appropriate interpretation of the index and field content. Thus we expect the mean lattice action to be equal to the number of degrees of freedom $< S >= 2L^2$ for a lattice of linear size L (the two counts the two real degrees of freedom at each lattice point in either boson or fermion sector). Likewise we expect the Ward identity based on the variation δ_1 to be exact for arbitrary lattice spacing.

$$\left\langle \psi_i^\alpha \overline{\psi}_j^\beta \right\rangle + \left\langle N_j^\beta x_i^\alpha \right\rangle = 0 \tag{10}$$

Similarly we expect the following three Ward identities to be satisfied as $a \to 0$.

$$\begin{aligned}
0 &= \left\langle i\gamma_3^{\alpha\gamma}\psi_i^\gamma \overline{\psi}_j^\beta \right\rangle + \left\langle i\gamma_3^{\beta\gamma}\overline{N}_j^\gamma x_i^\alpha \right\rangle \\
0 &= \left\langle \gamma_1^{\alpha\gamma}\overline{\psi}_i^\gamma \psi_j^\beta \right\rangle + \left\langle Q_j^\beta x_i^\alpha \right\rangle \\
0 &= \left\langle \gamma_2^{\alpha\gamma}\overline{\psi}_i^\gamma \psi_j^\beta \right\rangle + \left\langle i\gamma_3^{\beta\gamma}\overline{Q}_j^\gamma x_i^\alpha \right\rangle
\end{aligned} \tag{11}$$

5 Numerical Results

To check these conclusions we have chosen to simulate the model for $W'(\phi) = m\phi + g\phi^2$. We have used a hybrid monte carlo algorithm [9] to handle the integration over the pseudofermion fields. In order to reduce the computation time for large lattices we have implemented a refinement of this algorithm using Fourier acceleration techniques. Details are given in [6] and more recently [10]. In the latter paper we show that the autocorrelation time for SUSY quantum mechanics is drastically reduced - the dynamical critical exponent z is reduced from $z \sim 2$ for the usual HMC algorithm to $z \sim 0$ with fourier acceleration. In the Wess Zumino case the gains are also large.

Weak Coupling

In order to compare our results with other continuum and perturbative calculations we simulated the model initially at zero and small coupling g. We show data for $m = 10$ $g = 0$ and $g = 3$ obtained from 1×10^6 HMC trajectories at $L = 4$, and $L = 8$, 2×10^5 HMC trajectories at $L = 16$ and 2×10^4 HMC trajectories at $L = 32$. To take the continuum limit we imagine holding the physical size of the lattice fixed at unity (we are neglecting finite size effects sinhce our bare masses are relatively large). This allows

L	$g = 0.0$	$g = 3.0$
4	32.01(4)	31.93(6)
8	127.98(6)	127.97(7)
16	512.5(3)	512.0(3)
32	2048(1)	2046(3)

Table 1: Mean total action vs lattice size

us to extract the lattice spacing $a = \frac{1}{L}$. Since our lattice action contains only dimensionless quantities the bare physical couplings g and m must be translated to bare lattice quantities $g^L = g/L$, $m^L = m/L$ in the lattice action. The continuum limit is then reached by simply taking $L \to \infty$.

Table 1. shows the mean action as a function of lattice size for both $g = 0$ and $g = 3$. As is evident the mean action is close to the predicted value of $2L^2$ consistent with a non-breaking of SUSY (this is expected since the Witten index for this model $\Delta = 2$).

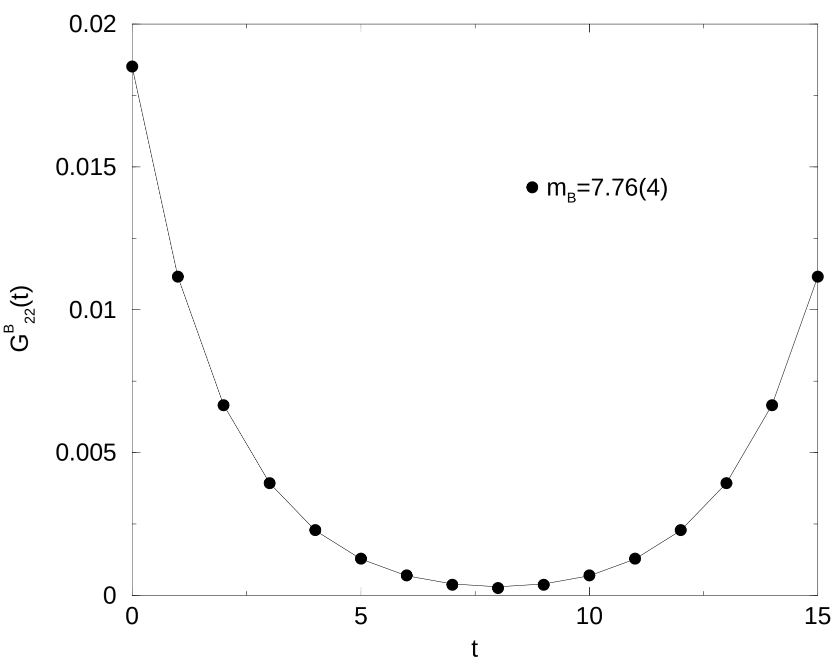

Figure 1: Boson Correlator at $L = 16$ and $m = 10.0$, $g = 3.0$

To extract information on the spectrum of the model we have studied zero momentum correlation functions which are given by averaging the fields transverse to the direction of propagation.

$$G^B_{\alpha\beta}(t)=\frac{1}{L^2}\sum_{j,j'}\left<x_\alpha(0,j)x_\beta(t,j')\right>_c$$

and

$$G^F_{\alpha\beta}(t)=\frac{1}{L^2}\sum_{j,j'}\left<\overline{\psi}_\alpha(0,j)\psi_\beta(t,j')\right>$$

On account of the periodic boundary conditions we expect the boson correlator $G^B(t)=A(t-L/2)$ where A is a symmetric function of its argument. Conversely the fermionic correlator can be expected to take the form

$$G^F_{\alpha\beta}(t)=k(I_{\alpha\beta}B(t-L/2)+\lambda\gamma^t_{\alpha\beta}C(t-L/2))$$

where $B(x)$ and $C(x)$ are symmetric and antisymmetric functions of their arguments, λ is a numerical coefficient and γ^t is the gamma matrix appropriate to the t-direction. For large x we expect a single mass state to dominate in which case $A(x),B(x)\rightarrow\cosh{(m^L_g x)}$ and $C(x)\rightarrow\sinh{(m^L_g x)}$. These latter functional forms were found to yield good fits over the whole range of parameters studied. The parameter m^L_g corresponds to the massgap of the model expressed in lattice units. To convert this value to physical units we merely have to divide by the lattice spacing a, $m_g=\frac{m^L_g(a)}{a}$.

Fig. 1 and fig. 2 show $G^B_{\alpha\beta}(t)$ and $G^F_{\alpha\beta}(t)$ for $L=16$, $g=3$ $\alpha=\beta=2$ and t lying along the 1-direction. This choice of time direction implies that the fermion correlator will be purely diagonal with $G^F_{11}(t)=\cosh{(m^L_g(t-L/2))}+\lambda\sinh{(m^L_g(t-L/2))}$ and $G^F_{22}(t)=\cosh{(m^L_g(t-L/2))}-\lambda\sinh{(m^L_g(t-L/2))}$. For weak coupling we find that the numerical value of λ extracted from the fit is consistent with unity which would be expected for a free theory as $a\rightarrow 0$. Notice that although the lattice action does not satisfy reflection positivity there is no sign of a problem in the correlation functions at weak coupling.

In Table 2. we show the results for the massgaps in physical units m_g as a function of the lattice spacing. It is clear that the boson and fermion masses are degenerate within statistical errors and increase smoothly with decreasing lattice spacing.

Fig. 3 is a plot of physical (fermion) mass $m_g(a)$ extracted from the simulations as a function of lattice spacing. For small $g/m=g^L/m^L$ we

L	m_B	m_F
4	5.09(2)	4.95(8)
8	6.52(2)	6.44(5)
16	7.76(4)	7.75(6)
32	8.29(19)	8.33(30)

Table 2: Physical massgaps m_g vs lattice size for $m = 10.0$ and $g = 3.0$

expect perturbation theory to provide a good approximation. The one loop result for the massgap is

$$m_g^{\rm pert} = m(1 - \frac{2}{3\sqrt{3}}\left(\frac{g}{m}\right)^2)$$

which yields $m_g^{\rm pert} = 9.65$ for $g = 3.0$. Notice that since this theory is finite there is no need to introduce a scale dependent renormalized mass – the physical massgap of the theory is a finite function of the bare parameters in physical units. It is encouraging that reasonable extrapolations of $m_g(a)$ to $a = 0$ are consistent with the one loop result. These numerical results are also consistent with ones which were previously obtained using a stochastic approach based on the Nicolai map [11].

To understand whether the continuum limit will describe an $N = 2$ supersymmetry we have also checked the four Ward identities written down in the last section. We again choose to average the correlations transverse to a chosen t-direction ($t = 1$ as before). Each Ward identity then yields two independent relations between components of boson and fermion correlators. Fig. 4 shows a plot of $-G^F_{22}(t)$ and $< x_2(0)N_2(t) >$ versus time t. The first Ward identity requires the sum of these two curves to vanish – clearly to a very good statistical accuracy the numerical data support this conclusion. The first Ward identity also predicts a relationship between $G^F_{11}(t)$ and $< x_1(0)N_1(t) >$ which we also observe to be true within (small) statistical errors. Thus, as expected, the existence of the exact SUSY eqn. 3 leads to a Ward identity relating correlation functions which we observe to be accurately satisfied on the lattice.

We have also examined the other Ward identities corresponding to the other three broken symmetries – fig. 5 plots $G^F_{22}(t)$ vs $< x_1(0)\overline{N}_1(t) >$. Again, if the 2nd Ward identity were to hold exactly the sum of these two curves would again vanish – and it appears that the data are consistent with this. Indeed, we have found that each of these three Ward identities is also satisfied within statistical error at this (weak) coupling. The explanation for this seems to lie in the magnitude of the symmetry breaking - as we have argued the breaking effects come in at $O(ga^2) = g/L^3$ yielding corrections to the broken Ward identities which are too small to be resolved over our statistical errors.

Strong Coupling

Having checked by explicit simulation that this lattice model appears to possess the correct supersymmetric structure at weak coupling we have extended our simulations to strong coupling. This allows us to probe directly the non-perturbative structure of the theory. Classically the model has two vacua - corresponding to $\phi = 0$ and $\phi = -m/g$. These vacua are separated classically by a barrier of height $m^2 \left(m/4g\right)^2$. For small g we are effectively confined to the $\phi = 0$ well but for large g/m we would expect both vacua to be sampled. In addition since the terms in the action that violate reflection positivity are proportional to g we might wonder whether a sensible continuum limit even exists for strong coupling.

We have examined this issue by simulations at $m = 5$ and $g = 2.5$, $g = 5.0$ and $g = 10.0$ for lattices from $L = 8$ through $L = 32$ as before and with similar statistics. The choice of a smaller bare mass parameter m reduces the barrier height and allows our simulation to more effectively tunnel between the two classical vacua. For large g we were forced to refine our Hybrid Monte Carlo scheme to eliminate problems stemming from large pseudofermion forces occasionally encountered in the vicinity of such tunneling configurations of the boson field. Essentially an entire trajectory is abandoned as soon as a force component larger than some threshold is seen - the trajectory is restarted with new momenta. This process increases the autocorrelation time of the algorithm but for the parameters at which we performed simulations the effect was not overly severe.

The results for $g = 2.5$ are similar to those obtained in the previous section and will not be examined further. In all cases we have observed that the mean action $< S >= 2L^2$ independent of g and m. This is again evidence that supersymmetry is not broken even outside of perturbation theory. We have also seen that typical configurations extend over a region of field space encompassing both classical minima – indeed we have found that $< \mathrm{Re}\phi >= -m/2g$ very accurately.

The correlation functions $G^B_{22}(t)$ and $G^F_{22}(t)$ for $g = 5.0$ $(g/m = 1)$ are shown in fig. 6 and fig. 7. We again choose time along the 1-direction. The boson is again accurately fitted by a simple hyperbolic cosine function

L	m_B	m_F
8	$--$	$--$
16	4.35(7)	5.5(4)
32	6.0(2)	4.4(7)

Table 3: Physical massgaps vs lattice size for $m = 5.0$ and $g = 5.0$

and yields a physical mass of $m_B = 4.35(7)$ at this lattice spacing. The fermionic correlator is a little more complicated – at this coupling the fits favor a signal which is predominantly given by a hyperbolic sine function with a small admixture of hyperbolic cosine (typically $\lambda \sim 4/5$). The use of a three parameter fit yields a larger error in the fermion mass estimate – $m_F = 5.5(4)$. The results for all the lattice sizes are summarized in table. 3.

It is not clear whether the discrepancy between boson and fermion massgaps is significant or merely reflects the large errors in the fermion mass determination. More interestingly, the gaps in the table for $L = 8$ arise because it was not possible to extract a mass from the small lattice $L = 8$ - the signal descends into noise after just one timeslice. This is not true for smaller g and may indicate a problem with reflection positivity at this lattice spacing. A similar problem occurs for $L = 16$ when g is increased to $g = 10.0$. Fig 8 shows a plot of the bosonic correlator there. We conjecture the oscillations visible in the signal are a signal for a mass spectrum which is not real positive. This might indicate that the problem could indeed be attributed to the lack of reflection positivity in the lattice action. However, even if this were the case, the problem appears to diminish with lattice spacing – the correlators for $L = 32$ at this *same* coupling $g = 10$ exhibited none of these problems and allowed fits for both boson and fermion massgaps $m_B = 4.7(1)$ and $m_F = 4.9(7)$ for $L = 32$. Interestingly, in the region of parameter space where the two-point functions show this oscillatory behavior we have also observed that the sign of the fermion determinant may fluctuate also. In practice the sign changes were relatively infrequent and their effects could be taken into account by reweighting the measured observables in the usual manner. However, this effect also disappeared with decreasing lattice spacing at fixed coupling.

Similar results were obtained at other values of the bare parameters. Hence we speculate that, while problems associated with a lack of reflection positivity may be evident on coarse lattices, these effects disappear with decreasing lattice spacing. Thus a well-defined continuum limit may be defined for all finite g.

Finally we have examined the Ward identities. Again, the presence of an exact symmetry yields a relationship between boson and fermion correlators for arbitrarily large g which is exhibited in fig. 9 which shows the same correlators deriving from the first Ward identity now for $m = 5$, $g = 5$ and $L = 16$. The middle curve (diamonds) shows the sum of the two contributions which is seen to be consistent with zero for all t within errors. Contrast this with fig. 10 which exhibits the bosonic $< x_1(0)Q_1(t) >$ and fermionic $G^F_{11}(t)$ contributions to the third Ward identity. The middle (diamond) curve is no longer zero and indeed shows a marked variation with t. Similar effects are seen in the fourth Ward identity.

6 Discussion and Conclusions

In this paper we have studied a lattice version of the two-dimensional Wess-Zumino model with $N = 2$ supersymmetry. The lattice action we use was first derived in [3],[4] and follows from a discretization of the continuum Nicolai map for the model. We have rederived it in a different way by requiring that the lattice field theory model exhibit a single parameter SUSY-like invariance. This approach has the advantage that is allows us to identify the other broken invariances which would yield a full $N = 2$ SUSY in the naive continuum limit. From the form of those transformations we have derived a set of Ward identities which would be satisfied in the continuum limit. We furthermore argue that the presence of one exact symmetry (together with the finiteness of the continuum theory) guarantees that the full symmetry is restored *without* fine tuning in the continuum limit.

These conclusions have been checked by an explicit numerical simulation of the Euclidean lattice theory in which the boson and fermion massgaps and a set of supersymmetric Ward identities were computed at a variety of lattice spacings. We utilized a Fourier accelerated Hybrid Monte Carlo algorithm to handle the fermionic integrations.

At weak coupling we were able to extract boson and fermion masses and verify their equality within statistical errors. We also found that all the Ward identities were satisfied to high precision. We have argued that the small magnitude $O(ga^2)$ of the symmetry breaking effects places the corrections within the statistical noise inherent in our calculation. Most importantly, the numerical results show no sign of any problems stemming from the lack of reflection positivity in the lattice action.

At strong coupling we found difficulties in extracting masses and interpreting the theory for coarse lattices but, at least for the parameters we studied, these effects seemed to go away on finer lattices. Our simulations, while efficiently sampling the classical vacua of the model, show no evidence for supersymmetry breaking – the mean action remained at $2L^2$ and the Ward identities corresponding to the exact symmetry were still satisfied within errors. However at strong coupling we did observe clear corrections to some of the other approximate Ward identities.

Of course the interesting question is whether one can generalize any of these ideas to gauge models in higher dimensions. A Nicolai map is known for the continuum $N = 2$ super Yang Mills model in two dimensions [12] (indeed it can be obtained by dimensional reduction of a map for $N = 1$ super Yang Mills in four dimensions). Unfortunately, a naive transcription to the lattice is problematic since the map utilizes an explicit noncompact formulation for the gauge field. Replacing continuum derivatives by finite differences as for the Wess Zumino model would then lead to an action which was not gauge invariant.

7 Acknowledgments

Simon Catterall was supported in part by DOE grant DE-FG02-85ER40237

References

[1] The Quantum Theory of Fields Vol. III Supersymmetry, S. Weinberg, Cambridge University Press (2000).

[2] P. Dondi and H. Nicolai, Nuovo Cimento 41A (1977)
S. Elitzur, E. Rabinovici and A. Schwimmer, Phys. Lett B119 (1982) 165.
J. Bartels and J. Bronzan, Phys. Rev. D28 (1983) 818.
T. Banks and P. Windey, Nucl. Phys. B198 (1982) 226.
I. Montvay, Nucl. Phys. B466 (1996) 259.
M Golterman and D. Petcher, Nucl. Phys. B319 (1989) 307.
W. Bietenholz, Mod. Phys. Lett A14 (1999) 51.

[3] N. Sakai and M. Sakamoto, Nucl. Phys. B229 (1983) 173.

[4] S. Elitzur and A. Schwimmer, Nucl. Phys. B226 (1983) 109.

[5] H. Nicolai, Phys. Lett. B89 (1980) 341.

[6] S. Catterall and E. Gregory, Phys. Lett. B487 (2000) 349.

[7] E. Witten, Nucl. Phys. B185 (1981) 513.

[8] S. Cecotti and L. Giradello, Phys. Lett. B110, (1982) 39.

[9] S. Duane, A. Kennedy, B. Pendleton and D. Roweth, Phys. Lett. B195B (1987) 216.

[10] S. Catterall and S. Karamov, 'Testing a Fourier Accelerated Hybrid Monte Carlo Algorithm', hep-lat/0112025.

[11] M. Beccaria, G. Curci and E. D'Ambrosio, Phys. Rev. D58 (1998) 065009.

[12] V. de Alfaro, S. Fubini, G. Furlan and G. Veneziano, Nucl. Phys. B255 (1985) 1.

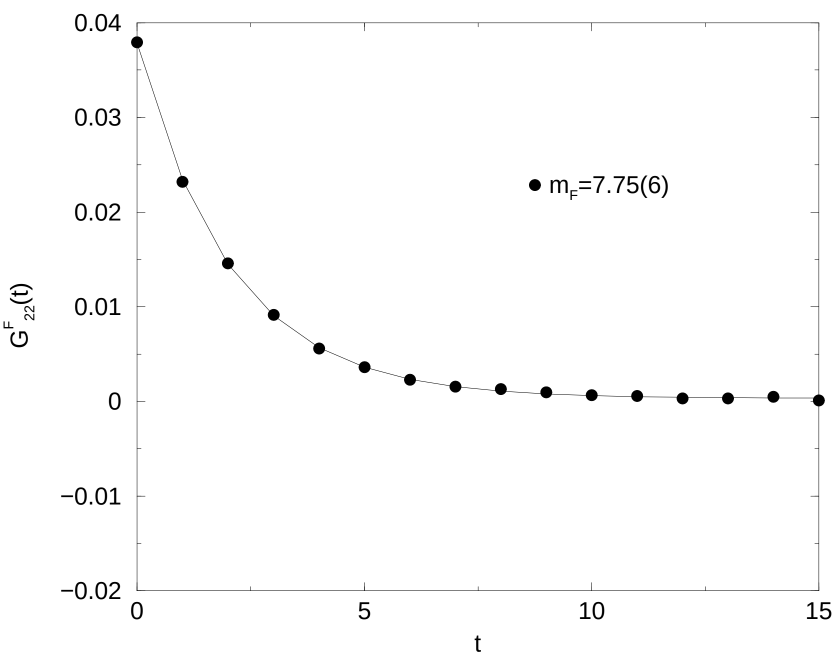

Figure 2: Fermion Correlator at $L = 16$ and $m = 10.0$, $g = 3.0$

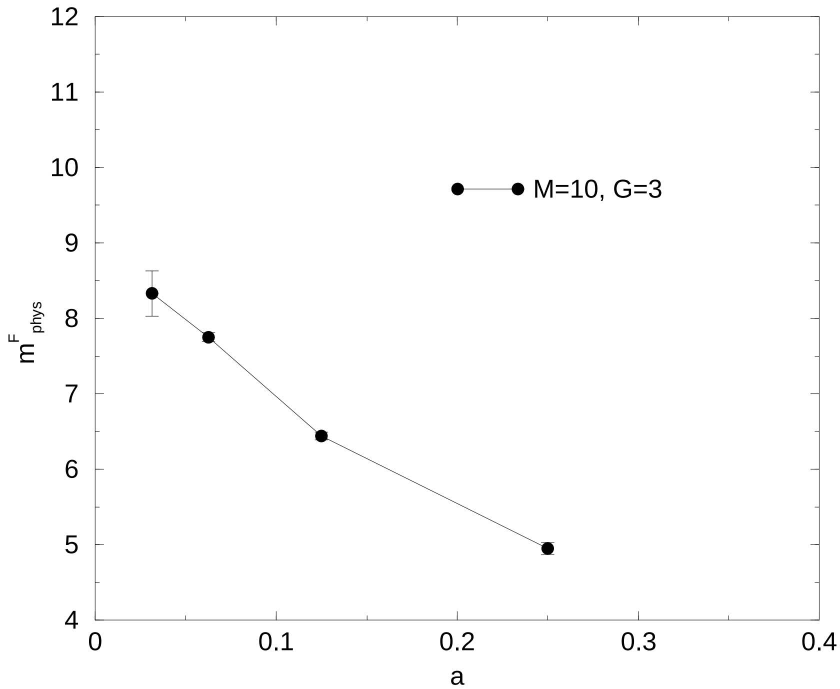

Figure 3: Massgaps vs lattice spacing $a = 1/L$ for $m = 10.0$, $g = 3.0$

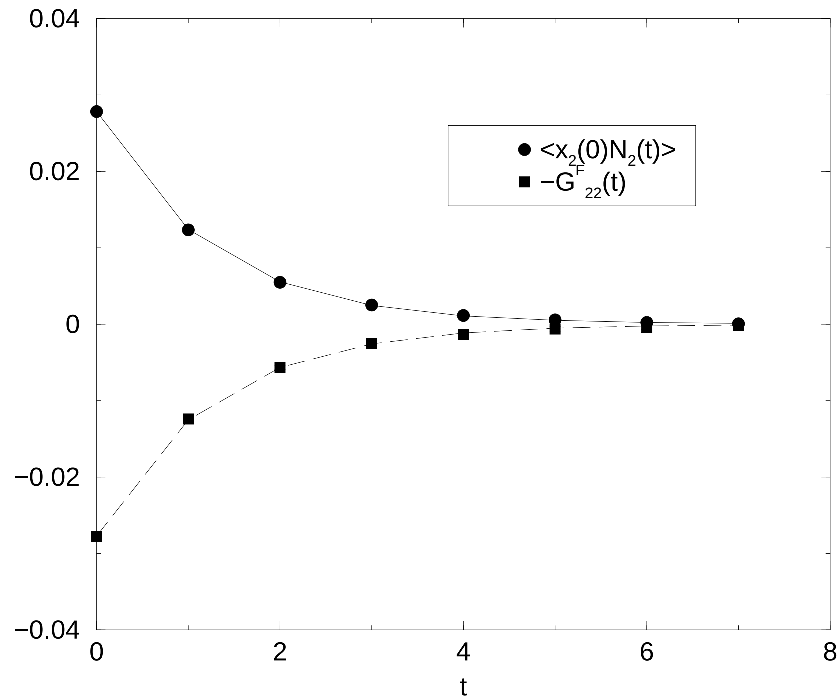

Figure 4: Fermionic and Bosonic Contributions to 1st Ward identity for $m = 10.0$, $g = 3.0$

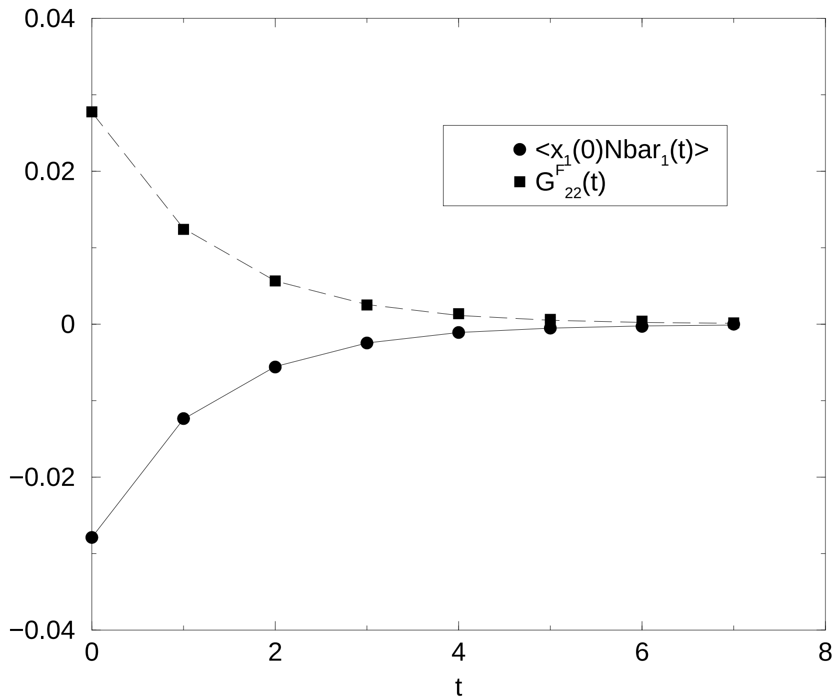

Figure 5: Fermionic and Bosonic Contributions to 2nd Ward identity for $m = 10.0$, $g = 3.0$

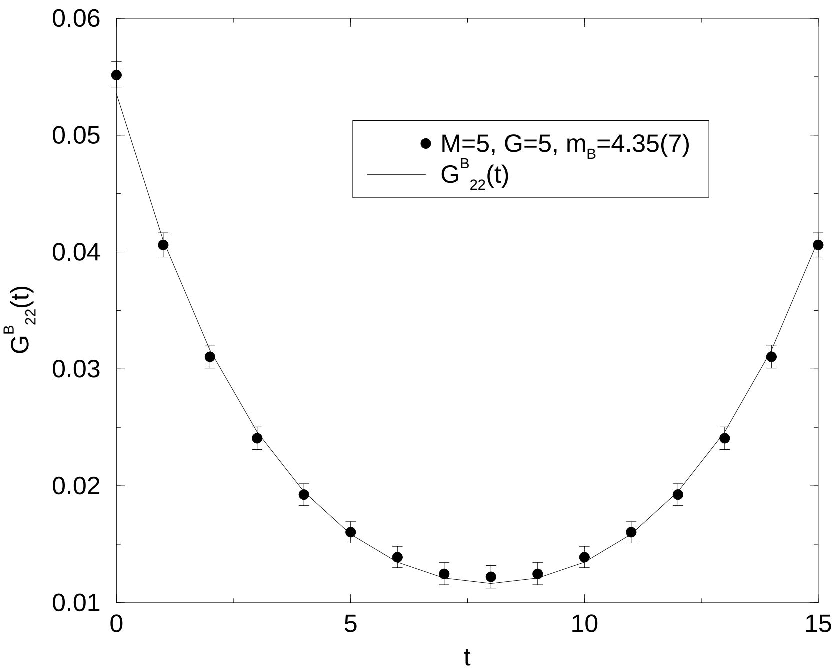

Figure 6: Bosonic Correlator $m = 5$, $g = 5$, $L = 16$

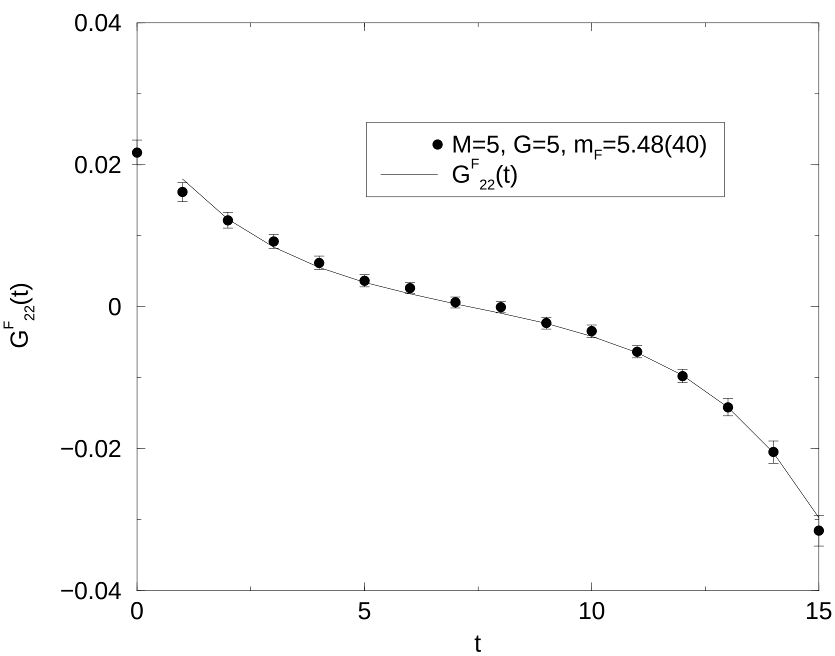

Figure 7: Fermionic Correlator $m = 5$, $g = 5$, $L = 16$

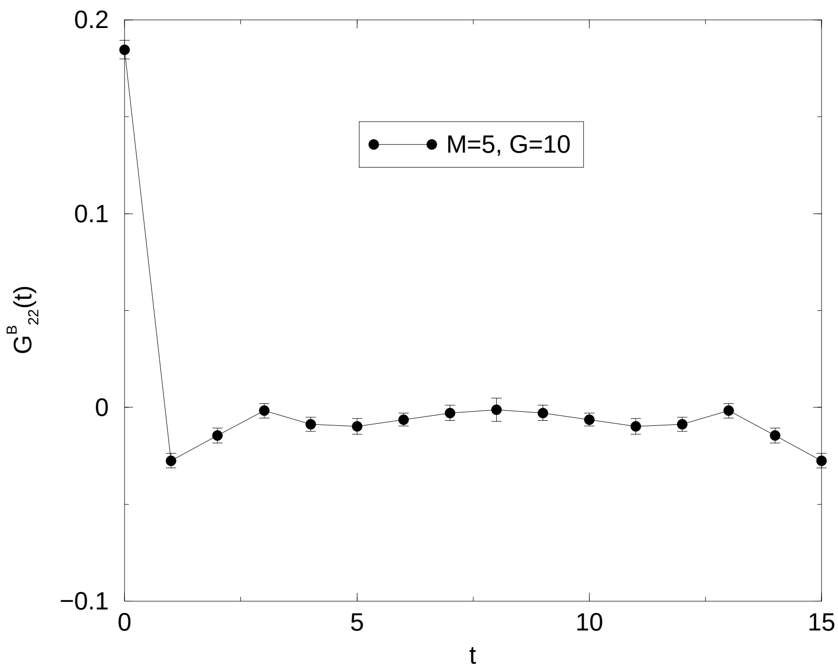

Figure 8: Bosonic Correlator $m = 5$, $g = 10$, $L = 16$

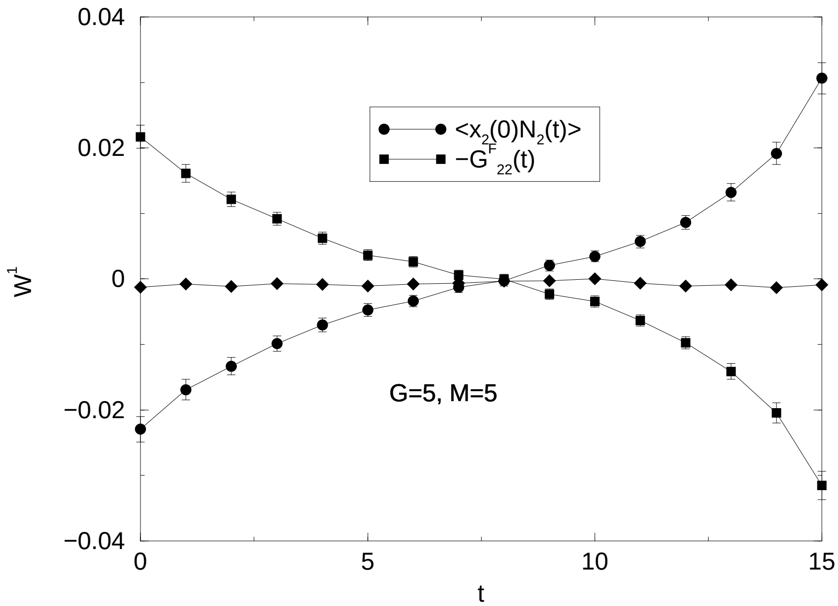

Figure 9: Fermionic and Bosonic Contributions to 1st Ward identity for $m = 5.0$, $g = 5.0$ $L = 16$

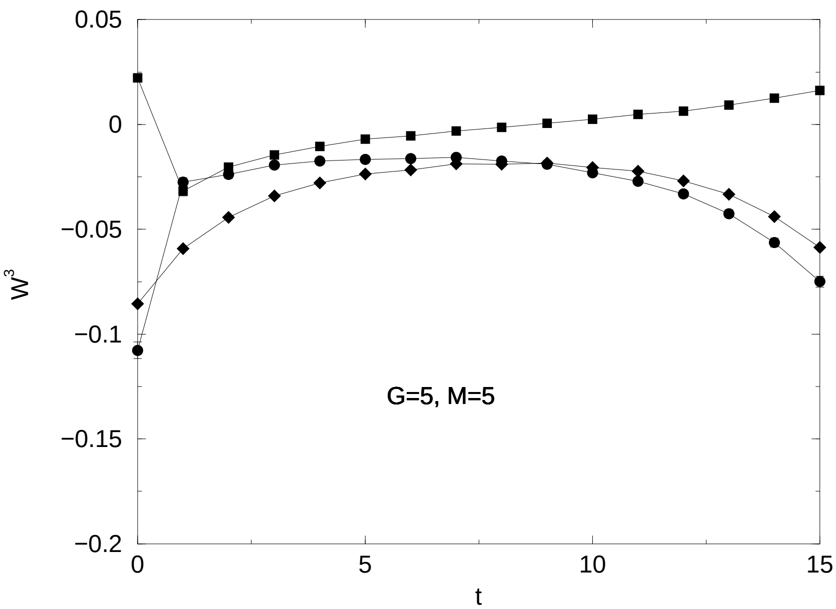

Figure 10: Fermionic and Bosonic Contributions to 3rd Ward identity for $m = 5.0$, $g = 5.0$ $L = 16$