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Data types as lattices: retractions, closures, and projections

Luis E. Sanchis
Syracuse University

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0.1 This paper considers the mathematical principles of lattice theory oriented toward the theory of computation. This relatively new direction can be traced back to the explanation of recursive definitions as fixed point of monotonic (actually continuous) operators. The usual operational explanation (Kleene's first recursion theorem) is replaced by a pure lattice theoretical existence theorem.

Another problem for which the lattice approach provided a significant clarification was the so-called self-application of functions. Introduced first in some formal systems of λ-calculus and combinatory logic it was accepted later as a proper procedure for the definition of algorithms in programming languages, the implication being then that there existed a clear operational meaning for such procedure. Again the discovery by Scott of models in which such self-application was available provided a mathematical meaning for an operational notion. But it is important to notice that-contrary to the situation for recursive definitions-it is not clear whether the mathematical notion of self-application corresponds to the operational.

More recently (see [7]) the lattice approach has been found useful for the definition of data structures. In all these applications a number of constructions appear frequently: retractions, projection, representations.

0.2 We attempt here a systematic treatment of the lattice theory involving the definition of data structures. We consider monotonic rather than continuous functions and show that a number of results appear quite natural in this more general setting. Then we try to characterize whenever possible the situation arising by assuming continuity.

0.3 Most of the notation we use is standard in lattice theory or in the work of Scott. What we call here a lattice is usually called a complete lattice. We use the shorter expression only because this is the only type of lattice considered in the paper. The name representation originates from Reynolds (see for instance [4]) who actually refers only to what we call continuous representation. It should be noted that the general notion of representation is quite old and apparently was introduced by Ore in [3] as Galois connection. The formulation is not exactly the same but it is clear that the notions are equivalent. In Everett [2] the expression Galois correspondence is used.

In place of continuous lattice we use injective lattice; a term which is also used by Scott. Since the qualification continuous is used in many places in this paper we think it is wiser to avoid using the same term with a different connotation. Finally compactly generated lattices are known from Crowley [1] and appear as algebraic lattices in Scott [7] with some extra restrictions.

1. Definitions and notation.

1.1 Domains. A domain is a non empty set D with a partial order on D. Such partial order will be denoted in the form $x \sqsubseteq y$. We use D to denote both the set and the partial order. If D' is another partial order we write $x \sqsubseteq' y$ to indicate the relation in D'; but if there is no danger of confusion we may write just $x \sqsubseteq y$.

If D is a domain and D' is a subset of D then we consider D' as a partial order where the order relation is the restriction to D' of the order relation in D. We call then D' a subdomain of D.

If D is a domain and $X \subseteq D$ is such that X is not empty and whenever $x \in X$ and $y \in X$ there is $v \in X$ such that $x \sqsubseteq v$ and $y \sqsubseteq v$ then we say that X is a directed subset of D.

1.2 Lattices. If D is a domain and $X \subseteq D$ then the notion of upper bound of X (u.b. of X) and lower bound of X (l.b. of X) is defined in the usual way. In case there is a least upper bound of X (l.u.b. of X) it is denoted $\bigsqcup X$; and in case there is a greatest lower bound of X (g.l.b. of X) it is denoted $\bigsqcap X$. The domain D is called a lattice in case $\bigsqcup X$ and $\bigsqcap X$ exists for every $X \subseteq D$. It is well known that in order for D to be a lattice it is sufficient that $\bigsqcup X$ (or of $\bigsqcup X$) exists for every $X \subseteq D$.

If D is a lattice, D' is a subdomain of D and D' is itself a lattice then D' is called a sublattice of D. Notice that this does not mean that for any $X \subseteq D'$ the l.u.b. in D is the same as the l.u.b. in D'. Whenever we have to distinguish we shall note by

$\bigsqcup'X$ and $\bigsqcap'X$ the l.u.b. and g.l.b. of X in D', respectively. Note that $\bigsqcup X = \bigsqcup' X$ if and only if $\bigsqcup X \in D'$; and similarly for $\bigsqcap X$. A sublattice of D' of D is said to be $\bigsqcup$-closed in case that for every $X \subseteq D'$ we have $\bigsqcup X \in D'$ and is said to be $\bigsqcap$-closed in case that $\bigsqcap X \in D'$ for every $X \subseteq D'$. Finally if D' is a sublattice of D such that whenever $X \subseteq D'$ and X is directed then $\bigsqcup X \in D'$ we say that D' is a sublattice of finite character in D.

In a lattice D the element $\bigsqcup D$ is denoted by $\top$ and the element $\bigsqcap D$ is denoted by $\bot$.

1.3 Functional domains. Let D and D' be lattices. Then $D \longrightarrow D'$ denotes the set of all functions which are defined for every $x \in D$ and the value is always in D'. If f and g are elements of $D \longrightarrow D'$ we define $f \sqsubseteq g$ to hold exactly when $f(x) \sqsubseteq g(x)$ for every $x \in D$. This is easily seen to be a partial order. Furthermore $D \longrightarrow D'$ is a lattice where the lattice operations are defined as follows. If $F \subseteq D \longrightarrow D'$ then

$$\bigsqcup F\ (x) = \bigsqcup\{f(x) : f \in F\}$$

$$\bigsqcap F\ (x) = \bigsqcap\{f(x) : f \in F\}$$

1.3.1 A function $f \in D \longrightarrow D'$ is monotonic in case that whenever $x \sqsubseteq y$ then $f(x) \sqsubseteq f(y)$. The set of all monotonic functions is denoted by $D \xrightarrow{m} D'$ and it is both $\bigsqcup$-closed and $\bigsqcap$-closed sublattice of $D \longrightarrow D'$.

1.3.2 If $f \in D \longrightarrow D'$ and $X \subseteq D$ we put $f(X) = \{f(x) : x \in X\}$. The function f is continuous in case that $f(\bigsqcup X) = \bigsqcup f(X)$ whenever $X \subseteq D$ is directed. The set of all continuous functions is denoted

$D \xrightarrow{c} D'$; it is a $\bigsqcup$-closed sublattice of $D \xrightarrow{m} D'$.

1.3.3 In case $f(\bigsqcup X) = \bigsqcup f(X)$ holds for arbitrary $X \subseteq D$ the function f is called additive. The set of all additive functions is denoted $D \xrightarrow{a} D'$ and it is a $\bigsqcup$-closed sublattice of $D \xrightarrow{m} D'$.

1.3.4 Finally if $f(\bigsqcap X) = \bigsqcap f(X)$ for arbitrary $X \subseteq D$ then f is a coadditive function. The set of all coadditive functions is denoted $D \xrightarrow{ca} D'$. It is a $\bigsqcap$-closed sublattice of $D \xrightarrow{m} D'$.

1.3.5 We note the following property of functions $f \in D \longrightarrow D'$. If $D_1 \subseteq D$ is $\bigsqcup$-closed ($\bigsqcap$-closed) sublattice of D and f is additive (coadditive) then $f(D_1)$ is $\bigsqcup$-closed ($\bigsqcap$-closed) sublattice of D'. Now let us put $R(f) = f(D)$. Then $R(f)$ is $\bigsqcup$-closed ($\bigsqcap$-closed) sublattice of D' whenever f is an additive (coadditive) function.

1.3.6 There are other classes of functions we may want to consider but in general they do not form sublattices of $D \longrightarrow D'$. If f satisfies the condition that $x \sqsubseteq y$ if and only if $f(x) \sqsubseteq f(y)$ then f is said to be strictly monotonic. If f is strictly monotonic then it is 1-1. Furthermore if R(f) is $\bigsqcup$-closed ($\bigsqcap$-closed) (of finite character) in D' then f is additive (coadditive)(continuous).

If $f \in D \longrightarrow D'$ is strictly monotonic and onto then f is called an isomorphism of D onto D'. Such f is both additive and coadditive. The inverse of an isomorphism is also an isomorphism. Finally note the following property. If f is 1-1 and additive or coadditive then f is strictly monotonic.

1.3.7 The identity function on a lattice D will be denoted by I_D or simply by I when there is no risk of confusion.

1.3.8 Composition of functions is defined in the usual way. If $f \in D \longrightarrow D'$ and $g \in D' \longrightarrow D''$ then $g \circ f \in D \longrightarrow D''$. All the lattices introduced in 1.3.1, 1.3.2, 1.3.3, 1.3.4, and 1.3.6 are closed under composition.

2. Retraction, closures and projections.

In this section we study sublattices of a lattice D induced by monotonic functions satisfying some conditions. First we prove some results that hold for arbitrary monotonic functions. If $f \in D \longrightarrow D$ we put $\mathrm{Fix}(f) = \{x : f(x) = x\}$.

2.1.1 Lemma. Let $f \in D \xrightarrow{m} D$ and put $D_1 = \{x : f(x) \sqsubseteq x\}$. Then D_1 is $\sqcap$-closed sublattice of D. If f is continuous then D_1 is of finite character in D.

Take $X \subseteq D_1$ to show $\sqcap X \in D_1$. For every $x \in X$ we have $\sqcap X \sqsubseteq x$ hence $f(\sqcap X) \sqsubseteq f(x) \sqsubseteq x$. It follows that $f(\sqcap X) \sqsubseteq \sqcap X$, so $\sqcap X \in D_1$. Assume now that $X \subseteq D_1$ is directed. Then $f(\sqcup X) = \sqcup f(X) \sqsubseteq \sqcup X$. Hence $\sqcup X \in D_1$.

2.1.2 Lemma. Let $f \in D \xrightarrow{m} D$ and put $D_2 = \{x : x \sqsubseteq f(x)\}$. Then D_2 is $\sqcup$-closed sublattice of D.

This is the dual of the corresponding part in 2.1.1.

2.1.3 Corollary. Let $f \in D \xrightarrow{m} D$ and D_1 and D_2 be defined as in 2.1.1 and 2.1.2. Then $\mathrm{Fix}(f) = D_1 \cap D_2$ is $\sqcup$-closed sublattice of D_1 and also $\sqcap$-closed sublattice of D_2. If f is continuous then Fix(f) is of finite character in D.

The first part follows from 2.1.1 and 2.1.2. If f is continuous and $X \subseteq \mathrm{Fix}(f)$ is directed then $f(\sqcup X) = \sqcup f(X) = \sqcup X$.

2.2.1 A function $f \in D \xrightarrow{m} D$ such that $f \circ f = f$ is called a retraction in D. Note that in case f is a retraction then $\mathrm{Fix}(f) = R(f)$. If f is a retraction such that $I \sqsubseteq f$ then f is called a closure in D. And in case that $f \sqsubseteq I$ then f is called a projection in D. If f is a retraction (closure) (projection) in D and f is continuous then f is a continuous retraction (closure) (projection) in D.

2.2.2 Theorem. If f is a retraction (closure) (projection) in D then R(f) is a sublattice ($\sqcap$-closed sublattice)($\sqcup$-closed sublattice) of D. If f is continuous then R(f) is of finite character in D.

These are easy consequences of 2.1.1, 2.1.2 and 2.1.3.

2.2.3 Theorem. Let f be a closure in D such that R(f) is of finite character in D. Then f is continuous.

Take $X \subseteq D$ directed, to show $F(\bigsqcup X) \sqsubseteq \bigsqcup f(X)$. Note that f(X) is also directed and since $f(X) \subseteq R(f)$ we have $f(\bigsqcup f(X)) = \bigsqcup f(X)$. Now if $x \in X$ then $x \sqsubseteq f(x) \sqsubseteq \bigsqcup f(X)$, hence $\bigsqcup X \sqsubseteq \bigsqcup f(X)$ so $f(\bigsqcup X) \sqsubseteq \bigsqcup f(X)$.

2.3 If D' is a sublattice of a lattice D such that there is a retraction (closure) (projection) f and D' = R(f) then D' is called a retraction (closure) (projection) in D. In case f is continuous then it is called a continuous retraction (closure) (projection) in D.

2.3.1 Theorem. If D' is a sublattice of D then D' is a retraction in D. If D' is $\sqcap$-closed ($\sqcup$-closed) then D' is a closure (projection) in D.

There are many retractions f such that D' = R(f). The maximal one (in the ordering of $D \longrightarrow D$) is the function

$$f_1(y) = \sqcap'\{x \in D' : y \sqsubseteq x\}$$

which is easily seen to be a retraction such that $D' = R(f_1)$. If D' is $\sqcap$-closed then f_1 is a closure. The minimal retraction is the following:

$$f_2(y) = \sqcup'\{x \in D' : x \sqsubseteq y\}$$

and in case D' is $\sqcup$-closed f_2 is a projection.

2.3.2 As mentioned above in general there are many retractions f such that $D' = R(f)$. But in case D' is $\sqcap$-closed ($\sqcup$-closed) the closure (projection) f such that $D' = R(f)$ is unique.

Note also that from 2.2.3 it follows that D' is a continuous closure in D if and only if D' is $\sqcap$-closed and of finite character in D.

2.4 We consider now some examples. If D is a lattice and $v \in D$ then $[v) = \{x : v \sqsubseteq x\}$ and $(v] = \{x : x \sqsubseteq v\}$. We denote by $D \xrightarrow{r} D$, $D \xrightarrow{cr} D$, $D \xrightarrow{cl} D$, $D \xrightarrow{ccl} D$, $D \xrightarrow{p} D$ and $D \xrightarrow{cp} D$ the set of all retractions, continuous retractions, closures, continuous closures, projections and continuous projections in D, respectively.

2.4.1 The sets $[v)$ and $(v]$ are both of finite character in D. The set $[v)$ is $\sqcap$-closed hence it is a continuous closure in D. The set $(v]$ $\sqcup$-closed so it is a projection in D.

2.4.2 Let $q(f) = f \circ f$ be a function on functions $f \in D \longrightarrow D$. Clearly $q \in (D \xrightarrow{m} D) \xrightarrow{m} (D \xrightarrow{m} D)$ hence the set $D_1 = \{f \in D \xrightarrow{m} D : q(f) \sqsubseteq f\}$ is $\sqcap$-closed in $D \xrightarrow{m} D$. It follows that $D \xrightarrow{cl} D = D_1 \cap [I)$ is also $\sqcap$-closed in $D \xrightarrow{m} D$ hence it is a closure in this set.

2.4.3 A similar argument shows that $D \xrightarrow{p} D$ is a projection in $D \xrightarrow{m} D$.

2.4.4 We may restrict the operation $q(f)$ to continuous f. In this case we have $q \in (D \xrightarrow{c} D) \xrightarrow{c} (D \xrightarrow{c} D)$. Now the set $D_1 = \{f \in D \xrightarrow{c} D : q(f) \sqsubseteq f\}$ is not only $\sqcap$-closed but also of finite character. Hence $D \xrightarrow{ccl} D = D_1 \cap [I)$ is a continuous closure in $D \xrightarrow{c} D$.

2.4.5 The dual argument shows only that $D \xrightarrow{cp} D$ is a projection in $D \xrightarrow{c} D$.

2.4.6 The argument given in [7], Theorem 5.5 shows that the unique continuous closure V in $D \xrightarrow{c} D$ such that $R(V) = C \xrightarrow{ccl} D$ is given by the following expression:

$$V(v) = \lambda x.\ Y(\lambda y.\ x \sqcup f(y))$$

where Y is the fixed point operator.

3. Connections and Representations.

In the preceding section we have introduced several relations between lattices assuming in general that one lattice is a sublattice of the other. In applications we have lattices which are not so related and the obvious approach is to extend the notions via isomorphisms. What we shall do is rather to generalize the relations and show that they can be reduced to the originals up to isomorphism. The generalizations take the form of connections and representations between lattices, and provide an extremely useful tool to study the relations. It is essentially a factorization technique in which different factors represent different aspects of the total relation.

3.1 Let D and D' be lattices. A connection between D and D' is a pair of functions (f,g) such that $f \in D \xrightarrow{m} D'$, $g \in D' \xrightarrow{m} D$, $f \circ g \circ f = f$, and $g \circ f \circ g = g$. Note that the two last conditions are satisfied in case $f \circ g = I$ or $g \circ f = I$. Note also that in case (f,g) is a connection between D and D' then (g,f) is a connection between D' and D.

If follows immediately from the definition that whenever (f,g) is a connection between D and D' then $g \circ f$ is a retraction in D and $f \circ g$ is a retraction in D'.

3.1.1 Theorem. Let (f,g) be a connection between D and D'. Then f restricted to R(g) is an isomorphism of R(g) onto R(f), and g restricted to R(f) is the inverse isomorphism of R(f) onto R(g).

Since $R(f) = R(f \circ g)$ it follows that f restricted to $R(g)$ is onto $R(f)$. By assumption f is monotonic; so we need to prove only that whenever $y_1 \in R(g)$, $y_2 \in R(g)$ and $f(y_1) \sqsubseteq f(y_2)$ then $y_1 \sqsubseteq y_2$. Let $g(x_1) = y_1$ and $g(x_2) = y_2$. Then $y_1 = g(x_1) = g(f(g(x_1))) \sqsubseteq g(f(g(x_2))) = g(x_2) = y_2$.

3.1.2 Theorem. Let D and D' be lattices; let D_1 be a sublattice of D and D_1' be a sublattice of D'; let h be an isomorphism of D_1 onto D_1'. Then there is a connection (f,g) between D and D' such that $R(f) = D_1'$, $R(g) = D_1$, for $x \in D_1$ $f(x) = h(x)$ and for $y \in D_1'$ $g(y) = h^{-1}(y)$.

We shall denote by $\bigsqcup_1$ and $\bigsqcap_1$ the join and meet operation in D_1 and D_1' respectively. We define then f and g as follows:

$$f(x) = \bigsqcap_1 \{y \in D_1' : x \sqsubseteq h^{-1}(y)\}$$

$$g(y) = \bigsqcup_1 \{x \in D_1 : h(x) \sqsubseteq y\}$$

It is clear that $R(f) \subseteq D_1'$ and $R(g) \subseteq D_1$. It is easy to verify that for $x \in D_1$ $g(h(x)) = x$ and for $y \in D_1'$ $f(h^{-1}(y)) = y$, hence $f(x) = f(h^{-1}(h(x))) = h(x)$ and $g(y) = g(h(h^{-1}(y))) = h^{-1}(y)$. We have also

$$f(g(f(x))) = f(h^{-1}(f(x))) = f(x)$$

$$g(f(g(y))) = g(h(g(y))) = g(y)$$

3.1.3 Theorem. Let (f,g) be a connection between D and D'. The following conditions are equivalent:

i) $f \circ g = I$ $(g \circ f = I)$

ii) f is onto D' (f is 1-1)

iii) g is 1-1 (g is onto)

That i) implies ii) and iii) is clear. From $f \circ g \circ f = f$ it follows that ii) implies i), and from $g \circ f \circ g = g$ that iii) implies i).

3.2 A connection (f,g) between D and D' is continuous in case both f and g are continuous functions. In case $f \circ g = I$ the connection (f,g) is called a retraction of D' into D and it is called a continuous retraction in case both f and g are continuous functions. If there is a (continuous) retraction of D' into D we shall say that D' is a (continuous) retraction of D.

This notation is consistent with that of 2.2.1. For suppose that $f \in D \xrightarrow{m} D$ so we may consider $f \in D \xrightarrow{m} R(f)$. Define $g(y) = y$ for $y \in R(f)$, hence $g \in R(f) \xrightarrow{m} D$. It follow that f is a (continuous) retraction in D if and only if (f,g) is a (continuous) retraction of R(f) into D. The argument is straightforward but note that $g \in R(f) \xrightarrow{c} D$ if and only if R(f) is of finite character in D. Note also that in case R(f) is of finite character in D and $f \in D \xrightarrow{c} R(f)$ then $f \in D \xrightarrow{c} D$.

3.3 A representation between lattices D and D' is a pair of function (f,g) such that $f \in D \xrightarrow{m} D'$, $g \in D' \xrightarrow{m} D$, $I \sqsubseteq g \circ f$, and $f \circ g \sqsubseteq I$. A representation is a connection because from $I \sqsubseteq g \circ f$ it follows that $f \sqsubseteq f \circ g \circ f$ and $g \sqsubseteq g \circ f \circ g$, and from $f \circ g \sqsubseteq I$ it follows that $f \circ g \circ f \sqsubseteq f$ and $g \circ f \circ g \sqsubseteq g$.

The study of representations is facilitated by introducing the following two operators $\phi(g)$ and $\psi(f)$

$$\phi \in (D' \xrightarrow{m} D) \xrightarrow{m} (D \xrightarrow{m} D')$$
$$\psi \in (D \xrightarrow{m} D') \xrightarrow{m} (D' \xrightarrow{m} D)$$

which are defined as follows:

$$\phi(g) = \lambda x.\ \bigsqcap\{y : x \sqsubseteq g(y)\}$$
$$\psi(f) = \lambda y.\ \bigsqcup\{x : f(x) \sqsubseteq y\} \quad .$$

It is easy to check that $g \sqsubseteq \psi(\phi(g))$ and $\phi(\psi(f)) \sqsubseteq f$ but (ϕ,ψ) is not a representation since neither ϕ nor ψ is monotonic.

3.3.1 Theorem. Let $f \in D \xrightarrow{m} D'$ and $g \in D' \xrightarrow{m} D$. The following conditions are equivalent:

i) (f,g) is a representation between D and D'.

ii) For arbitrary $x \in D$ and $y \in D'$: $f(x) \sqsubseteq y$ if and only if $x \sqsubseteq g(y)$.

iii) $\phi(g) = f$ and $\psi(f) = g$.

iv) $f \in D \xrightarrow{a} D'$ and $\psi(f) = g$.

v) $f \sqsubseteq \phi(g)$ and $\psi(f) \sqsubseteq g$.

vi) $g \in D' \xrightarrow{ca} D$ and $\phi(g) = f$.

The implications from i) to ii) and from ii) to iii) are easy. Assume that iii) holds to prove f is additive. Let $X \subseteq D$; then for $x \in X$ we have $f(x) \sqsubseteq \bigsqcup f(X)$. Since $\psi(f) = g$ this means $x \sqsubseteq g(\bigsqcup f(X))$. This holds for every $x \in X$ so $\bigsqcup X \sqsubseteq g(\bigsqcup f(X))$. But now since $\phi(g) = f$ we have $f(\bigsqcup X) \sqsubseteq \bigsqcup f(X)$. So f is additive.

Assume now iv) holds to prove $f(x) \sqsubseteq \phi(g)(x)$ for arbitrary $x \in D$. From $\psi(f) = g$ and the additivity of f we get $f(g(y)) \sqsubseteq y$ for arbitrary $y \in D'$. Now if $x \sqsubseteq g(y)$ then $f(x) \sqsubseteq y$. It follows that $f(x) \sqsubseteq \phi(g)(x)$.

Assume v) to prove vi). Consider $Y \subseteq D'$. If $y \in Y$ then $\sqcap g(Y) \sqsubseteq g(y)$. Then from $f \sqsubseteq \phi(g)$ it follows that $f(\sqcap g(Y)) \sqsubseteq y$. Hence we have $f(\sqcap g(Y)) \sqsubseteq \sqcap Y$ and from $\psi(f) \sqsubseteq g$ we get that $\sqcap g(Y) \sqsubseteq g(\sqcap Y)$, so g is coadditive. In order to prove that $\phi(g) \sqsubseteq f$ note that for arbitrary $x \in D$ and the definition of ψ we have $x \sqsubseteq \psi(f)(f(x))$; hence from $\psi(f) \sqsubseteq g$ we have $x \sqsubseteq g(f(x))$. But this implies that $\phi(g) \sqsubseteq f$.

If we assume vi) then from $\phi(g) = f$ we get immediately that $f \circ g \sqsubseteq I$. Using also the coadditivity of g we get

$$g(f(x)) = \sqcap \{g(y) : x \sqsubseteq g(y)\}$$

hence $I \sqsubseteq g \circ f$.

3.3.2 Some consequences of the definition and Theorem 3.3.1 are the following. If (f,g) is a representation between D and D' then $g \circ f$ is a closure in D and $f \circ g$ is a projection in D'. Hence $R(g) = R(g \circ f)$ is $\sqcap$-closed in D and $R(f) = R(f \circ g)$ is $\sqcup$-closed in D'. By 3.1.1 f restricted to R(g) is an isomorphism onto R(f). Furthermore the function f is additive and the function g is coadditive and one of them determines uniquely the other.

3.3.3 Theorem. Let D and D' be lattices. Let D_1 be a $\sqcap$-closed sublattice of D and D_1' be a $\sqcup$-closed sublattice of D'. Let h be an isomorphism of D_1 onto D_1'. Then there is a unique representation (f,g) between D and D' such that $R(f) = D_1'$, $R(g) = D_1$, for $x \varepsilon D_1$ $f(x) = h(x)$ and for $y \varepsilon D_1'$ $g(y) = h^{-1}(y)$.

We define f and g exactly as in Theorem 3.1.2 and we need only to show that $I \sqsubseteq g \circ f$ and $f \circ g \sqsubseteq I$. Since D_1' is $\bigsqcup$-closed in D' it follows that $h \varepsilon D_1 \xrightarrow{a} D'$; in the same way since D_1 is $\bigsqcap$-closed in D we have $h^{-1} \varepsilon D_1' \xrightarrow{ca} D$. Hence

$$f(g(y)) = h(g(y)) = \bigsqcup\{h(x): x \varepsilon D_1 \wedge h(x) \sqsubseteq y\}$$

so clearly $f(g(y)) \sqsubseteq y$. Also

$$g(f(x)) = h^{-1}(f(x)) = \bigsqcap\{h^{-1}(y) : y \varepsilon D_1' \wedge x \sqsubseteq h^{-1}(y)\}$$

so clearly $x \sqsubseteq g(f(x))$.

To prove the uniqueness note that if (f_1,g_1) is another representation such that $f(x) = f_1(x)$ for $x \varepsilon D_1$ and $g(y) = g_1(y)$ for $y \varepsilon D_1'$, then from $x \sqsubseteq g(f(x))$ we get $f_1(x) \sqsubseteq f_1(g(f(x)) = f(g(f(x))) = f(x)$ and similarly for the function g using $f(g(y)) \sqsubseteq y$.

3.4 A representation (f,g) between D and D' is continuous in case $g \varepsilon D' \xrightarrow{c} D$. If g is 1-1 the representation is called a closure of D' into D and D' is a closure of D. If g is continuous and 1-1 then (f,g) is a continuous closure of D' into D and D' is a continuous closure of D. If f is 1-1 then the representation is called a projection of D into D' and D is a projection of D'. In case f is 1-1 and g is continuous then (f,g) is a continuous projection of D into D' and D is a continuous projection of D'. Note that Theorem 3.1.3 applies to representations.

3.4.1 This notation is consistent with the notation of 2.2.1. For let $f \varepsilon D \xrightarrow{m} D$ so we may consider $f \varepsilon D \xrightarrow{m} R(f)$. Define $g(y) = y$ for $y \varepsilon R(f)$ so $g \varepsilon R(f) \xrightarrow{m} D$. Then we have: i) f is a (continuous) closure in D if and only if (f,g) is a (continuous) closure of R(f) into D. ii) f is a (continuous) projection in D if and only (g,f) is a (continuous) projection of R(f) into D.

3.4.2 The lattices that are closure (projections) of a given lattice are characterized by 3.1.1 and 3.3.3 as exactly those lattices which are isomorphic to a $\sqcap$-closed ($\sqcup$-closed) sublattice of D. We can characterize the continuous closures of D as follows.

3.4.3 Theorem. A lattice D' is a continuous closure of a lattice D if and only if D' is isomorphic to some $\sqcap$-closed sublattice of D of finite character in D.

This is essentially a consequence of Theorem 2.2.3.

3.5 Let (f,g) be a representation between D and D'. We can define two operators $\sigma(h)$ and $\tau(j)$ where $h \varepsilon D \xrightarrow{m} D$ and $j \varepsilon D' \xrightarrow{m} D'$

$$\sigma(h) = f \circ h \circ g$$
$$\tau(j) = g \circ j \circ f \quad .$$

Then it can be easily checked that (σ,τ) is a representation between $D \xrightarrow{m} D$ and $D' \xrightarrow{m} D'$. We call (σ,τ) the representation induced by (f,g).

Assume now that (f,g) is a continuous representation between D and D'. Then whenever h and j are continuous $\sigma(h)$ and $\tau(j)$ are also continuous. Furthermore it is possible to check that τ is also continuous when restricted to continuous functions. This means that (σ,τ) is a continuous representation between $D \xrightarrow{c} D$ and $D' \xrightarrow{c} D'$. It is called the continuous representation induced by (f,g).

3.5.1 Theorem. Let (f,g) be a (continuous) representation between D and D' and let (σ,τ) be the (continuous) representation induced by (f,g). If j is a (continuous) closure in D' then $\tau(j)$ is a (continuous) closure in D and the function g maps $R(j)$ onto $R(\tau(j))$.

From $I \sqsubseteq j$ it follows easily that $I \sqsubseteq \tau(I) \sqsubseteq \tau(j)$. Furthermore $\tau(j) \circ \tau(j) = g \circ j \circ f \circ g \circ j \circ f \sqsubseteq g \circ j \circ j \circ f = g \circ j \circ f$. To prove that g maps $R(j)$ onto $R(\tau(j))$ take $y \in R(j)$, so $j(y) = y$. Then $\tau(j)(g(y)) = g(j(f(g(y)))) \sqsubseteq g(j(y)) = g(y)$, hence $g(y) \in R(\tau(j))$. Conversely let $x \in R(\tau(j))$ then $j(f(x)) \in R(j)$ and $g(j(f(x))) = \tau(j)(x) = x$.

3.5.2 Theorem. Let (f,g) be a (continuous) representation between D and D' and let (σ,τ) be the (continuous) representation induced by (f,g). If h is a (continuous) projection in D then $\sigma(h)$ is a (continuous) projection in D' and the function f maps $R(h)$ onto $R(\sigma(h))$.

This is the dual of Theorem 3.5.1 and the proof is similar.

Note that in the situation of Theorem 3.5.1 in case (f,g) is a closure and j is a closure in D' then g is an isomorphism of $R(j)$ onto $R(\tau(j))$. Similar remark for Theorem 3.5.2.

3.6 We shall consider now some examples of these notions. If D is a set then $P(D)$ is the power set of D considered as a lattice under the inclusion relation.

3.6.1 Let D be some lattice and $D_1 \subseteq D$ be some subset of D. We can define a representation (f,g) between $P(D_1)$ and D as follows: for $X \subseteq D_1$ define $f(X) = \bigsqcup X$ and for $y \in D$ define $g(y) = \{x \in D_1 : x \sqsubseteq y\}$. This representation is a closure if and only D_1 is a set of generators for D, namely in the case for every $y \in D$ there is some $X \subseteq D_1$ such that $y = \bigsqcup X$. Also (f,g) is continuous if and only if every element $x \in D_1$ satisfies the following condition: if $Y \subseteq D$ is directed and $x \sqsubseteq \bigsqcup Y$ then there is some $y \in Y$ such that $x \sqsubseteq y$. An element like this will be called compact in the next section.

3.6.2 If $D_1 \subseteq D$ note that $P(D_1) = (D_1]$ in $P(D)$ (see 2.4) so it is $\bigsqcup$-closed in $P(D)$. The corresponding projection is $f(X) = X \cap D_1$ for $X \subseteq D$ and it is a continuous function so $P(D_1)$ is a continuous projection in $P(D)$. Since $P(D_1)$ is not $\sqcap$-closed it is not a closure in $P(D)$; but it is actually a continuous closure of $P(D)$ given by the closure (f,g) where f is defined above and $g(Y) = Y \cup (D - D_1)$ for $Y \subseteq D_1$.

3.6.3 Let D be a lattice and consider the function $K \in D \longrightarrow (D \xrightarrow{c} D)$ defined $K(y) = \lambda x.y$. This function is both additive and coadditive, and 1-1. Hence $(K, \psi(K))$ is a projection of D into $D \xrightarrow{c} D$. Since $\psi(K)(j) = j(\bot)$ it follows that the projection is continuous. On the other hand $(\phi(K),K)$ is a continuous closure of D into $D \xrightarrow{c} D$. It is easy to check that $\phi(K)(j) = j(\top)$.

3.6.4 Let $K' \in D \longrightarrow (D \xrightarrow{a} D)$ be defined as follows: $K'(y)(x) = y$ in case $x = \bot$; $K'(y)(x) = \top$ in case $x \neq \bot$. This function is not additive but it is coadditive and continuous. Hence $(\phi(K'),K')$ is a continuous closure of D into $D \xrightarrow{a} D$. It is easy to check that $\phi(K')(j) = j(\bot)$.

3.6.5 Consider the lattice $P(\omega)$ where ω is the set of non-negative integers. The function graph $\in (P(\omega) \xrightarrow{c} P(\omega)) \longrightarrow P(\omega)$ defined in [7] is 1-1 and both additive and coadditive. It follows that $(\phi(\text{graph}), \text{graph})$ is a continuous closure of $P(\omega) \xrightarrow{c} P(\omega)$ into $P(\omega)$, and $(\text{graph}, \psi(\text{graph}))$ is a projection of $P(\omega) \xrightarrow{c} P(\omega)$ into $P(\omega)$. We shall show later that it is impossible to improve the latter relation to a continuous projection.

3.6.6 As a final example we define the following continuous closure in $P(\omega)$. $f(X) = \{x : x + 1 \in X\}$ and $g(Y) = \{0\} \cup \{x + 1 : x \in Y\}$. This transformation is used in the definition of $\boxplus$ in [7], page 619.

4.1 Injective and compactly generated lattices. In this section we consider retraction, closures and projections involving lattices satisfying special conditions. These conditions amount to requiring that every element in the lattice is the join of elements that are compact; in some cases we require the compactness property to be absolute, in others only relative to the element generated.

4.1.1 Let D be a lattice and let u and y be elements of D. The element u is compact relative to y in case that whenever $X \subseteq D$ is directed and $y \sqsubseteq \bigsqcup X$ then there is some $x \in X$ such that $u \sqsubseteq x$. We use the notation $u \prec y$ to denote that u is compact relative to y. An element $x \in D$ such that $x \prec x$ holds is called compact.

We define the following sets. If $y \in D$ then $B_D(y) = \{u : u \prec y\}$ and $C_D(y) = \{x : x \text{ is compact and } x \sqsubseteq y\}$. The set $\bar{C}_D = C_D(\top)$ is the set of all compact elements in D.

4.1.2 We note several properties of these notions that follow easily from the definitions. i) the relation $\prec$ is transitive; ii) if $u \prec y$ then $u \sqsubseteq y$; iii) if $u \prec y$, $v \sqsubseteq u$ and $y \sqsubseteq x$ then $v \prec x$; iv) $C_D(y) \subseteq B_D(y)$; v) $C_D \in D \xrightarrow{c} P(D)$; vi) $B_D \in D \xrightarrow{m} P(D)$.

Note that the relation $\prec$ is equivalent to the similar notated relation defined in [6] only for the lattices called continuous in that reference.

4.1.3 A lattice D such that for every $y \in D$ $y = \bigsqcup B_D(y)$ is called injective. In case $y = \bigsqcup C_D(y)$ for every $y \in D$ the lattice is called compactly generated. Clearly a compactly generated lattice is injective. The lattice D is $\prec$-well founded in case that there is in D no infinite sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $x_{n+1} \prec x_n$ and $x_n \neq x_{n+1}$.

4.1.4 Theorem. If a lattice D is injective and <-well founded then it is compactly generated.

Assume D is not compactly generated and let $D_1 = \{y : y \neq \bigsqcup C_D(y)\}$. Let y be a minimal element in D_1 relative to $\prec$. Since $y = \bigsqcup B_D(y)$ there is at least one $x \in B_D(y)$ such that $x \in D_1$. Since $x \neq y$ this contradicts the minimality of y.

4.1.5 We give some examples of these notions. Let R be the set of all real numbers with the usual order and two extra elements $\bot$ and $\top$. Here $x \prec y$ means $x = \bot$ or $x < y$. Since the only compact element is $\bot$ this lattice is not compactly generated but it is injective.

Let D be some set. Then P(D) is compactly generated and $\prec$-well founded. In this lattice $X \prec Y$ means X is finite and $X \subseteq Y$.

Let D be an infinite set and D' be the sublattice of P(D) consisting of all finite subsets of D plus the set D itself. Here $X \prec Y$ holds if and only if $X = \bot = \emptyset$, so it is not an injective lattice. Note that D' is (trivially) a $\prec$-well founded lattice.

Finally let D be the nonegative integers where the order is defined $n \sqsubseteq m$ if and only $n \geq m$, plus one extra element $\bot$. In this lattice every element is compact so it is compactly generated, but it is not $\prec$-well founded.

4.2 We want to investigate to what extent injective and compactly generated lattices are related to continuous representations. In one direction this is clarified by the next two theorems.

4.2.1 Theorem. Let (f,g) be a representation between lattices D and D' where D is injective and for every $x \in D$ $f(B_D(x)) \subseteq B_{D'}(f(x))$. Then $g \in D' \xrightarrow{c} D$.

Let $Y \subseteq D'$ be directed and assume $v \in B_D(g(\bigsqcup Y))$. It follows that $f(v) \prec f(g(\bigsqcup Y)) \sqsubseteq \bigsqcup Y$ hence $f(v) \sqsubseteq y$ for some $y \in Y$. This implies $v \sqsubseteq g(y)$ hence $g(\bigsqcup Y) \sqsubseteq \bigsqcup g(Y)$.

4.2.2 Theorem. Let (f,g) be a representation between lattices D and D' where D is compactly generated and for every $x \in D$ $f(C_D(x)) \subseteq C_{D'}(f(x))$. Then $g \in D' \xrightarrow{c} D$.

The proof is similar.

4.3 We consider now the problem in the other direction, namely assuming a connection or representation is continuous in which way are related the compact elements of the lattices. The key result is given by the following lemma.

4.3.1 Lemma. Let (f,g) be a continuous connection satisfying the conditions $f \circ g \sqsubseteq I$. Then $f(B_D(g(y))) \subseteq B_{D'}(y)$ for arbitrary $y \in D'$.

Take $x \prec g(y)$ to show $f(x) \prec y$. Let $Y \subseteq D'$ be directed and $y \sqsubseteq \bigsqcup Y$. Then $g(y) \sqsubseteq g(\bigsqcup Y) = \bigsqcup g(Y)$. Hence there is $v \in Y$ such that $x \sqsubseteq g(v)$; it follows $f(x) \sqsubseteq f(g(v)) \sqsubseteq v$.

4.3.2 Corollary. If D is an injective lattice and D' is a continuous retraction of D then D' is injective.

Let (f,g) be the continuous retraction of D' into D. Since $f \circ g = I$ the preceding theorem applies. If $y \in D'$ then $g(y) = \bigsqcup B_D(g(y))$ hence $y = f(g(y)) = \bigsqcup f(B_D(g(y)) \sqsubseteq \bigsqcup B_D{}'(y)$.

4.3.3 Theorem. A lattice D is injective if and only if it is a continuous retraction of P(D).

The lattice P(D) is injective so in case D is a continuous retraction of P(D) 4.3.2 applies. Conversely assume D is injective and we define $f(X) = \bigsqcup X$ for $X \subseteq D$ then (f, B_D) is a continuous retraction of D into P(D). Clearly $f(B_D(y)) = y$ and f is additive, so we need only to show B_D is continuous. For this it is sufficient to prove that $B_D(\bigsqcup Y) \subseteq \bigcup\{B_D(y) : y \in Y\}$ whenever $Y \subseteq D$ is directed. But $\bigsqcup Y = \bigsqcup\{v :$ There is $y \in Y$ and $v \in B_D(y)\}$ which is also directed set, hence if $u \prec \bigsqcup Y$ then $u \sqsubseteq v$ such that $v < y$ for some $y \in Y$. It follows that $u \in B_D(y)$.

4.3.4 Theorem. Let (f,g) be a continuous representation between D and D'. Then $f(B_D(x)) \subseteq B_{D'}(f(x))$ and $f(C_D(x)) \subseteq C_{D'}(f(x))$ for every $x \in D$.

Assume $u \prec x$; then $u \prec g(f(x))$ hence by 4.3.1 $f(u) \prec f(x)$. Similar argument if $u \prec u \sqsubseteq x$.

4.3.5 Corollary. Let (f,g) be a continuous closure of D' into D. If D is compactly generated then D' is also compactly generated. Furthermore $f(C_D(g(y))) = C_{D'}(y)$ for $y \in D'$.

Let $y \in D'$ and $f(x) = y$. Then $f(x) = f(\bigsqcup C_D(x)) = \bigsqcup f(C_D(x)) \sqsubseteq \bigsqcup C_{D'}(f(x)) = \bigsqcup C_{D'}(y)$.

Now to prove $f(C_D(g(f(x)))) = C_{D'}(f(x))$ we need to consider only the inclusion from right to left. Let $v \in C_{D'}(f(x))$; then $g(v) \sqsubseteq g(f(x))$ and $v = f(g(v)) = \bigsqcup f(C_D(g(v)))$. Since v is compact this means $v = f(u)$ for some $u \in C_D(g(v))$, but then $u \in C_D(g(f(x)))$.

4.3.6 Theorem. A lattice D is compactly generated if and only if it is a continuous closure of P(D).

Since P(D) is compactly generated in one direction the equivalence follows from 4.3.5. In the other direction if we define $f(X) = \bigsqcup X$ for $X \subseteq C_D$ then (f, C_D) is a continuous closure of D into $P(\bar{C}_D)$ (see 3.6.1). But $P(\bar{C}_D)$ is a continuous closure of P(D) as explained in 3.6.2.

4.4 We have obtained characterizations of injective and compactly generated lattices in terms of continuous retractions and continuous closures. We present now a partial result that relates $\prec$-well founded lattices and continuous projections.

4.4.1 Theorem. Let D be a continuous projection of D'. If D' is $\prec$-well founded then D is also $\prec$-well founded.

Let (f,g) be a continuous projection of D into D'. Then Theorem 4.3.4 applies and f is 1-1. It follows that any infinite descending sequence in D will induce a corresponding infinite descending sequence in D'. Hence D is $\prec$-well founded.

4.4.2 Corollary. If D is a continuous projection of P(D) then D is compactly generated and $\prec$-well founded.

4.5 We collect in this last section some results on the lattice of continuous functions. This matter has been considered in [6] so here we give only an outline of the arguments.

4.5.1 First note that whenever D and D' are both injective lattices then $D \xrightarrow{c} D'$ is also an injective lattice. To show this define functions $f_{u,v}$ where $u \in D$ and $v \in D'$. If $x \in D$ then $f_{u,v}(x) = v$ in case $u \prec x$ and $f_{u,v}(x) = \bot$ otherwise. Since D is injective it follows that $f_{u,v} \in D \xrightarrow{c} D'$. Furthermore if $f \in D \xrightarrow{c} D'$ and $v \prec f(u)$ then $f_{u,v} \prec f$. Now it is easy to prove, using the fact that D' is also injective that for any such function f we have $f = \bigsqcup\{f_{u,v} : v \prec f(u)\}$.

4.5.2 A similar argument shows that whenever D and D' are compactly generated then $D \xrightarrow{c} D'$ is also compactly generated. To prove this define functions $g_{u,v}$ where $u \in D$ and $v \in D'$. We put $g_{u,v}(x) = v$ in case $u \sqsubseteq x$ and $g_{u,v}(x) = \bot$ otherwise. It follows that in case $f \in D \xrightarrow{c} D'$ and $v \sqsubseteq f(u)$ then $g_{u,v} \sqsubseteq f$. Furthermore if u is compact in D and v is compact in D' then $g_{u,v}$ is compact in $D \xrightarrow{c} D'$. Now it is easy to show (using the fact that both D and D' are compactly generated) that whenever $f \in D \xrightarrow{c} D'$ then $f = \bigsqcup\{g_{u,v} : v \sqsubseteq f(u) \wedge u \in \bar{C}_D \wedge v \in \bar{C}_{D'}\}$.

4.5.3 Since $D \xrightarrow{ccl} D$ is a continuous closure of $D \xrightarrow{c} D$ (see 2.4.4) it follows that in case D is injective (compactly generated) lattice then $D \xrightarrow{ccl} D$ is also injective (compactly generated) lattice.

4.5.4 For $D \xrightarrow{cp} D$ we have a weaker result. Suppose D is compactly generated and whenever $u \in \bar{C}_D$ and $u' \sqsubseteq u$ then $u' \in \bar{C}_D$. Then $D \xrightarrow{cp} D$ is compactly generated. To prove this define functions h_u for $u \in D$ such that $h_u(x) = u$ in case $u \sqsubseteq x$ and

$h_u(x) = \bot$ otherwise. Such functions are always projections. But if u is compact in D then $h_u \in D \xrightarrow{cp} D$ and also it is compact in this lattice. Note that $D \xrightarrow{cp} D$ is $\sqcup$-closed in $D \xrightarrow{c} D$ (see 2.4.5). It is sufficient to prove now that whenever $f \in D \xrightarrow{cp} D$ then $f = \sqcup\{h_u : u \in \bar{C}_D \wedge h_u \sqsubseteq f\}$. For this purpose we show that for arbitrary $x \in D$ and $v \in C_D(f(x))$ there is $u \in \bar{C}_D$ and $h_u \sqsubseteq f$ such that $v \sqsubseteq h_u(x)$. Since $f(x) = \sqcup f(C_D(x))$ there is $u \in C_D(x)$ such that $v \sqsubseteq f(u) \sqsubseteq u$ so $u' = f(u)$ is also compact in D. Obviously we have $v \sqsubseteq h_{u'}(x)$ so we need only to show $h_{u'} \sqsubseteq f$. Now if for some $y \in D$ we have $u' \sqsubseteq y$ and $h_{u'}(y) = u'$ then $u' = f(u') \sqsubseteq f(y)$.

4.5.5 Finally we consider a lattice D in which there is an infinite sequence of elements $u_1, \ldots, u_i, \ldots$ where each u_i is compact, $u_i \sqsubseteq u_{i+1}$ and $u_i \neq u_{i+1}$. Let v some fixed compact element in D, $v \neq \bot$ and put $g_i = g_{u_i,v}$ (see 4.5.2). These functions are all compact in $D \xrightarrow{c} D$, $g_{i+1} \sqsubseteq g_i$ and $g_{i+1} \neq g_i$ hence $D \xrightarrow{c} D$ is not $\prec$-well founded. In particular $P(\omega) \xrightarrow{c} P(\omega)$ is not a continuous projection of $P(\omega)$.

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