Entire Pluricomplex Green Functions and Lelong Numbers of Projective Currents

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Abstract. Let T be a positive closed current of bidimension (1,1) and unit mass on the complex projective space $\mathbb{P}^n$. We prove that the set $V_\alpha(T)$ of points where T has Lelong number larger than α is contained in a complex line if $\alpha \geq 2/3$, and $|V_\alpha(T) \setminus L| \leq 1$ for some complex line L if $1/2 \leq \alpha < 2/3$. We also prove that in dimension 2 and if $2/5 \leq \alpha < 1/2$, then $|V_\alpha(T) \setminus C| \leq 1$ for some conic C.

1. Introduction

Let T be a positive closed current of bidimension (1,1) on the complex projective space $\mathbb{P}^n$, which has mass $\|T\| = 1$ (see Section 2 for the defintion of $\|T\|$). Siu's theorem [S] states that the upper level sets $E_\alpha(T)$ of the Lelong numbers $\nu(T, \cdot)$ of T,

$$E_\alpha(T) := \{p \in \mathbb{P}^n : \nu(T,p) \geq \alpha\},$$

are analytic subvarieties of $\mathbb{P}^n$ of dimension at most one. Hence by Chow's theorem, $E_\alpha(T)$ are algebraic varieties.

We consider here the following upper level sets of Lelong numbers of T:

$$V_\alpha(T) := \{p \in \mathbb{P}^n : \nu(T,p) > \alpha\},\ \alpha < 1.$$

Given a finite set S, we denote by $|S|$ the number of points of S. The main results of this paper are the following theorems.

Theorem 1.1. *(i) If $\alpha \geq 2/3$, the set $V_\alpha(T)$ is either a complex line or an at most countable subset of a complex line.*

(ii) If $1/2 \leq \alpha < 2/3$, the set $V_\alpha(T)$ is either a complex line, or a countable subset of a complex line, or a finite set such that $|V_\alpha(T) \setminus L| \leq 1$ for some complex line L.

Theorem 1.2. *If $n = 2$ and $2/5 \leq \alpha < 1/2$, the set $V_\alpha(T)$ is one of the following: a conic, the union of a complex line and an at most countable subset of a complex line, a countable subset of a conic, a finite set such that $|V_\alpha(T) \setminus C| \leq 1$ for some conic C.*

The proofs of these theorems require the construction of entire plurisubharmonic functions of logarithmic growth and with logarithmic poles in some special finite subsets of $\mathbb{C}^n$. In certain cases such functions were constructed in [Co] and [CN].

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If $D \subseteq \mathbb{C}^n$ is an open set, we denote by $PSH(D)$ the class of plurisubharmonic functions in D. A function $u \in PSH(D)$ is said to have a logarithmic pole of weight $\alpha > 0$ at $p \in D$, if

$$u(z) = \alpha \log \|z - p\| + O(1)$$

holds for $z \in D$ near p. We say that $u \in PSH(\mathbb{C}^n)$ has logarithmic growth if

$$\gamma_u := \limsup_{\|z\| \to +\infty} \frac{u(z)}{\log \|z\|} < +\infty.$$

Let S be a finite subset of $\mathbb{C}^n$. If $u \in PSH(\mathbb{C}^n)$ has logarithmic growth, it is locally bounded and maximal on $\mathbb{C}^n \setminus S$, and it equals $-\infty$ on S, we call u an entire pluricomplex Green function of S. We let $E(S)$ be the class of entire pluricomplex Green functions of S with logarithmic poles of weight 1 at all points of S. Moreover, we let $\widetilde{E}(S) \subset PSH(\mathbb{C}^n)$ be the class of plurisubharmonic functions with logarithmic growth, which are locally bounded on $\mathbb{C}^n \setminus S$ and have logarithmic poles of weight 1 at the points of S. These classes were introduced in [CN], where we defined and studied two affine invariants $\gamma(S) \geq \widetilde{\gamma}(S)$ of the set S,

$$\gamma(S) = \inf\{\gamma_u : \, u \in E(S)\}, \; \widetilde{\gamma}(S) = \inf\{\gamma_u : \, u \in \widetilde{E}(S)\}.$$

These numbers are connected to the singular degree of S introduced in [W] and studied in [W] and [Ch].

We note that entire pluricomplex Green functions were generally considered in the case when they are locally bounded, for example when dealing with the extremal function of a compact set, while pluricomplex Green functions with logarithmic poles were considered mostly in the case of bounded domains, by prescribing their boundary values to be 0 (see e.g. the survey [B]). For our purposes, it is useful to combine these two cases, and consider entire plurisubharmonic functions with finitely many logarithmic poles, as in [Co] and [CN].

Theorems 1.1 and 1.2, together with two other related results, are proved in Section 3. We also give examples of currents which show that the results of these theorems are sharp. In Proposition 2.1 from Section 2 we show the connection between the Lelong numbers of projective currents at the points of a finite set S, and the growth constants γ_u of functions $u \in PSH(\mathbb{C}^n)$ with logarithmic poles in S. The entire pluricomplex Green functions which we need to prove the results from Section 3 are constructed in Section 2.

2. Entire pluricomplex Green functions

We denote by $[z : t]$, $(z, t) \in \mathbb{C}^n \times \mathbb{C}$, $(z, t) \neq (0, 0)$, the homogeneous coordinates on $\mathbb{P}^n$, and we use the standard embedding $\mathbb{C}^n \equiv \{[z : 1] \in \mathbb{P}^n : \, z \in \mathbb{C}^n\}$. Let ω be the standard Kähler form on $\mathbb{P}^n$, corresponding to the Fubini-Study metric. Then

$$\omega \mid_{\mathbb{C}^n} = dd^c V, \; V(z) := \log \sqrt{1 + \|z\|^2},$$

where $d^c = (\partial - \overline{\partial})/(2\pi i)$. If T is a positive closed current on $\mathbb{P}^n$ of bidimension (l,l), its mass is given by

$$\|T\| = \int_{\mathbb{P}^n} T \wedge \omega^l.$$

The following simple result is analogous to [CN, Theorem 3.4].

Proposition 2.1. *Let $S = \{p_1, \dots, p_k\} \subset \mathbb{C}^n$ and let T be a positive closed current on $\mathbb{P}^n$ of bidimension (l,l). If $u \in PSH(\mathbb{C}^n)$ has logarithmic growth, it is locally bounded outside a finite set, and $u(z) \leq \alpha_j \log \|z - p_j\| + O(1)$ for z near p_j, where $\alpha_j > 0$, $1 \leq j \leq k$, then*

$$\sum_{j=1}^{k} \alpha_j^l \nu(T, p_j) \leq \gamma_u^l \|T\|.$$

In particular, $\sum_{j=1}^{k} \nu(T, p_j) \leq \widetilde{\gamma}(S)^l \|T\|$.

Proof. Since u is locally bounded outside a finite set, the positive measure $T \wedge (dd^c u)^l$ is well defined on $\mathbb{C}^n$, by [D, Proposition 2.1]. Demailly's first comparison theorem for Lelong numbers with weights [D, Theorem 5.1] implies that

$$T \wedge (dd^c u)^l(\{p_j\}) \geq \alpha_j^l \nu(T, p_j).$$

Using [CN, Proposition 3.2] we obtain

$$\sum_{j=1}^{k} \alpha_j^l \nu(T, p_j) \leq \int_{\mathbb{C}^n} T \wedge (dd^c u)^l \leq \gamma_u^l \int_{\mathbb{C}^n} T \wedge (dd^c V)^l \leq \gamma_u^l \|T\|.$$

□

We now construct entire pluricomplex Green functions of small growth, for special finite subsets S of $\mathbb{C}^n$. We denote by $m_j(S)$ the maximum number of points of S which are contained in an algebraic curve of degree j.

Lemma 2.2. *If $S \subset \mathbb{C}^n$, $|S| \in \{3, 4\}$, $m_1(S) = 2$, then $\gamma(S) = 2$.*

Proof. If S is contained in a two-dimensional complex plane, the lemma follows from analogous results in $\mathbb{C}^2$ [CN]. Otherwise, we can assume that $S \subset \mathbb{C}^3 \times \{0\}$ consists of the points $(0,0,0,0)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$. Let

$$u(z) = \frac{1}{2} \log \left(\sum_{j=1}^{3} |P_j(z_1, z_2, z_3)|^2 + \sum_{k=4}^{n} |z_k|^2 \right), \ z = (z_1, \dots, z_n),$$

where $P_j(z_1, z_2, z_3) = z_j(z_1 + 2z_2 + 3z_3 - j)$. Then $\gamma_u = \gamma(S) = 2$. □

In the next three results, we consider the case of sets $S \subset \mathbb{C}^2$ with 7 or 8 elements.

Proposition 2.3. *Let $S \subset \mathbb{C}^2$ be such that $|S| = 7$ and $m_2(S) = 5$. Then S has an entire pluricomplex Green function u with $\gamma_u = 4$, such that u has logarithmic poles of weight 2 at 3 of the points of S, and of weight 1 at the remaining 4 points of S.*

Proof. We show that we can find 3 points of S, $\zeta_1, \zeta_2, \zeta_3$, with the following property: There exist two polynomials P_1, P_2 of degree 4, with no common factors, such that $P_1(S) = P_2(S) = 0$ and both P_1, P_2 vanish to second order at each point ζ_j. By Bezout's theorem it follows that $S = \{P_1 = P_2 = 0\}$, and the intersection numbers $(P_1 \cdot P_2)_{\zeta_j} = 4$ and $(P_1 \cdot P_2)_\zeta = 1$ at all other points $\zeta \in S$. By [CN, Theorem 4.1], the function $u = \frac{1}{2}\log(|P_1|^2 + |P_2|^2)$ has the desired properties.

Let $S = \{p_1, \dots, p_7\}$. The vector space of polynomials of degree at most 4, which vanish on S and to second order at 3 given points of S, has dimension at least 2.

Case 1. $m_1(S) = 2$, $m_2(S) = 5$. Let C_1, C_2 be quadratic polynomials vanishing at p_1, p_2, p_3, p_4, p_5, respectively at p_1, p_2, p_3, p_6, p_7. Then C_j are irreducible, the conics $C_j = 0$ are smooth algebraic curves, C_1 does not vanish at p_6, p_7, and C_2 does not vanish at p_4, p_5. Let $P_1 = C_1C_2$ and P_2 be a polynomial of degree 4 vanishing on S and to second order at p_1, p_2, p_3, such that P_1, P_2 are linearly independent. If C_1 divides P_2, then $P_2 = C_1C$, where C is a quadratic polynomial vanishing at p_1, p_2, p_3, p_6, p_7, so $C = \alpha C_2$. It follows that P_1, P_2 have no common factors.

Case 2. $m_1(S) = 3$, $m_2(S) = 5$. We may assume that p_1, p_2, p_3 lie on a complex line l, hence p_4, p_5, p_6, p_7 lie outside of l and are in general position, since $m_2(S) = 5$.

(i) Assume first that $\{p_1, p_2, p_3\}$ is the only 3 point subset of S contained in a complex line. Let C_1, C_2 be irreducible quadratic polynomials vanishing at p_4, p_5, p_6, p_1, p_2, and respectively at p_4, p_5, p_6, p_3, p_7. We let $P_1 = C_1C_2$ and continue as in Case 1.

(ii) Assume that, after relabelling if necessary the points of S, p_1, p_4, p_5 are also contained in a complex line, so p_2, p_3, p_6, p_7 are in general position. We show that there exist 3 points of S, p_i, p_j, p_k, with the following property (P): p_i, p_j, p_k are not contained in a complex line and the polynomial $L_{ij}L_{jk}L_{ki}$ does not vanish at any of the remaining 4 points of S. Then we can continue as in (i), with the points p_4, p_5, p_6 replaced by p_i, p_j, p_k (although the corresponding conics may now be reducible).

If $(L_{24}L_{25}L_{34}L_{35})(p_j) \neq 0$ for $j = 6, 7$, then the points p_2, p_4, p_6 have the above property. Otherwise, we may assume that $L_{24}(p_6) = 0$. Since $m_2(S) = 5$, we have $L_{35}(p_7) \neq 0$. If $L_{35}(p_6) \neq 0$ then p_3, p_5, p_6 verify property (P). Finally, we assume $L_{24}(p_6) = L_{35}(p_6) = 0$. If $L_{25}(p_7) \neq 0$ or $L_{34}(p_7) \neq 0$, then the points p_2, p_5, p_7, respectively p_3, p_4, p_7, have property (P). If $L_{25}(p_7) = L_{34}(p_7) = 0$, then p_1, p_6, p_7 verify property (P). □

Proposition 2.4. *Let $A \subset \mathbb{C}^2$ with $|A| = 7$, $m_1(A) \leq 3$, $m_2(A) = 6$, and let Γ be the conic with $|A \cap \Gamma| = 6$. Let $q \notin A \cup \Gamma$. Then $m_1(A \cup \{q\}) \leq 4$ and we have:*

(i) If $m_1(A \cup \{q\}) \leq 3$ there exists $u \in PSH(\mathbb{C}^2)$ with $\gamma_u = 3$, such that u is locally bounded outside a finite set and $u(z) \leq \log \|z - p\| + O(1)$ near each point $p \in A \cup \{q\}$.

(ii) If $m_1(A \cup \{q\}) = 4$ there exists a subset S of $A \cup \{q\}$ with 7 elements, which has an entire pluricomplex Green function as in the conclusion of Proposition 2.3.

Proof. Let $\{p_1\} = A \setminus \Gamma$, $\{p_2, \dots, p_7\} = A \cap \Gamma = A'$, and C be a quadratic polynomial defining Γ. We have either $m_1(A') = 2$ and C is irreducible, or else $C = l_1l_2$, where l_j are linear polynomials and $|\{l_j = 0\} \cap A'| = 3$, $\{l_1 = 0\} \cap \{l_2 = 0\} \cap A = \emptyset$. In the latter case, we can assume that p_2, p_4, p_6 lie on the line $l_1 = 0$ and p_3, p_5, p_7 on the line $l_2 = 0$. In either case, the conic Γ is smooth at the points p_j, $j \geq 2$.

(i) If $m_1(A \cup \{q\}) \leq 3$, let L be a linear polynomial with $L(p_1) = L(q) = 0$, and let $P_1 = LC$. There exists a polynomial P_2 of degree 3 which is zero on $A \cup \{q\}$ and such that P_1, P_2 are linearly independent. Assume that $P_2 = LC'$, for some quadratic polynomial C'. Since L vanishes at most at one of the points $p_2, \dots, p_7$, and the remaining 5 points determine Γ uniquely, it follows that $C' = \alpha C$, a contradiction. Similarly one shows that C (or l_j if C is reducible) cannot divide P_2, so P_1, P_2 have no common factors. We let $u = \frac{1}{2} \log(|P_1|^2 + |P_2|^2)$.

(ii) If $m_1(A \cup \{q\}) = 4$ then the complex line determined by p_1, q must contain two points of Γ. After relabelling points, we can assume that p_1, p_2, p_3, q are contained in a complex line. Let L_{jk} be a linear polynomial such that $L_{jk}(p_j) = L_{jk}(p_k) = 0$. Since $m_1(A) \leq 3$, we have for each $j \geq 2$ that L_{1j} can vanish at most at one other point p_k ($k \geq 2$, $k \neq j$).

Case 1. If some polynomial L_{1j}, $j \geq 4$, does not vanish at any other point p_k, then there must exist another polynomial L_{1i}, $i \geq 4$, with the same property. Note that when $C = l_1 l_2$ then such i exists so that p_i, p_j are not both contained in a line $l_k = 0$. We let $S = A$, $P_1 = L_{1i} L_{1j} C$, and P_2 be a polynomial of degree 4 vanishing on S and to second order at p_1, p_i, p_j, such that P_1, P_2 are linearly independent. A direct application of Bezout's theorem shows that P_1, P_2 have no common factors. By the considerations at the beginning of the proof of Proposition 2.3, $u = \frac{1}{2} \log(|P_1|^2 + |P_2|^2)$ is the desired pluricomplex Green function of S.

Case 2. If each polynomial L_{1j}, $j \geq 4$, vanishes at some other point p_k, $k \geq 4$, then after relabelling points we have $L_{14}(p_5) = L_{16}(p_7) = 0$. If $S = (A \cup \{q\}) \setminus \{p_2\}$ then $m_2(S) = 5$, and the conclusion follows from Proposition 2.3. Indeed, the points $p_3, \dots, p_7$ determine Γ, which does not contain p_1, q, and p_1, p_4, p_5, p_6, p_7 determine the conic $L_{45} L_{67} = 0$, which does not contain p_3, q. If $S \setminus \{p_j\}$, $j \geq 4$, is contained in a conic and without loss of generality $j = 4$, then this conic is defined by $L_{23} L_{67} = 0$. But $L_{23} L_{67}(p_5) \neq 0$. □

3. Upper level sets of Lelong numbers

Throughout this section, T is a positive closed current of bidimension $(1,1)$ on $\mathbb{P}^n$, normalized by $\|T\| = 1$. Then the Lelong numbers $\nu(T, p) \leq 1$, at every point $p \in \mathbb{P}^n$. We will need the following simple lemma:

Lemma 3.1. *If S is an at most countable subset of $\mathbb{P}^n$, then there exists a hyperplane H which does not intersect S.*

Proof. We show by induction on k, $0 \leq k \leq n-1$, that there exists a k-dimensional plane P_k which does not intersect S. This is clear for $k = 0$, and assume such P_k exists for $k < n-1$. The family of $(k+1)$-dimensional planes P which contain P_k is uncountable, and the sets $P \setminus P_k$ are pairwise disjoint. Since S is at most countable and it does not intersect P_k, it follows that there is a $(k+1)$-dimensional plane $P_{k+1} \supset P_k$ which does not intersect S. □

Recall that for a finite subset S of $\mathbb{P}^n$, $m_j(S)$ denotes the maximum number of points in S which are contained in an algebraic curve of degree j. We now proceed with the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. (i) Assume that some set $E_\beta(T)$, $\beta > \alpha$, has dimension one, so it contains an algebraic curve C. Then $T = \beta[C] + R$, where $[C]$ is the current of integration along C and R is a $(1,1)$ bidimensional positive closed current on $\mathbb{P}^n$. The degree of C (see e.g. [LG]) is $\|[C]\| \leq 1/\beta < 2$, so C is a complex line. Since $\nu(R,p) \leq \|R\| = 1 - \beta < 1/3$ at all $p \in \mathbb{P}^n$, it follows that $V_\alpha(T) = C$.

If all sets $E_\beta(T)$, $\beta > \alpha$, have dimension 0, then $V_\alpha(T)$ is at most countable. By Lemma 3.1 there is a hyperplane H so that $V_\alpha(T) \subset \mathbb{P}^n \setminus H$, so we may assume $V_\alpha(T) \subset \mathbb{C}^n$. If $V_\alpha(T)$ is not contained in a complex line, there exists $S \subseteq V_\alpha(T)$ with $|S| = 3$ and $m_1(S) = 2$, so $\gamma(S) = 2$ by Lemma 2.2. Proposition 2.1 yields the following contradiction:

$$3\alpha < \sum_{p \in S} \nu(T,p) \leq \gamma(S)\|T\| = 2.$$

(ii) Arguing as in the proof of (i), we have either that $V_\alpha(T)$ is a complex line, or it is at most countable and contained in $\mathbb{C}^n$.

If $V_\alpha(T)$ is countable, then $E_{1/2}(T)$ must contain an algebraic curve C. As before, $T = \frac{1}{2}[C] + R$, so the degree of C is at most 2. If C has degree 2 then $R = 0$, so $\nu(T,p) > 1/2$ only at the singular points of C and $V_{1/2}(T)$ is a finite set. We conclude that C is a complex line. Since $\|R\| = 1/2$, we have $V_\alpha(T) \subset C$.

We assume finally that $V_\alpha(T)$ is a finite set not contained in a complex line and with at least 4 elements. So there exists $S = \{p_1, p_2, p_3, p_4\} \subset V_\alpha(T)$ such that p_1, p_2, p_3 are not contained in a complex line. If $m_1(S) = 2$ then Lemma 2.2 and Proposition 2.1 imply $4\alpha < 2$, a contradiction. So, after relabelling points, p_4 lies on the complex line L determined by p_1, p_2. If there exists $p \in V_\alpha(T) \setminus L$, $p \neq p_3$, then at least one of the following sets S,

$$\{p_1, p_2, p_3, p\}, \ \{p_1, p_4, p_3, p\}, \ \{p_2, p_4, p_3, p\},$$

has $m_1(S) = 2$, and we reach a contradiction as above. Therefore $|V_\alpha(T) \setminus L| = 1$. □

Proof of Theorem 1.2. As in the previous proof, there are two possibilities: some set $E_\beta(T)$, $\beta > \alpha$, contains an algebraic curve C, or $V_\alpha(T)$ is at most countable and contained in $\mathbb{C}^2$.

In the first case, $T = \beta[C] + R$, for some positive closed (1,1) current R on $\mathbb{P}^2$, and the degree of C is at most 2. If C is a conic then $\|R\| < 1/5$ and $V_\alpha(T) = C$. If C is a complex line and $R = 0$ then $V_\alpha(T) = C$. Otherwise $0 < \|R\| = 1 - \beta < 3/5$ and $V_\alpha(T) = C \cup V_\alpha(R)$. Since $\alpha/\|R\| \geq 2/3$, $V_\alpha(R)$ is a complex line or an at most countable subset of a complex line, by Theorem 1.1.

We assume next that $V_\alpha(T)$ is countable, so $E_{2/5}(T)$ contains an algebraic curve C of degree at most 2, and $T = \frac{2}{5}[C] + R$. If C is a conic then $V_\alpha(T) \subset C$. If C is a complex line then $\|R\| = 3/5$, so $V_\alpha(R)$ is contained in a complex line since $\alpha/\|R\| \geq 2/3$, and $V_\alpha(T) \subset C \cup V_\alpha(R)$.

We assume finally that $V_\alpha(T)$ is a finite set and $|V_\alpha(T)\setminus C|>1$, for any conic C. Then $V_\alpha(T)$ has a subset A with $|A|=7$. We have the following possibilities:

Case 1. $m_2(A)=5$. Then A has an entire pluricomplex Green function u as in Proposition 2.3. Proposition 2.1 applied to u and T implies $10\alpha<4$, a contradiction.

Case 2. $m_1(A)\leq 3$, $m_2(A)=6$. Let C be the conic with $|A\cap C|=6$. Since $|V_\alpha(T)\setminus C|>1$, there is $q\in V_\alpha(T)\setminus(A\cup C)$. Applying Proposition 2.1 with the functions provided by Proposition 2.4 yields $8\alpha<3$, or $10\alpha<4$, a contradiction.

Case 3. $m_1(A)=2$ and A is contained in a conic C. There exists $q\in V_\alpha(T)\setminus C$. If $p\in A$, then $S=(A\cup\{q\})\setminus\{p\}$ verifies the hypotheses of Case 2.

Case 4. $m_1(A)\geq 3$. We show that $V_\alpha(T)$ has a subset S with $|S|=7$, $m_1(S)\leq 3$, $m_2(S)\leq 6$, so we are back to Case 1 or Case 2. Let L_{jk} be the complex line determined by the points p_j,p_k. Let p_1,p_2,p_3 be points of A so that $p_3\in L_{12}$. As $|V_\alpha(T)\setminus C|>1$ for any conic C, there exist points $p_4,\dots,p_7\in V_\alpha(T)\setminus L_{12}$, with $p_6,p_7\notin L_{45}$. Let $S_1=\{p_1,\dots,p_7\}$. Then $m_2(S_1)\leq 6$, so $m_1(S_1)\leq 4$. If $m_1(S_1)\leq 3$ we let $S=S_1$. If $m_1(S_1)=4$ then, after relabelling points, $p_6,p_7\in L_{14}$. Hence there exists $p_8\in V_\alpha(T)\setminus(L_{12}\cup L_{14})$, $p_8\neq p_5$. The point p_8 can lie on at most one of the lines L_{5j}, $j=4,6,7$. If $p_8\in L_{5j}$ for some $j\in\{4,6,7\}$, we let $S=(S_1\cup\{p_8\})\setminus\{p_j\}$. Otherwise, we let $S=(S_1\cup\{p_8\})\setminus\{p_7\}$. $\Box$

It is not difficult to see that all cases described in Theorems 1.1 and 1.2 can occur. We present here a few examples in $\mathbb{P}^2$.

Example 3.2. Let L_1,L_2,L_3 be complex lines with $L_1\cap L_2\cap L_3=\emptyset$, and let $\{p_1\}=L_2\cap L_3$, $\{p_2\}=L_1\cap L_3$, $\{p_3\}=L_1\cap L_2$. If $T=\frac{1}{3}\sum_{j=1}^3[L_j]$ then $E_{2/3}(T)=\{p_1,p_2,p_3\}$ is not contained in a complex line, so the value $\alpha=2/3$ in Theorem 1.1 (i) is the best possible.

Example 3.3. Let $L_1,\dots,L_4$ be complex lines so that no 3 of them pass through the same point, let $\{p_{jk}\}=L_j\cap L_k$, $1\leq j<k\leq 4$, and $S=\{p_{jk}\}$, $|S|=6$. If $T=\frac{1}{4}\sum_{j=1}^4[L_j]$ then $E_{1/2}(T)=S$ has $m_1(S)=3$, so the value $\alpha=1/2$ in Theorem 1.1 (ii) is sharp. Moreover, $E_{1/2}(T)$ is not contained in a conic.

Example 3.4. If C is a conic and $T=[C]/2$ then $E_{1/2}(T)\setminus L$ is an uncountable set, for every complex line L.

Example 3.5. Let L_j be complex lines containing the point q, L be a complex line with $q\notin L$, and $\{p_j\}=L\cap L_j$. If $m\geq 2$ and

$$T=\frac{m-1}{2m}\,[L]+\frac{m+1}{2m^2}\,\sum_{j=1}^m[L_j],$$

then $V_{1/2}(T)=\{p_1,\dots,p_m,q\}$ has m points on L.

Example 3.6. Let L_j,p_j be as in Example 3.2, and let $p_4\notin L_1\cup L_2\cup L_3$. Let L_4,L_5,L_6 be the complex lines determined by the points p_4 and respectively p_1,p_2,p_3, and let $\{p_5\}=L_1\cap L_4$, $\{p_6\}=L_2\cap L_5$, $\{p_7\}=L_3\cap L_6$. Finally, let l_1,l_2,l_3 be the complex lines determined by pairs of points from $\{p_5, p_6, p_7\}$. If $S = \{p_1, \dots, p_7\}$ and

$$T = \frac{1}{15}\left(2\sum_{j=1}^{6}[L_j] + \sum_{j=1}^{3}[l_j]\right),$$

then $m_2(S) = 5$, $E_{2/5}(T) = S$, so the value $\alpha = 2/5$ in Theorem 1.2 is sharp.

Example 3.7. Let C be a conic and let $p_j \in C$, $j \geq 0$, be distinct points so that $p_j \to p_0$ as $j \to \infty$. Let L_j be a complex line passing through p_j and some given point $q \notin C$. If $\epsilon_j > 0$, $\sum_{j=0}^{\infty} \epsilon_j = 1/5$, and

$$T = \frac{2}{5}\,[C] + \sum_{j=0}^{\infty} \epsilon_j [L_j],$$

then $V_{2/5}(T)$ is a countable subset of C.

We saw in Examples 3.3 and 3.4 that for $\beta < 1/2$ one can have $|V_\beta(T) \setminus L| > 1$ for every complex line L. But if T has "large" Lelong numbers at two points, then the conclusion of Theorem 1.1 (ii) still holds for some values $\beta < 1/2$.

Theorem 3.8. *Assume that $\alpha > 1/2$ and there are points $q_1, q_2 \in \mathbb{P}^n$ so that $\nu(T, q_j) \geq \alpha$, $j = 1, 2$. If $\beta = (2-\alpha)/3$ then $|V_\beta(T) \setminus L| \leq 1$ for some complex line L which contains at least one of the points q_1, q_2.*

Proof. Let L_1 be the complex line determined by q_1, q_2. If $|V_\beta(T) \setminus L_1| > 1$, let $p_1, p_2 \in V_\beta(T) \setminus L_1$ and let L_2 be the complex line determined by p_1, p_2. We choose $1/2 < \alpha' < \alpha$ so that $\nu(T, p_j) > (2-\alpha')/3$ and we consider the current

$$R = \frac{2\alpha' - 1}{2\alpha'}\,[L_2] + \frac{1}{2\alpha'}\,T.$$

Then $\nu(R, q_j) \geq \alpha/(2\alpha') > 1/2$ and

$$\nu(R, p_j) > \frac{2\alpha' - 1}{2\alpha'} + \frac{2-\alpha'}{6\alpha'} > \frac{1}{2}\,,$$

for $j = 1, 2$. Theorem 1.1 (ii) implies that one of the points q_1, q_2, say without loss of generality q_1, lies on L_2. We show that $V_\beta(T) \setminus \{q_2\} \subseteq L_2$. If not, there exists $p_3 \in V_\beta(T) \setminus \{q_2\}$, $p_3 \notin L_2$. If L_{jk} denotes the complex line determined by p_j, p_k, let

$$R = \frac{2\alpha' - 1}{6\alpha'}\,([L_{12}] + [L_{23}] + [L_{13}]) + \frac{1}{2\alpha'}\,T,$$

where $1/2 < \alpha' < \alpha$ is such that $\nu(T, p_j) > (2-\alpha')/3$ for $j = 1, 2, 3$. Then $\nu(R, q_j) > 1/2$ and

$$\nu(R, p_j) > \frac{2(2\alpha' - 1)}{6\alpha'} + \frac{2-\alpha'}{6\alpha'} = \frac{1}{2}\,.$$

If $S = \{p_1, p_2, p_3, q_1, q_2\}$ then $m_1(S) = 3$, which contradicts Theorem 1.1 (ii). $\square$

Example 3.9. Let L_1, L_2 be complex lines in $\mathbb{P}^n$ which intersect at the point q, and let $T = \frac{1}{2}([L_1] + [L_2])$. Then $\nu(T, q) = 1$ and $E_{1/2}(T) \setminus L$ is uncountable for every complex line L. So the assumption in Theorem 3.8 on the existence of two points where T has large Lelong numbers, is necessary.

Our last result shows that the complex lines from the conclusion of Theorem 1.1 are determined by 3 points where T has "small" Lelong numbers.

Proposition 3.10. *Let $\alpha \geq 1/2$ and assume that $E_{1-\alpha}(T)$ contains the points p_1, p_2, p_3 which lie on a complex line $L \subset \mathbb{P}^n$. Then $|V_\alpha(T) \setminus L| \leq 1$. Moreover, if $\alpha \geq 2/3$ then either $V_\alpha(T) \subseteq L$ or else $|V_\alpha(T)| \leq 2$.*

Proof. If $|V_\alpha(T) \setminus L| > 1$, let $q_1, q_2 \in V_\alpha(T) \setminus L$, and let $\alpha' > \alpha$ be chosen so that $\nu(T, q_j) > \alpha'$, $j = 1, 2$. Then

$$R = \frac{2\alpha' - 1}{2\alpha'} [L] + \frac{1}{2\alpha'} T$$

has Lelong numbers larger than $1/2$ at the points of the set $S = \{p_1, p_2, p_3, q_1, q_2\}$. As $m_1(S) = 3$, this contradicts Theorem 1.1 (*ii*).

We assume next that $\alpha \geq 2/3$ and there exists $q \in V_\alpha(T) \setminus L$. By what we already proved, we have $V_\alpha(T) \setminus \{q\} \subseteq L$. Theorem 1.1 (*i*) implies that $V_\alpha(T) \subseteq L'$, where L' is a complex line containing q. It follows that $|V_\alpha(T)| \leq 2$. □

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