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SMOOTH SUBMANIFOLDS INTERSECTING ANY ANALYTIC CURVE IN A DISCRETE SET

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ABSTRACT. We construct examples of C^∞ smooth submanifolds in $\mathbb{C}^n$ and $\mathbb{R}^n$ of codimension 2 and 1, which intersect every complex, respectively real, analytic curve in a discrete set. The examples are realized either as compact tori or as properly imbedded Euclidean spaces, and are the graphs of quasianalytic functions. In the complex case, these submanifolds contain real n-dimensional tori or Euclidean spaces that are not pluripolar while the intersection with any complex analytic disk is polar.

1. INTRODUCTION

If a real analytic $(2n-2)$-dimensional submanifold R in $\mathbb{C}^n$ either intersects every complex analytic disk in a discrete set or contains the disk, then R is a complex submanifold. A natural question arises: is this true when R is merely smooth?

The main result of this paper is:

Theorem 1.1. *There exists a smooth, compact manifold R in $\mathbb{C}^n$, diffeomorphic to a $(2n-2)$-dimensional torus, which intersects every analytic disk in a discrete set. Moreover, R contains a smooth submanifold M, diffeomorphic to an n-dimensional torus, which is not pluripolar.*

Since R does not contain any analytic disk, it is not a complex manifold. Thus in the category of smooth manifolds the discreteness of intersections with analytic disks does not imply that a submanifold is complex.

To explain the last statement of the theorem, we recall that a set K in $\mathbb{C}^n$ is *pluripolar* if there is a plurisubharmonic function $u \not\equiv -\infty$ which is equal to $-\infty$ on K. If $n = 1$, pluripolar sets are classical polar sets. In general, it is hard to detect whether a set is pluripolar.

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In 1971 L. I. Ronkin introduced in [R] the notion of Γ-capacity of a set, which is computed in terms of the capacities of intersections of that set with complex lines. Ch. Kiselman [Ki] and, independently, A. Sadullaev [Sa] constructed a real algebraic surface in $\mathbb{C}^2$ which intersects any complex line in at most 4 points. So the Γ-capacity of this set is 0, while the set is not pluripolar.

The question whether the polarity of intersections of a set implies the pluripolarity of that set when complex lines are replaced by one-dimensional varieties of higher degree was open since that time. It was posed as an open problem by E. Bedford in his survey [B] as follows:

Let $E \subset \mathbb{C}^n$. Suppose $E \cap V$ is polar in V for each germ V of an irreducible, one-dimensional complex variety $V \subset \mathbb{C}^n$. Is E pluripolar in $\mathbb{C}^n$?

So the last part of our theorem answers this question in the negative.

The submanifolds in our examples are graphs of quasianalytic functions (see Section 2) that belong to Denjoy–Carleman classes (see [BM2]). Quasianalytic functions are smooth, and, by Lemma 2.2, among the quasianalytic functions of one real variable there exist functions which coincide with analytic functions on at most a discrete set. This property is fundamental in our constructions.

In [CLP] we proved that quasianalytic *curves* are pluripolar. Raising the dimension of our submanifolds to at least n we obtain nonpluripolar examples.

Using the same ideas as in the proof of Theorem 1.1, we construct in Theorem 3.1 a quasianalytic function on $\mathbb{R}^n$ whose graph intersects any real analytic curve in a discrete set and, consequently, does not contain any analytic curve. It serves as an example of an "extremely" smooth function which is not *arc-analytic* anywhere (see [BM1]).

We would like to thank Al Taylor for introducing us to the notion of quasianalytic functions.

2. Basic definitions and facts

Let $f : [a, b] \to \mathbb{R}$ be a C^∞ function and let

$$M_j(f) = \sup_{a \le x \le b} |f^{(j)}(x)|.$$

Given an increasing sequence $\{M_j\}$ which is logarithmically convex, the class $C^\#\{M_j\}$ consists of all smooth functions $f : [a, b] \to \mathbb{R}$ satisfying the estimate $M_j(f) \le C^j M_j$ for all j, with a constant C depending on f. Note that if $f, g \in C^\#\{M_j\}$ then clearly $f + g \in C^\#\{M_j\}$.

The proof of the following lemma can be found in [H, Prop. 8.4.1].

Lemma 2.1. *Let $\{M_j\}$ be a sequence such that $M_j^{1/j} > j$ is increasing and let $f \in C^{\#}\{M_j\}$ on some interval $[a,b]$. If $g : [\alpha,\beta] \to [a,b]$ is a real analytic function, then the function $h = f \circ g \in C^{\#}\{M_j\}$ on $[\alpha,\beta]$.*

Let

$$\tau(r) = \inf_{j\geq 1} \frac{M_j}{r^j}$$

be the associated function to the sequence $\{M_j\}$. We denote by $\nu(r)$ the largest integer such that $\tau(r) = M_{\nu(r)}/r^{\nu(r)}$. The following lemma is a slight extension of a result by S. Mandelbrojt in [M, Ch.VI.41].

Lemma 2.2. *Let $\{M_j\}$ be a sequence such that $C^j = o(M_j)$ for every constant $C > 1$. Suppose*

$$\liminf_{r\to\infty} \frac{\nu(r)}{\log r} \geq A > 0.$$

There exists a smooth periodic function $f \in C^{\#}\{M_j\}$ on $[0,2\pi]$ with the following property: for every sequence $\{N_j\}$ such that

$$\lim_{j\to\infty} \frac{{N_j}^{1/j}}{M_j^{1/j}} = 0,$$

f does not belong to $C^{\#}\{N_j\}$ on any interval.

Proof. Following [M, Ch.VI.41], we introduce

$$a_n = \frac{\tau(2^n)}{2^n}$$

and for $k = 1, 2, \ldots$ we define

$$f_k(x) = \sum_{n=k}^{\infty} a_n \cos 2^n x.$$

Since $\tau(2^n) \leq M_j 2^{-jn}$ for all $j = 1, 2, \ldots$, we have $a_n \leq M_j 2^{-(j+1)n}$ and

$$|f_k^{(j)}(x)| \leq \sum_{n=k}^{\infty} a_n 2^{jn} \leq M_j \sum_{n=k}^{\infty} 2^{-n} \leq M_j.$$

Thus each of the functions $f_k \in C^{\#}\{M_j\}$ on $[0,2\pi]$.

We show that $f = f_1$ is the desired function. Suppose that f_1 is in $C^{\#}\{N_j\}$ on some interval $[\alpha,\beta]$ in $[0,2\pi]$. We may assume that $\alpha = 2\pi m 2^{-k_0}$ and $\beta = 2\pi(m+1)2^{-k_0}$ for some appropriate choice of integers m and k_0 and we write $f_1 = g_{k_0} + f_{k_0}$, where

$$g_{k_0}(x) = \sum_{n=1}^{k_0-1} a_n \cos 2^n x.$$

We have $|g_{k_0}^{(j)}(x)| \le C^j$ for all $j = 1, 2, \dots$ and $x \in [0, 2\pi]$, where C is a constant depending on k_0. Since

$$\lim_{j\to\infty} \frac{N_j^{1/j}}{M_j^{1/j}} = 0,$$

we conclude that the function $f_{k_0} \in C^{\#}\{L_j\}$ on $[\alpha, \beta]$, where $L_j = N_j + C^j$ and

$$\lim_{j\to\infty} \frac{L_j^{1/j}}{M_j^{1/j}} = 0.$$

However, this function is periodic with period $2^{-k_0+1}\pi$. Hence $f_{k_0} \in C^{\#}\{L_j\}$ on $[0, 2\pi]$.

Since

$$f_{k_0}^{(j)}(x) = \sum_{n=k_0}^{\infty} a_n 2^{jn} \phi_n(x)$$

where $\phi_n(x)$ is either $\pm\cos 2^n x$ or $\pm\sin 2^n x$,

$$|a_n| = \left| \frac{1}{\pi 2^{nj}} \int_0^{2\pi} f_{k_0}^{(j)}(x)\phi_n(x)\, dx \right| \le \frac{2R^j L_j}{2^{nj}} .$$

Here $R \ge 1$ depends on k_0.

If $j_n = \nu(2^n)$, then $a_n = M_{j_n} 2^{-(j_n+1)n}$ by the definition of a_n. By the previous inequality, this implies that

$$M_{j_n}^{1/j_n} \le C_1 L_{j_n}^{1/j_n} 2^{n/j_n}$$

or

$$\lim_{n\to\infty} \frac{L_{j_n}^{1/j_n}}{M_{j_n}^{1/j_n}} \ge \frac{1}{C_1} \liminf_{n\to\infty} 2^{-n/j_n}.$$

By hypothesis

$$\liminf_{n\to\infty} \frac{j_n}{n} \ge B = A \log 2 > 0;$$

thus we get

$$\lim_{n\to\infty} \frac{L_{j_n}^{1/j_n}}{M_{j_n}^{1/j_n}} \ge \frac{2^{-1/B}}{C_1} > 0.$$

This contradiction proves the lemma. □

The class $C^{\#}\{M_j\}$ on $[a, b]$ is called *quasianalytic* if every function in $C^{\#}\{M_j\}$ which vanishes to infinite order at some point in $[a, b]$ must be identically equal to 0. A smooth function f is called *quasianalytic in the sense of Denjoy* if the class $C^{\#}\{M_j(f)\}$ is quasianalytic. The

Denjoy-Carleman theorem (see e.g. [T, 3.10(12)]) states that the class $C^{\#}\{M_j\}$ is quasianalytic if

$$\sum_{j=1}^{\infty} \frac{1}{M_j^{1/j}} = \infty.$$

3. Examples

We introduce the notation

$$\log_1 x = \log x,\ \log_k x = \log_{k-1}(\log x),$$

and consider the sequences $\{M_j^{(k)}\}$, where $M_j^{(k)} = (j \log_k j)^j$ and $k > 0$. These sequences are logarithmically convex and it is easy to see that they verify the hypotheses of Lemmas 2.1 and 2.2. Moreover, the classes $C^{\#}\{M_j^{(k)}\}$ are quasianalytic.

Our constructions rely on the following examples of smooth hypersurfaces in $\mathbb{R}^m$, diffeomorphic to $\mathbb{R}^{m-1}$, which intersect any real analytic curve in a discrete set. Let $f_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m-1$, be smooth periodic functions such that $f_k \in C^{\#}\{M_j^{(k)}\} \setminus C^{\#}\{M_j^{(k+1)}\}$ on any interval. Such functions exist by Lemma 2.2.

Theorem 3.1. *The hypersurface $H \subset \mathbb{R}^m$, defined by the equation*

$$x_m = \sum_{j=1}^{m-1} f_j(x_j),$$

intersects any real analytic curve in a discrete set.

Proof. Let $\gamma(t) = (\gamma_1(t), \dots, \gamma_m(t))$ be a real analytic curve defined for t in an open interval I about 0 with $\gamma(0) = x_0 = (x_{01}, \dots, x_{0m}) \in H$ and such that there exists a sequence $t_s \to 0$, $t_s \neq 0$, with $\gamma(t_s) \in H$. It suffices to show that $\gamma(t) = x_0$ for all $t \in I$.

We have

$$\gamma_m(t_s) = \sum_{j=1}^{m-1} f_j(\gamma_j(t_s)),\ \forall\, s \geq 1. \tag{1}$$

By Lemma 2.1, the right side is quasianalytic; since the left side is real analytic, it follows that (1) holds for all $t \in I$. We show by induction on $k = 1, \dots, m-1$ that all functions γ_k are identically constant. Then γ_m is also identically constant by (1), and we are done.

If the function γ_1 is not identically constant, then it has a real analytic inverse function on some interval J. For $u \in J$, we have by

(1)

$$f_1(u) = \gamma_m(\gamma_1^{-1}(u)) - \sum_{j=2}^{m-1} f_j(\gamma_j(\gamma_1^{-1}(u))).$$

By Lemma 2.1 and the choice of $f_1 \in C^{\#}\{M_j^{(1)}\} \setminus C^{\#}\{M_j^{(2)}\}$ this is impossible. Hence $\gamma_1(t) \equiv x_{01}$. We assume by induction that for $2 \le k \le m-1$ we have $\gamma_j(t) \equiv x_{0j}$ for all $j \le k-1$. Equation (1) becomes

$$\gamma_m(t) - \sum_{j=1}^{k-1} f_j(x_{0j}) = \sum_{j=k}^{m-1} f_j(\gamma_j(t)).$$

A similar argument shows, since $f_k \in C^{\#}\{M_j^{(k)}\} \setminus C^{\#}\{M_j^{(k+1)}\}$, that $\gamma_k(t) \equiv x_{0k}$. □

Theorem 3.2. *There exists a smooth submanifold R in $\mathbb{C}^n$, diffeomorphic to $\mathbb{R}^{2n-2}$, which intersects every analytic disk in a discrete set. Moreover, R contains a smooth submanifold M, diffeomorphic to $\mathbb{R}^n$, which is not pluripolar.*

Proof. Using Lemma 2.2, we choose smooth periodic functions f_k, $g_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, n-1$, such that

$$f_k \in C^{\#}\{M_j^{(2k-1)}\} \setminus C^{\#}\{M_j^{(2k)}\},\ g_k \in C^{\#}\{M_j^{(2k)}\} \setminus C^{\#}\{M_j^{(2k+1)}\},$$

on any interval.

Suppose that in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ we have coordinates

$$z = (z_1, z_2, \dots, z_n),\ z_j = x_j + iy_j = (x_j, y_j).$$

Consider the real hypersurface

$$S = \{z \in \mathbb{C}^n :\ y_n = F(z_1, \dots, z_{n-1}, x_n)\},$$

where $F : \mathbb{R}^{2n-1} \to \mathbb{R}$ is a stricly convex real analytic function (i.e., its Hessian is positive definite at any point). We define the submanifold $R \subset S$ by the equation

$$x_n = \sum_{j=1}^{n-1} (f_j(x_j) + g_j(y_j)).$$

Clearly R is diffeomorphic to $\mathbb{R}^{2n-2}$. Suppose, for the sake of obtaining a contradiction, that $z_0 \in R$, $\phi = (\phi_1, \dots, \phi_n) : U \to \mathbb{C}^n$ is an analytic disk with $\phi(0) = z_0$, and there is a sequence $\zeta_s \to 0$ in U such that the points $\phi(\zeta_s) \in R$ are distinct. The set of $\zeta = x + iy \in U$ with $\phi(\zeta) \in S$ is defined by the equation $\Phi(x, y) = 0$, where

$$\Phi(x, y) = \mathbf{Im}\, \phi_n(\zeta) - F(\phi_1(\zeta), \dots, \phi_{n-1}(\zeta), \mathbf{Re}\, \phi_n(\zeta)).$$

The function Φ is real analytic and not identically 0, since S does not contain any analytic disk.

It follows from the Weierstrass Preparation Theorem and Theorem 3.2.5 in [KP] that there are integers k, $l > 0$ such that the solutions of either of the equations $\Phi(t^k, y) = 0$, $\Phi(-t^l, y) = 0$ in some neighborhood of the origin are graphs of finitely many real analytic functions $y = \alpha(t)$, defined in open intervals about 0, or $\{t = 0\}$ (see also [BM1]). We conclude that there exist a sequence $t_s > 0$, $t_s \to 0$, and a real analytic curve $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ defined for t in an interval about 0 (where $\gamma_1(t)$ is one of the functions t^k, $-t^l$, or 0), such that $\gamma(0) = 0$ and

$$\phi(\gamma_1(t_s) + i\gamma_2(t_s)) \in R, \ \forall\, s \geq 1.$$

By passing to a subsequence, we can assume $\zeta_s = \gamma_1(t_s) + i\gamma_2(t_s)$.

If $\psi_j(t) = \mathbf{Re}\, \phi_j(\gamma_1(t) + i\gamma_2(t))$, $\eta_j(t) = \mathbf{Im}\, \phi_j(\gamma_1(t) + i\gamma_2(t))$, for $j = 1, \ldots, n$, we have

$$\psi_n(t_s) = \sum_{j=1}^{n-1} \left(f_j(\psi_j(t_s)) + g_j(\eta_j(t_s))\right).$$

Hence the hypersurface $H \subset \mathbb{R}^{2n-1}$ defined by the equation

$$x_n = \sum_{j=1}^{n-1} (f_j(x_j) + g_j(y_j))$$

intersects the real analytic curve

$$\Psi(t) = (\psi_1(t), \eta_1(t), \ldots, \psi_{n-1}(t), \eta_{n-1}(t), \psi_n(t))$$

at the points $\Psi(t_s)$. By Theorem 3.1 the functions ψ_n, ψ_j, η_j, $1 \leq j \leq n-1$, are constant, so $\phi(\zeta_s) = (\eta, \beta + i\delta_s)$ for some $\eta \in \mathbb{C}^{n-1}$, $\beta \in \mathbb{R}$, and a real sequence δ_s. Since $\phi(\zeta_s) \in S$ we have $\phi(\zeta_s) = z_0$, a contradiction. This shows that R intersects any analytic disk in a discrete set.

We conclude the proof with the construction of a submanifold M of R with the desired properties. We choose F so that it has a minimum point at the origin and $F(0) = 0$. The functions f_j, g_j we select from Lemma 2.2 can be chosen so that $f_j(0) = g_j(0) = 0$ and $f_1'(0) = 1$, $g_1'(0) = 0$. Let $M \subset R$ be defined by $y_2 = \cdots = y_{n-1} = 0$. Clearly M is diffeomorphic to $\mathbb{R}^n$ and $0 \in M$. The tangent space T_0M is given by the equations $y_2 = \cdots = y_n = 0$, $x_n = x_1 + \sum_{j=2}^{n-1} f_j'(0)x_j$, so $T_0M \cap iT_0M = \{0\}$. Hence M is not pluripolar, since it is generating at 0 (see [P] and [Sa]). $\square$

We now proceed with the proof of Theorem 1.1 stated in the Introduction. We need the following lemma.

Lemma 3.3. *Let G and H be real valued, real analytic functions, defined in open intervals about θ_0 and 0. Assume that $k < \infty$ is the vanishing order of $G - G(\theta_0)$ at θ_0 and that $H(t_j) = G(\theta_j)$, where $t_j > 0$, $t_j \to 0$, $\theta_j \to \theta_0$. Then there is an analytic function h defined in an open interval about 0 and a subsequence $\{j_n\}$ so that $\theta_{j_n} = h(t_{j_n}^{1/k})$.*

Proof. Without loss of generality we can assume $\theta_0 = G(\theta_0) = 0$. We write $G(\theta) = \pm\theta^k G_1(\theta)$, where $G_1(0) > 0$. Since $|\theta_j G_1^{1/k}(\theta_j)| = |H(t_j)|^{1/k}$, there exists a choice of signs such that for infinitely many j

$$\theta_j G_1^{1/k}(\theta_j) = \pm(\pm H(t_j))^{1/k},$$

where $\pm H(t_j) > 0$. Let $H_1(t)$ be the analytic branch of the function $(\pm H(t^k))^{1/k}$ which is positive for $t > 0$. Then $\theta_j G_1^{1/k}(\theta_j) = \pm H_1(t_j^{1/k})$. Finally, let $h(t) = g(\pm H_1(t))$, where g is the inverse of the analytic function $\theta \to \theta G_1^{1/k}(\theta)$. □

Proof of Theorem 1.1. Let $\{a_k\}$ be any sequence such that $a_1 > 0$ and $a_1 + \cdots + a_k < a_{k+1}$ holds for all $k \geq 1$. We define inductively maps

$$T^k : \mathbb{R}^k \to \mathbb{R}^{k+1},\ T^k = (T_1^k, \ldots, T_{k+1}^k),$$

whose image is a k-dimensional torus embedded in $\mathbb{R}^{k+1}$. Let

$$T^1(\theta_1) = (T_1^1(\theta_1), T_2^1(\theta_1)) = (a_1 \sin\theta_1, a_1 \cos\theta_1).$$

Given T^k we define T^{k+1} by

$$\begin{aligned}
T_j^{k+1}(\theta_1, \ldots, \theta_k, \theta_{k+1}) &= T_j^k(\theta_1, \ldots, \theta_k), \qquad j = 1, \ldots, k,\\
T_{k+1}^{k+1}(\theta_1, \ldots, \theta_k, \theta_{k+1}) &= (T_{k+1}^k(\theta_1, \ldots, \theta_k) + a_{k+1}) \sin\theta_{k+1},\\
T_{k+2}^{k+1}(\theta_1, \ldots, \theta_k, \theta_{k+1}) &= (T_{k+1}^k(\theta_1, \ldots, \theta_k) + a_{k+1}) \cos\theta_{k+1}.
\end{aligned}$$

It follows by induction on k that $|T_j^k(\theta_1, \ldots, \theta_k)| \leq a_1 + \cdots + a_k$ for $j \leq k+1$, and that T^k is injective on $[0, 2\pi)^k$.

We use notation similar to that used in the proof of Theorem 3.2. Let $f_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, 2n-2$, be smooth periodic functions such that $f_k \in C^{\#}\{M_j^{(k)}\} \setminus C^{\#}\{M_j^{(k+1)}\}$ on any interval. Let R be the submanifold of a sphere $S \subset \mathbb{C}^n$ of sufficiently large radius r, given by

the image of the map $z = F(\theta_1, \ldots, \theta_{2n-2})$ defined by

$$x_j = T^{2n-2}_{2j-1}(\theta_1, \ldots, \theta_{2n-2}),\ y_j = T^{2n-2}_{2j}(\theta_1, \ldots, \theta_{2n-2}),\ j \leq n-1, \tag{2}$$

$$x_n = T^{2n-2}_{2n-1}(\theta_1, \ldots, \theta_{2n-2}) + \sum_{j=1}^{2n-2} f_j(\theta_j), \tag{3}$$

$$y_n = (r^2 - x_1^2 - y_1^2 - \cdots - x_{n-1}^2 - y_{n-1}^2 - x_n^2)^{1/2}. \tag{4}$$

Let X denote the image of the map $z = G(\theta_1, \ldots, \theta_{2n-2})$ defined by

$$x_j = \left(\sum_{k=1}^{2j-2} a_k \cos\theta_k \ldots \cos\theta_{2j-2} + a_{2j-1} \right) \sin\theta_{2j-1}, \tag{5}$$

$$y_j = \left(\sum_{k=1}^{2j-1} a_k \cos\theta_k \ldots \cos\theta_{2j-1} + a_{2j} \right) \sin\theta_{2j}, \tag{6}$$

$$x_n = \left(\sum_{k=1}^{2n-3} a_k \cos\theta_k \ldots \cos\theta_{2n-3} + a_{2n-2} \right) \cos\theta_{2n-2}, \tag{7}$$

$$y_n = \left(r^2 - x_1^2 - y_1^2 - \cdots - x_{n-1}^2 - y_{n-1}^2 - x_n^2 \right)^{1/2}. \tag{8}$$

Here $1 \leq j \leq n-1$ and we used the explicit formulas defining T^{2n-2}. Then X is diffeomorphic to the $(2n-2)$-dimensional torus. By choosing f_j with sufficiently small C^1-norm, the map F is a C^1-perturbation of the map G, hence R is also diffeomorphic to the $(2n-2)$-dimensional torus.

Suppose, for the sake of obtaining a contradiction, that there are $z_0 \in R$ and a non-constant analytic disk $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbb{C}^n$ with $\phi(0) = z_0$ and with the following property: there exist sequences of points $\zeta_s \neq 0$, $\zeta_s \to 0$ in U, and $\Theta_s = (\theta_{1s}, \ldots, \theta_{(2n-2)s}) \to \Theta_0 = (\theta_{10}, \ldots, \theta_{(2n-2)0})$ in $\mathbb{R}^{2n-2}$, such that $F(\Theta_s) = \phi(\zeta_s)$.

The set $\{\zeta = x + iy \in U : \phi(\zeta) \in S\}$ is defined by the equation

$$\Phi(x, y) := \sum_{j=1}^{n} |\phi_j(x + iy)|^2 - r^2 = 0.$$

The function Φ is real analytic and is not identically equal to 0, since S does not contain any analytic disk. Hence there exist a real analytic curve γ defined in an interval about 0, with $\gamma(0) = 0$, and a sequence $t_s > 0$, $t_s \to 0$, such that $\Phi(\gamma(t)) \equiv 0$ and $\gamma(t_s) = \zeta_s$ (after passing to a subsequence of $\{\zeta_s\}$, if necessary).

We let $\psi_j(t) = \mathbf{Re}\,\phi_j(\gamma(t))$, $\eta_j(t) = \mathbf{Im}\,\phi_j(\gamma(t))$, $j = 1, \ldots, n$, and we write the equations (2)-(3) corresponding to $\phi(\gamma(t_s)) = F(\Theta_s)$. Equation (2) for x_1 becomes $\psi_1(t_s) = a_1 \sin\theta_{1s}$. By Lemma 3.3, there exist an integer $N_1 \geq 1$ and a real analytic function h_1 near 0 so that $\theta_{1s} = h_1(t_s^{1/N_1})$ for a sequence of integers $s \to \infty$. Changing variables $t = u_1^{N_1}$ and letting $u_{1s} = t_s^{1/N_1}$, we obtain from equation (2) for y_1 that

$$\eta_1(u_{1s}^{N_1}) = (a_1 \cos(h_1(u_{1s})) + a_2) \sin\theta_{2s}.$$

Now Lemma 3.3 implies the existence of an integer $N_2 \geq 1$ and a real analytic function h_2 near 0 such that $\theta_{2s} = h_2(u_{1s}^{1/N_2})$ for infinitely many s. Next we change variables $t = u_1^{N_1} = u_2^{N_1 N_2}$, $u_{2s} = t_s^{1/(N_1 N_2)}$ and consider equation (2) for x_2:

$$\psi_2(u_{2s}^{N_1 N_2}) = (a_1 \cos(h_1(u_{2s}^{N_2})) \cos(h_2(u_{2s})) + a_2 \cos(h_2(u_{2s})) + a_3) \sin\theta_{3s}.$$

Continuing like this, we conclude that there are an integer $N \geq 1$, real analytic functions g_j near 0, $j = 1, \ldots, 2n-2$, and a sequence of integers $s \to \infty$, such that $\theta_{js} = g_j(u_s)$, where $u_s = t_s^{1/N}$. Using (3) we get

$$\psi_n(u_s^N) - T_{2n-1}^{2n-2}(g_1(u_s), \ldots, g_{2n-2}(u_s)) = \sum_{j=1}^{2n-2} f_j(g_j(u_s)).$$

Hence the hypersurface $H \subset \mathbb{R}^{2n-1}$ defined by the equation

$$\theta_{2n-1} = \sum_{j=1}^{2n-2} f_j(\theta_j)$$

intersects the real analytic curve

$$\Psi(u) = \big(g_1(u), \ldots, g_{2n-2}(u), \psi_n(u^N) - T_{2n-1}^{2n-2}(g_1(u), \ldots, g_{2n-2}(u))\big)$$

at the points $\Psi(u_s)$. By Theorem 3.1 all the functions g_j are identically constant; thus $F(\Theta_s) = \phi(\zeta_s) = z_0$, a contradiction. Hence R intersects any analytic disk in a discrete set.

We proceed now with the construction of a submanifold M of R with the desired properties. Recall that R was defined as the image of the map F in (2)-(4), which is a C^1-perturbation of the map G given by (5)-(8). Let $M \subset R$ and $Y \subset X$ be the submanifolds defined by $x_1 = \cdots = x_{n-2} = 0$, i.e., by taking $\theta_{2j-1} = 0$, $j = 1, \ldots, n-2$, in the formulas (2)-(4) and in (5)-(8). Since Y is diffeomorphic to the n-torus, so is M. We check that Y is generating at the point P corresponding to $\theta_{2k} = 0$, $k = 1, \ldots, n-2$, $\theta_{2n-3} = 0$ and $\theta_{2n-2} = \pi/2$. Indeed, P has $\mathbb{R}^{2n}$-coordinates $P = (0, \ldots, 0, a, 0, b)$ for some a, b, and the tangent space $T_P Y$ is given by $x_1 = \cdots = x_{n-2} = y_{n-1} = y_n = 0$. We conclude

that M is generating at the point P' corresponding to the same values of parameters, so that M is not pluripolar (see [P], [Sa]). $\square$

Remark. Let $D \subset \mathbb{C}^n$ be any bounded pseudoconvex domain with real analytic boundary. By [DF] the boundary of D does not contain any non-constant analytic disk. The construction in the proof of Theorem 1.1 shows the existence of smooth nonpluripolar compact submanifolds M and R of ∂D which are diffeomorphic to an n-dimensional torus and a $(2n-2)$-dimensional torus, and which intersect any analytic disk in a discrete set.

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