

Syracuse University

**SURFACE**

---

Electrical Engineering and Computer Science -  
Technical Reports

College of Engineering and Computer Science

---

10-1969

## A Note on the Free Distance of a Convolutional Code

Alexander Miczo  
*Syracuse University*

Follow this and additional works at: [https://surface.syr.edu/eecs\\_techreports](https://surface.syr.edu/eecs_techreports)



Part of the [Computer Sciences Commons](#)

---

### Recommended Citation

Miczo, Alexander, "A Note on the Free Distance of a Convolutional Code" (1969). *Electrical Engineering and Computer Science - Technical Reports*. 14.

[https://surface.syr.edu/eecs\\_techreports/14](https://surface.syr.edu/eecs_techreports/14)

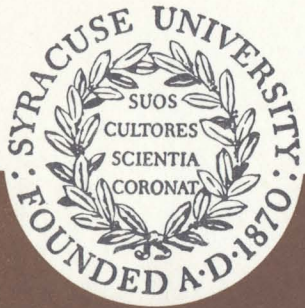
This Report is brought to you for free and open access by the College of Engineering and Computer Science at SURFACE. It has been accepted for inclusion in Electrical Engineering and Computer Science - Technical Reports by an authorized administrator of SURFACE. For more information, please contact [surface@syr.edu](mailto:surface@syr.edu).

A NOTE ON THE FREE DISTANCE OF A CONVOLUTIONAL CODE

ALEXANDER MICZO

LUTHER D. RUDOLPH

OCTOBER, 1969



SYSTEMS AND INFORMATION SCIENCE  
SYRACUSE UNIVERSITY

A NOTE ON THE FREE DISTANCE OF A CONVOLUTIONAL CODE

Alexander Miczo

Luther D. Rudolph

Systems and Information Science

Syracuse University

Syracuse, N. Y. 13204<sup>10</sup>

October, 1969

## Abstract

A counterexample to a conjecture on the number of constraint lengths required to achieve the free distance of a rate  $1/n$  systematic convolutional code is presented.

### Footnotes

This work was supported by the National Science Foundation under Grant GK-4737.

<sup>1</sup>D.J. Costello, "A Construction Technique for Random - Error - Correcting Convolutional Codes," IEEE Trans. Information Theory, IT-15, pp. 631-636, September 1969.

A rate  $1/n$  systematic convolutional code is the row space of a generator matrix of the form shown in Figure 1, where

$$\underline{g} = (1, g_0^{(2)}, \dots, g_0^{(n)}, 0, g_1^{(2)}, \dots, g_1^{(n)}, \dots, 0, g_m^{(2)}, \dots, g_m^{(n)}).$$

A code word  $t$  is thus defined by

$$\underline{t} = \underline{i}G$$

where  $\underline{i} = (i_0, i_1, \dots)$  is the input sequence. Let  $\underline{i}_j = (i_0, i_1, \dots, i_j)$ .

$G_j$  denotes the matrix consisting of the first  $(j+1)n$  columns of  $G$ .

Costello<sup>1</sup> defines the order  $j$  column distance,  $d_j$ , to be

$$d_j = \min_{i_0 \neq 0} W_H(\underline{i}_j G_j)$$

where  $W_H(x)$  is the Hamming weight of  $x$ . He then defines the free distance to be

$$d_{\text{free}} = \lim_{j \rightarrow \infty} d_j.$$

Since  $d_j$  is a monotonically increasing function of  $j$  and  $d_{\text{free}}$  is upper bounded by  $W_H(\underline{g})$ , we have

$$d_j \leq d_{\text{free}} \leq W_H(\underline{g}) \quad j = 0, 1, \dots$$

For a systematic code, there exists an  $L$  such that  $d_j = d_{\text{free}}$  for all  $j \geq L$ . Costello showed that  $L \leq (n-1)(m+1)m$ . If an algorithm for computing the free distance of a given code were dependent on this bound, it would probably be impractical for all but small codes. Costello conjectured that the bound could be improved to  $L = 2m$ .

This, however, is not the case. In fact there exists no fixed integer  $s$  such that  $L = sm$  for all  $m$ , as we shall now show.

For simplicity, we will consider only rate  $1/2$  binary codes. It will be apparent that our result extends to rate  $1/n$  codes. The generator matrix of a rate  $1/2$  systematic code can be written in the form shown in Figure 2. The weight of a code word  $\underline{t}$  is then given by

$$W_H(\underline{t}) = W_H(\underline{i}) + W_H(\underline{i}G^{(2)}).$$

Consider now a code of odd memory order  $m$  in which the subgenerator  $\underline{g}^{(2)} = (g_0^{(2)}, g_1^{(2)}, \dots, g_m^{(2)})$  is constrained as follows:  $g_i^{(2)} = g_{i+\frac{m+1}{2}}^{(2)}$  for  $i = 0, 1, \dots, \frac{m-1}{2}$ . In this case, the matrix  $G^{(2)}$  is of the form shown in Figure 3. The column distance of the code generated is bounded by

$$d_{\frac{km+k-2}{2}} < W_H(\underline{g}') + k \quad k = 1, 2, \dots$$

This can be seen by considering the code word constructed from the rows of  $G$  that correspond to the shaded blocks of  $G^{(2)}$ . Let  $k^*$  denote the smallest integer for which

$$W_H(\underline{g}') + k^* = d_{\text{free}}.$$

Then

$$L \geq \frac{k^*m + k^* - 2}{2} \geq \frac{k^*}{2} m \quad \text{for } k^* > 1.$$

Now suppose it is possible to find a class of codes for which  $W_H(\underline{g}')$  is an increasing function of  $m$  and for which  $d_{\text{free}} = 2W_H(\underline{g}') + 1$ .

Then

$$k^* = d_{\text{free}}^{-W_H(\underline{g}')} = W_H(\underline{g}') + 1$$

and

$$L \geq \frac{W_H(\underline{g}') + 1}{2} m, \quad ,$$

which shows that there exists no fixed integer  $s$  such that  $L = sm$  for all  $m$ . We now present such a class.

The generator polynomial for the  $k^{\text{th}}$  code in the class is defined by

$$\begin{aligned} \underline{g}'_k(x) &= \underline{g}'_{k-1}(x) + x^{6\phi_{k-1}^2} \\ \phi_k &= \deg(\underline{g}'_k(x)) + 1 \\ \underline{g}_k^{(2)}(x) &= \underline{g}'_k(x) (1 + x^{2\phi_k}) \end{aligned}$$

where  $\underline{g}'_1(x) = 1$ . (Note that this construction inserts 0's between the two copies of  $\underline{g}'$ . This is not inconsistent with above; see Figure 4.)

Theorem

$$d_{\text{free}_k} = 2W_H(\underline{g}'_k) + 1 \quad \text{for } k = 1, 2, \dots$$

Proof

For  $k = 1$ ,  $\underline{g}'_1(x) = 1$ ,  $\phi_1 = 1$  and  $\underline{g}_1^{(2)}(x) = 1 + x^2$ . The reader may easily verify that the free distance of the rate 1/2 binary systematic code with  $\underline{g}^{(2)} = 101$  is

$$d_{\text{free}_1} = 2W_H(\underline{g}'_1) + 1 = 3.$$

Now assume that  $d_{\text{free}_k} = 2W_H(\underline{g}'_k) + 1$ . We must show that



$d_{\text{free}_{k+1}} = 2W_H(\underline{g}'_{k+1})+1$ . Since  $W_H(\underline{g}'_{k+1}) = W_H(\underline{g}'_k)+1$  by construction, this amounts to showing that  $d_{\text{free}_{k+1}} = d_{\text{free}_k} + 2$ . Suppose  $\underline{t}_{k+1}$  is a minimum weight code word in the  $(k+1)$ st code. The corresponding code word in the  $k^{\text{th}}$  code is  $\underline{t}_k = \underline{i}G_k$ . We claim that  $W_H(\underline{t}_{k+1}) \geq W_H(\underline{t}_k)+2$ . This is most easily seen by reference to Figure 4. If  $\underline{t}_{k+1}$  is to have minimum weight in the code, then it cannot be the sum of two disjoint code words. This requires that at least one out of every  $\phi_k$  rows of  $G_k$  be included in the sum,  $\underline{i}G_k$ . There are two cases to consider.

(1) Suppose that  $\underline{t}_{k+1}$  is formed from some combination of the first  $5\phi_k^2$  rows of  $G_{k+1}$ . In this case, the 1 added in going from  $\underline{g}'_k$  to  $\underline{g}'_{k+1}$  cannot be cancelled because of the spacing allowed. Hence  $\underline{t}_{k+1} = \underline{i}G_{k+1}$  will have at least two more 1's than  $\underline{t}_k = \underline{i}G_k$ .

(2) Suppose on the other hand that  $\underline{t}_{k+1}$  is formed from some combination of rows that includes a row beyond the first  $5\phi_k^2$  rows of  $G_{k+1}$ . In the case, the assumption that  $\underline{t}_{k+1}$  has minimum weight requires that at least  $5\phi_k^2/\phi_k = 5\phi_k$  rows be included. But then

$$W_H(\underline{t}_{k+1}) \geq W_H(\underline{i}) \geq 5\phi_k \geq 5W_H(\underline{g}'_k) \geq 2W_H(\underline{g}'_k)+3.$$

Therefore  $d_{\text{free}_{k+1}} = d_{\text{free}_k} + 2$  in either case and the proof is complete.

We have shown here that  $L$  increases more rapidly than  $m$ , and it seems unlikely that  $L$  increases as rapidly as  $m^2$ . This would appear to leave  $m \log m$  as the next most likely candidate.

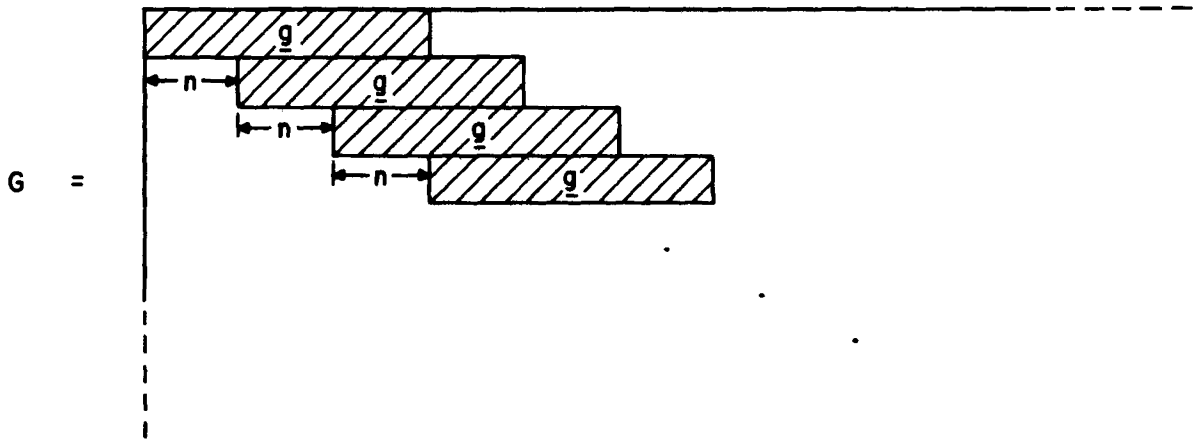


Figure 1

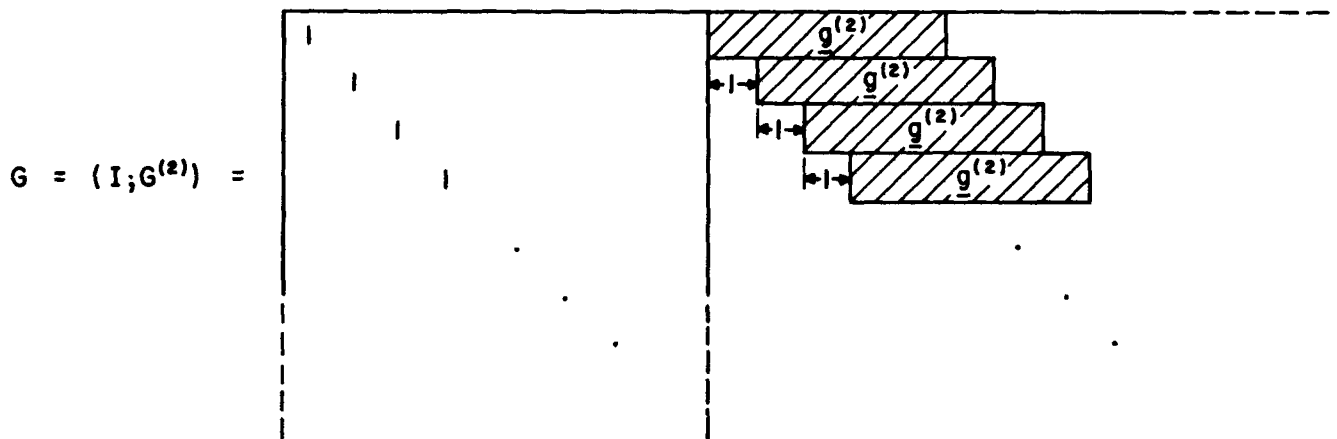


Figure 2

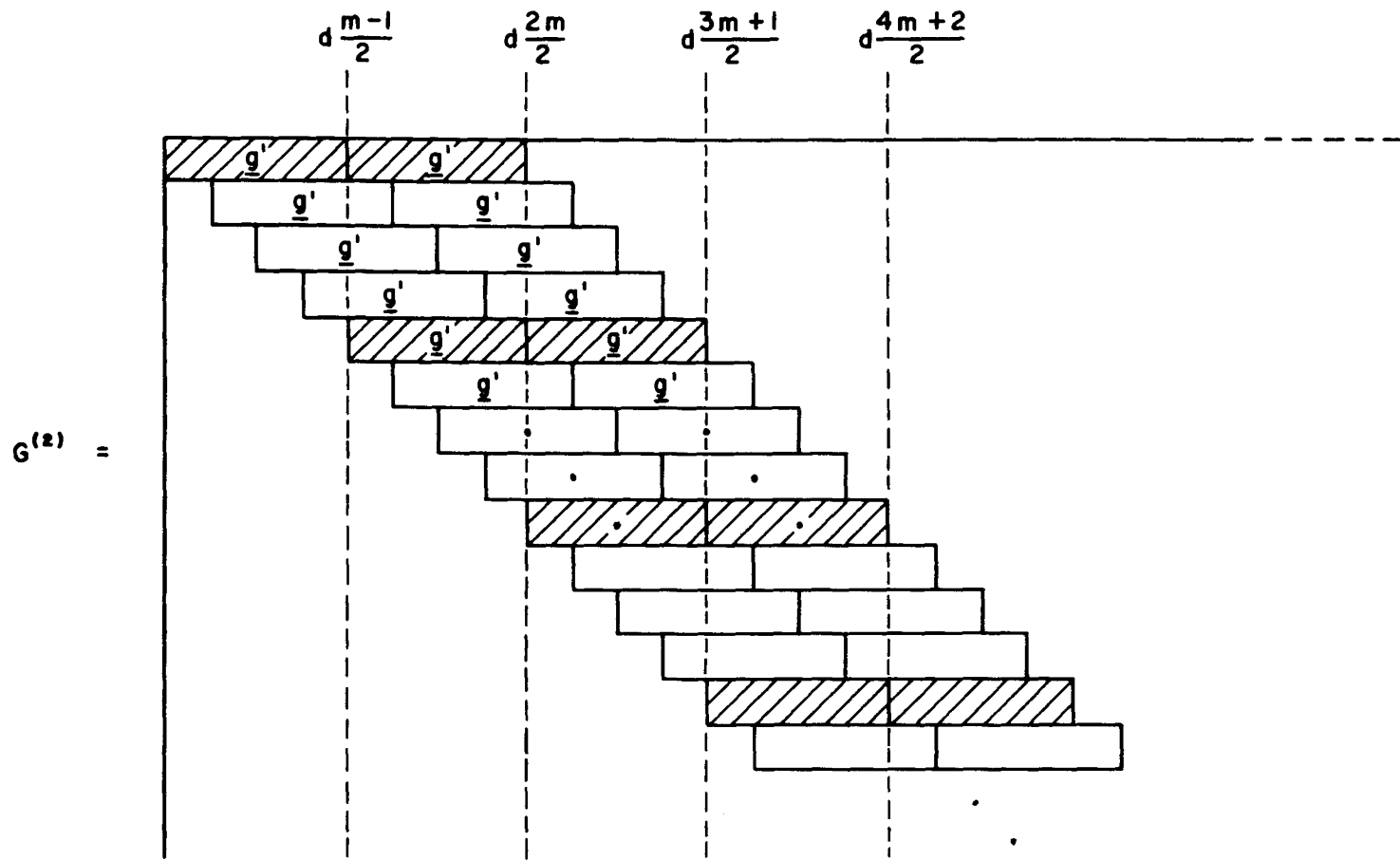


Figure 3

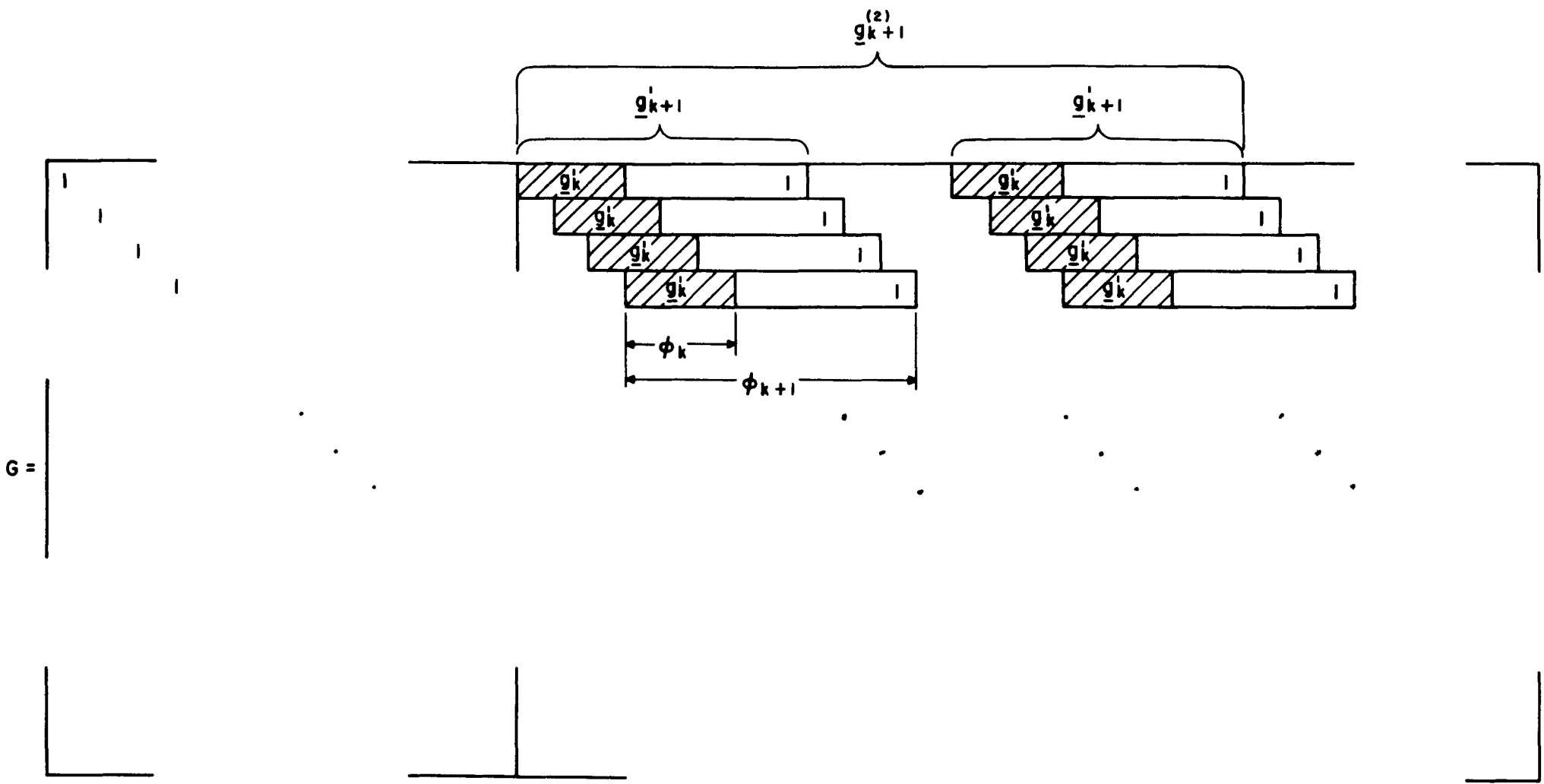


Figure 4