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Dan Coman
Syracuse University

Evgeny A. Poletsky
Syracuse University

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TRANSCENDENCE MEASURES AND ALGEBRAIC GROWTH OF ENTIRE FUNCTIONS

DAN COMAN AND EVGENY A. POLETSKY

ABSTRACT. In this paper we obtain estimates for certain transcendence measures of an entire function f . Using these estimates, we prove Bernstein, doubling and Markov inequalities for a polynomial $P(z, w)$ in \mathbb{C}^2 along the graph of f . These inequalities provide, in turn, estimates for the number of zeros of the function $P(z, f(z))$ in the disk of radius r , in terms of the degree of P and of r .

Our estimates hold for arbitrary entire functions f of finite order, and for a subsequence $\{n_j\}$ of degrees of polynomials. But for special classes of functions, including the Riemann ζ -function, they hold for all degrees and are asymptotically best possible. From this theory we derive lower estimates for a certain algebraic measure of a set of values $f(E)$, in terms of the size of the set E .

1. INTRODUCTION

In recent years there was a significant interest in the behavior of a polynomial P along an algebraic subvariety X of \mathbb{R}^n or \mathbb{C}^n . This started with the paper [FN] of Fefferman and Narasimhan, where they obtained local *doubling inequalities*, which bound the ratio of the uniform norms of P on two concentric balls in X , in terms of the degrees of P and X , and of the ratio of the radii of these balls.

Later, these inequalities were improved in papers of Brudnyi [Br] and Roytwarf and Yomdin [RY], and they were applied to questions from analytic geometry, pseudodifferential operators, to Hilbert's 16th problem, and so on.

Much earlier, Tijdeman [Ti1] studied the behavior of a polynomial $P(z, w)$ in \mathbb{C}^2 along the graph of the exponential function $w = e^z$. In this situation, he obtained global doubling inequalities and estimates for the number of zeros of the function $P(z, e^z)$ in a disk of radius r . He used these results in [Ti2] to get new advancements in transcendental number theory. The proofs in [Ti2] involved transcendence measures

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of numbers, which were studied extensively in transcendental number theory.

Transcendence measures appear quite naturally when transcendental objects are investigated. In general, if B is a subring of a commutative ring A , then an element $\omega \in A$ is called *transcendental* over B if $P(\omega) \neq 0$, for any non-trivial polynomial $P \in B[x]$. For example, if $A = \mathbb{C}$ and $B = \mathbb{Z}$ we get the transcendental numbers, and if A is the ring of entire functions and $\mathbb{C}[z]$ the ring of polynomials in \mathbb{C} , we get the entire transcendental functions.

If A and B are normed rings and the algebra $B[x]$ is graded, i.e., there is an increasing sequence of sets $B_n[x]$ such that $\bigcup_{n \geq 0} B_n[x] = B[x]$, then we can measure the transcendence of ω . For this, we define a suitable norm $h(P)$ of $P \in B[x]$, and let the *transcendence measure* $\tau(\omega, n, H)$ of ω be the infimum of $\|P(\omega)\|$ over all polynomials $P \in B_n[x]$ with $1 \leq h(P) \leq H$.

In our papers [CP1] and [CP2], we started to study *transcendence measures* of an entire function f . A transcendence measure can be defined by

$$E_n(f) = \sup\{\|P\|_{\Delta^2} : P \in \mathbb{C}[z, w], \deg P \leq n, \|P(z, f(z))\|_{\Delta} \leq 1\}.$$

Here Δ is the closed unit disk in \mathbb{C} and the norms are uniform norms. Since f is usually fixed, we write $E_n = E_n(f)$, and let $e_n = \log E_n$. In [CP1] we proved that

$$e_n(e^z) = \frac{1}{2} n^2 \log n + O(n^2).$$

This transcendence measure is closely connected with the following aspects of analysis and geometry:

1) Polynomial estimates on \mathbb{C}^2 : if $P(z, w)$ is a polynomial of degree n and $|P(z, f(z))| \leq 1$ on the unit disk Δ , then

$$|P(z, w)| \leq E_n \exp(n \max\{\log^+ |z|, \log^+ |w|\});$$

2) Polynomial estimates along the graph of f : if

$$m(r) = m(r, f) = \max\{\log^+ |f(z)| : |z| = r\},$$

$P_f(z) = P(z, f(z))$, $\Delta_r = \{z \in \mathbb{C} : |z| \leq r\}$ and

$$m_n(r) = m_n(r, f) = \sup\{\log \|P_f\|_{\Delta_r} : \deg P \leq n, \|P_f\|_{\Delta} \leq 1\},$$

then

$$m_n(r) \leq \frac{2e_n}{\log t_n} \log r, \quad 1 \leq r \leq t_n,$$

where t_n is defined by $nm(t_n) = e_n$. The functions $m_n(r, f)$, $r > 1$, can also be considered as transcendence measures of f , by using $\|P_f\|_{\Delta_r}$ as the norm of a polynomial $P(z, w)$.

3) Estimates on the number of zeros: if $Z_n(r) = Z_n(r, f)$ is the maximum number of zeros of the function P_f in the disk Δ_r when $\deg P \leq n$, then $Z_n(r) \leq 2m_n(3r)$. The number $Z_n(r)$ gives the maximum number of intersection points of an algebraic variety of degree n with the graph of f in \mathbb{C}^2 lying over Δ_r .

These connections were proved in [CP2], where we also found an approach to estimate e_n for general transcendental functions. It allowed us to handle, in particular, the class of functions $f(z) = e^{P(z)}$, where P is a polynomial.

For any transcendental function f one has [CP1, Proposition 1.3]

$$m_n(r, f) \geq \frac{n^2 + 3n}{2} \log r, \quad r \geq 1.$$

Using the transcendence measure $m_n(e, f)$, we define the *lower order of transcendence* as

$$\underline{\tau}(f) = \sup \left\{ \tau : \liminf_{n \rightarrow \infty} \frac{m_n(e, f)}{n^\tau} > 0 \right\},$$

and the *upper order of transcendence* as

$$\overline{\tau}(f) = \inf \left\{ \tau : \limsup_{n \rightarrow \infty} \frac{m_n(e, f)}{n^\tau} < \infty \right\}.$$

Since $m_n(e, f) \geq n^2/2$, we have $\underline{\tau}(f) \geq 2$. If $f(z) = e^z$ then $\lim_{n \rightarrow \infty} m_n(e)/n^2 = 1/2$ [CP1, Theorem 1.2]. More generally, if $f(z) = e^{P(z)}$, for some polynomial P , then $\underline{\tau}(f) = \overline{\tau}(f) = 2$ [CP2, Theorem 5.1]. For $\tau \geq 3$, we constructed examples of entire functions of order 1 and type $1/e$ with $\tau - 1 \leq \overline{\tau}(f) \leq \tau$ [CP2, Corollary 6.2]. In all these examples, $\underline{\tau}(f) = 2$. Whether this was true in general remained unsettled until the present paper.

The approach in [CP2] was based on estimates of $e_n(f)$ in terms of the n -th diameter of the set of preimages of a point on the unit circle. The n -th diameter $d_n(F)$ of a set F is the minimal sum of radii of n disks covering F . In [So], using the theory of Dufresnoy, Sodin gave lower bounds for the smallest number of disks of radius R^α , $\alpha < 1$, needed to cover the set $f^{-1}(\{0, 1\}) \cap \Delta_R$, when f is a function of finite positive order ρ . Applied to our problem, his result leads to only exponential estimates for e_n . In Section 3, using the Ahlfors theory of covering surfaces and certain results of Dufresnoy, we obtain the necessary estimates for the n -th diameter. The results we need from these theories are recalled in Section 2.

The estimates for the n -th diameter allow us to obtain several results, which can be summarized in the following theorem. In this theorem, the second inequality is usually called a *Bernstein* inequality, the third – a *Bezout* inequality, the fourth – a *doubling* inequality, and the fifth – a *Markov* inequality. Bernstein and Markov inequalities have been extensively studied and have wide applications, for example in approximation theory (see e.g. [BBLT] and references therein).

Theorem 1.1. *For any entire function f of finite order $\rho > 0$, there exist sequences of integers $\{n_j\}$ and $\epsilon_j > 0$, $\epsilon_j \rightarrow 0$, such that*

$$e_{n_j} \leq C_1 n_j^2 \log n_j, \quad m_{n_j}(r) \leq C_2 n_j^2 \log r, \quad 1 \leq r \leq \frac{1}{2} n_j^{1/\rho - \epsilon_j}.$$

For every $r \geq 1$ there exists an integer j_r such that if $j \geq j_r$ then

$$Z_{n_j}(r) \leq C_3 n_j^2, \quad \frac{M(2r, P_f)}{M(r, P_f)} \leq 2^{an_j^2}, \quad M(r, P'_f) \leq C_4 n_j^2 \frac{M(r, P_f)}{r},$$

where P is a polynomial of degree at most n_j .

Here $M(r, F) = \max\{|F(z)| : |z| = r\}$ and the constants are effectively computed and depend only on ρ . A sequence of integers $\{n_j\}$ for which the above theorem holds will be called a *fundamental sequence* for f . It follows from this theorem that $\underline{\tau}(f) = 2$ for all entire functions of finite positive order.

Theorem 1.1 is proved in Sections 4, 5 and 6, where we also show that for entire functions with a covering system of admissible intervals $I(R, \alpha, \beta, \gamma, C)$ (see Section 4), the inequalities in Theorem 1.1 hold for all n sufficiently large. Again, all constants are effectively computed. The only change is that one should substitute $n^{1+1/\gamma}$ instead of n_j^2 .

In Section 7 we give three sufficient criteria for classes of functions to have a covering system of admissible intervals $I(R, \alpha, \beta, \gamma, C)$. The first one states that if $A_1 m(r, f) \leq m(kr, f) \leq A_2 m(r, f)$, for some constants $A_1, A_2, k > 1$, then the function f has a covering system of admissible intervals $I(R, \alpha, \beta, 1, C)$. This class includes all functions $f(z) = \sum_{j=1}^m p_j(z) e^{q_j(z)}$, where p_j and q_j are polynomials, and, as shown in Section 8, the Riemann ζ -function and the function ξ . It follows that for such functions Theorem 1.1 holds for all n sufficiently large.

The second criterion can be applied when we know that $m(r, f) \leq r^{\phi(r)}$ and $r^{\phi(r)-\rho}$ is a slowly increasing function (see Theorem 7.3). Finally, Corollary 7.4 gives a criterion based on the behavior of the Taylor coefficients of f , similar to the formulas for the order and type of f .

In Section 9 we introduce and study an *extremal function* $W^*(z)$, related to Bernstein inequalities, and we prove that $W^*(z) = \frac{1}{2} \log^+ |z|$ when $f(z) = e^z$.

In Section 11 we address a problem posed by Mahler in [M]: given an entire transcendental function f , describe, or at least find properties of, the set of algebraic numbers where the values of f are also algebraic. There are many results claiming that this set is finite when either f is a special function, or when *all* the derivatives of f take algebraic values on this set and their algebraic measure satisfies some growth conditions (see, e.g., [Sc], [St], [La], [W]). But a general entire function may take algebraic values on any set of algebraic numbers (see [M] and [GS]), in particular, on any algebraic number field K of degree σ . So it is interesting to look at the *algebraic growth characteristic* $\mathbf{a}_K(s, r, m)$ of f , defined as the smallest algebraic measure of the first m derivatives of f on sets $E \subset \Delta_r \cap K$ with $|E| \geq s$ (see Section 11). The following theorem, proved in Section 11, gives lower bounds for this characteristic.

Theorem 1.2. *If f is an entire function of finite positive order then*

$$\limsup_{s \rightarrow \infty} \frac{\mathbf{a}_K(s, r, m)}{s^{1/2} \log s} \geq Cm^{1/2}.$$

If f has a covering system of admissible intervals $I(R, \alpha, \beta, \gamma, C)$, then for all s sufficiently large

$$\mathbf{a}_K(s, r, m) \geq c(ms)^{1/\tau} \log \frac{ms}{a} - C_K, \quad \tau = 1 + \frac{1}{\gamma}.$$

Let $I_K(A)$ be the set of algebraic integers in K whose algebraic measure does not exceed A . Then it is possible that $f(I_K(A))$ lies in some $I_K(B)$, like in the theorems of Polya and Gelfond (see [GS]), where $K = \mathbb{Q}$ or $\mathbb{Q}[i]$. Of course, B can be large, simply due to the growth of f . However, if $m(r, f) \leq r^{\phi(r)}$ we prove in Section 11 the following theorem:

Theorem 1.3. *If f is an entire function of order $0 < \rho < \sigma/2$ then*

$$\liminf_{A \rightarrow \infty} \frac{|I_K(A) \cap f^{-1}(I_K(\exp A^{\phi(A)}))|}{|I_K(A)|} = 0.$$

This theorem tells us that, with probability close to 1, the algebraic measure of $f(z)$ for $z \in I_K(A)$ growth faster than f .

To prove these and other theorems, we combine our Bezout inequalities with the standard machinery based on Siegel's lemma. This is developed in Section 10 and gives lower bounds for the algebraic measure of arguments and values of f on a set $E \subset K$.

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2. CHARACTERISTICS OF ENTIRE FUNCTIONS

For an entire function f on \mathbb{C} we let

$$u_f(z) = \frac{1}{2} \log(1 + |f|^2).$$

This is a subharmonic function with Laplacian

$$\Delta u_f(z) = 2\rho_f^2 = \frac{2|f'|^2}{(1 + |f|^2)^2},$$

where ρ_f is the absolute value of the spherical derivative of f .

For a domain $D \subset \mathbb{C}$ with piecewise analytic boundary let

$$L(D) = L_f(D) = 2 \int_{\partial D} \rho_f |dz|,$$

$$S(D) = S_f(D) = \frac{1}{\pi} \int_D \rho_f^2 d\lambda = \frac{1}{2\pi} \int_D \Delta u_f,$$

where λ is the Lebesgue measure on \mathbb{C} . If $D = \Delta_r$ then

$$L(D) = L(r) = 2 \int_{|z|=r} \rho_f |dz|, \quad S(D) = S(r) = \frac{1}{\pi} \int_{|z|\leq r} \rho_f^2 d\lambda$$

The following result of Ahlfors, with improvements by Dufresnoy, can be found in Chapters 5 and 6 of [H] and [D, Theorem A₁, p. 190]:

Theorem 2.1. *Let f be an entire holomorphic function and let D be a domain in \mathbb{C} with piecewise analytic boundary and with Euler–Poincaré characteristic χ . If f does not assume in D the values $a \neq b$, where $|a| = |b| = 1$, then*

$$S(D) \leq \chi + 1 + \frac{3}{2\delta_0} L(D),$$

where δ_0 is the spherical distance between a and b . Moreover, if $z \in D$ and $\text{dist}(z, \partial D) = r$, then

$$\rho_f(z) \leq \frac{e^{36\pi^2/\delta_0^2}}{r}.$$

Here the Euler characteristic equals -2 for the sphere, -1 for the disk, and $\chi \geq 0$ for multiply connected domains.

The function

$$T_0(r) = T_0(r, f) = \int_0^r \frac{S(t)}{t} dt$$

is called the Ahlfors–Shimizu characteristic of f . If

$$m_0(r) = m_0(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + |f(re^{i\theta})|^2} d\theta,$$

then (see [H, §1.5])

$$T_0(r) = m_0(r) - \log \sqrt{1 + |f(0)|^2}.$$

We let

$$M(r) = M(r, f) = \max\{|f(z)| : |z| = r\},$$

$$m(r) = m(r, f) = \log^+ M(r, f),$$

$$T(r) = T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta.$$

By [H, Theorem 1.6] and [H, p. 13] we have

$$(1) \quad T(r) \leq m(r) \leq \frac{R+r}{R-r} T(R), \quad 0 \leq r < R,$$

$$(2) \quad |T(r) - T_0(r) - \log^+ |f(0)|| \leq \frac{\log 2}{2}.$$

The following relations between $L(r)$, $S(r)$ and $T_0(r)$ will be important in the sequel. If $k > 1$ then

$$(3) \quad T_0(kr) \geq \int_r^{kr} \frac{S(t)}{t} dt \geq S(r) \log k.$$

Moreover, Hölder's inequality implies for all r

$$(4) \quad L^2(r) \leq 8\pi^2 r S'(r).$$

Note that if the function $S(r)$ is bounded then f is a polynomial. Hence if f is transcendental, $\epsilon > 0$, $k > 1$, we can define $r_0 = r_0(f, \epsilon, k)$ by

$$S(r_0) = \frac{8\pi^2}{\epsilon^2 \log k}.$$

Lemma 2.2. *If $r \geq r_0$ then there exists $r' \in (r, kr)$ so that $L(r') \leq \epsilon S(r')$.*

Proof. Assuming that $L(t) > \epsilon S(t)$ for $t \in (r, kr)$, we have by (4) $\epsilon^2 S^2(t) < 8\pi^2 t S'(t)$. Hence

$$\frac{\epsilon^2 \log k}{8\pi^2} = \frac{\epsilon^2}{8\pi^2} \int_r^{kr} \frac{dt}{t} < \int_r^{kr} \frac{S'(t)}{S^2(t)} \leq \frac{1}{S(r_0)},$$

a contradiction. □

We will need the following facts about functions of finite order. Recall that (see [Le, Th.I.16]) if $\theta(r)$, $r > 0$, is a positive function with

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log \theta(r)}{\log r} < \infty,$$

then θ has a *proximate* order $\rho(r)$ with the following properties:

- (i) $\lim_{r \rightarrow \infty} \rho(r) = \rho$;
- (ii) $\theta(r) \leq r^{\rho(r)}$, and $\theta(r_n) = r_n^{\rho(r_n)}$ for some sequence $r_n \rightarrow \infty$;
- (iii) the function $\psi(r) = r^{\rho(r)-\rho}$ is *slowly increasing*, i.e.,

$$\lim_{r \rightarrow \infty} \frac{\psi(kr)}{\psi(r)} = 1$$

uniformly on each interval $0 < a \leq k \leq b < \infty$. If $r^{\rho(r)-\rho}$ is a slowly increasing function, then for every $\epsilon > 0$ and every $0 < a < b < \infty$ there is r_0 such that

$$(5) \quad (1 - \epsilon)k^\rho r^{\rho(r)} < (kr)^{\rho(kr)} < (1 + \epsilon)k^\rho r^{\rho(r)},$$

for $a \leq k \leq b$ and $r \geq r_0$.

3. ESTIMATES FOR THE n -TH DIAMETER

We will need the following lemma:

Lemma 3.1. *Let u be a non-negative upper bounded subharmonic function in the disk Δ_R . If $R' = R/2$ and $\Gamma = \Delta_{R'} \cap \Delta(a, r)$, where $r < 3R/4$, then*

$$\frac{1}{2\pi} \int_{\Gamma} \Delta u \leq \frac{m_0}{\log \frac{3R}{4r}},$$

where $m_0 = \sup\{u(z) : |z| < R\}$.

Proof. If $\Gamma \neq \emptyset$, there exists $b \in \Delta_{R'}$ such that $\Gamma \subset \Delta(b, r)$. Since

$$0 \leq u(b) \leq m_0 + \frac{1}{2\pi} \int_{\Delta_R} \log \left| \frac{R(z-b)}{R^2 - \bar{b}z} \right| \Delta u(z)$$

and

$$\left| \frac{R(z-b)}{R^2 - \bar{b}z} \right| \leq \frac{rR}{R^2 - R'^2} = \frac{4r}{3R}$$

for $z \in \Gamma$, we obtain

$$-m_0 \leq \frac{\log \frac{4r}{3R}}{2\pi} \int_{\Gamma} \Delta u(z).$$

□

We introduce the constants

$$\Lambda(\delta_0) = \left(4 + \frac{48\pi e^{36\pi^2/\delta_0^2}}{\delta_0} \right)^{-1}, \quad \Lambda = \Lambda(1).$$

The following theorem shows that the set of preimages of two points cannot be covered by a limited number of disks of small radius.

Theorem 3.2. *Let f be an entire transcendental function and let $a, b \in \mathbb{C}$, $|a| = |b| = 1$, and δ_0 be the spherical distance between a and b . If $L(R) \leq \delta_0 S(R)/6$ and if the set*

$$E = \{z \in \mathbb{C} : |z| \leq R + r, z \in f^{-1}(\{a, b\})\}$$

is covered by $n \leq \Lambda(\delta_0)S(R)$ disks of radius r , then

$$\log \frac{3R}{4r} \leq 4n \frac{m(2R)}{S(R)}.$$

Proof. We assume at first that $3nr > 2e^{-2}R$, so

$$\log \frac{3R}{4r} < 2 + \log \frac{9}{8} + \log n.$$

Using (3) and (2) we get

$$(6) \quad S(R) \leq \frac{T_0(2R)}{\log 2} \leq \frac{m(2R)}{\log 2} + \frac{1}{2},$$

so

$$4n \frac{m(2R)}{S(R)} \geq 4n \log 2 \left(1 - \frac{1}{2S(R)}\right).$$

Since $n \leq \Lambda(\delta_0)S(R) \leq S(R)/4$, we obtain

$$4n \frac{m(2R)}{S(R)} \geq 4n \log 2 \left(1 - \frac{1}{8n}\right) \geq 2 + \log \frac{9}{8} + \log n,$$

for all $n \geq 1$. This proves Theorem 3.2 in the case $3nr > 2e^{-2}R$.

We assume in the remainder of the proof that $3nr \leq 2e^{-2}R$. Suppose that the set E can be covered by n disks $\Delta(a_j, r)$ such that $nr = d$. We claim that there are disjoint disks $\Delta(b_j, t_j)$, $1 \leq j \leq k \leq n$, whose union contains all disks $\Delta(a_j, 2r)$ and so that $\sum t_j \leq 2d$. For this, we note that if the disks $\Delta(a_1, 2r_1)$ and $\Delta(a_2, 2r_2)$ are not disjoint, then there is a point z such that the disk $\Delta(z, 2(r_1 + r_2))$ contains both these disks. Now a simple induction proves our claim.

We consider those disks $F_j = \Delta(b_j, t_j)$, $j = 1, \dots, l$ which intersect Δ_R . Let $\Gamma_j = F_j \cap \Delta_R$ and let $D = \Delta_R \setminus \cup_{j=1}^l F_j$. It follows that

$$\Delta_R = D \cup \bigcup_{j=1}^l \Gamma_j,$$

thus

$$S(R) = S(D) + \sum_{j=1}^l S(\Gamma_j).$$

By (6) and the assumption that $S(R) \geq \Lambda^{-1}(\delta_0)$, it follows that $m(2R) \geq \sqrt{2}$. Since $\log(1+x^2) \leq 4 \log x$ when $x \geq \sqrt{2}$, we get by Lemma 3.1 (with $u = \log \sqrt{1+|f|^2}$)

$$S(\Gamma_j) \leq 2 \frac{m(2R)}{\log \frac{3R}{2t_j}}.$$

Hence

$$\sum_{j=1}^l S(\Gamma_j) \leq 2m(2R) \sum_{j=1}^l \frac{1}{\log \frac{3R}{2t_j}}.$$

Since the sum of $2t_j/(3R)$ does not exceed $4d/(3R) \leq e^{-2}$ and the function $-1/\log x$ is concave on $(0, e^{-2})$ we conclude that

$$\sum_{j=1}^l \frac{1}{\log \frac{3R}{2t_j}} \leq \frac{l}{\log \frac{3Rl}{4d}}.$$

As the function $x/\log ax$ is increasing when $x > e/a$ we have

$$\frac{l}{\log \frac{3Rl}{4d}} \leq \frac{n}{\log \frac{3R}{4r}}.$$

Thus

$$\sum_{j=1}^l S(\Gamma_j) \leq \frac{2nm(2R)}{\log \frac{3R}{4r}}.$$

Note that the Euler characteristic χ_0 of D verifies $\chi_0 \leq n-1$, since the domain D is bounded by at most $n+1$ Jordan curves. Moreover, we have

$$L(\partial D) \leq L(R) + \sum_{j=1}^l L(\gamma_j),$$

where $\gamma_j = \Delta_R \cap \partial \Delta(b_j, t_j)$. Thus Theorem 2.1 implies that

$$S(D) \leq n + hL(R) + h \sum_{j=1}^l L(\gamma_j),$$

where $h = 3/(2\delta_0)$. If $z \in \gamma_j$, then f does not take the values a and b in the disc $\Delta(z, r)$, so by Theorem 2.1 $\rho_f(z) \leq h_1/r$, where $h_1 = e^{36\pi^2/\delta_0^2}$. Hence $L(\gamma_j) \leq 4\pi h_1 t_j/r$ and

$$\sum_{j=1}^l L(\gamma_j) \leq \frac{8\pi h_1 d}{r} = 8\pi h_1 n.$$

We conclude that $S(D) \leq (1 + 8\pi h h_1)n + hL(R)$, so

$$S(R) \leq (1 + 8\pi h h_1)n + hL(R) + \frac{2nm(2R)}{\log \frac{3R}{4r}}.$$

If $n \leq \Lambda(\delta_0)S(R)$ and $L(R) \leq \delta_0 S(R)/6$, then

$$\log \frac{3R}{4r} \leq \frac{4nm(2R)}{S(R)}.$$

□

For a set $G \subset \mathbb{C}$ and an integer $n \geq 1$ we introduced in [CP2] the n -th diameter of G as

$$\text{diam}_n(G) = \inf \left\{ r_1 + \cdots + r_k : k \leq n, G \subset \bigcup_{j=1}^k C_j(r_j) \right\},$$

where $C_j(r_j)$ are closed disks of radii $r_j > 0$.

Given a non-constant entire function f we denote in the sequel by $n_0 = n_0(f)$ the maximum of the numbers $|f^{-1}(w) \cap \Delta_2|$ when $|w| = 1$.

Corollary 3.3. *In the assumptions of Theorem 3.2, let*

$$F = \{z \in \mathbb{C} : 2 \leq |z| \leq R + 1, z \in f^{-1}(\{a, b\})\}.$$

If $L(R) \leq \delta_0 S(R)/6$, $n \leq \Lambda(\delta_0)S(R) - 2n_0$, and $d_n = \text{diam}_n(F) < 1$, then

$$\log \frac{R}{d_n} \leq 4(n + 2n_0) \frac{m(2R)}{S(R)} + \log \frac{4}{3}.$$

Proof. If $\epsilon > 0$ and $d_n + \epsilon < 1$, we can cover F by n disks of radius $d_n + \epsilon$. The number of points of $f^{-1}(\{a, b\}) \cap \Delta_2$ does not exceed $2n_0$. We cover them with $2n_0$ disks of radius $d_n + \epsilon$. Since $d_n + \epsilon < 1$, we apply Theorem 3.2 and then let $\epsilon \rightarrow 0$. □

Let

$$D_n(\theta, r) = \{z \in \mathbb{C} : 2 \leq |z| \leq r, f(z) = e^{i\theta}\}$$

and

$$d_n(\theta, r) = \min\{1, \text{diam}_n(D_n(\theta, r))\}.$$

Corollary 3.4. *Let f be an entire transcendental function. If $L(R) \leq S(R)/6$ and $n \leq \frac{1}{2} \Lambda S(R) - n_0$, then*

$$\log \frac{R}{d_n(\theta, R+1)} \leq \max \left\{ 8(n + n_0) \frac{m(2R)}{S(R)} + \log 3, \log(2R) \right\}$$

for all $e^{i\theta}$ in an arc of length $l > \pi$ in $\partial\Delta$.

Proof. Suppose that the spherical distance between $a = e^{i\phi}$ and $b = e^{i\psi}$ is at least $\delta_0 = 1$. Let $m = 2n$. If F is as in Corollary 3.3, then

$$\text{diam}_m(F) \leq d_n(\phi, R+1) + d_n(\psi, R+1).$$

Hence if $\text{diam}_m(F) < 1$ we have by Corollary 3.3

$$-\log \left(\frac{d_n(\phi, R+1)}{R} + \frac{d_n(\psi, R+1)}{R} \right) \leq 4(m+2n_0) \frac{m(2R)}{S(R)} + \log \frac{4}{3}.$$

If $0 < \alpha \leq \beta$, then $\log(\alpha + \beta) \leq \log \beta + \log 2$. Thus

$$\log \frac{R}{\max\{d_n(\phi, R+1), d_n(\psi, R+1)\}} \leq 8(n+n_0) \frac{m(2R)}{S(R)} + \log 3.$$

If $\text{diam}_m(F) \geq 1$ then

$$\max\{d_n(\phi, R+1), d_n(\psi, R+1)\} \geq 1/2.$$

Consequently, if the estimate in the statement of the corollary fails for some $e^{i\phi}$, then it must hold for all $e^{i\psi}$ lying at spherical distance at least 1 from $e^{i\phi}$. Since the set of such $e^{i\psi}$ is an arc of length greater than π , the corollary follows. \square

4. GENERAL ESTIMATES FOR e_n AND $m_n(r)$

Let f be an entire transcendental function and recall that $n_0 = n_0(f)$ is the maximum of the numbers $|f^{-1}(w) \cap \Delta_2|$, when $|w| = 1$. In the following lemma, the estimates on the n -th diameter obtained in the previous section, combined with results from [CP2], lead to estimates of the transcendence measures e_n and $m_n(r)$ in terms of $m(r)$.

Lemma 4.1. *Let $R_0 = R_0(f)$ be the largest among the unique solutions of the equations:*

$$R = 64, \quad S(R) = \frac{288\pi^2}{\log(4/3)}, \quad m(R) = 4 \log^+ R, \quad m(4R) = 36.$$

If $R > R_0$ and $n \leq \frac{1}{2} \Lambda S(R) - n_0$, then

$$\begin{aligned} e_n &\leq 2nm(4R) \log R, \\ m_n(r) &\leq 3nm(4R) \log r, \quad 1 \leq r \leq R. \end{aligned}$$

Proof. Using Lemma 2.2 with $k = 4/3$ and $\epsilon = 1/6$ we can find, for all $R > R_0$, a radius $R' \in (R, 4R/3)$ so that $L(R') \leq S(R')/6$. Since $n + n_0 \leq \frac{1}{2} \Lambda S(R)$, we have by Corollary 3.4 with $r = R' + 1$, that for $e^{i\theta}$ in a set of length $l > \pi$ in $\partial\Delta$

$$\log \frac{R'}{d_n(\theta, r)} \leq \max \{4\Lambda m(2r) + \log 3, \log(2r)\}.$$

Since $er < 3R'$, the latter inequality implies

$$\log \frac{36er}{d_n(\theta, r)} \leq 5 + \max \{4\Lambda m(2r) + \log 3, \log(2r)\}.$$

Theorem 4.2 in [CP2] asserts that if for some $r \geq 2$ one has $d_n(\theta, r) \geq a$ on a set $E \subset \partial\Delta$ of length l , then

$$e_n \leq n \max\{m(er), \log(er)\} \log r + n \log(er) \left(\log \frac{36er}{a} + \frac{4\pi}{l} \right).$$

Suppose that $4\Lambda m(2r) + \log 3 \geq \log(2r)$. Since $m(er) \geq \log(er)$, the above estimate yields

$$e_n \leq nm(er) \log r + n \log(er)(11 + 4\Lambda m(2r)).$$

Since $er < 4R$, $R > 64$, $m(4R) > 36$ and $\Lambda < e^{-300}$ we have

$$\begin{aligned} e_n &\leq \left(1 + 4\Lambda + \frac{11}{m(4R)} \right) nm(4R) \log(4R) \\ &\leq \frac{4}{3} \left(1 + 4\Lambda + \frac{11}{36} \right) nm(4R) \log R < 2nm(4R) \log R. \end{aligned}$$

If $4\Lambda m(2r) + \log 3 < \log(2r)$, then using in addition that $4 \log(4R) \leq m(4R)$, we get

$$\begin{aligned} e_n &\leq nm(er) \log r + n \log(er)(9 + \log(2r)) \\ &\leq \left(1 + \frac{9}{m(4R)} + \frac{1}{4} \right) nm(4R) \log(4R) \leq 2nm(4R) \log R. \end{aligned}$$

Thus $e_n \leq 2nm(4R) \log R$. By [CP2, §4 (5)] we have for $1 \leq r \leq R$

$$m_n(r) \leq \frac{e_n + nm(R)}{\log R} \log r \leq 3nm(4R) \log r.$$

□

The above lemma shows that estimates for e_n and m_n require knowledge of the relationship between $m(4R)$ and $S(R)$. The following theorem shows the kind of hypotheses on $m(4R)$ and $S(R)$ needed to get good estimates on e_n and m_n .

We denote by $R_1(f)$ the maximum of $R_0(f)$ and the solution of the equation

$$T_0(r) = (3 \log 2)/2 + 3 \log^+ |f(0)|.$$

We call an interval

$$I(R, \alpha, \beta, \gamma, C) = \left[\beta S^\gamma(R), \frac{1}{2} \Lambda S(R) - n_0 \right]$$

admissible if $R > R_1(f)$, $\alpha, \beta > 0$, $0 < \gamma \leq 1$, $\beta S^\gamma(R) \leq \frac{1}{2} \Lambda S(R) - n_0 - 1$, $S(R) \geq R^\alpha$ and $m(4R) \leq CS(R)$. We let $I(\alpha, \beta, \gamma, C)$ be the union of all admissible intervals $I(R, \alpha, \beta, \gamma, C)$.

Theorem 4.2. *If $n \in I(\alpha, \beta, \gamma, C)$, then*

$$e_n \leq \frac{2C}{\alpha\gamma\beta^{1/\gamma}} n^{1+1/\gamma} \log \frac{n}{\beta}.$$

If R is so that $n \in I(R, \alpha, \beta, \gamma, C)$, then $2\Lambda m(2R) \geq n$ and

$$m_n(r) \leq \frac{3C}{\beta^{1/\gamma}} n^{1+1/\gamma} \log r, \quad 1 \leq r \leq R.$$

Proof. By the properties of admissible intervals we have

$$m(4R) \leq CS(R) \leq C \left(\frac{n}{\beta} \right)^{1/\gamma}$$

and

$$R \leq S^{1/\alpha}(R) \leq \left(\frac{n}{\beta} \right)^{1/(\alpha\gamma)}.$$

By Lemma 4.1

$$e_n \leq \frac{2C}{\alpha\gamma\beta^{1/\gamma}} n^{1+1/\gamma} \log \frac{n}{\beta}$$

and

$$m_n(r) \leq \frac{3C}{\beta^{1/\gamma}} n^{1+1/\gamma} \log r, \quad 1 \leq r \leq R.$$

By (1) and (2) we have

$$T_0(r) - \frac{\log 2}{2} \leq m(r) \leq 3T_0(2r) + \frac{3 \log 2}{2} + 3 \log^+ |f(0)|.$$

So if $r \geq R_1(f)$ then

$$(7) \quad \frac{1}{2} T_0(r) \leq m(r) \leq 4T_0(2r).$$

By (3) $S(R) \log 2 \leq T_0(2R)$, so

$$(8) \quad S(R) \leq \frac{2m(2R)}{\log 2}.$$

Therefore $n \leq \Lambda S(R)/2 \leq 2\Lambda m(2R)$, and the proof is complete. \square

The following corollary establishes a case when polynomial estimates for e_n hold for all n .

Corollary 4.3. *Suppose that there is a sequence of admissible intervals $I(R_j, \alpha, \beta, \gamma, C)$ such that $R_j \rightarrow \infty$ and $\beta S^\gamma(R_{j+1}) \leq \Lambda S(R_j)/2 - n_0$, $j \geq 1$. Then the conclusions of Theorem 4.2 hold for all $n \geq \beta S^\gamma(R_1)$.*

A system of admissible intervals satisfying the hypotheses of this corollary will be called a *covering system*.

5. THE LOWER ORDER OF TRANSCENDENCE

In order to apply Theorem 4.2 effectively we need information on the set $I(\alpha, \beta, \gamma, C)$. The main goal of this section is to establish that for every entire function of finite positive order we can find α , β and C such that the set $I(\alpha, \beta, 1, C)$ is unbounded. Then Theorem 4.2 will imply that the lower order of transcendence $\underline{\lambda}(f)$ of any entire function f of finite positive order is 2. Our first step is to study the ratio of $T_0(r)$ and $S(r)$.

Lemma 5.1. *If $AT_0(r_1/k) \leq T_0(r_1)$, where $k > 1$ and $A > 1$, then there is $r \in (r_1/k, r_1)$ such that $cS(r) \geq T_0(r_1)$, where*

$$c = \frac{A \log k}{\log A}.$$

Proof. Let us take r_2 such that $T_0(r_1) = AT_0(r_2)$. Then $r_1/k \leq r_2 < r_1$. If

$$T_0(t) > \frac{\log k}{\log A} S(t)$$

on (r_2, r_1) , then

$$\log A = \int_{r_2}^{r_1} \frac{T_0'(t)}{T_0(t)} dt = \int_{r_2}^{r_1} \frac{S(t)}{tT_0(t)} dt < \frac{\log A}{\log k} \int_{r_2}^{r_1} \frac{dt}{t} \leq \log A.$$

Hence there is $r \in (r_2, r_1)$ such that

$$\frac{\log k}{\log A} S(r) \geq T_0(r) \geq T_0(r_2) = \frac{T_0(r_1)}{A}.$$

□

Next we need the ratio $m(4r)/S(r)$ to be bounded above for some numbers r . The following lemma provides sufficient conditions for such values of r .

Lemma 5.2. *Suppose that for some $k > 1$ and $r_1 > kR_1(f)$ there are constants $A_1 > 8$ and $A_2 > 1$ such that $A_1m(r_1/k) \leq m(r_1)$ and $m(8r_1) \leq A_2m(r_1)$. Then there is $r \in (r_1/k, 2r_1)$ such that $CS(r) \geq m(8r_1)$, where*

$$C = \frac{A_1 A_2 \log(2k)}{2 \log(A_1/8)}.$$

Proof. By (7) it follows that

$$\frac{1}{2}T_0(r_1/k) \leq m(r_1/k) \leq \frac{m(r_1)}{A_1} \leq \frac{4T_0(2r_1)}{A_1}.$$

By Lemma 5.1 there is $r \in (r_1/k, 2r_1)$, such that $c_1S(r) \geq T_0(2r_1)$, where

$$c_1 = \frac{A_1 \log(2k)}{8 \log(A_1/8)}.$$

Hence

$$c_1S(r) \geq T_0(2r_1) \geq \frac{m(r_1)}{4} \geq \frac{m(8r_1)}{4A_2}.$$

□

As we will now see, the ratio $m(4r)/S(r)$ is bounded near points where $m(r)$ is close to its proximate order.

Lemma 5.3. *Suppose that $m(r) \leq r^{\phi(r)}$, where $\lim_{r \rightarrow \infty} \phi(r) = \rho$ and the function $r^{\phi(r)-\rho}$ is slowly increasing. Let $0 < a \leq 1$ and $k^\rho > 16/a$. If r_1 is sufficiently large and $m(r_1) \geq ar_1^{\phi(r_1)}$, then there is $r \in (r_1/k, 2r_1)$ such that $CS(r) \geq m(8r_1)$, where*

$$C = \frac{(8k)^\rho \log(2k)}{2 \log(ak^\rho/16)}.$$

Proof. “Sufficiently large” in the statement of the lemma means $r_1 > kr_0$, where $r_0 > R_1(f)$ is a number such that

$$\frac{1}{2}b^\rho r^{\phi(r)} < (br)^{\phi(br)} < 2b^\rho r^{\phi(r)}$$

holds for $k^{-1} \leq b \leq 8$ and $r \geq r_0$ (see (5)). Then

$$m(r_1/k) \leq \left(\frac{r_1}{k}\right)^{\phi(r_1/k)} \leq 2k^{-\rho} r_1^{\phi(r_1)} \leq \frac{2}{ak^\rho} m(r_1),$$

and

$$m(8r_1) \leq (8r_1)^{\phi(8r_1)} \leq 2^{3\rho+1} r_1^{\phi(r_1)} \leq \frac{2^{3\rho+1}}{a} m(r_1).$$

The conclusion follows by Lemma 5.2, if we let $A_1 = ak^\rho/2$ and $A_2 = 2^{3\rho+1}/a$. □

In the following theorem we prove that $\underline{\tau}(f) = 2$. Note that we also give effective estimates on the “type” of growth of e_n and $m_n(r)$.

Theorem 5.4. *Let f be an entire function of finite order $\rho > 0$. There exist sequences of integers $n_j \nearrow \infty$ and $\epsilon_j \rightarrow 0$, $\epsilon_j > 0$, such that*

$$\frac{n_j^2 \log n_j}{2\rho + 1} \leq e_{n_j} \leq \frac{8^{\rho+3}(\rho + 5)}{\Lambda \rho^2} n_j^2 \log n_j,$$

$$m_{n_j}(r) \leq \frac{8^{\rho+3}(\rho + 5)}{\Lambda \rho} n_j^2 \log r, \quad 1 \leq r \leq \frac{1}{2} n_j^{1/\rho - \epsilon_j}.$$

Proof. Let $\rho(r)$ be a proximate order for $m(r)$. By its definition there exists a sequence $R'_j \rightarrow \infty$ such that $m(R'_j) = (R'_j)^{\rho(R'_j)}$. Take $a = 1$ and $k = 2^{5/\rho}$. By Lemma 5.3 there exist, for all j sufficiently large, numbers $R_j \in (R'_j/k, 2R'_j)$ such that

$$CS(R_j) \geq m(8R'_j) \geq m(4R_j),$$

where

$$(9) \quad C = \frac{2^{3\rho+4}(\rho + 5)}{\rho}.$$

Since for j large

$$m(8R'_j) > (R'_j)^{\rho(R'_j)} > (R_j/2)^{3\rho/4},$$

we see that $S(R_j) \geq R_j^{\rho/2}$. Also $S(R_j) \geq 6(n_0 + 1)/\Lambda$ when j is sufficiently large.

Hence the intervals $I_j = I(R_j, \rho/2, \Lambda/3, 1, C)$ are admissible and there exists a sequence of integers $n_j \in I_j$. By Theorem 4.2

$$e_{n_j} \leq \frac{3 \cdot 2^{3\rho+6}(\rho + 5)}{\Lambda \rho^2} n_j^2 \log \frac{3n_j}{\Lambda} \leq \frac{2^{3\rho+9}(\rho + 5)}{\Lambda \rho^2} n_j^2 \log n_j,$$

for all j sufficiently large. Moreover

$$m_{n_j}(r) \leq \frac{2^{3\rho+8}(\rho + 5)}{\Lambda \rho} n_j^2 \log r, \quad 1 \leq r \leq R_j,$$

and

$$n_j \leq 2\Lambda m(2R_j) \leq 2\Lambda(2R_j)^{\rho(2R_j)}.$$

By (5) there is a sequence of positive $\epsilon_j \rightarrow 0$ such that

$$n_j \leq 2\Lambda(2R_j)^{\rho(2R_j)} \leq 4\Lambda 2^\rho R_j^{\rho(R_j)} \leq 2^\rho R_j^{1/(1/\rho - \epsilon_j)}.$$

Hence $R_j \geq n_j^{1/\rho - \epsilon_j}/2$.

For the lower estimate on e_{n_j} , we take r with $n_j = m(r) < r^{\rho+1/4}$, so $\log r > \log n_j/(\rho + 1/4)$. By [CP2, §4 (3)] and [CP2, Corollary 2.6]

$$e_{n_j} \geq \frac{n_j^2}{2} \log r - n_j m(r) \geq \frac{n_j^2 \log n_j}{2\rho + 1/2} - n_j^2 \geq \frac{n_j^2 \log n_j}{2\rho + 1}.$$

□

6. DOUBLING INEQUALITIES

In this section we prove *doubling inequalities*, which provide upper bounds for the ratio $M(2r, F)/M(r, F)$, where $F(z) = P(z, f(z))$ and P is a polynomial. For $f(z) = e^z$ such inequalities were obtained by Tijdeman in [Ti1].

We will need a simple lemma, whose proof is contained in the proof of Theorem 2.2 of [CP2].

Lemma 6.1. *If $r < s$ and an entire function f has m zeros in Δ_r , then*

$$\frac{M(s, f)}{M(r, f)} \geq \left(\frac{r^2 + s^2}{2rs} \right)^m.$$

First, we reduce the problem of doubling inequalities to the problem of obtaining estimates for the transcendence measures m_n of dilations $f_r(z) = f(rz)$ of f . Let $r \geq 1$, let $P(z, w)$ be a polynomial of degree at most n and let $F(z) = P(z, f(z))$. If $P_r(z, w) = P(rz, w)$, then $\|P_r(z, f_r(z))\|_{\Delta} = M(r, F)$, while $\|P_r(z, f_r(z))\|_{\Delta_2} = M(2r, F)$. Hence

$$\frac{M(2r, F)}{M(r, F)} \leq e^{m_n(2, f_r)},$$

and we have to estimate $m_n(2, f_r)$.

Theorem 6.2. *Let f be an entire transcendental function of finite positive order ρ . There exists a sequence of integers $\{n_j\}$ increasing to infinity with the following property: For every $r \geq 1$ there is an integer j_r such that*

$$\frac{M(2r, F)}{M(r, F)} \leq 2^{an_j^2}, \quad a = \frac{8^{\rho+3}(\rho+5)}{\Lambda\rho}, \quad j \geq j_r,$$

where $F(z) = P(z, f(z))$ and $P(z, w)$ is a polynomial of degree at most n_j .

Proof. Let us denote by n_r the maximum of the numbers $|f_r^{-1}(w) \cap \Delta_2|$, when $|w| = 1$. Let w_r be a point where this maximum is achieved and let $g_r(z) = f(rz) - w_r$. By Lemma 6.1

$$\left(\frac{5}{4} \right)^{n_r} \leq \frac{M(4, g_r)}{M(2, g_r)}.$$

Since f is not constant, there exists $\epsilon > 0$ such that $M(2, g_r) \geq \epsilon$ for every $r \geq 1$. Since $M(4, g_r) \leq M(4r) + 1$ it follows that

$$(10) \quad n_r \leq C_1 m(4r) - 1,$$

where C_1 is a constant depending only on f .

Let $I = I(R, \alpha, \beta, \gamma, C)$ be an admissible interval for f . From the definition of the number $R_1(f)$ in Section 4 it follows that $R_1(f_r) \leq R_1(f)$. Note that $m(t, f_r) = m(rt)$ and $S_{f_r}(t) = S(rt)$. Hence, if $R' = R/r$, then $S_{f_r}(R') \geq R'^\alpha$ and $m(4R', f_r) \leq CS_{f_r}(R')$. Therefore, the interval $I' = (R', \alpha, \beta, \gamma, C)$ is admissible for f_r if $R' \geq R_1(f)$ and

$$\frac{\Lambda}{2} S(R) - n_r - 1 \geq \beta S^\gamma(R).$$

Since f has finite positive order ρ , by the proof of Theorem 5.4 there is a sequence R_j increasing to infinity such that the intervals

$$I_j = I(R_j, \rho/2, \Lambda/3, 1, C) = \left[\frac{\Lambda S(R_j)}{3}, \frac{\Lambda S(R_j)}{2} - n_0 \right]$$

are admissible, where C is defined in (9). For every $r \geq 1$ let j_r be the smallest integer such that $R_{j_r} > rR_1(f)$ and

$$\frac{\Lambda}{10} S(R_{j_r}) \geq C_1 m(4r).$$

Then

$$\frac{\Lambda}{2} S(R_{j_r}) - n_r - 1 \geq \frac{2\Lambda}{5} S(R_{j_r}) \geq \frac{\Lambda}{3} S(R_{j_r})$$

and the intervals

$$I'_j = I(R_j/r, \rho/2, \Lambda/3, 1, C) = \left[\frac{\Lambda S(R_j)}{3}, \frac{\Lambda S(R_j)}{2} - n_r \right]$$

are admissible for f_r when $j \geq j_r$.

Let j_0 be the smallest integer so that $S(R_{j_0}) \geq \max\{15/\Lambda, 10n_0/\Lambda\}$. Then for $j \geq j_0$ the intervals $I''_j = [\Lambda S(R_j)/3, 2\Lambda S(R_j)/5]$ contain an integer n_j and $I''_j \subset I_j$. Moreover, if $j \geq j_r$ then $I''_j \subset I'_j$, so by Theorem 4.2

$$m_{n_j}(2, f_r) \leq \frac{9C}{\Lambda} n_j^2 \log 2 \leq an_j^2 \log 2.$$

Consequently $M(2r, F)/M(r, F) \leq 2^{an_j^2}$, for all $j \geq j_r$. \square

Remark. With the notations of the above proof, since $n_j \in I''_j \subset I_j$ it follows that the conclusions of Theorem 5.4 hold for the sequence $\{n_j\}$ constructed in Theorem 6.2. A sequence of integers $\{n_j\}$ increasing to infinity for which the conclusions of both Theorems 5.4 and 6.2 are valid, will be called a *fundamental sequence* for f .

In the following theorem we prove doubling inequalities for functions which possess a covering system of admissible intervals.

Theorem 6.3. *Let f be an entire transcendental function which has a covering system of admissible intervals $I_j = I(R_j, \alpha, \beta, \gamma, C)$. For every $r \geq 1$ there exists an integer j_r such that*

$$\frac{M(2r, F)}{M(r, F)} \leq \begin{cases} \exp(3nm(4R_{j_r}) \log 2), & \text{if } n < \beta S^\gamma(R_{j_r})/2, \\ \exp(3C(2\beta^{-1})^{1/\gamma} n^{1+1/\gamma} \log 2), & \text{if } n \geq \beta S^\gamma(R_{j_r})/2, \end{cases}$$

where $F(z) = P(z, f(z))$ and $P(z, w)$ is a polynomial of degree at most n .

Proof. Let j_r be the smallest integer such that

$$(11) \quad R_{j_r} > rR_1(f), \quad \Lambda m(4R_{j_r}) \geq 4CC_1m(4r),$$

where C_1 is the constant from (10). By (10) and the properties of admissible intervals we have for $j \geq j_r$

$$\frac{\Lambda}{4} S(R_j) \geq \frac{\Lambda}{4C} m(4R_j) \geq C_1m(4r) \geq n_r + 1,$$

so

$$\frac{\Lambda}{2} S(R_j) - n_r - 1 \geq \frac{\Lambda}{4} S(R_j) \geq \frac{\beta}{2} S^\gamma(R_j).$$

Moreover, since I_j form a covering system we have

$$\frac{\beta}{2} S^\gamma(R_{j+1}) \leq \frac{\Lambda}{4} S(R_j) - \frac{n_0}{2} \leq \frac{\Lambda}{2} S(R_j) - n_r.$$

Thus the intervals $I'_j = I(R_j/r, \alpha, \beta/2, \gamma, C)$, $j \geq j_r$, form a covering system of admissible intervals for f_r . By Corollary 4.3

$$m_n(2, f_r) \leq 3C(2\beta^{-1})^{1/\gamma} n^{1+1/\gamma} \log 2,$$

when $n \geq \beta S^\gamma(R_{j_r})/2$.

If

$$n < \beta S^\gamma(R_{j_r})/2 \leq \frac{\Lambda}{2} S(R_{j_r}) - n_r,$$

then by Lemma 4.1 $m_n(2, f_r) \leq 3nm(4R_{j_r}) \log 2$. \square

Let us denote by $Z_n(r, f) = Z_n(r)$ the maximal number of zeros of $P(z, f(z))$ in Δ_r , when $P(z, w)$ is a polynomial of degree at most n . In Corollary 2.6 of [CP2] we proved that $Z_n(r) \leq 2m_n(3r)$. Now we can improve this estimate.

The first result gives an estimate on $Z_n(r)$ for all transcendental functions of finite positive order. Note that the constant a depends only on the order ρ of f .

Corollary 6.4. *If $\{n_j\}$ is a fundamental sequence for f then $Z_{n_j}(r) \leq 4an_j^2$, for $r \geq 1$ and $j \geq j_r$.*

Proof. Let $P(z, w)$ be a polynomial of degree n_j such that the number of zeros of $F(z) = P(z, f(z))$ in Δ_r equals $Z_{n_j}(r)$. Then by Theorem 6.2 and Lemma 6.1

$$\left(\frac{5}{4}\right)^{Z_{n_j}(r)} \leq \frac{M(2r, F)}{M(r, F)} \leq 2^{an_j^2}$$

when $j \geq j_r$. Hence

$$Z_{n_j}(r) \leq \frac{an_j^2 \log 2}{\log(5/4)} \leq 4an_j^2.$$

□

The second corollary provides estimates on $Z_n(r)$ for all n and has a similar proof.

Corollary 6.5. *In the assumptions of Theorem 6.3 we have*

$$Z_n(r) \leq \begin{cases} 10nm(4R_{j_r}), & \text{if } n < \beta S^\gamma(R_{j_r})/2, \\ 10C(2\beta^{-1})^{1/\gamma} n^{1+1/\gamma}, & \text{if } n \geq \beta S^\gamma(R_{j_r})/2. \end{cases}$$

Doubling inequalities lead to tangential Markov inequalities, which provide upper estimates for the derivative of the function $F(z) = P(z, f(z))$, where $P(z, w)$ is a polynomial of degree n . As before, we give two versions of such inequalities: one for general entire functions and another for functions with a covering system of admissible intervals.

Theorem 6.6. *Let $\{n_j\}$ be a fundamental sequence for f . For every $r \geq 1$ there is an integer j_r such that*

$$M(r, F') \leq \frac{eaM(r, F)n_j^2}{r},$$

where $F(z) = P(z, f(z))$, $P(z, w)$ is a polynomial of degree n_j , and $j \geq j_r$.

Proof. For $r \geq 1$ let j_r be the integer from Theorem 6.2. Let $F(z) = P(z, f(z))$, where $P(z, w)$ is a polynomial of degree n_j and $j \geq j_r$. Since $m(r, F)$ is a convex increasing function of $\log r$

$$|F(z)| \leq M(r, F) \exp\left((m(t, F) - m(r, F)) \frac{\log(|z|/r)}{\log(t/r)}\right), \quad r \leq |z| \leq t.$$

Let $b > 1$ be such that

$$\frac{m(t, F) - m(r, F)}{\log(t/r)} \leq b.$$

The function $h(x) = e^{b \log(1+x)}/x$ attains its minimum value when $x = x_b = 1/(b-1)$, and $h(x_b) < eb$. Therefore, if $r(1+x_b) \leq t$ and $|z| = r$, then by the Cauchy estimates

$$|F'(z)| \leq \frac{M(r, F)}{rx_b} e^{b \log(1+x_b)} \leq \frac{ebM(r, F)}{r}.$$

Taking $t = 2r$, we have by Theorem 6.2 $(m(2r, F) - m(r, F))/\log 2 \leq b = an_j^2$ and $1 + x_b \leq 2$. Thus

$$|F'(z)| \leq \frac{eaM(r, F)n_j^2}{r}.$$

□

The following theorem provides estimates on $M(r, F')$ for all n and has a similar proof.

Theorem 6.7. *In the assumptions of Theorem 6.3 we have*

$$M(r, F') \leq \frac{3eC2^{1/\gamma}n^{1+1/\gamma}M(r, F)}{\beta^{1/\gamma}r},$$

where $F(z) = P(z, f(z))$, $P(z, w)$ is a polynomial of degree n , and $n \geq \beta S^\gamma(R_{j_r})/2$.

7. SPECIAL CLASSES OF FUNCTIONS

In this section we find sufficient conditions for a function f to have estimates of the form $e_n = O(n^\tau \log n)$ for some $\tau \geq 2$. These conditions are imposed on the growth of f and are easy to verify. We start with the class of entire functions f whose growth satisfies the following inequalities: There exist constants $A_2 > A_1 > 1$ and $k > 1$ such that

$$(12) \quad A_1 m(r) \leq m(kr) \leq A_2 m(r)$$

for all r sufficiently large. These are functions of finite positive order and this class includes, for example, all functions

$$f(z) = \sum_{j=1}^m p_j(z) e^{q_j(z)},$$

where p_j and q_j are polynomials. Moreover we show in the next section that the Riemann ζ -function and the function ξ are also in this class.

Theorem 7.1. *Let f be an entire function of order ρ which satisfies (12) for all r sufficiently large. Then, for all n sufficiently large,*

$$e_n \leq K_1 n^2 \log n, \quad m_n(r) \leq K_2 n^2 \log r, \quad 1 \leq r \leq n^{1/\rho - \epsilon_n}/2,$$

where the constants K_1, K_2 depend only on A_1, A_2, k , and $\epsilon_n > 0, \epsilon_n \rightarrow 0$.

Proof. Inequalities (12) imply that

$$A_1^j m(r) \leq m(k^j r) \leq A_2^j m(r), \quad d_1 r^{\rho_1} \leq m(r) \leq d_2 r^{\rho_2},$$

where

$$\rho_1 = \frac{\log A_1}{\log k}, \quad \rho_2 = \frac{\log A_2}{\log k}, \quad d_1 = \frac{m(1)}{A_1}, \quad d_2 = A_2 m(1).$$

Thus f is a function of finite positive order $\rho \in [\rho_1, \rho_2]$, and we may assume that (12) holds with constants $k \geq 8$ and $A_1 > 8$. Then $A_1 m(r/k) \leq m(r)$ and $m(8r) \leq A_2 m(r)$.

For every r sufficiently large there is, by Lemma 5.2, $r' \in (r/k, 2r)$ such that $CS(r') \geq m(4r')$, where

$$C = \frac{A_1 A_2 \log(2k)}{2 \log(A_1/8)}.$$

In particular, for all j sufficiently large, there is $R_j \in ((2k)^j, (2k)^{j+1})$ such that $CS(R_j) \geq m(4R_j)$. Since the order of f is ρ we may assume that $S(R_j) \geq R_j^{\rho/2}$.

Using (8) and (12) we get

$$S(R_{j+1}) \leq 3m(2R_{j+1}) \leq 3m(8k^2(2k)^j) \leq 3A_2^3 m(R_j) \leq 3CA_2^3 S(R_j).$$

Hence $S(R_{j+1})/S(R_j) \leq M = 3CA_2^3$.

Let j be so large that $S(R_j) \geq 6(n_0 + 1)/\Lambda$ and let $\beta = \Lambda/(3M)$. Then

$$\beta S(R_j) \leq \beta S(R_{j+1}) \leq \frac{\Lambda}{3} S(R_j) \leq \frac{\Lambda}{2} S(R_j) - n_0 - 1,$$

so the intervals $I_j = I(R_j, \rho/2, \beta, 1, C)$ form a covering system of admissible intervals, starting with some j sufficiently large. The theorem now follows from Corollary 4.3. If $n \in I_{j_n}$ then the fact that $R_{j_n} \geq n^{1/\rho - \epsilon_n}/2$ can be proved exactly like the similar statement in Theorem 5.4. \square

The functions f satisfying (12) have covering systems of admissible intervals. Hence they also satisfy the hypotheses of Theorem 6.3. Moreover, in this case we can get better estimates on the integers j_r from Theorem 6.3.

Corollary 7.2. *Let f be an entire function of order ρ which satisfies (12) for all r sufficiently large. Then there is a constant $a > 1$ such that $Z_n(r) \leq a(nm(ar) + n^2)$, for all $n \geq 1$ and $r \geq 1$.*

Proof. Fix $r \geq 1$, let $I_j = I(R_j, \rho/2, \beta, 1, C)$ be the covering system of admissible intervals from the proof of Theorem 7.1, and recall that $R_j \in ((2k)^j, (2k)^{j+1})$. By Corollary 6.5 we have $Z_n(r) \leq 10nm(4R_{j_r}) + An^2$

for all $n \geq 1$, where A is a constant and j_r is defined in (11) as the smallest integer such that $R_{j_r} > rR_1(f)$ and $\Lambda m(4R_{j_r}) \geq 4CC_1 m(4r)$.

We fix j_0, j_1 such that

$$(2k)^{j_0} \geq R_1(f), A_1^{j_0} \geq 4CC_1/\Lambda, r \in [(2k)^{j_1}, (2k)^{j_1+1}].$$

Then $R_{j_0+j_1+1} > rR_1(f)$ and

$$m(4R_{j_0+j_1+1}) \geq A_1^{j_0} m(4(2k)^{j_1+1}) > \frac{4CC_1}{\Lambda} m(4r).$$

Consequently, $j_r \leq j_0 + j_1 + 1$, $R_{j_r} \leq (2k)^{j_0+j_1+2} \leq (2k)^{j_0+2} r$, and the corollary follows. \square

Given an entire function f , it is frequently known that f verifies a growth condition $m(r) \leq r^{\phi(r)}$, where $\lim_{r \rightarrow \infty} \phi(r) = \rho$ and the function $r^{\phi(r)-\rho}$ is slowly increasing. In the remainder of this section, we denote by r_n the unique solution of the equation $r^{\phi(r)} = n$. Our next theorem shows that in this case there are estimates $e_n = O(n^\tau \log n)$, provided that

$$m(r_{n_j}) \geq ar_{n_j}^{\phi(r_{n_j})} = an_j$$

holds for a ‘‘slow growing’’ subsequence n_j .

Theorem 7.3. *In the above setting, assume there is an increasing sequence of integers n_j such that $n_{j+1}^\gamma \leq bn_j$ and $m(r_{n_j}) \geq an_j$, where $0 < \gamma \leq 1$, $b > 0$, $0 < a \leq 1$. Then there exists a sequence of positive $\epsilon_n \rightarrow 0$, such that the estimates*

$$e_n \leq \frac{4C(3M)^{1/\gamma}}{\rho\gamma\Lambda^{1/\gamma}} n^{1+1/\gamma} \log \frac{3Mn}{\Lambda},$$

$$m_n(r) \leq \frac{3C(3M)^{1/\gamma}}{\Lambda^{1/\gamma}} n^{1+1/\gamma} \log r, \quad 1 \leq r \leq \frac{1}{2} n^{1/\rho-\epsilon_n},$$

hold for all n sufficiently large, where

$$C = \frac{2^{3\rho+4}}{a} \left(1 + \frac{1}{\rho} \log_2(32/a) \right), \quad M = \frac{2^{(2\rho+3)\gamma} C b}{a}.$$

Proof. We let $s_j = r_{n_j}$. By Lemma 5.3 with $k = (32/a)^{1/\rho}$ and j sufficiently large, there is $R_j \in (s_j/k, 2s_j)$ such that

$$CS(R_j) \geq m(8s_j) \geq m(4R_j),$$

where

$$C = \frac{2^{3\rho+4}}{a} \left(1 + \frac{1}{\rho} \log_2(32/a) \right).$$

We may assume that

$$(4s_j)^{\phi(4s_j)} \leq 2 \cdot 4^\rho s_j^{\phi(s_j)} = 2^{2\rho+1} n_j.$$

Using this and (8) we get

$$S(R_j) \leq \frac{2m(2R_j)}{\log 2} \leq 4m(4s_j) \leq 2^{2\rho+3}n_j.$$

Since $CS(R_j) \geq m(8s_j) \geq an_j$ it follows that

$$\frac{S^\gamma(R_{j+1})}{S(R_j)} \leq \frac{2^{(2\rho+3)\gamma}Cn_{j+1}^\gamma}{an_j} \leq \frac{2^{(2\rho+3)\gamma}Cb}{a} = M.$$

Moreover,

$$S(R_j) \geq \frac{a}{C} s_j^{\phi(s_j)} \geq \frac{a}{C} (R_j/2)^{3\rho/4} \geq R_j^{\rho/2},$$

when j is sufficiently large, and if $\beta = \Lambda/(3M)$ then

$$\max\{\beta S^\gamma(R_j), \beta S^\gamma(R_{j+1})\} \leq \frac{\Lambda}{3} S(R_j) \leq \frac{\Lambda}{2} S(R_j) - n_0 - 1.$$

So the intervals $I_j = I(R_j, \rho/2, \beta, \gamma, C)$ form a covering system of admissible intervals, starting with some j sufficiently large. The theorem now follows from Corollary 4.3. If $n \in I_{j_n}$ then the fact that $R_{j_n} \geq n^{1/\rho-\epsilon_n}/2$ can be proved exactly like the similar statement in Theorem 5.4. \square

This theorem has a corollary which allows us to estimate e_n and m_n using the behavior of the Taylor coefficients of f .

Corollary 7.4. *Suppose that for an entire function*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

we have $m(r) \leq r^{\phi(r)}$, where $\lim_{r \rightarrow \infty} \phi(r) = \rho$ and the function $r^{\phi(r)-\rho}$ is slowly increasing. Let r_n be defined by $r_n^{\phi(r_n)} = n$. If there is an increasing sequence of integers n_j such that

$$n_{j+1}^\gamma \leq bn_j, \quad \log |c_{n_j}| \geq an_j - n_j \log r_{n_j},$$

where $0 < \gamma \leq 1$, $b > 0$, $0 < a \leq 1$, then the estimates on e_n and $m_n(r)$ from Theorem 7.3 hold for all n sufficiently large.

Proof. This follows from Theorem 7.3, since by the Cauchy inequalities we have $m(r_{n_j}) \geq \log |c_{n_j}| + n_j \log r_{n_j} \geq an_j$. \square

As an example we take the entire function

$$f(z) = \sum_{j=1}^{\infty} (z/n_j)^{n_j},$$

where $n_1 \geq 2$, $n_{j+1} = n_j^{\tau-1}$ and $\tau > 2$. This function was studied in Section 6 of [CP2], where it was shown that there are constants C_1 and

C_2 such that $e_n \leq C_1 n^\tau \log n$ and $m_n(r) \leq C_2 n^\tau \log r$ for $1 \leq r \leq n$. This was a result of quite elaborate estimates. Since f is a function of order 1 and type $1/e$, we have $m(r) \leq 2r/e$ for r large. Taking $r_{n_j} = en_j/2$ and $a = \log(e/2)$ we get

$$\log c_{n_j} = -n_j \log n_j = an_j - n_j \log r_{n_j}.$$

So Corollary 7.4 applies with $\gamma = 1/(\tau - 1)$ and $b = 1$.

8. THE FUNCTIONS ζ AND ξ

The Riemann ζ -function is holomorphic in \mathbb{C} except at $z = 1$, where it has a simple pole (see e.g. [T, Theorem 2.1]). The function ξ is defined by

$$\xi(z) = \frac{z(z-1)}{2} \pi^{-z/2} \Gamma\left(\frac{z}{2}\right) \zeta(z),$$

where Γ is the Euler Gamma function (see [T, (2.1.12)]). Then ξ is an entire function of order 1 [T, Theorem 2.12].

For the convenience of the reader, we include the proof of the following proposition.

Proposition 8.1. *There exist positive constants $c_1 < c_2$, $d_1 < d_2$ such that for all $r \geq 2$ we have*

$$\begin{aligned} c_1 r \log r &\leq m(r, \zeta) \leq c_2 r \log r, \\ d_1 r \log r &\leq m(r, \xi) \leq d_2 r \log r. \end{aligned}$$

Proof. If $x = \mathbf{Re} z > 0$, then

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad |\Gamma(z)| \leq \Gamma(x).$$

We let $\mu(r)$ be the supremum of $\log |\Gamma(z)|$ when $|z| = r$ and $x \geq 1/2$. Then by Stirling's formula $C_1 r \log r \leq \mu(r) \leq C_2 r \log r$ for $r \geq 2$.

For $x > 0$ one has (see [T, (2.1.4)])

$$\zeta(z) = z \int_1^\infty \frac{[t] - t + \frac{1}{2}}{t^{z+1}} dt + \frac{1}{z-1} + \frac{1}{2}.$$

Hence for $x \geq 1/2$ and $|z-1| > 2$ we have

$$|\zeta(z)| \leq \frac{|z|}{2} \int_1^\infty \frac{1}{t^{x+1}} dt + 1 \leq |z| + 1,$$

while $\zeta(x) = \sum_{k=1}^\infty k^{-x} > 1$ for $x > 1$. To estimate $\zeta(z)$ for $\mathbf{Re} z \leq 1/2$ we use the functional equation (see [T, Theorem 2.1])

$$\zeta(z) = 2^z \pi^{z-1} \sin \frac{\pi z}{2} \Gamma(1-z) \zeta(1-z).$$

We conclude that $m(r, \zeta) \leq c_2 r \log r$. But for odd integers $n > 0$ we have $|\zeta(-n)| \geq 2^{-n} \pi^{-n-1} n!$ and therefore $m(n, \zeta) \geq c'_1 n \log n$. This implies $m(r, \zeta) \geq c_1 r \log r$, $c_1 > 0$.

Using the definition of $\xi(z)$ we have for $|z| = r$ with $\mathbf{Re} z \geq 1/2$

$$|\xi(z)| \leq \frac{r(r+1)^2}{2} \Gamma\left(\frac{r}{2}\right), \quad \xi(r) \geq \frac{r(r-1)}{2} \pi^{-r/2} \Gamma\left(\frac{r}{2}\right).$$

Since $\xi(z) = \xi(1-z)$ (see [T, (2.1.13)]) we obtain for $|z| = r$ with $\mathbf{Re} z \leq 1/2$

$$|\xi(z)| \leq \frac{(r+1)(r+2)^2}{2} \Gamma\left(\frac{r+1}{2}\right).$$

So we see that $d_1 r \log r \leq m(r, \xi) \leq d_2 r \log r$. □

By Proposition 8.1 the function ξ verifies condition (12) if k is chosen sufficiently large, so Theorem 7.1 and Corollary 7.2 hold for ξ . Since ζ is meromorphic, the quantity $m_n(r)$ is not well defined for ζ . We have the following:

Theorem 8.2. *There exists a constant $C > 0$ such that for every integer $n \geq 1$ and every $r \geq 1$ we have*

$$\frac{n^2 + 3n}{2} \leq Z_n(r, \zeta) \leq C(nr \log r + n^2).$$

Proof. Since ζ is holomorphic near 0, we can find, by a simple dimension argument, a non-trivial polynomial $P(z, w)$ of degree at most n such that $P(z, \zeta(z))$ has a zero of order at least $(n^2 + 3n)/2$ at 0 (see the proof of Theorem 2.5 in [CP2]). This implies the lower estimate on Z_n .

The function $\tilde{\zeta}(z) = (z-1)\zeta(z)$ is entire. Proposition 8.1 implies that

$$c_1 r \log r \leq m(r, \tilde{\zeta}) \leq c'_2 r \log r.$$

By Corollary 7.2 it follows that there exists a constant $C > 0$ such that

$$Z_n(r, \tilde{\zeta}) \leq C(nr \log r + n^2),$$

for all $n \geq 1$ and $r \geq 1$. Note that if $P(z, w)$ is a polynomial of degree at most n , then there exists a polynomial $Q(z, w)$ of degree at most $2n$ such that $(z-1)^n P(z, \zeta(z)) = Q(z, \tilde{\zeta}(z))$. Hence $Z_n(r, \zeta) \leq Z_{2n}(r, \tilde{\zeta})$, and the proof is complete. □

9. EXTREMAL FUNCTIONS

If $K \subset \mathbb{C}^2$ is a compact set, the extremal function V_K of K (also called the pluricomplex Green function of K with pole at infinity) is

defined by

$$V_K(z, w) = \sup \frac{1}{\deg P} \log |P(z, w)|,$$

where the supremum is taken over all polynomials P such that $\|P\|_K \leq 1$. Then either V_K is finite at every point, or $V_K \equiv \infty$, and the latter occurs if and only if K is pluripolar (see e.g. [K, Ch. 5]).

Let f be an entire transcendental function and let

$$K = \{(z, f(z)) : |z| \leq 1\}.$$

Then K is pluripolar and $V_K \equiv \infty$. Using our estimates on $m_n(r)$, it is still possible to define a meaningful extremal function of K along the graph of f . This relates to Sadullaev's result on the existence of extremal functions for non-pluripolar subsets of algebraic varieties [Sa].

We assume in this section that f is an entire transcendental function which verifies

$$m_n(r) \leq C_f n^2 \log r, \quad 1 \leq r \leq r_n, \quad n \geq 1,$$

where r_n is a sequence increasing to infinity and C_f is a constant depending on f . Classes of such functions are constructed in Section 7. Let us define

$$W_n(z) = \sup \log |P(z, f(z))|,$$

where the supremum is taken over all polynomials P of degree at most n which verify $|P(z, f(z))| \leq 1$ on Δ . The functions W_n are non-negative, continuous and subharmonic on \mathbb{C} , and $W_n \equiv 0$ on Δ .

Next we define

$$W(z) = \limsup_{n \rightarrow \infty} \frac{1}{n^2} W_n(z),$$

and we let W^* denote the upper semicontinuous regularization of W . We have the following:

Proposition 9.1. *The function W^* is non-negative subharmonic on \mathbb{C} , $W^* \equiv 0$ on Δ , and for all $r \geq 1$*

$$\frac{1}{2} \log r \leq \max\{W^*(z) : |z| = r\} \leq C_f \log r.$$

If $f(z) = e^z$ then $W^(z) = \frac{1}{2} \log^+ |z|$ for all $z \in \mathbb{C}$.*

Proof. By the proofs of Theorem 2.5 and Corollary 2.6 of [CP2] there exists, for each $n \geq 1$, a non-trivial polynomial $P_n(z, w)$ of degree n , such that the function $F_n(z) = P_n(z, f(z))$ verifies $M(1, F_n) = 1$ and

$$\frac{n^2 + 3n}{2} \log r \leq \log M(r, F_n) \leq m_n(r),$$

for all $r \geq 1$. Note that, in particular, this implies $C_f \geq 1/2$ for any f .

Our assumption on the growth of $m_n(r)$ implies that the family of subharmonic functions W_n/n^2 is locally upper bounded, hence W^* is a non-negative subharmonic function on \mathbb{C} which verifies

$$W^*(z) \leq C_f \log^+ |z|.$$

Suppose that for some $r > 1$ we have

$$\max\{W^*(z) : |z| = r\} < \frac{1}{2} \log r.$$

The Hartogs Lemma implies that for n large and for all z with $|z| = r$

$$\log |F_n(z)| \leq W_n(z) < \frac{n^2}{2} \log r.$$

This contradicts the above lower estimate on $\log M(r, F_n)$.

In the case of the function $f(z) = e^z$ it was proved in [CP1] that

$$m_n(r) \leq \left(\frac{n^2}{2} + o(n^2) \right) \log r, \quad 1 \leq r \leq n.$$

The preceding argument shows that now $W^*(z) \leq \frac{1}{2} \log^+ |z|$. We conclude that the equality must hold, by applying the maximum principle on $\mathbb{C} \setminus \Delta$ to the subharmonic function $W^*(z) - \frac{1}{2} \log |z| \leq 0$. \square

10. ESTIMATES FOR ALGEBRAIC MEASURES

Throughout Sections 10 and 11, K is an algebraic extension of degree σ of the field \mathbb{Q} of rational numbers and f is, unless otherwise specified, an entire transcendental function of finite positive order ρ . Without loss of generality we may assume that $M(r, f) \geq r$ for $r \geq 1$.

For an algebraic number ζ , we define its norm $\|\zeta\|$ as the maximum of the absolute values of its conjugates. Then $\|\zeta_1 \zeta_2\| \leq \|\zeta_1\| \|\zeta_2\|$ and $\|\zeta_1 + \zeta_2\| \leq \|\zeta_1\| + \|\zeta_2\|$ (see [M, p. 62]).

If $P(\zeta_1, \dots, \zeta_n)$ is a polynomial with algebraic coefficients, then its *height* $h(P)$ is defined as the maximum of the norms of its coefficients.

If $\omega_1, \dots, \omega_\sigma$ is a basis for the ring I_K of algebraic integers in K , then any $\zeta \in I_K$ can be written as

$$(13) \quad \zeta = p_1 \omega_1 + \dots + p_\sigma \omega_\sigma,$$

where p_1, \dots, p_σ are rational integers. If $\|\zeta\| = \max\{|p_1|, \dots, |p_\sigma|\}$, then (see [M, p. 62]) there are constants γ_1 and γ_2 depending only on K such that $\gamma_1 \|\zeta\| \leq \|\zeta\| \leq \gamma_2 \|\zeta\|$.

Given a natural number d we denote by $I_K(d)$ the set of numbers $z \in K$ such that $dz \in I_K$, and by $I_K(d, A)$ the set of $z \in I_K(d)$ with $\|z\| \leq A$. Let $N_K(d, A, r)$ be the number of points in $I_K(d, A) \cap \Delta_r$.

Lemma 10.1. *There exist constants c' and c'' depending only on K with the following properties: If K is real and $r \leq A$ then*

$$c'd^\sigma A^{\sigma-1}r \leq N_K(d, A, r) \leq c''d^\sigma A^{\sigma-1}r.$$

If K is complex and $r \leq A$ then

$$c'd^\sigma A^{\sigma-2}r^2 \leq N_K(d, A, r) \leq c''d^\sigma A^{\sigma-2}r^2.$$

Moreover, in both cases, if $r > A$ then

$$c'(dA)^\sigma \leq N_K(d, A, r) \leq c''(dA)^\sigma.$$

Proof. We will consider only the complex case, so $\sigma \geq 2$. The real case can be considered in a similar manner.

We assume at first that $d = 1$. If N is the number of points in $I_K(1, A)$, then by (13) $c_1A^\sigma \leq N \leq c_2A^\sigma$, where the constants c_1 and c_2 depend only on K . Moreover $I_K(1, A) \subset \Delta_A$, so the lemma is proved for $r > A$.

Suppose that $r \leq A$. There exists an absolute constant c_3 such that Δ_A can be covered by $c_3A^2r^{-2}$ disks of radius $r/2$. Therefore there is a disk D of radius $r/2$ containing at least $c_4A^{\sigma-2}r^2$ points of $I_K(1, A/2)$, where c_4 depends only on K . Let z_0 be one of this points. If $z \in I_K(1, A/2) \cap D$, then $z - z_0 \in I_K(1, A) \cap \Delta_r$. Consequently, $N_K(1, A, r) \geq c'A^{\sigma-2}r^2$, where c' depends only on K .

Now let $N = N_K(1, A, r)$ and let $\omega_1, \dots, \omega_\sigma$ be a basis for I_K over \mathbb{Z} . Since $I_K \not\subset \mathbb{R}$, we may assume that $\omega_1/\omega_2 \notin \mathbb{R}$. Then it is easy to see that there is a constant c_5 depending only on K such that the disk Δ_A contains at least c_5A^2 points z of the form $z = p_1\omega_1 + p_2\omega_2$, where $p_1, p_2 \in \mathbb{Z}$ and $|p_1|, |p_2| \leq A/\gamma_2$, so $\|z\| \leq A$. Moreover, there is a constant c_6 depending only on K and at least $c_6A^2r^{-2}$ disjoint disks of radius r centered at these points. Hence each of these disks contains at least N points from $I_K(1, 2A)$. It follows that $c_6NA^2r^{-2} \leq c_22^\sigma A^\sigma$, so $N \leq c''A^{\sigma-2}r^2$.

If $d > 1$ then we note that $z \in I_K(1, dA, dr)$ if and only if $z/d \in I_K(d, A, r)$, hence $N_K(d, A, r) = N_K(1, dA, dr)$. \square

We say that a function f takes values at z in $I_K(d)$ with multiplicity m if the numbers $z, f(z), \dots, f^{(m-1)}(z)$ belong to $I_K(d)$. In this case we define $\|f(z)\|_m$ as the maximum of $\|z\|, \|f(z)\|, \dots, \|f^{(m-1)}(z)\|$.

In this setting, we have the following lemma (see Ch. 1, §2 and Ch. 2, §2 in [G]).

Lemma 10.2. *Let f be a holomorphic function in a neighborhood of z_0 , which takes values at z_0 in $I_K(d)$ with multiplicity m , and such that $\|f(z_0)\|_m \leq A$, $A \geq 1$. If $P(z, w)$ is a polynomial of degree n with*

coefficients in I_K and of height h , and if $F(z) = P(z, f(z))$, then for $k \leq m - 1$ we have $F^{(k)}(z_0) \in I_K(d^n)$ and

$$\|g_{ij}^{(k)}(z_0)\| \leq A^{i+j}(i+j)^k,$$

where $g_{ij}(z) = z^i f^j(z)$, $i+j \leq n$. Moreover, if $F^{(k)}(z_0) \neq 0$ then

$$|d^n F^{(k)}(z_0)| \geq (hd^n A^n (n+1)^{k+2})^{-\sigma+1}.$$

Proof. If

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j,$$

then the k -th derivative a_{jk} of $f^j(z)$ at z_0 is

$$a_{jk} = k! \sum_{i_1+\dots+i_j=k} a_{i_1} \cdots a_{i_j} = k! \sum_{i_1+\dots+i_j=k} \frac{f^{(i_1)}(z_0) \cdots f^{(i_j)}(z_0)}{i_1! \cdots i_j!}.$$

Hence $d^j a_{jk} \in I_K$. Since

$$j^k = \sum_{i_1+\dots+i_j=k} \frac{k!}{i_1! \cdots i_j!},$$

we see that $\|a_{jk}\| \leq j^k A^j$.

If $g_{ij}(z) = z^i f^j(z)$, then

$$g_{ij}^{(k)}(z_0) = \sum_{p=0}^{\min\{i,k\}} \binom{k}{p} \frac{i!}{(i-p)!} z_0^{i-p} a_{j,k-p}.$$

Thus $d^{i+j} g_{ij}^{(k)}(z_0) \in I_K$. Moreover,

$$\begin{aligned} \|g_{ij}^{(k)}(z_0)\| &\leq \sum_{p=0}^{\min\{i,k\}} \binom{k}{p} \frac{i!}{(i-p)!} j^{k-p} A^{i+j-p} \\ &\leq A^{i+j} \sum_{p=0}^k \binom{k}{p} i^p j^{k-p} = A^{i+j} (i+j)^k. \end{aligned}$$

Hence $d^n F^{(k)}(z_0) \in I_K$ and

$$\|F^{(k)}(z_0)\| \leq \frac{(n+1)(n+2)}{2} h A^n n^k \leq h A^n (n+1)^{k+2}.$$

Since $c_1 = d^n F^{(k)}(z_0) \in I_K$, the number ν of its conjugates c_2, \dots, c_ν does not exceed σ and

$$\left| \prod_{i=1}^{\nu} c_i \right| \geq 1$$

when $c_1 \neq 0$. Note that $|c_i| \leq \|c_i\| = \|c_1\|$. Consequently,

$$|c_1| \geq (hd^n A^n (n+1)^{k+2})^{-\sigma+1}.$$

□

The following result is a consequence of C. L. Siegel's lemma adapted for our purposes.

Lemma 10.3. *Suppose that there are l points z_1, \dots, z_l in Δ_r , $r \geq 1$, such that, for $1 \leq q \leq l$, f takes values at z_q in $I_K(d_q)$ with multiplicity m_q and $\|f(z_q)\|_{m_q} \leq A$, $A \geq 1$. If $\nu = \sum_{q=1}^l m_q < N$, where $N = (n+1)(n+2)/2$, and $m_q \leq m$, $d_q \leq d$, then there is a polynomial $P(z, w)$ of degree n with coefficients $c_{ij} \in I_K$ and of height*

$$h(P) \leq H_n = C_1 (C_2 d^n A^n (n+1)^{m+1})^{\nu/(N-\nu)},$$

where C_1, C_2 are constants depending only on K , with the following properties: The function

$$F(z) = P(z, f(z)) = \sum_{i+j=0}^n c_{ij} z^i f^j(z) \neq 0$$

and for $t \geq 2r$

$$\|F\|_{\Delta_{2r}} \leq (n+1)^2 H_n \left(\frac{4r}{t}\right)^\mu M^n(t, f),$$

where $\mu \geq \nu$ is the number of zeros of F in Δ_r .

Proof. By Lemma 10.2 $d_q^n g_{ij}^{(k)}(z_q)$, where $g_{ij}(z) = z^i f^j(z)$, is an algebraic integer and

$$\|d_q^n g_{ij}^{(k)}(z_q)\| \leq d_q^n A^n n^k \leq d^n A^n n^{m-1}.$$

Let us consider the system of ν equations

$$\sum_{i+j=0}^n c_{ij} d_q^n g_{ij}^{(k)}(z_q) = 0, \quad 1 \leq q \leq l, \quad 0 \leq k \leq m_q - 1,$$

with N unknowns c_{ij} . By [M, p. 63] there are constants C_1 and C_2 depending only on K such that this system has a non-trivial solution in I_K with

$$\|c_{ij}\| \leq C_1 (C_2 N d^n A^n n^{m-1})^{\nu/(N-\nu)} \leq H_n.$$

Since $\|P\|_{\Delta^2} \leq (n+1)^2 H_n$, by the Bernstein–Walsh inequality

$$|P(z, w)| \leq (n+1)^2 H_n \exp(n \max\{\log^+ |z|, \log^+ |w|\}),$$

so $\|F\|_{\Delta_t} \leq (n+1)^2 H_n M^n(t, f)$.

The function F has $\mu \geq \nu$ zeros in Δ_r , so by Lemma 6.1

$$\|F\|_{\Delta_{2r}} \leq \left(\frac{4r}{t}\right)^\mu \|F\|_{\Delta_t} \leq (n+1)^2 H_n \left(\frac{4r}{t}\right)^\mu M^n(t, f).$$

□

Suppose that for a set $E \subset \mathbb{C}$ and for some integer $m \geq 1$ we have $z, f(z), \dots, f^{(m-1)}(z) \in K$ for all $z \in E$. Then for $z \in E$ we let d_z be the smallest natural number such that f takes values at z in $I_K(d_z)$ with multiplicity m . We set

$$\|f\|_{E,m} = \max\{1, \sup_{z \in E} \|f(z)\|_m\}, \quad d(E, m) = \sup_{z \in E} d_z,$$

$$\mathcal{A}_K(E, m) = d(E, m) \|f\|_{E,m}.$$

The number $\mathcal{A}_K(E, m)$ will be called the *algebraic measure of order m* of the function f on E . If for some $z \in E$ we have $z \notin K$ or $f^{(j)}(z) \notin K$ for some $j < m$, then we set $\mathcal{A}_K(E, m) = \infty$. Note also that if a set E is infinite then $\mathcal{A}_K(E, m) = \infty$ for every $m \geq 1$.

Throughout the rest of this section and in Section 11 we will assume that $m(r) \leq r^{\phi(r)}$, where $\lim_{r \rightarrow \infty} \phi(r) = \rho$ and the function $r^{\phi(r)-\rho}$ is slowly increasing. We denote by r_n the unique solution of the equation $r^{\phi(r)} = n$. The following result is the main tool in the forthcoming estimates of the algebraic measure.

Theorem 10.4. *There exists a constant C_K depending only on K with the following property: If $n \geq 1$, $1 \leq r \leq r_n/4$ and $E \subset \Delta_r$, then there are integers $k \geq 0$ and μ such that*

$$(k+1)|E| > n^2/4, \quad \max\{n^2/4, k|E|\} \leq \mu \leq Z_n(r),$$

$$C_K \mathcal{A}_K^{2\sigma}(E, k+1) \geq \left(\frac{r}{k(n+1)^{2\sigma-1}}\right)^{k/n} \exp\left(\frac{\mu}{n} \log \frac{r_n}{4e^4 r}\right).$$

In the above statement we let $k^k = 1$ if $k = 0$.

Proof. We may assume that E is finite. Let $E = \{z_1, \dots, z_l\}$ and $\nu = [n^2/4] + 1$. Note that $\nu/(N - \nu) \leq 1$ and by Theorem 2.5 and Corollary 2.6 in [CP2] we have $\nu \leq (n^2 + 3n)/2 \leq Z_n(r)$.

Let $m = [\nu/l]$. If $\mathcal{A}_K(E, m+1) = \infty$, then we take $k = m$ and $\mu = \nu$ and the proof is finished. Otherwise, we let $A_1 = \|f\|_{E, m+1}$ and $d_1 = d(E, m+1)$. We have $\nu = ml + p$, $0 \leq p \leq l-1$. Applying Lemma 10.3 with the above points, with $m_q = m+1$ when $1 \leq q \leq p$ and $m_q = m$ when $p+1 \leq q \leq l$, and with this value of ν , we construct a non-trivial polynomial $P(z, w)$ of degree n with coefficients in I_K and with height

$$h(P) \leq h = C_1 C_2 d_1^n A_1^n (n+1)^{m+2}$$

such that

$$\|F\|_{\Delta_{2r}} \leq (n+1)^2 h \left(\frac{4r}{t}\right)^\mu M^n(t, f),$$

where $\mu \geq n^2/4$ is the number of zeros of F in Δ_r and $t \geq 2r$.

There exist q and k , $1 \leq q \leq l$, $0 \leq k \leq \mu/l$, such that $F^{(k)}(z_q) \neq 0$. We may assume that $\mathcal{A}_K(E, k+1) < \infty$. Clearly, $k \geq m$ and therefore $kl > \nu - l > n^2/4 - l$. Moreover $A = \|f\|_{E, k+1} \geq A_1$, $d = d(E, k+1) \geq d_1$, so

$$h \leq C_1 C_2 d^n A^n (n+1)^{k+2}.$$

By Lemma 10.2

$$\begin{aligned} |d^n F^{(k)}(z_q)| &\geq (hd^n A^n (n+1)^{k+2})^{-\sigma+1} \\ &\geq (C_1 C_2 d^{2n} A^{2n} (n+1)^{2k+4})^{-\sigma+1}. \end{aligned}$$

By Cauchy's inequalities

$$\begin{aligned} |d^n F^{(k)}(z_q)| &\leq d^n k! \frac{\|F\|_{\Delta_{2r}}}{r^k} \leq d^n (n+1)^2 h \left(\frac{k}{r}\right)^k \left(\frac{4r}{t}\right)^\mu M^n(t, f) \\ &\leq C_1 C_2 d^{2n} A^{2n} (n+1)^{k+4} \left(\frac{k}{r}\right)^k \left(\frac{4r}{t}\right)^\mu M^n(t, f). \end{aligned}$$

We obtain

$$\begin{aligned} (C_1 C_2 d^{2n} A^{2n} (n+1)^{2k+4})^{-\sigma+1} &\leq \\ C_1 C_2 d^{2n} A^{2n} (n+1)^{k+4} \left(\frac{k}{r}\right)^k \left(\frac{4r}{t}\right)^\mu M^n(t, f). \end{aligned}$$

Since $A \geq 1$ this implies

$$(C_1 C_2)^\sigma (d^{2n} A^{2n})^\sigma \geq \left(\frac{r}{k}\right)^k \left(\frac{t}{4r}\right)^\mu M^{-n}(t, f) (n+1)^{-k(2\sigma-1)-4\sigma}.$$

Let $C_K = (16C_1 C_2)^\sigma$. Taking the n th root and using the inequality $(n+1)^{-1/n} \geq 1/2$, we get

$$C_K \mathcal{A}^{2\sigma}(E, k+1) \geq \left(\frac{r}{k(n+1)^{2\sigma-1}}\right)^{k/n} \left(\frac{t}{4r}\right)^{\mu/n} M^{-1}(t, f).$$

Let $t = r_n$. Since $\mu \geq n^2/4$, $4r \leq r_n$ and $M(r_n, f) \leq e^n$, we get

$$\left(\frac{r_n}{4r}\right)^{\mu/n} M^{-1}(r_n, f) \geq \left(\frac{r_n}{4r}\right)^{\mu/n} e^{-4\mu/n} = \exp\left(\frac{\mu}{n} \log \frac{r_n}{4e^4 r}\right),$$

so

$$C_K \mathcal{A}_K^{2\sigma}(E, k+1) \geq \left(\frac{r}{k(n+1)^{2\sigma-1}}\right)^{k/n} \exp\left(\frac{\mu}{n} \log \frac{r_n}{4e^4 r}\right).$$

□

11. ALGEBRAIC GROWTH OF TRANSCENDENTAL FUNCTIONS

Let f be an entire function of finite order ρ and K be an algebraic number field of degree $\sigma = [K : \mathbb{Q}]$. As in Section 10 we assume, without loss of generality, that $m(r) \leq r^{\phi(r)}$, where $\phi(r) \rightarrow \rho$ and $r^{\phi(r)-\rho}$ is a slowly increasing function. Recall the definition of the sequence $\{r_n\}$ by the equations $r_n^{\phi(r_n)} = n$.

Given a transcendental function f we define the *algebraic growth characteristic* of f on K by

$$\mathbf{a}_K(s, r, m) = \inf\{\log \mathcal{A}_K(E, m) : E \subset \Delta_r, |E| \geq s\}.$$

Due to our knowledge of the behavior of $Z_n(r)$ we are now able to get estimates for $\mathbf{a}_K(s, r, m)$. The first series of results applies to general transcendental functions. We recall that when $\sigma > 2$, then for every $\epsilon > 0$ there is an entire function f of order smaller than ϵ such that $f(K) \subset I_K$ (see [GS, Satz 1]). Moreover, one can find such a function so that $f^{(m)}(K) \subset K$ for all m (see [GS, Satz 2]).

Our first theorem shows that when m and r are fixed, the algebraic growth characteristic exceeds $s^{1/2} \log s$, at least for a subsequence of integers s .

Theorem 11.1. *If f has finite order $\rho > 0$ then for all $m, r \geq 1$*

$$\limsup_{s \rightarrow \infty} \frac{\mathbf{a}_K(s, r, m)}{s^{1/2} \log s} > \frac{2^{-3\rho/2-10}}{\sigma} \left(\frac{\Lambda m}{\rho(\rho+5)} \right)^{1/2}.$$

Proof. By Corollary 6.4 there is a fundamental sequence of integers $\{n_j\}$ for f with the following property: For every $r \geq 1$ there is an integer j_r such that $Z_{n_j}(r) \leq an_j^2$ for $j \geq j_r$, where

$$a = \frac{2^{3\rho+11}(\rho+5)}{\Lambda\rho}.$$

We may assume that $4r \leq r_{n_j}$ when $j \geq j_r$. Let $s_j = an_j^2/m + 1$ and E be a subset of Δ_r with $|E| \geq s_j$. If k is the integer from Theorem 10.4 corresponding to $n = n_j, r, E$, then $m|E| > Z_{n_j}(r) \geq k|E|$, so $m \geq k + 1$. It follows from Theorem 10.4 that

$$C_K \mathcal{A}_K^{2\sigma}(E, m) \geq (m(n_j + 1)^{2\sigma-1})^{-m/n_j} \exp\left(\frac{n_j}{4} \log \frac{r_{n_j}}{4e^4 r}\right).$$

For all j sufficiently large (depending on r, m) we have

$$(m(n_j + 1)^{2\sigma-1})^{-m/n_j} \geq 1/2.$$

Since $r_{n_j}^{\phi(r_{n_j})} = n_j$ and $\phi(r_{n_j}) \rightarrow \rho$, we conclude that there is a sequence of positive $\epsilon_j \rightarrow 0$ such that

$$\frac{n_j}{4} \log \frac{r_{n_j}}{4e^{4r}} \geq \frac{(1 - \epsilon_j)n_j}{4\rho} \log n_j.$$

It follows that

$$2\sigma \mathbf{a}_K(s_j, r, m) + \log(2C_K) \geq \frac{(1 - \epsilon_j)n_j}{4\rho} \log n_j.$$

Since $n_j = (m(s_j - 1)/a)^{1/2}$ we see that

$$\limsup_{s \rightarrow \infty} \frac{\mathbf{a}_K(s, r, m)}{s^{1/2} \log s} \geq \frac{(m/a)^{1/2}}{16\sigma\rho} > \frac{2^{-3\rho/2-10}}{\sigma} \left(\frac{\Lambda m}{\rho(\rho + 5)} \right)^{1/2}.$$

□

Remark: It is interesting to note that the value of \limsup in the above theorem is achieved on a sequence $\{s_j\}$ depending only on f and m , $s_j = an_j^2/m + 1$.

As mentioned above, there are functions whose derivatives of all orders map K into K . So it is interesting to estimate the algebraic growth of such functions on the sets $I_K(d, A)$. The number dA can be viewed as the algebraic measure of the set $I_K(d, A)$, while $\mathcal{A}_K(I_K(d, A), m)$ is the algebraic measure of the set of values of f and its derivatives on $I_K(d, A)$. We introduce

$$\eta_K(\lambda, r, m) = \inf \{ \log \mathcal{A}_K(I_K(d, A) \cap \Delta_r, m) : dA \geq \lambda \}.$$

The following result describes the growth of $\eta_K(\lambda, r, m)$.

Corollary 11.2. *If $\sigma \geq 3$, then there is a constant c' depending only on K such that for $r, m \geq 1$*

$$\limsup_{\lambda \rightarrow \infty} \frac{\eta(\lambda, r, m)}{\lambda^{\sigma/2-1} \log \lambda} \geq \frac{\sigma - 2}{\sigma} 2^{-3\rho/2-10} \left(\frac{c' \Lambda m}{\rho(\rho + 5)} \right)^{1/2}.$$

Proof. By Lemma 10.1, $|I_K(d, A) \cap \Delta_r| \geq c'(dA)^{\sigma-2}$. By Theorem 11.1, let s_j be a sequence such that

$$\limsup_{j \rightarrow \infty} \frac{\mathbf{a}_K(s_j, r, m)}{s_j^{1/2} \log s_j} \geq \frac{2^{-3\rho/2-10}}{\sigma} \left(\frac{\Lambda m}{\rho(\rho + 5)} \right)^{1/2}.$$

We define λ_j by $s_j = c'\lambda_j^{\sigma-2}$. If $dA \geq \lambda_j$ then $\eta(\lambda_j, r, m) \geq \mathbf{a}_K(s_j, r, m)$, so

$$\limsup_{j \rightarrow \infty} \frac{\eta(\lambda_j, r, m)}{\lambda_j^{\sigma/2-1} \log \lambda_j} \geq \sqrt{c'}(\sigma - 2) \limsup_{j \rightarrow \infty} \frac{\mathbf{a}_K(s_j, r, m)}{s_j^{1/2} \log s_j},$$

and the conclusion follows. \square

Let $I_K(A) = I_K(1, A)$ be the set of algebraic integers $z \in I_K$ of norm $\|z\| \leq A$. Clearly, $I_K(A) \subset \Delta_A$. In our next theorem we estimate the number of points $z \in I_K(A)$ which are mapped to points of I_K of smallest possible norm A' . Since $|z| \leq A$ and $|f(z)| \leq \|f(z)\|$, it is natural to expect, due to the growth of f , that $A' \geq \exp(A^{\phi(A)})$. We will prove that if $\rho < \sigma/2$ then the proportion of points of $I_K(A_j)$ which are mapped by f into $I_K(\exp A_j^{\phi(A_j)})$ tends to 0, for a certain sequence $A_j \rightarrow \infty$.

To this end, we need the following version of Theorem 10.4, which provides upper bounds for $|E|$ if the algebraic measure of order 1 of f on E is bounded above by certain quantities.

Proposition 11.3. *There exists a constant C_K depending only on K such that if $n \geq 1$, $1 \leq r \leq r_n/4$, $E \subset \Delta_r$ and*

$$C_K \mathcal{A}_K^{2\sigma}(E, 1) < \exp\left(\frac{n}{4} \log \frac{r_n}{4e^4 r}\right),$$

then $|E| \leq Z_n(r)$.

Proof. If $|E| > Z_n(r)$ and k is the integer from Theorem 10.4, then $k|E| \leq Z_n(r)$ implies $k = 0$. Since $\mu \geq n^2/4$, we reach a contradiction with the conclusion of Theorem 10.4. \square

We have the following theorem:

Theorem 11.4. *If f is an entire function of order $0 < \rho < \sigma/2$ then*

$$\liminf_{A \rightarrow \infty} \frac{|I_K(A) \cap f^{-1}(I_K(\exp A^{\phi(A)}))|}{|I_K(A)|} = 0.$$

Proof. By Theorem 5.4 and Corollary 2.6 in [CP2], there exists a fundamental sequence $\{n_j\}$ for f and positive numbers $\epsilon_j \rightarrow 0$ such that

$$Z_{n_j}(r) \leq a n_j^2 \log 3r, \quad 1 \leq r \leq n_j^{1/\rho - \epsilon_j} / 6,$$

where $a = 2^{3\rho+10}(\rho+5)/(\Lambda\rho)$.

Let $A_j = n_j^{1/((1+\epsilon)\rho)}$, where $\epsilon > 0$ is chosen so that $\sigma > 2(1+\epsilon)\rho$, and let $E_j = I_K(A_j) \cap f^{-1}(I_K(\exp A_j^{\phi(A_j)}))$. Then

$$\mathcal{A}_K(E_j, 1) \leq \exp A_j^{\phi(A_j)} = \exp n_j^{\phi(A_j)/((1+\epsilon)\rho)},$$

and for j sufficiently large

$$Z_{n_j}(A_j) \leq a n_j^2 \log 3A_j = a A_j^{2(1+\epsilon)\rho} \log 3A_j.$$

Since $r_{n_j} = n_j^{1/\phi(r_{n_j})}$ and $\phi(r_{n_j}) \rightarrow \rho$, we have $A_j < r_{n_j}/(8e^4)$ for all j sufficiently large, hence

$$\frac{n_j}{4} \log \frac{r_{n_j}}{4e^4 A_j} > \frac{n_j}{4} \log 2.$$

As $\phi(A_j)/((1+\epsilon)\rho) \rightarrow 1/(1+\epsilon)$ as $j \rightarrow \infty$, we conclude that for all j sufficiently large we have

$$C_K \mathcal{A}_K^{2\sigma}(E_j, 1) < \exp\left(\frac{n_j}{4} \log \frac{r_{n_j}}{4e^4 A_j}\right).$$

Proposition 11.3 implies that $|E_j| \leq Z_{n_j}(A_j)$, so

$$A_j^{-\sigma} |E_j| \leq a A_j^{2(1+\epsilon)\rho-\sigma} \log 3A_j \rightarrow 0$$

as $j \rightarrow \infty$. The theorem follows by Lemma 10.1, as $|I_K(A_j)| \geq c' A_j^\sigma$, with a constant c' depending only on K . \square

The theorems of Polya and Gelfond state that if an entire transcendental function takes integer values at all integer points, or Gaussian integer values at all Gaussian integer points, then its order is at least 1, and respectively at least 2. Using Theorem 11.4, we can obtain asymptotic estimates for the number of integer (or Gaussian integer) points in the disk of radius A , where a function f takes integer (respectively Gaussian integer) values.

Corollary 11.5. *Let K be either \mathbb{Q} or $\mathbb{Q}(i\sqrt{p})$, where $p > 0$ is a square free integer. If f is an entire function of order $0 < \rho < \sigma/2$ then*

$$\liminf_{A \rightarrow \infty} \frac{|I_K \cap f^{-1}(I_K) \cap \Delta_A|}{|I_K \cap \Delta_A|} = 0.$$

Proof. Note that for $z \in K$ we have $\|z\| = |z|$, so

$I_K \cap \Delta_A = I_K(A)$, $I_K \cap f^{-1}(I_K) \cap \Delta_A = I_K(A) \cap f^{-1}(I_K(\exp A^{\phi(A)}))$, for every $A > 0$. The conclusion now follows from Theorem 11.4. \square

We conclude by considering entire transcendental functions which have a covering system of admissible intervals $I(R_j, \alpha, \beta, \gamma, C)$ (see Corollary 4.3). Classes of such functions were constructed in Section 7. In this case we can estimate $\mathbf{a}_K(s, r, m)$ for fixed values of r, m and for all s sufficiently large. Let $\tau = 1 + 1/\gamma$.

Theorem 11.6. *Let f be as above and let $m, r \geq 1$. There exist positive constants a depending only on f , and C'_K depending only on K , such that*

$$\mathbf{a}_K(s, r, m) \geq \frac{(ms)^{1/\tau}}{64\sigma\rho\tau a^{1/\tau}} \log \frac{ms}{a} - C'_K,$$

for all s sufficiently large.

Proof. By Corollary 6.5 there is n_r such that

$$Z_n(r) \leq an^\tau, \quad a = 10C(2\beta^{-1})^{1/\gamma},$$

when $n \geq n_r$. We fix $n_0 = n_0(m, r) \geq n_r$ such that

$$(m(n+1)^{2\sigma-1})^{-m/n} \geq 1/2, \quad 4e^4r \leq n^{1/(4\rho)}, \quad r_n \geq n^{1/(2\rho)},$$

for $n \geq n_0$.

Let $s > a(2n_0)^\tau/m$, and let E be a subset of Δ_r with $|E| \geq s$. If

$$n = \left[\left(\frac{ms}{a} \right)^{1/\tau} \right] - 1,$$

then $n > n_0$ and $m|E| > Z_n(r)$. Applying Theorem 10.4 as in the proof of Theorem 11.1, it follows that

$$2C_K \mathcal{A}_K^{2\sigma}(E, m) \geq \exp\left(\frac{n}{4} \log \frac{r_n}{4e^4r}\right) \geq \exp\left(\frac{n \log n}{16\rho}\right).$$

Since

$$n \log n \geq \frac{1}{2\tau} \left(\frac{ms}{a} \right)^{1/\tau} \log \frac{ms}{a},$$

we obtain

$$2C_K \mathcal{A}_K^{2\sigma}(E, m) \geq \exp\left(\frac{1}{32\rho\tau} \left(\frac{ms}{a} \right)^{1/\tau} \log \frac{ms}{a}\right),$$

so

$$\mathbf{a}_K(s, r, m) \geq \frac{(ms)^{1/\tau}}{64\sigma\rho\tau a^{1/\tau}} \log \frac{ms}{a} - \frac{\log(2C_K)}{2\sigma}.$$

□

We remark that versions of Theorem 11.4 and Corollary 11.5 can be stated for functions f as in Theorem 11.6, by requiring that $\rho < \sigma/\tau$ and replacing the “lim inf” in the conclusion by “lim”.

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DEPARTMENT OF MATHEMATICS, 215 CARNEGIE HALL, SYRACUSE UNIVERSITY, SYRACUSE, NY 13244-1150, USA. E-MAIL: DCOMAN@SYR.EDU, EAPOLETS@SYR.EDU