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Stable Algebras of Entire Functions

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DOMAINS OF DEFINITION OF MONGE-AMPÈRE OPERATORS ON COMPACT KÄHLER MANIFOLDS

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Abstract. Let (X,ω) be a compact Kähler manifold. We introduce and study the largest set $DMA(X,\omega)$ of ω-plurisubharmonic (psh) functions on which the complex Monge-Ampère operator is well defined. It is much larger than the corresponding local domain of definition, though still a proper subset of the set $PSH(X,\omega)$ of all ω-psh functions.

We prove that certain twisted Monge-Ampère operators are well defined for all ω-psh functions. As a consequence, any ω-psh function with slightly attenuated singularities has finite weighted Monge-Ampère energy.

Introduction

It is well known that the complex Monge-Ampère operator is not well defined for arbitrary plurisubharmonic (psh) functions. Bedford and Taylor [BT3] found a way to define it for locally bounded psh functions. Later the definition was extended to classes of unbounded functions (see [Sib], [D2], [FS]). Whenever defined, the Monge-Ampère operator was shown to be continuous along decreasing sequences, but it is discontinuous along sequences in L^p. The natural domain of definition of the Monge-Ampère operator on open sets in $\mathbb{C}^n$ was recently characterized in [C2], [Bl1], [Bl2].

We consider here the problem of defining Monge-Ampère operators on a compact Kähler manifold X of complex dimension n. Let $PSH(X,\omega)$ denote the set of ω-plurisubharmonic (ω-psh) functions on X. Here, and throughout the paper, ω is a fixed Kähler form on X. Recall that an upper semicontinuous function $\varphi \in L^1(X)$ is called ω-psh if the current $\omega_\varphi := \omega + dd^c\varphi$ is positive. Motivated by the results of [BT3], [C2], [Bl1], [Bl2], it is natural to define the domain of the Monge-Ampère operator as follows:

Definition. *Let $DMA(X,\omega)$ be the set of functions $\varphi \in PSH(X,\omega)$ for which there is a positive Radon measure $MA(\varphi)$ with the following property: If $\{\varphi_j\}$ is any sequence of bounded ω-psh functions decreasing to φ then $(\omega + dd^c\varphi_j)^n \to MA(\varphi)$, in the weak sense of measures. We set*

$$\omega_\varphi^n = (\omega + dd^c\varphi)^n := MA(\varphi).$$

According to this definition, $DMA(X,\omega)$ is the largest set of ω-psh functions on which the Monge-Ampère operator $(\omega + dd^c\cdot)^n$ can be defined so that it is continuous with respect to decreasing sequences of bounded ω-psh functions. It includes all the classes in which the operator was previously

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defined, either as a consequence of the local theory, or genuinely in the compact setting (the class $\mathcal{E}(X,\omega)$ from [GZ2]). The set $DMA(X,\omega)$ is a proper subset of $PSH(X,\omega)$ (see Examples 1.3 and 1.4). We show in Proposition 1.1 that the operator is continuous under decreasing sequences in its domain. Moreover, if $\varphi \in DMA(X,\omega)$ then we prove in Proposition 1.2 that the set of points where φ has positive Lelong number is at most countable.

There are several properties of $DMA(X,\omega)$ that we expect to hold. We discuss these in section 1.2, and we introduce a few subclasses of interest, especially the class $\widehat{DMA}(X,\omega)$ (Definition 1.7): a function $\varphi \in PSH(X,\omega)$ belongs to this class if it is in $DMA(X,\omega)$ and moreover for any sequence of bounded ω-psh function φ_j decreasing towards φ, and for any "test function" $u \in PSH(X,\omega) \cap L^\infty(X)$,

$$\int_X u(\omega + dd^c\varphi_j)^n \longrightarrow \int_X u(\omega + dd^c\varphi)^n.$$

This convergence property interpolates inbetween the two natures of the Monge-Ampère measure $(\omega + dd^c\varphi)^n$: on one hand it is stronger than the weak convergence in the sense of positive Radon measures (any smooth test function is $C\omega$-psh for some constant $C > 0$), on the other hand it is weaker than the convergence in the sense of Borel measures. We show (Theorem 1.9) that a generalized comparison principle holds in $\widehat{DMA}(X,\omega)$, and that all concrete classes under consideration are subsets of this class (see Corollary 2.3 and Theorem 3.2).

In [GZ2] a class $\mathcal{E}(X,\omega) \subset PSH(X,\omega)$ was introduced, on which the Monge-Ampère operator is well defined and continuous along decreasing sequences, hence $\mathcal{E}(X,\omega) \subset DMA(X,\omega)$. Defining this class requires that one works globally on a compact manifold, hence many of its properties have no analogue in the local context (see [GZ2]). We study in Section 2 more general classes $\mathcal{E}(T,\omega)$ of ω-psh functions with finite energy with respect to a closed positive current T. These help us in studying twisted Monge-Ampère operators, which happen to be well defined in all of $PSH(X,\omega)$ (Theorems 2.4 and 2.5). As a consequence, we show that any ω-psh function with slightly attenuated singularities has finite energy (Corollary 2.6):

Theorem. *Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a smooth convex increasing function, with $\chi'(-1) \leq 1$, $\chi'(-\infty) = 0$. Fix $\varphi \in PSH(X,\omega)$ with $\sup_X \varphi \leq -1$. Then*

$$\chi \circ \varphi \in \mathcal{E}(X,\omega) \subset DMA(X,\omega),$$

hence its Monge-Ampère measure does not charge pluripolar sets.

Note that it is necessary to slightly attenuate the singularities of φ (condition $\chi'(-\infty) = 0$), since functions in $\mathcal{E}(X,\omega)$ have zero Lelong numbers at all points (see Lemma 3.5).

In Sections 3 and 4 we consider the set $DMA_{loc}(X,\omega)$ on which the Monge-Ampère operator is defined as a consequence of the local theory ([Bl1], [Bl2]). We prove in Theorem 3.2 that this local domain can be characterized in terms of energy classes. Moreover, $DMA_{loc}(X,\omega)$ is a proper subset of $DMA(X,\omega)$ and consists of functions whose gradient is square integrable (Proposition 4.2). In Proposition 4.6 we show that ω-psh functions, bounded in a neighborhood of an ample divisor, belong to $DMA_{loc}(X,\omega)$.

In Proposition 4.1 we prove that the measure $\chi''(\varphi)d\varphi \wedge d^c\varphi \wedge \omega^{n-1}$ has density in $L^1(X)$ for every $\varphi \in PSH(X,\omega)$, where χ is any convex increasing function. It is an interesting problem to study the stability of subclasses in $DMA(X,\omega)$ under standard geometric constructions. We show that $DMA_{loc}(X,\omega)$ is not preserved by blowing up, but behaves well under blowing down.

We conclude this paper by analyzing further the connection between $DMA(X,\omega)$ and various energy classes in some concrete cases of Kähler surfaces. The motivation comes from the fact that energy classes have important properties, such as convexity and stability under taking maximum. Hence an equivalent description of $DMA(X,\omega)$ in terms of such energy classes would be very useful. Let $\mathcal{E}(\omega,\omega)$ be the class of ω-psh functions on X such that the trace measure $\omega_\varphi \wedge \omega$ does not charge the set $\{\varphi = -\infty\}$. Then both $DMA_{loc}(X,\omega)$ and $\mathcal{E}(X,\omega)$ are contained in $\mathcal{E}(\omega,\omega)$. We give some evidence that $\mathcal{E}(\omega,\omega)$ might be equal to $DMA(X,\omega)$ when $\dim_{\mathbb{C}} X = 2$.

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1. General properties and examples

1.1. **Lelong numbers and other constraints.** It is important to require in the definition that the convergence holds on *any* sequence of bounded ω-psh functions (see Example 1.3). This allows us to show that the operator is continuous under decreasing sequences in $DMA(X,\omega)$. We have in fact the following more general property.

Proposition 1.1. *Let $\epsilon_j \geq 0$, $\epsilon_j \to 0$, and let $\varphi_j \in DMA(X,(1+\epsilon_j)\omega)$ be a decreasing sequence towards $\varphi \in DMA(X,\omega)$. Then*

$$((1+\epsilon_j)\omega + dd^c\varphi_j)^n \longrightarrow (\omega + dd^c\varphi)^n$$

in the weak sense of measures.

Proof. We first assume that all $\epsilon_j = 0$. Fix χ a test function and set, for an integer $k > 0$, $\varphi_j^k := \max(\varphi_j, -k) \in PSH(X,\omega) \cap L^\infty(X)$. Since $\varphi_j \in DMA(X,\omega)$, we can find an increasing sequence k_j so that

$$\left| \langle (\omega + dd^c\varphi_j^{k_j})^n, \chi \rangle - \langle (\omega + dd^c\varphi_j)^n, \chi \rangle \right| \leq 2^{-j}.$$

Thus $\tilde{\varphi_j} := \max(\varphi_j, -k_j)$ is a sequence of bounded ω-psh functions decreasing towards φ, hence $\langle \omega^n_{\tilde{\varphi}_j}, \chi \rangle \to \langle \omega^n_\varphi, \chi \rangle$. The desired convergence follows.

In the general case, subtracting a constant we may assume that $\varphi_1 < 0$. The sequence of measures $((1+\epsilon_j)\omega + dd^c\varphi_j)^n$ has bounded mass, so by passing to a subsequence we may assume that it converges weakly to a measure μ. By taking another subsequence we may assume that ϵ_j decreases to 0. Then $\varphi_j' = \varphi_j/(1+\epsilon_j) \in DMA(X,\omega)$ is a decreasing sequence to φ, so $\omega^n_{\varphi_j'} \to \omega^n_\varphi$. We conclude that $\mu = \omega^n_\varphi$. $\square$

If $\varphi \in PSH(X,\omega)$, let $\nu(\varphi,x)$ be the Lelong number of φ at $x \in X$, and

$$E_\varepsilon(\varphi) = \{x \in X \,/\, \nu(\varphi,x) \geq \varepsilon\}\ ,\ E^+(\varphi) = \bigcup_{\varepsilon > 0} E_\varepsilon(\varphi).$$

Proposition 1.2. *If $\varphi \in DMA(X,\omega)$ then $\omega_\varphi^n(\{p\}) \geq \nu^n(\varphi,p)$, for all $p \in X$. Moreover, the set $E^+(\varphi)$ is at most countable and*

$$\sum_{x\in E^+(\varphi)} \nu^n(\varphi,x) \leq \int_X \omega^n.$$

Proof. Observe that the measure ω_φ^n has at most countably many atoms. We are going to show that if $\nu(\varphi,x)>0$ then ω_φ^n has an atom at x. Let $g=\chi \log dist(\cdot,x)$, where $\chi\geq 0$ is a smooth cut off function such that $\chi=1$ in a neighborhood of x. Since ω is Kähler, $dd^c g\geq 0$ near x, and g is smooth on $X\setminus\{x\}$, we can find $\varepsilon_0>0$ such that $\varepsilon_0 g\in PSH(X,\omega)$. Consider

$$\varphi_j := \max(\varphi, \varepsilon_0 g - j) \in PSH(X,\omega)\cap L^\infty_{loc}(X\setminus\{x\}).$$

Proposition 4.6 from Section 4 implies $\varphi_j\in DMA(X,\omega)$. Since φ_j is in the local domain of definition of the Monge-Ampère operator, we have by [D2]

$$\omega^n_{\varphi_j}(\{x\}) \geq \nu^n(\varphi_j,x) = (\min(\varepsilon_0,\nu(\varphi,x)))^n.$$

Using Proposition 1.1, we infer that ω_φ^n has a Dirac mass at x.

By [D1], there exist sequences $\varepsilon_j,\delta_j\searrow 0$, and $\psi_j\in PSH(X,(1+\delta_j)\omega)$, $\psi_j\searrow\varphi$, such that ψ_j is smooth in $X\setminus E_{\varepsilon_j}(\varphi)$ and $\sup_{x\in X}|\nu(\psi_j,x)-\nu(\varphi,x)|\to 0$. Using Siu's theorem [Siu] and the previous discussion, we conclude that $E_{\varepsilon_j}(\varphi)$ is a finite set and

$$(1+\delta_j)^n\int_X\omega^n \geq \sum_{x\in E^+(\varphi)}\nu^n(\psi_j,x),\ ((1+\delta_j)\omega+dd^c\psi_j)^n(\{p\})\geq\nu^n(\psi_j,p),$$

where $p\in X$. We conclude by Proposition 1.1, letting $j\to+\infty$. □

Proposition 1.2 provides examples of functions not in $DMA(X,\omega)$.

Example 1.3. *Let $X=\mathbb{P}^2$, ω be the Fubini-Study Kähler form, and $\varphi\in PSH(\mathbb{P}^2,\omega)$ be so that $\omega_\varphi = d^{-1}[\mathcal{C}]$, where $[\mathcal{C}]$ is the current of integration along an algebraic curve $\mathcal{C}$ of degree $d\geq 1$. By Proposition 1.2, $\varphi\notin DMA(\mathbb{P}^2,\omega)$ since $E^+(\varphi)$ is not countable. Alternatively, we can construct two sequences of functions in $DMA(\mathbb{P}^2,\omega)$ decreasing to φ with constant Monge-Ampère measures that are different. Indeed, if L is a generic line, $\varphi_j^L := \max(\varphi, u_L - j)\in PSH(\mathbb{P}^2,\omega)$, where $\omega+dd^c u_L=[L]$, then φ_j^L is continuous outside the finite set $L\cap\mathcal{C}$, and $\varphi_j^L\searrow\varphi$. Hence $\varphi_j^L\in DMA(\mathbb{P}^2,\omega)$ and*

$$(\omega+dd^c\varphi_j^L)^2 = \omega_\varphi\wedge[L] = \frac{1}{d}\sum_{p\in L\cap\mathcal{C}}\delta_p$$

is independent of j. Here δ_p is the Dirac mass at p, and the first equality follows easily since the currents involved have local potentials which are pluriharmonic away from their $(-\infty)$-locus. Using sequences $\varphi_j^L,\varphi_j^{L'}$, for lines $L\neq L'$, we conclude by Proposition 1.1 that $\varphi\notin DMA(\mathbb{P}^2,\omega)$.

The previous construction can be generalized to exhibit examples of functions $\varphi\notin DMA(X,\omega)$ with zero Lelong numbers at all but one point.

Example 1.4. *Assume $\varphi\in PSH(\mathbb{P}^2,\omega)$ is such that $\{\varphi=-\infty\}$ is a closed proper subset of $\mathbb{P}^2$, and the positive current ω_φ is supported on $\{\varphi=-\infty\}$. We claim that $\varphi\notin DMA(\mathbb{P}^2,\omega)$. Indeed, let $p\notin\{\varphi=-\infty\}$ and q_1,q_2 be* *distinct points in* $\{\varphi = -\infty\}$. *The lines* $L_k = (pq_k)$ *intersect* $\{\varphi = -\infty\}$ *in compact subsets of* $\mathbb{C} \subset \mathbb{P}^1 \simeq L_k$, *hence both sequences*

$$\varphi_j^{L_k} := \max(\varphi, u_{L_k} - j) \in DMA(\mathbb{P}^2, \omega), \text{ where } \omega + dd^c u_{L_k} = [L_k],$$

decrease towards φ. *Now* $(\omega + dd^c \varphi_j^{L_k})^2 = \omega_\varphi \wedge [L_k]$ *are distinct measures (independent of* j*), hence* $\varphi \notin DMA(\mathbb{P}^2, \omega)$.

A concrete function φ as in Example 1.4 is obtained as follows. Fix $a \in \mathbb{P}^2$ and a line $L \simeq \mathbb{P}^1$, $a \notin L$. Let π denote the projection from a onto L: this is a meromorphic map which is holomorphic in $\mathbb{P}^2 \setminus \{a\}$. Fix ν a probability measure on L with no atom, whose logarithmic potential is $-\infty$ exactly on the support of ν (one can consider for instance the Evans potential of a compact polar Cantor set). Then $\pi^*\nu = \omega_\varphi$, where $\varphi \in PSH(\mathbb{P}^2, \omega)$ satisfies our assumptions. Note that $\nu(\varphi, x) = 0$, $x \neq a$, and $\nu(\varphi, a) = 1$. Further examples can be obtained by considering functions φ such that ω_φ is a laminar current, whose transverse measure is an Evans measure.

1.2. **Special subclasses.** There are several properties of $DMA(X, \omega)$ that we expect to hold. We analyze here some of these and introduce interesting subclasses of $DMA(X, \omega)$.

1.2.1. *Intermediate Monge-Ampère operators.* It is natural to expect that if a function $\varphi \in PSH(X, \omega)$ has a well defined Monge-Ampère measure $(\omega + dd^c\varphi)^n$, then the currents $(\omega + dd^c\varphi)^\ell$ are also well defined for $1 \leq \ell \leq n$. Unfortunately we are unable to prove this (except of course in dimension 2), hence the following:

Definition 1.5. *We let* $DMA_\ell(X, \omega)$, *where* $1 \leq \ell \leq n$, *be the set of functions* $\varphi \in PSH(X, \omega)$ *for which there is a positive closed current* $MA^\ell(\varphi)$ *of bidegree* (ℓ, ℓ) *with the following property: If* $\{\varphi_j\}$ *is any sequence of bounded* ω*-psh functions decreasing to* φ *then* $\omega_{\varphi_j}^\ell \to MA^\ell(\varphi)$, *in the weak sense of currents. We set* $\omega_\varphi^\ell = (\omega + dd^c\varphi)^\ell := MA^\ell(\varphi)$. *We also set*

$$DMA_{\leq k}(X, \omega) = \bigcap_{\ell=1}^{k} DMA_\ell(X, \omega).$$

We let the reader check that Proposition 1.1 holds for $\varphi \in DMA_\ell(X, \omega)$ and $1 \leq \ell \leq n$. Clearly

$$DMA_{\leq n}(X, \omega) \subseteq DMA_n(X, \omega) = DMA(X, \omega),$$

with equality when $n = 2$ since $\varphi \longmapsto \omega_\varphi$ is well defined for all ω-psh functions. We expect the equality to hold also when $n \geq 3$.

It is clear from the definition that $\varphi \in DMA(X, \omega)$ if and only if $\lambda\varphi \in DMA(X, \lambda\omega)$, where $\lambda > 0$. One would expect moreover that, if $\varphi \in DMA(X, \omega)$, then $\lambda\varphi \in DMA(X, \omega)$ for $0 \leq \lambda \leq 1$, as the function $\lambda\varphi$ is slightly less singular than φ. It is also natural to expect that the class $DMA(X, \omega)$ is stable under taking maximum:

$$\varphi \in DMA(X, \omega) \text{ and } \psi \in PSH(X, \omega) \overset{?}{\Longrightarrow} \max(\varphi, \psi) \in DMA(X, \omega).$$

If this property holds, then applying it to $\psi = \lambda\varphi \le 0$, $0 \le \lambda \le 1$, shows that $\lambda\varphi = \max(\varphi, \lambda\varphi) \in DMA(X,\omega)$ as soon as $\varphi \in DMA(X,\omega)$. An alternative and desirable property is

$$DMA(X,\omega) \stackrel{?}{=} DMA(X,\omega') \cap PSH(X,\omega),$$

when $\omega \le \omega'$. All these properties are related to convexity properties of $DMA(X,\omega)$.

1.2.2. *The non pluripolar part.* It was observed in [GZ2] that if φ is ω-psh and $\varphi_j := \max(\varphi, -j)$, then

$$j \longmapsto \mu_j(\varphi) := \mathbf{1}_{\{\varphi > -j\}}(\omega + dd^c\varphi_j)^n$$

is an increasing sequence of positive Radon measures, of total mass uniformly bounded above by $\int_X \omega^n$. Thus $\{\mu_j(\varphi)\}$ converges to a positive Radon measure $\mu(\varphi)$ on X. It is generally expected (see [BT4] for similar considerations in the local context) that $\mu(\varphi)$ should correspond to the non pluripolar part of $(\omega + dd^c\varphi)^n$, whenever the latter makes sense. We can justify this expectation in two special cases:

Proposition 1.6. *Fix $\varphi \in DMA(X,\omega)$. Then*

$$\mu(\varphi) \le \mathbf{1}_{\{\varphi > -\infty\}}(\omega + dd^c\varphi)^n,$$

with equality if $\exp\varphi$ is continuous, or if ω_φ^n is concentrated on $\{\varphi = -\infty\}$.

Proof. We can assume wlog that $\varphi \le 0$. Set $u_s := \max(\varphi/s+1, 0)$. Note that u_s are bounded ω-psh functions which increase towards $\mathbf{1}_{\{\varphi > -\infty\}}$. Moreover $\{u_s > 0\} = \{\varphi > -s\}$ and $u_s = 0$ elsewhere. We infer, when $j > s$, that

$$u_s(\omega + dd^c\varphi_j)^n = u_s \mathbf{1}_{\{\varphi > -j\}}(\omega + dd^c\varphi_j)^n,$$

where $\varphi_j = \max(\varphi, -j)$ are the canonical approximants.

When e^φ is continuous, then so is u_s, hence passing to the limit yields $u_s(\omega + dd^c\varphi)^n = u_s\mu(\varphi)$. Letting $s \to +\infty$, we infer $\mu(\varphi) = \mathbf{1}_{\{\varphi > -\infty\}}\mu(\varphi) = \mathbf{1}_{\{\varphi > -\infty\}}(\omega + dd^c\varphi)^n$.

In the general case, we only get $u_s\mu(\varphi) \le u_s(\omega + dd^c\varphi)^n$, since u_s is upper-semi-continuous. This yields $\mu(\varphi) = \mathbf{1}_{\{\varphi > -\infty\}}\mu(\varphi) \le \mathbf{1}_{\{\varphi > -\infty\}}(\omega + dd^c\varphi)^n$, whence equality if $(\omega + dd^c\varphi)^n$ is concentrated on $\{\varphi = -\infty\}$. □

To overcome this difficulty in the general case, we introduce interesting subclasses of $DMA(X,\omega)$:

Definition 1.7. *Fix $1 \le \ell \le n$. We let $\widehat{DMA}_\ell(X,\omega)$ (resp. $\widehat{DMA}_{\le \ell}(X,\omega)$) denote the set of functions $\varphi \in DMA_\ell(X,\omega)$ (resp. $DMA_{\le \ell}(X,\omega)$) such that for any sequence $\varphi_j \in PSH(X,\omega) \cap L^\infty(X)$ decreasing to φ,*

$$\int_X u(\omega + dd^c\varphi_j)^\ell \wedge \omega^{n-\ell} \longrightarrow \int_X u(\omega + dd^c\varphi)^\ell \wedge \omega^{n-\ell},$$

for all $u \in PSH(X,\omega) \cap L^\infty(X)$.

Note that this convergence property is stronger than the usual convergence in the weak sense of Radon measures: any smooth test function is $C\omega$-psh for some constant $C > 0$. The following corollary can be proved exactly like Proposition 1.6.

Corollary 1.8. *If* $\varphi \in \widehat{DMA}(X,\omega)$ *then*

$$\mu(\varphi) = \mathbf{1}_{\{\varphi > -\infty\}}(\omega + dd^c\varphi)^n.$$

Moreover, if $\varphi_j = \max(\varphi, -j)$ *and* $B \subset \{\varphi > -\infty\}$ *is any Borel set then*

$$\int_B \omega_\varphi^n = \lim_{j \to +\infty} \int_{B \cap \{\varphi > -j\}} \omega_{\varphi_j}^n.$$

As a consequence, we obtain the following generalized comparison principle:

Theorem 1.9. *Let* $\varphi, \psi \in PSH(X,\omega)$ *and set* $\varphi \vee \psi := \max(\varphi, \psi)$*. Then*

$$\mathbf{1}_{\{\varphi > \psi\}}\mu(\varphi) = \mathbf{1}_{\{\varphi > \psi\}}\mu(\varphi \vee \psi).$$

Moreover if $\varphi, \psi \in \widehat{DMA}(X,\omega)$ *then*

$$\int_{\{\varphi < \psi\}} \omega_\psi^n \leq \int_{\{\varphi < \psi\} \cup \{\varphi = -\infty\}} \omega_\varphi^n.$$

Proof. Set $\varphi_j = \max(\varphi, -j)$ and $\psi_j = \max(\psi, -j)$. Recall from [BT4] that the desired equality is known for bounded psh functions,

$$\mathbf{1}_{\{\varphi_j > \psi_{j+1}\}}(\omega + dd^c\varphi_j)^n = \mathbf{1}_{\{\varphi_j > \psi_{j+1}\}}(\omega + dd^c \max(\varphi_j, \psi_{j+1}))^n.$$

Observe that $\{\varphi > \psi\} \subset \{\varphi_j > \psi_{j+1}\}$, hence

$$\begin{aligned}\mathbf{1}_{\{\varphi > \psi\}} \cdot \mathbf{1}_{\{\varphi > -j\}}(\omega + dd^c\varphi_j)^n &= \mathbf{1}_{\{\varphi > \psi\}} \cdot \mathbf{1}_{\{\varphi > -j\}}(\omega + dd^c \max(\varphi, \psi, -j))^n \\ &= \mathbf{1}_{\{\varphi > \psi\}} \cdot \mathbf{1}_{\{\varphi \vee \psi > -j\}}(\omega + dd^c \max(\varphi \vee \psi, -j))^n.\end{aligned}$$

Note that the sequence of measures $\mathbf{1}_{\{\varphi > -j\}}(\omega + dd^c\varphi_j)^n$ converges in the strong sense of Borel measures towards $\mu(\varphi)$ and the sequence $\mathbf{1}_{\{\varphi \vee \psi > -j\}}(\omega + dd^c(\varphi \vee \psi, -j))^n$ converges in the strong sense of Borel measures towards $\mu(\varphi \vee \psi)$. Hence, since $\{\varphi > \psi\} \subset \{\varphi > -\infty\} \subset \{\varphi \vee \psi > -\infty\}$, it follows from Corollary 1.8 that

$$\mathbf{1}_{\{\varphi > \psi\}}\mu(\varphi) = \mathbf{1}_{\{\varphi > \psi\}}\mu(\varphi \vee \psi).$$

Now let $\varphi, \psi \in \widehat{DMA}(X,\omega)$ and assume first that ψ is bounded. Then it follows from Corollary 1.8 that

$$\int_{\{\varphi < \psi\}} (\omega + dd^c\psi)^n = \int_{\{\varphi < \psi\}} (\omega + dd^c \max(\psi, \varphi))^n.$$

Since $\int_X (\omega + dd^c \max(\psi, \varphi))^n = \int_X (\omega + dd^c\varphi)^n$ it follows that

$$\int_{\{\varphi < \psi\}} (\omega + dd^c\psi)^n \leq \int_X (\omega + dd^c\varphi)^n - \int_{\{\varphi > \psi\}} (\omega + dd^c\varphi)^n = \int_{\{\varphi \leq \psi\}} (\omega + dd^c\varphi)^n.$$

If $\psi \in \widehat{DMA}(X,\omega)$ is not bounded, we apply the previous inequality to φ and $\psi_j := \max(\psi, -j)$ for $j \in \mathbb{N}$. Then we get for any $j \in \mathbb{N}$

$$\int_{\{\varphi < \psi_j\}} (\omega + dd^c\psi_j)^n \leq \int_{\{\varphi \leq \psi_j\}} (\omega + dd^c\varphi)^n.$$

Since $\{\varphi < \psi\} \cap \{\psi > -j\} \subset \{\varphi < \psi_j\}$ for any j and $\{\varphi \leq \psi_j\}$ is a decreasing sequence of Borel sets converging to the Borel set $\{\varphi \leq \psi\}$, by taking the limit in the previous inequalities and applying Corollary 1.8, we obtain

$$\int_{\{\varphi<\psi\}} (\omega + dd^c\psi)^n \leq \int_{\{\varphi\leq\psi\}} (\omega + dd^c\varphi)^n.$$

Now take $0 < \varepsilon < 1$ and apply the previous result with $\varphi + \varepsilon$ and ψ. Then let $\varepsilon \to 0$ to obtain the required inequality. $\square$

Remark 1.10. *Let $\varphi \in PSH(X, \omega)$ and $\varphi_j := \max(\varphi, -j)$. It follows as in* [GZ2] *that $\mu_j^\ell(\varphi) = \mathbf{1}_{\{\varphi>-j\}}(\omega + dd^c\varphi_j)^\ell$ is an increasing sequence of positive currents of mass at most $\int_X \omega^n$, so it converges in the strong sense to a positive current $\mu^\ell(\varphi)$ on X. We leave it to the reader to check that Proposition 1.6, Corollary 1.8 and Theorem 1.9 have analogues in the classes $DMA_\ell(X, \omega)$. For instance, if $\varphi \in DMA_\ell(X, \omega)$ and e^φ is continuous, or if $\varphi \in \widehat{DMA}_\ell(X, \omega)$, then $\mu^\ell(\varphi) = \mathbf{1}_{\{\varphi>-\infty\}}\omega_\varphi^\ell$. Moreover, if $\varphi, \psi \in \widehat{DMA}_\ell(X, \omega)$ then*

$$\int_{\{\varphi<\psi\}} \omega_\psi^\ell \wedge \omega^{n-\ell} \leq \int_{\{\varphi<\psi\}\cup\{\varphi=-\infty\}} \omega_\varphi^\ell \wedge \omega^{n-\ell}.$$

Remark 1.11. *Observe that in dimension $n = 1$,*

$$PSH(X, \omega) = DMA(X, \omega) = \widehat{DMA}(X, \omega).$$

The latter equality follows easily by integrating by parts and using the monotone convergence theorem. We expect that the equality $\widehat{DMA}(X, \omega) = DMA(X, \omega)$ continues to hold in higher dimension.

1.2.3. *The class $\mathcal{E}(X, \omega)$.* We consider here

$$\mathcal{E}(X, \omega) = \left\{\varphi \in PSH(X, \omega) \,/\, \mu(\varphi)(X) = \int_X \omega^n\right\}.$$

Equivalently, $\varphi \in \mathcal{E}(X, \omega)$ if and only if $(\omega + dd^c\varphi_j)^n(\{\varphi \leq -j\}) \to 0$. This class of functions was studied in [GZ2], where it was shown that

(*i*) $\mathcal{E}(X, \omega) \subset DMA(X, \omega)$;

(*ii*) $\mathcal{E}(X, \omega)$ is convex and stable under maximum;

(*iii*) $\mathcal{E}(X, \omega)$ is the largest subclass of $DMA(X, \omega)$ on which the comparison principle holds.

The rough idea is that a ω-psh function φ should belong to $\mathcal{E}(X, \omega)$ iff it belongs to $DMA(X, \omega)$ and its Monge-Ampère measure does not charge pluripolar sets. This is partly justified by the following result:

Proposition 1.12. *A function φ belongs to $\mathcal{E}(X, \omega)$ if and only if it belongs to $\widehat{DMA}(X, \omega)$ and $(\omega + dd^c\varphi)^n$ does not charge pluripolar sets.*

Proof. The inclusion $\mathcal{E}(X, \omega) \subset \widehat{DMA}(X, \omega)$ will follow from Theorem 2.1 below. Assume conversely that $\varphi \in \widehat{DMA}(X, \omega)$ is such that $(\omega + dd^c\varphi)^n$ does not charge pluripolar sets. Then

$$\mu(\varphi) = \mathbf{1}_{\{\varphi>-\infty\}}(\omega + dd^c\varphi)^n = (\omega + dd^c\varphi)^n$$

has full mass, hence $\varphi \in \mathcal{E}(X, \omega)$. $\square$

Remark 1.13. *The previous characterization of $\mathcal{E}(X,\omega)$ is related to the question of uniqueness of solutions to the equation $(\omega+dd^c\cdot)^n=\mu$. Indeed assume $\varphi \in DMA(X,\omega)$ is such that $\mu := (\omega+dd^c\varphi)^n$ does not charge pluripolar sets. It was shown in* [GZ2] *that there exists $\psi \in \mathcal{E}(X,\omega)$ so that $\mu=(\omega+dd^c\psi)^n$. It is expected that the solution ψ is unique up to an additive constant. If such is the case, then $\varphi \equiv \psi +$ constant belongs to $\mathcal{E}(X,\omega)$.*

2. Finite energy classes

In this section we establish further properties of the class $\mathcal{E}(X,\omega)$ and we consider energy classes with respect to a fixed current T.

2.1. **Weighted energies.** Let T be a positive closed current of bidimension (m,m) on X. In the sequel T will be of the form $T=(\omega+dd^cu_{m+1})\wedge\cdots\wedge(\omega+dd^cu_n)$, where $u_j \in PSH(X,\omega)\cap L^\infty(X)$. Let $\chi:\mathbb{R}\to\mathbb{R}$ be a convex increasing function such that $\chi(-\infty)=-\infty$. Following [GZ2] we consider

$$\mathcal{E}_\chi(T,\omega) := \left\{ \varphi \in PSH(X,\omega) \,/\, \sup_j \int_X (-\chi)\circ\varphi_j\, \omega_{\varphi_j}^m \wedge T < +\infty \right\},$$

where $\varphi_j := \max(\varphi,-j)$ denote the canonical approximants of φ. We let the reader check that [GZ2] can be adapted line by line, showing that the Monge-Ampère measure $(\omega+dd^c\varphi)^m\wedge T$ is well defined for $\varphi \in \mathcal{E}_\chi(T,\omega)$ and that $\chi\circ\varphi \in L^1((\omega+dd^c\varphi)^m\wedge T)$.

When $m=n$, i.e. $T=[X]$ is the current of integration along X, then the classes $\mathcal{E}_\chi(T,\omega)=\mathcal{E}_\chi(X,\omega)$ yield the following alternative description of $\mathcal{E}(X,\omega)$ (see [GZ2, Proposition 2.2]):

$$\mathcal{E}(X,\omega) = \bigcup_{\chi\in\mathcal{W}^-} \mathcal{E}_\chi(X,\omega),\ \mathcal{E}_\chi(X,\omega) := \left\{\varphi \in \mathcal{E}(X,\omega) :\, \chi\circ\varphi \in L^1(\omega_\varphi^n)\right\},$$

where $\mathcal{W}^- = \{\chi:\mathbb{R}\to\mathbb{R} \,/\, \chi \text{ convex, increasing, } \chi(-\infty)=-\infty\}$. We set

$$\mathcal{E}(T,\omega) := \bigcup_{\chi\in\mathcal{W}^-} \mathcal{E}_\chi(T,\omega).$$

Theorem 2.1. *Fix $\chi \in \mathcal{W}^-$. Let φ_j be a sequence of ω-psh functions decreasing towards $\varphi \in \mathcal{E}_\chi(T,\omega)$. Then $\varphi_j \in \mathcal{E}_\chi(X,\omega)$ and*

$$\int_X (-\chi)\circ\varphi_j\,(\omega+dd^c\varphi_j)^m\wedge T \longrightarrow \int_X (-\chi)\circ\varphi\,(\omega+dd^c\varphi)^m\wedge T.$$

Moreover for any $u \in PSH(X,\omega)\cap L^\infty(X)$,

$$\int_X u(\omega+dd^c\varphi_j)^m\wedge T \longrightarrow \int_X u(\omega+dd^c\varphi)^m\wedge T.$$

Proof. When φ_j is the canonical sequence of approximants, the theorem follows by a similar argument as in [GZ2, Theorem 2.6]. Moreover, it also follows from [GZ2, Theorem 2.6] that the first convergence in the statement holds for an arbitrary decreasing sequence, if there exists a weight function $\tilde\chi$ such that $\chi=o(\tilde\chi)$ and $\varphi \in \mathcal{E}_{\tilde\chi}(T,\omega)$. It turns out that such a weight always exists. Indeed, since $\chi(\varphi) \in L^1(\omega_\varphi^m\wedge T)$, it follows from standard measure theory arguments that there exists a convex increasing function $h:\mathbb{R}^+\to\mathbb{R}^+$ such that $\lim_{t\to+\infty} h(t)/t=+\infty$ and $h(-\chi(\varphi)) \in L^1(\omega_\varphi^m\wedge T)$.

Note that we can assume that h has polynomial growth, which implies that $\varphi \in \mathcal{E}_{\tilde{\chi}}(T,\omega)$, where $\tilde{\chi} := -h(-\chi) \in \mathcal{W}^- \cup \mathcal{W}_M^+$ (see [GZ2] for the definition of this latter class).

We now prove the second assertion. Set $\varphi_j^k := \max(\varphi_j, -k)$ and $\varphi^k := \max(\varphi, -k)$. Then

$$\begin{aligned}\int_X u\,\omega_{\varphi_j}^m \wedge T - u\,\omega_\varphi^m \wedge T &= \int_X u\,\omega_{\varphi_j}^m \wedge T - \int_X u\,\omega_{\varphi_j^k}^m \wedge T \\ &+ \int_X u\,\omega_{\varphi_j^k}^m \wedge T - \int_X u\,\omega_{\varphi^k}^m \wedge T \\ &+ \int_X u\,\omega_{\varphi^k}^m \wedge T - \int_X u\,\omega_\varphi^m \wedge T.\end{aligned}$$

It suffices to prove that $\int_X u\,\omega_{\varphi_j}^m \wedge T - \int_X u\,\omega_{\varphi_j^k}^m \wedge T$ converges to 0 as $k \to +\infty$ uniformly in j. This is a consequence of the following estimate,

$$\left| \int_X u\,\omega_{\varphi_j}^m \wedge T - \int_X u\,\omega_{\varphi_j^k}^m \wedge T \right| \leq \frac{2\|u\|_{L^\infty(X)}}{|\chi(-k)|} \int_X (-\chi(\varphi_j))\omega_{\varphi_j}^m \wedge T.$$

The latter integral is uniformly bounded since $\varphi \in \mathcal{E}_\chi(T,\omega)$. □

Let $\mathcal{E}^1(\omega^{n-p},\omega)$ denote the class $\mathcal{E}_\chi(T,\omega)$ for $T = \omega^{n-p}$ and $\chi(t) = t$.

Corollary 2.2. *Let $1 \leq p \leq n-1$ and let $\{\varphi_j\}$ be a decreasing sequence of ω-psh functions converging to $\varphi \in \mathcal{E}^1(\omega^{n-p},\omega)$. Then for any $u \in PSH(X,\omega) \cap L^\infty(X)$ we have*

$$\lim_{j\to+\infty} \int_X u\,\omega_{\varphi_j}^{p+1} \wedge \omega^{n-p-1} = \int_X u\,\omega_\varphi^{p+1} \wedge \omega^{n-p-1}.$$

In particular, $\mathcal{E}^1(\omega^{n-p},\omega) \subset \widehat{DMA}_{p+1}(X,\omega)$.

Proof. For simplicity, we consider the case $p = n-1$. The general case follows along the same lines. We want to prove that if $\varphi_j \searrow \varphi \in \mathcal{E}^1(\omega,\omega)$ then for any $u \in PSH(X,\omega) \cap L^\infty(X)$ we have

$$\lim_{j\to+\infty} \int_X u\,\omega_{\varphi_j}^n = \int_X u\,\omega_\varphi^n.$$

Observe that since $\omega_{\varphi_j}^n \to \omega_\varphi^n$ in the weak sense of Radon measures on X, the above equality holds when u is continuous on X.

We claim that for any $\psi \in \mathcal{E}^1(\omega,\omega)$, we have

$$(1) \quad \int_X (-u)\omega_\psi^n = \int_X (-u)\omega \wedge \omega_\psi^{n-1} + \int_X (-\psi)\omega_u \wedge \omega_\psi^{n-1} + \int_X \psi\omega \wedge \omega_\psi^{n-1}.$$

Observe that this identity is just integration by parts which clearly holds when u is a smooth test function on X. The identity also holds when $u, \psi \in PSH(X,\omega) \cap L^\infty(X)$ (see [GZ2]).

Fix u a bounded ω-psh function and set $\psi_j := \max\{\psi, -j\}$ for $j \in \mathbb{N}$. Applying (1) to u and ψ_j, it follows immediately that $\mathcal{E}^1(\omega_u,\omega) = \mathcal{E}^1(\omega,\omega)$. Hence the corollary follows at once from Theorem 2.1, as soon as (1) is established for $\psi \in \mathcal{E}^1(\omega,\omega)$.

To prove (1) we can assume that $u \leq 0$. Note that $\omega_{\psi_j}^n \to \omega_\psi^n$ weakly. Using the upper semicontinuity of u and applying (1) to u and ψ_j, it follows from Theorem 2.1 that

$$\begin{aligned}(2) \qquad &\int_X (-u)\omega_\psi^n \leq \liminf_{j\to+\infty} \int_X (-u)\omega_{\psi_j}^n = \\ &\int_X (-u)\omega\wedge\omega_\psi^{n-1} + \int_X (-\psi)\omega_u\wedge\omega_\psi^{n-1} + \int_X \psi\omega\wedge\omega_\psi^{n-1}.\end{aligned}$$

Next let $u_j \searrow u$ be a sequence of smooth ω-psh functions (see [BK]). Note that $u_j\,\omega_\psi^{n-1} \to u\,\omega_\psi^{n-1}$ in the weak sense of currents, hence $\omega_{u_j}\wedge\omega_\psi^{n-1} \to \omega_u\wedge\omega_\psi^{n-1}$ weakly in the sense of measure. Applying (1) to u_j and ψ, it follows by monotone convergence and the upper semicontinuity of ψ that

$$\begin{aligned}(3) \qquad &\int_X (-u)\omega_\psi^n = \\ &\int_X (-u)\omega\wedge\omega_\psi^{n-1} + \lim_{j\to+\infty}\int_X (-\psi)\omega_{u_j}\wedge\omega_\psi^{n-1} + \int_X \psi\omega\wedge\omega_\psi^{n-1} \geq \\ &\int_X (-u)\omega\wedge\omega_\psi^{n-1} + \int_X (-\psi)\omega_u\wedge\omega_\psi^{n-1} + \int_X \psi\omega\wedge\omega_\psi^{n-1}.\end{aligned}$$

The identity (1) now follows from the inequalities (2) and (3). □

Corollary 2.3. $\mathcal{E}(X,\omega) \subset \widehat{DMA}_{\leq n}(X,\omega)$.

Proof. Fix $\varphi \in \mathcal{E}(X,\omega)$. Then there exists $\chi \in \mathcal{W}^-$ such that $\varphi \in \mathcal{E}_\chi(X,\omega)$. Recall now that for any $1 \leq p \leq n-1$,

$$\mathcal{E}_\chi(X,\omega) \subset \mathcal{E}_\chi(\omega^p,\omega),$$

as follows from simple integration by parts (see [GZ2]). We can thus apply Theorem 2.1 several times to conclude. □

2.2. **Twisted Monge-Ampère operators.** We show here that certain Monge-Ampère operators with weights are always well defined.

Theorem 2.4. *Let $\eta : \mathbb{R} \to \mathbb{R}^+$ be a continuous function with $\eta(-\infty) = 0$ and $1 \leq \ell \leq n$. Let $\varphi \in PSH(X,\omega)$ and $\{\varphi_j\}$ be any sequence of bounded ω-psh functions decreasing to φ. Then the twisted Monge-Ampère currents*

$$M_\eta^\ell(\varphi_j) := \eta\circ\varphi_j(\omega + dd^c\varphi_j)^\ell$$

converge weakly towards a positive current $M_\eta^\ell(\varphi)$, which is independent of the sequence $\{\varphi_j\}$. The twisted Monge-Ampère operator M_η^ℓ is well defined on $PSH(X,\omega)$ and is continuous along any decreasing sequences of ω-psh functions. If $\varphi \in \widehat{DMA}_\ell(X,\omega)$ then

$$M_\eta^\ell(\varphi) = \eta\circ\varphi\,(\omega + dd^c\varphi)^\ell.$$

Proof. It is a standard fact that the operator $M_\eta^\ell(\varphi) = \eta\circ\varphi\,(\omega+dd^c\varphi)^\ell$ is well defined and continuous under decreasing sequences in the subclass of bounded ω-psh functions [BT3].

As it was observed in Remark 1.10, the sequence

$$k \longmapsto \mathbf{1}_{\{\varphi>-k\}}(\omega + dd^c\max(\varphi,-k))^\ell$$

is an increasing sequence of positive currents of total mass bounded by $\int_X \omega^n$. Hence this sequence converges in a strong sense to a positive current $\mu^\ell(\varphi)$ which puts no mass on pluripolar sets and satisfies (see [GZ2, Theorem 1.3])

$$\mathbf{1}_{\{\varphi>-k\}}(\omega + dd^c \max(\varphi, -k))^\ell = \mathbf{1}_{\{\varphi>-k\}}\mu^\ell(\varphi). \tag{4}$$

Since $\eta \circ \varphi$ is a bounded positive Borel function on X, we can define a positive current on X

$$M^\ell_\eta(\varphi) := \eta(\varphi)\, \mu^\ell(\varphi).$$

Note that when φ is bounded then $\mu^\ell(\varphi) = \omega^\ell_\varphi$ hence $M^\ell_\eta(\varphi) = \eta(\varphi)\ \omega^\ell_\varphi$.

To prove the continuity of this operator, let φ_j be any sequence of ω-psh functions decreasing to φ, and set $\varphi^k_j := \max(\varphi_j, -k)$, $\varphi^k := \max(\varphi, -k)$ for $j, k \in \mathbb{N}$.

Since the forms of type $(n-\ell, n-\ell)$ on X have a basis consisting of forms $h\Psi$, where h are smooth functions and Ψ are positive closed forms, it suffices to prove that $\eta(\varphi_j)\, \mu^\ell(\varphi_j) \wedge \Psi \to \eta(\varphi)\, \mu^\ell(\varphi) \wedge \Psi$ weakly as measures on X. For simplicity, we may assume that $\Psi = \omega^{n-\ell}$. Then

$$\begin{aligned}
&\int h\, \eta(\varphi_j)\, \mu^\ell(\varphi_j) \wedge \omega^{n-\ell} - \int h\, \eta(\varphi)\, \mu^\ell(\varphi) \wedge \omega^{n-\ell} \\
&= \int h\, \eta(\varphi_j)\, \mu^\ell(\varphi_j) \wedge \omega^{n-\ell} - \int h\, \eta(\varphi^k_j)\, \mu^\ell(\varphi^k_j) \wedge \omega^{n-\ell} \\
&+ \int h\, \eta(\varphi^k_j)\, \mu^\ell(\varphi^k_j) \wedge \omega^{n-\ell} - \int h\, \eta(\varphi^k)\, \mu^\ell(\varphi^k) \wedge \omega^{n-\ell} \\
&+ \int h\, \eta(\varphi^k)\, \mu^\ell(\varphi^k) \wedge \omega^{n-\ell} - \int h\, \eta(\varphi)\, \mu^\ell(\varphi) \wedge \omega^{n-\ell}.
\end{aligned}$$

We claim that the first term in the sum tends to 0 uniformly in j as $k \to +\infty$. Indeed, we have by (4)

$$\begin{aligned}
&\left| \int h\, \eta(\varphi_j)\, \mu^\ell(\varphi_j) \wedge \omega^{n-\ell} - \int h\, \eta(\varphi^k_j)\, \mu^\ell(\varphi^k_j) \wedge \omega^{n-\ell} \right| \\
&\le \int_{\{\varphi_j \le -k\}} |h|\, \eta(\varphi_j)\, \mu^\ell(\varphi_j) \wedge \omega^{n-\ell} + \int_{\{\varphi_j \le -k\}} |h|\, \eta(\varphi^k_j)\, \mu^\ell(\varphi^k_j) \wedge \omega^{n-\ell} \\
&\le 2\tilde{\eta}(-k) \|h\|_\infty \int_X \omega^n,
\end{aligned}$$

where $\tilde{\eta}(-k) := \sup\{\eta(s) :\ s \le -k\} \to 0$ as $k \to +\infty$. In the same way we see that the last term tends to 0 as $k \to +\infty$. Now for fixed k, it follows from the uniformly bounded case that the second term tends to 0 as $j \to +\infty$. The desired continuity result follows. □

Theorem 2.5. *Let $\eta : \mathbb{R} \to \mathbb{R}^+$ be an increasing function of class C^1 and $0 \le \ell \le n-1$. Let $\varphi \in PSH(X, \omega)$ and $\{\varphi_j\}$ be any sequence of bounded ω-psh functions decreasing to φ. Then the currents*

$$S^\ell_\eta(\varphi_j) := \eta' \circ \varphi_j\, d\varphi \wedge d^c\varphi_j \wedge (\omega + dd^c\varphi_j)^{n-\ell-1}$$

converge weakly towards a positive current $S^\ell_\eta(\varphi)$ which is independent of the sequence $\{\varphi_j\}$. The operator S^ℓ_η is well defined on $PSH(X, \omega)$ and is continuous along any decreasing sequences of ω-psh functions.

Proof. By subtracting a constant, we may assume that $\eta(-\infty) = 0$. Fix ℓ and $\varphi \in PSH(X, \omega)$. We can assume that $\varphi < 0$ and $\eta(0) = 1$.

Observe that if $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ is any C^1 function with $0 \le \theta \le 1$ then for any $u \in PSH(X, \omega) \cap L^\infty(X)$ and any positive closed current R on X of bidimension $(1, 1)$, we have

$$d(\theta(u) d^c u \wedge R) = \theta'(u) du \wedge d^c u \wedge R + \theta(u)\omega_u \wedge R - \theta(u)\omega \wedge R.$$

Using Stokes' formula and the fact that $0 \le \theta(u) \le 1$, we get the following uniform bound

$$\int_X \theta'(u)\, du \wedge d^c u \wedge R \le \int_X \omega \wedge R. \tag{5}$$

Assume that $\varphi \le 0$ and set $\varphi^k := \max(\varphi, -k)$ for $k \ge 0$. We want to prove that the sequence of positive currents

$$\mathbf{1}_{\{\varphi > -k\}} \eta'(\varphi^k)\, d\varphi^k \wedge d^c \varphi^k \wedge \omega_{\varphi^k}^{n-\ell-1}$$

converges to a positive current of bidimension (ℓ, ℓ) which will be denoted by $S_\eta^\ell(\varphi)$. It follows from the quasi-continuity of bounded psh functions [BT4] that for $j \ge k \ge 0$,

$$\mathbf{1}_{\{\varphi > -k\}} \eta'(\varphi^k) d\varphi^k \wedge d^c \varphi^k \wedge \omega_{\varphi^k}^{n-\ell-1} = \mathbf{1}_{\{\varphi > -k\}} \eta'(\varphi^j) d\varphi^j \wedge d^c \varphi^j \wedge \omega_{\varphi^j}^{n-\ell-1}.$$

This implies that

$$\mathbf{1}_{\{\varphi > -k\}} \eta'(\varphi^k) d\varphi^k \wedge d^c \varphi^k \wedge \omega_{\varphi^k}^{n-\ell-1}$$

is an increasing sequence of positive currents of bidimension (ℓ, ℓ), with uniformly bounded mass $\le \int_X \omega^n$ by (5). Therefore it converges in a strong sense to a positive current $S_\eta^\ell(\varphi)$ on X which satisfies the following equation

$$\mathbf{1}_{\{\varphi > -k\}} S_\eta^\ell(\varphi) = \mathbf{1}_{\{\varphi > -k\}} \eta'(\varphi^k) d\varphi^k \wedge d^c \varphi^k \wedge \omega_{\varphi^k}^{n-\ell-1}. \tag{6}$$

Observe that if φ is bounded then $S_\eta^\ell(\varphi) = \eta'(\varphi) d\varphi \wedge d^c \varphi \wedge \omega_\varphi^{n-\ell-1}$.

Now we want to prove that for any decreasing sequence $\{\varphi_j\}$ which converges to φ, the currents $S_\eta^\ell(\varphi_j)$ converge to the current $S_\eta^\ell(\varphi)$ in the sense of currents on X. As before it is enough to prove that the sequence of positive measures $S_\eta^\ell(\varphi_j) \wedge \omega^\ell$ converges weakly to the positive measure $S_\eta^\ell(\varphi) \wedge \omega^\ell$ on X. Let h be a continuous function on X. Proceeding as in the proof of the previous theorem, it suffices to show that

$$\left| \int h\, S_\eta^\ell(\varphi_j) \wedge \omega^\ell - \int h\, \eta'(\varphi_j^k) d\varphi_j^k \wedge d^c \varphi_j^k \wedge \omega_{\varphi_j^k}^{n-\ell-1} \wedge \omega^\ell \right| \to 0$$

as $k \to \infty$, uniformly in j, where $\varphi_j^k = \max(\varphi_j, -k)$. Indeed, if $\theta = \sqrt{\eta}$ then $\eta' = 2\sqrt{\eta}\,\theta'$, so it follows at once from the definition of S_η^ℓ that $S_\eta^\ell(\varphi_j) = 2\sqrt{\eta(\varphi_j)} S_\theta^\ell(\varphi_j)$. Since $\eta'(\varphi_j^k) \le 2\sqrt{\eta(-k)}\theta'(\varphi_j^k)$ on the set $\{\varphi_j \le -k\}$, we have by (6) that

$$\left|\int h\, S_\eta^\ell(\varphi_j)\wedge\omega^\ell - \int h\,\eta'(\varphi_j^k)d\varphi_j^k\wedge d^c\varphi_j^k\wedge\omega_{\varphi_j^k}^{n-\ell-1}\wedge\omega^\ell\right|$$
$$\leq 2\sqrt{\eta(-k)}\int_{\{\varphi_j\leq -k\}}|h|\, S_\theta^\ell(\varphi_j)\wedge\omega^\ell$$
$$+2\sqrt{\eta(-k)}\int_{\{\varphi_j\leq -k\}}|h|\,\theta'(\varphi_j^k)d\varphi_j^k\wedge d^c\varphi_j^k\wedge\omega_{\varphi_j^k}^{n-\ell-1}\wedge\omega^\ell$$
$$\leq 4\sqrt{\eta(-k)}\|h\|_\infty\int_X\omega^n,$$

which tends to 0 as $k\to+\infty$ uniformly in j. □

2.3. **Attenuation of singularities.** The goal of this section is to show that a very small attenuation of singularities transforms a function $\varphi\in PSH(X,\omega)$ into a function $\chi\circ\varphi\in\mathcal{E}(X,\omega)$. In particular most functions in the class $\mathcal{E}(X,\omega)$ do not belong to the local domain of definition $DMA_{loc}(X,\omega)$, which we consider in Section 3.

Let $\chi:\mathbb{R}^-\to\mathbb{R}^-$ be a convex increasing function of class C^2 and such that $\chi(-\infty)=-\infty$, $\chi'(-\infty)=0$ and $\chi'(-1)\leq 1$. Define for $x\leq -1$

$$\eta_\ell(x)=(\chi'(x))^\ell(1-\chi'(x))^{n-\ell}\ ,\ \varepsilon_\ell(x)=\int_0^{\chi'(x)}t^{n-\ell-1}(1-t)^\ell dt.$$

Corollary 2.6. *If $\varphi\in PSH(X,\omega)$, $\varphi\leq -1$, then $\chi\circ\varphi\in\mathcal{E}(X,\omega)$ and*

$$\omega_{\chi\circ\varphi}^n=\sum_{\ell=0}^n\binom{n}{\ell}M_{\eta_\ell}^\ell(\varphi)\wedge\omega^{n-\ell}+n\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}S_{\varepsilon_\ell}^\ell(\varphi)\wedge\omega^\ell, \tag{7}$$

where $MA_{\eta_\ell}^\ell$ and $S_{\varepsilon_\ell}^\ell$ are the operators defined in Theorems 2.4 and 2.5.

Proof. Note first that if φ is bounded then

$$\begin{aligned}(\omega+dd^c\chi\circ\varphi)^n &= \left[(1-\chi'(\varphi))\omega+\chi'(\varphi)\,\omega_\varphi\right]^n+\\ &\quad n\chi''(\varphi)\,d\varphi\wedge d^c\varphi\wedge\left[(1-\chi'(\varphi))\omega+\chi'(\varphi)\,\omega_\varphi\right]^{n-1},\end{aligned}$$

which equals the measure from (7). In the general case, let $\varphi^s=\max(\varphi,s)$, $s<0$. As $s\to-\infty$ we have, by the proofs of Theorems 2.4 and 2.5, that

$$\left(\varepsilon_\ell'(\varphi^s)\,d\varphi^s\wedge d^c\varphi^s\wedge\omega_{\varphi^s}^{n-\ell-1}\wedge\omega^\ell\right)(\{\varphi\leq s\})\to 0,$$
$$\left(\eta_\ell(\varphi^s)\,\omega_{\varphi^s}^\ell\wedge\omega^{n-\ell}\right)(\{\varphi\leq s\})\to 0.$$

Since $\max(\chi(\varphi),-j)=\chi(\varphi^{s_j})$, where $s_j=\chi^{-1}(-j)\to-\infty$ as $j\to+\infty$, we conclude by formula (7) applied to φ^{s_j} that

$$(\omega+dd^c\max(\chi(\varphi),-j))^n\,(\{\chi(\varphi)\leq -j\})\to 0,$$

so $\chi\circ\varphi\in\mathcal{E}(X,\omega)$. Moreover, using the continuity of the operators in Theorems 2.4 and 2.5, it follows that (7) holds for φ. □

Examples 2.7.

1) For $\chi(t)=-(-t)^p$, Corollary 2.6 shows that $-(-\varphi)^p\in\mathcal{E}(X,\omega)\subset DMA(X,\omega)$ for all $p<1$, although it usually does not have gradient in

$L^2(X)$ when $p \geq 1/2$. Functions with less attenuated singularities can be obtained by considering for instance $\chi(t) = t/\log(M-t)$.

2) Recall from Example 1.4 that there are functions $\varphi \notin DMA(X,\omega)$ such that the current ω_φ has support in $\{\varphi = -\infty\}$. The smoothing effect of the composition with χ is quite striking: indeed, by Corollary 2.6 and Proposition 4.1,

$$(\omega + dd^c\chi \circ \varphi)^n = [1-\chi'(\varphi)]^n\omega^n + n[1-\chi'(\varphi)]^{n-1}\chi''(\varphi)\, d\varphi \wedge d^c\varphi \wedge \omega^{n-1}$$

is a measure with density in $L^1(X,\omega^n)$.

3. The local vs. global domains of definition

Cegrell found in [C2] the largest class of psh functions on a bounded hyperconvex domain on which the Monge-Ampère operator is well defined, stable under maximum and continuous under decreasing limits. Later, Blocki gave in [Bl2] a complete characterization of the domain of definition of the Monge-Ampère operator on *any* open set in $\mathbb{C}^n$, $n \geq 2$. For an open subset $U \subset \mathbb{C}^n$, the domain of definition $\mathcal{D}(U) \subset PSH(U)$ of the Monge-Ampère operator on U is given by $(n-1)$ local boundedness conditions on weighted gradients [Bl2]. In particular, for $n = 2$, $\mathcal{D}(U) = PSH(U) \cap W^{1,2}_{loc}(U)$, where $W^{1,2}_{loc}(U)$ is the Sobolev space of functions in $L^2_{loc}(U)$ with locally square integrable gradient [Bl1].

We describe here the class of ω-psh functions on X which locally belong to the domain of definition of the Monge-Ampère operator. As we shall see, it is smaller than the global domain $DMA(X,\omega)$.

Definition 3.1. *Let $DMA_{loc}(X,\omega)$ be the set of functions $\varphi \in PSH(X,\omega)$ such that locally, on any small open coordinate chart $U \subset X$, the psh function $\varphi|_U + \rho_U \in \mathcal{D}(U)$, where ρ_U is a psh potential of ω on U.*

The goal of this section is to establish the following:

Theorem 3.2. *We have*

$$DMA_{loc}(X,\omega) = \bigcap_{1 \leq p \leq n-1} \mathcal{E}^p(\omega^p, \omega).$$

Moreover

$$DMA_{loc}(X,\omega) \subset \mathcal{E}^1(\omega,\omega) \subset \widehat{DMA}_{\leq n}(X,\omega).$$

Here $\mathcal{E}^p(\omega^p,\omega)$ denotes the class $\mathcal{E}_\chi(T,\omega)$ for $T = \omega^p$ and $\chi(t) = -(-t)^p$.

Proof. We first show that $DMA_{loc}(X,\omega) = \bigcap_{1 \leq p \leq n-1} \mathcal{E}^p(\omega^p,\omega)$. By [Bl2], a ω-psh function $\varphi \in DMA_{loc}(X,\omega)$ if and only if

$$\sup_j \int_X (-\varphi_j)^{p-1} d\varphi_j \wedge d^c\varphi_j \wedge \omega_{\varphi_j}^{n-p-1} \wedge \omega^p < +\infty,\ 1 \leq p \leq n-1, \tag{8}$$

where $\{\varphi_j\}$ is any sequence of bounded ω-psh functions decreasing to φ on X. For our first claim we need to prove that (8) is equivalent to the following

$$\sup_j \int_X (-\varphi_j)^p \omega_{\varphi_j}^{n-p} \wedge \omega^p < +\infty,\ 1 \leq p \leq n-1, \tag{9}$$

This follows by integration by parts. Indeed for $\psi \in PSH(X,\omega) \cap L^\infty(X)$, $\psi \leq -1$ and $1 \leq p \leq n-1$, we have

$$\begin{aligned} \int_X (-\psi)^p \omega_\psi^{n-p} \wedge \omega^p &= \int_X (-\psi)^p \omega_\psi^{n-p-1} \wedge \omega^{p+1} \\ &+ p \int_X (-\psi)^{p-1} d\psi \wedge d^c\psi \wedge \omega_\psi^{n-p-1} \wedge \omega^p. \end{aligned}$$

By iterating the above formula we get

$$\begin{aligned} \int_X (-\psi)^p \omega_\psi^{n-p} \wedge \omega^p &= \int_X (-\psi)^p \omega^n \\ &+ p \sum_{k=0}^{n-p-1} \int_X (-\psi)^{p-1} d\psi \wedge d^c\psi \wedge \omega_\psi^{n-p-k-1} \wedge \omega^{p+k}. \end{aligned}$$

This yields (8) $\iff$ (9), so $DMA_{loc}(X,\omega) = \bigcap_{1\leq p\leq n-1} \mathcal{E}^p(\omega^p,\omega)$.

By Corollary 2.2 we have $\mathcal{E}^1(\omega^p,\omega) \subset \widehat{DMA}_{n-p+1}(X,\omega)$, for any $1 \leq p \leq n-1$. Therefore the class

$$\mathcal{E}^1(\omega,\omega) := \left\{ \varphi \in PSH(X,\omega) \,/\, \sup_j \int_X |\varphi_j|(\omega + dd^c\varphi_j)^{n-1} \wedge \omega < +\infty \right\}$$

is contained in $\widehat{DMA}_{\leq n}(X,\omega)$ (here $\varphi_j := \max(\varphi, -j)$ denote as usually the canonical approximants), since

$$\mathcal{E}^1(\omega,\omega) = \bigcap_{1\leq p\leq n-1} \mathcal{E}^1(\omega^p,\omega).$$

This equality also shows that $DMA_{loc}(X,\omega) \subset \mathcal{E}^1(\omega,\omega)$.

$\square$

Remark 3.3. *A class similar to $\mathcal{E}^1(\omega,\omega)$ has been very recently considered by Y. Xing in* [X].

Note that in dimension $n = 2$, the class $\mathcal{E}^1(\omega,\omega) = DMA_{loc}(X,\omega)$ is simply the set of ω-psh function whose gradient is in $L^2(X)$. However when $n \geq 3$, the class $\mathcal{E}^1(\omega,\omega)$ is strictly larger than $DMA_{loc}(X,\omega)$, as the following example shows:

Example 3.4. *Observe that $\mathcal{E}^1(X,\omega) \subset \mathcal{E}^1(\omega,\omega)$. We are going to exhibit an example of a function φ such that $\varphi \in \mathcal{E}^1(X,\omega)$, but $\varphi \notin L^2(\omega_\varphi \wedge \omega^{n-1})$. This will show that $\varphi \notin DMA_{loc}(X,\omega)$ when $n \geq 3$.*

Assume $X = \mathbb{P}^{n-1} \times \mathbb{P}^1$ and $\omega(x,y) := \alpha(x) + \beta(y)$, where α is the Fubini-Study form on $\mathbb{P}^{n-1}$ and β is the Fubini-Study form on $\mathbb{P}^1$. Fix $u \in PSH(\mathbb{P}^{n-1},\alpha) \cap \mathcal{C}^\infty(\mathbb{P}^{n-1})$ and $v \in \mathcal{E}(\mathbb{P}^1,\beta)$. The function φ defined by $\varphi(x,y) := u(x) + v(y)$ for $(x,y) \in X$ belongs to $\mathcal{E}(X,\omega)$. Moreover $\omega_\varphi = \alpha_u + \beta_v$ and for any $1 \leq \ell \leq n$, we have

$$\omega_\varphi^\ell = \alpha_u^\ell + \ell\alpha_u^{\ell-1} \wedge \beta_v.$$

Therefore

$$\varphi \in L^p(\omega_\varphi^n) \iff v \in L^p(\beta_v) \iff \varphi \in L^p(\omega_\varphi^\ell \wedge \omega^{n-\ell}).$$

Thus choosing $v \in L^1(\omega_v) \backslash L^2(\omega_v)$, we obtain an example of a ω-psh function φ such that $\varphi \in \mathcal{E}^1(X,\omega) \subset \mathcal{E}^1(\omega,\omega)$ but $\varphi \notin DMA_{loc}(X,\omega)$.

We finally observe that there are functions in $DMA_{loc}(X,\omega)$ which do not belong to the class $\mathcal{E}(X,\omega)$, since the latter cannot have positive Lelong numbers.

Lemma 3.5. *Let $\varphi \in PSH(X,\omega)$ be such that $\varphi \geq c \log dist(\cdot, p)$, for some $c > 0$ and $p \in X$. Then $\varphi \in DMA_{loc}(X,\omega)$, and*

$$\varphi \in \mathcal{E}(X,\omega) \text{ if and only if } \nu(\varphi,p) = 0.$$

Proof. If $\varrho_p \in PSH(X,\omega)$ is a function comparable to $c \log dist(\cdot, p)$, then by Proposition 4.6, $\varphi, \varrho_p \in DMA_{loc}(X,\omega)$. We can assume without loss of generality that $\varrho_p \leq \varphi \leq 0$. Note that the positive Radon measure $\omega_\varphi \wedge \omega_{\varrho_p}^{n-1}$ is well defined on X and has a Dirac mass at p if and only if $\nu(\varphi,p) > 0$ (see [D2]).

It follows from the proof of Proposition 1.2 that if $\nu(\varphi,p) > 0$, then ω_φ^n has a Dirac mass at p, hence $\varphi \notin \mathcal{E}(X,\omega)$. If $\nu(\varphi,p) = 0$, then we can find a convex increasing function $\chi : \mathbb{R}^- \to \mathbb{R}^-$ such that $\chi(-\infty) = -\infty$ and $\int_X (-\chi) \circ \varrho_p \, \omega_\varphi \wedge \omega_{\varrho_p}^{n-1} < +\infty$. Using Stokes theorem (in the spirit of the fundamental inequality in [GZ2]), it follows that

$$\int_X (-\chi) \circ \varphi \, \omega_\varphi^n \leq 2^{n-1} \int_X (-\chi) \circ \varrho_p \, \omega_\varphi \wedge \omega_{\varrho_p}^{n-1} < +\infty,$$

hence $\varphi \in \mathcal{E}_\chi(X,\omega) \subset \mathcal{E}(X,\omega)$. $\square$

4. Sobolev classes

4.1. **Weighted gradients.** Let $\chi : \mathbb{R} \to \mathbb{R}$ be a convex increasing function of class C^2. If $\varphi \in PSH(X,\omega)$ is smooth then

$$\omega + dd^c \chi \circ \varphi = \chi'' \circ \varphi \, d\varphi \wedge d^c \varphi + \chi' \circ \varphi \, \omega_\varphi + (1 - \chi' \circ \varphi)\omega. \tag{10}$$

So if $\chi'(-1) \leq 1$ and $\varphi \leq -1$, then $\chi \circ \varphi \in PSH(X,\omega)$. It is well-known that (ω-)psh functions have gradient in $L^{2-\varepsilon}_{loc}$ for all $\varepsilon > 0$ [Hö], but in general not in L^2_{loc}. The previous computation indicates that a weighted version of the gradient is in $L^2(X)$.

We denote by $W^{1,2}(X,\omega)$ the set of functions $\varphi \in PSH(X,\omega)$ whose gradient is square integrable. Since ω is Kähler, we can in fact define, for $\varphi \in PSH(X,\omega)$, the function $|\nabla\varphi| = |\nabla\varphi|_\omega$ a.e. on X by

$$|\nabla\varphi|^2 := d\varphi \wedge d^c\varphi \wedge \omega^{n-1}/\omega^n.$$

Note that $\varphi \in W^{1,2}(X,\omega)$ if and only if $|\nabla\varphi| \in L^2(X,\omega^n)$.

Proposition 4.1. *Let $\chi : \mathbb{R} \to \mathbb{R}$ be a convex increasing function of class $\mathcal{C}^2$. Then for every $\varphi \in PSH(X,\omega)$,*

$$\int_X \chi'' \circ \varphi \, |\nabla\varphi|^2 \, \omega^n \leq \sup_X \chi'(\varphi) \int_X \omega^n.$$

In particular, if $\varphi \leq -1$ and $0 < p < 1/2$, then $-(-\varphi)^p \in W^{1,2}(X,\omega)$.

Proof. Let $M = \sup_X \chi'(\varphi)$ and $\varphi^j := \max(\varphi, -j)$ for $j \in \mathbb{N}$. It follows from (10) that

$$\int_X \chi'' \circ \varphi^j \, |\nabla\varphi^j|^2 \, \omega^n = \int_X \chi'' \circ \varphi^j \, d\varphi^j \wedge d^c\varphi^j \wedge \omega^{n-1} \leq M \int_X \omega^n.$$

This shows that the sequence $f_j := \chi''(\varphi^j)\,|\nabla\varphi^j|^2$ is bounded in $L^1(X,\omega^n)$. Since $\varphi_j \searrow \varphi$, we use [Hö, Theorem 4.1.8] to conclude, after taking a subsequence, that $f_j \to \chi''(\varphi)\,|\nabla\varphi|^2$ a.e. on X. The inequality in the statement now follows from Fatou's lemma.

For our second claim, set $\varphi_a := -(-\varphi)^a$, $0 < a < 1$. Applying (10) with $\chi(t) = -(-t)^a$ yields $\varphi_a \in PSH(X,\omega)$ and

$$\int_X (-\varphi)^{a-2} d\varphi \wedge d^c\varphi \wedge \omega^{n-1} \leq \frac{1}{a(1-a)} \int_X \omega^n.$$

If $p = a/2$ then $d\varphi_p \wedge d^c\varphi_p = p^2(-\varphi)^{2p-2} d\varphi \wedge d^c\varphi$, hence

$$\int_X d\varphi_p \wedge d^c\varphi_p \wedge \omega^{n-1} \leq \frac{p}{2(1-2p)} \int_X \omega^n.$$

$\square$

Note that using $\chi(t) = t/\log(M-t)$, for M large enough, one can improve the previous weighted L^2-bound to

$$\int_X (-\varphi)^{-1}[\log(M-\varphi)]^{-2}|\nabla\varphi|^2\,\omega^n < +\infty.$$

Observe also that this cannot be improved much farther: in the case when ω_φ is the current of integration along a hypersurface, $-(-\varphi)^{1/2}$ does not have square integrable gradient.

Next, we show that functions from the class $DMA_{loc}(X,\omega)$ satisfy stronger weighted gradient boundedness conditions.

Proposition 4.2. *If $\varphi \in DMA_{loc}(X,\omega)$, $\varphi \leq -1$, then*

$$\int_X (-\varphi)^{n-2} d\varphi \wedge d^c\varphi \wedge \omega^{n-1} \leq \frac{1}{n-1} \int_X (-\varphi)^{n-1}\omega_\varphi \wedge \omega^{n-1}.$$

In particular, $DMA_{loc}(X,\omega) \subset W^{1,2}(X,\omega)$.

Proof. If $\varphi_j := \max(\varphi, -j)$, $j \in \mathbb{N}$, then integrating by parts we get

$$(n-1)\int_X (-\varphi_j)^{n-2} d\varphi_j \wedge d^c\varphi_j \wedge \omega^{n-1} \leq \int_X (-\varphi_j)^{n-1}\omega_{\varphi_j} \wedge \omega^{n-1}.$$

The sequence of functions $f_j := (-\varphi_j)^{n-2}|\nabla\varphi_j|^2$ is thus uniformly bounded in $L^1(X,\omega^n)$. Since $\varphi_j \searrow \varphi$, the conclusion follows as in the previous proof by [Hö, Theorem 4.1.8] and by Fatou's lemma. $\square$

4.2. **Blowing up and down.** We saw in Section 2 that there are many functions $\varphi \in DMA(X,\omega)$ whose gradient does not belong to $L^2(X)$. This condition, although the best possible in the local two-dimensional theory, is thus not the right one from the global point of view. We show here that this condition does not behave well under a birational change of coordinates. For simplicity, and without loss of generality, we restrict ourselves to the two-dimensional local setting.

Let $\pi : Y \to B$ be the blow up at the origin of a ball $B \subset \mathbb{C}^2$, and let E be the exceptional divisor.

Lemma 4.3. *If $\delta > 0$ then $\pi^\star\left(L^{2+\delta}_{loc}(B)\right) \subset L^1_{loc}(Y)$, but $\pi^\star\left(L^2_{loc}(B)\right) \not\subset L^1_{loc}(Y)$.*

Proof. Let $z = (x, y) \in B$, and fix a coordinate chart in Y given by $(s, t) \to (s, st, [1 : t]) \in Y$. Then $\pi(s, t) = (x, y)$, $x = s$, $y = st$. For fixed positive constants M and a, let

$$\Delta = \{(s, t) : |s| < a,\ |t| < M\}\ ,\ K = \pi(\Delta) = \{(x, y) : |x| < a,\ |y| < M|x|\}.$$

Let $f \in L^{2+\delta}_{loc}(B)$, and $\widetilde{f} = \pi^\star f$, so $\widetilde{f}(s, t) = f(s, st)$. Note that the function $z \to |x|^{-2}$ is in $L^{2-\epsilon}(K)$ for every $\epsilon > 0$. We take $\gamma = \delta/(1 + \delta)$ and apply Hölder's inequality with conjugate exponents $2 + \delta$ and $2 - \gamma$:

$$\int_\Delta |\widetilde{f}| = \int_K \frac{|f|}{|x|^2} \leq \left(\int_K |f|^{2+\delta}\right)^{1/(2+\delta)} \left(\int_K \frac{1}{|x|^{4-2\gamma}}\right)^{1/(2-\gamma)} .$$

Hence $\widetilde{f} \in L^1_{loc}(Y)$. The function $f(z) = 1/(\|z\|^2 \log \|z\|)$ is in $L^2_{loc}(B)$, but $|\widetilde{f}(s, t)| \geq C|s|^{-2}(|\log |s|| + 1)^{-1}$ on Δ, for some $C > 0$. So $\widetilde{f} \notin L^1(\Delta)$. □

In dimension n, a similar proof shows that $\pi^\star \left(L^{n+\delta}_{loc}(B)\right) \subset L^1_{loc}(Y)$.

Example 4.4. $\pi^\star \left(PSH \cap W^{1,4}_{loc}(B)\right) \not\subset W^{1,2}_{loc}(Y)$. *Indeed, let*

$$u_\alpha(z) = -(-\log \|z\|)^\alpha\ ,\ 0 < \alpha < 1.$$

One checks easily that $u_\alpha \in W^{1,4}(B)$ *if* $\alpha < 3/4$. *Let* $\widetilde{u}_\alpha = \pi^\star u_\alpha$. *With the notation from the proof of Lemma 4.3, we have for* $|s|$ *small*

$$\widetilde{u}_\alpha(s, t) = -\left(-\log |s| - \log \sqrt{1 + |t|^2}\right)^\alpha\ ,\ \left|\frac{\partial \widetilde{u}_\alpha}{\partial s}(s, t)\right| \geq \frac{C}{|s|(-\log |s|)^{1-\alpha}}\ ,$$

for some constant $C > 0$. *So* $\widetilde{u}_\alpha \notin W^{1,2}_{loc}(\Delta)$ *if* $\alpha \geq 1/2$. *Note that if* $u \in PSH \cap W^{1,4+\delta}_{loc}(B)$, $\delta > 0$, *then, by the Sobolev embedding theorem,* u *is continuous and so is* $\pi^\star u$. *Hence* $\pi^\star u \in W^{1,2}_{loc}(Y)$ *(see* [Bl1]*).*

The previous example shows that the condition $\nabla\varphi \in L^2(X)$ does not behave well under blow-up. We show it behaves well under blowing down.

Proposition 4.5. *Let* T *be a positive closed current of bidegree (1,1) on* B, *and let* $R = \pi^\star T - \nu[E]$, *where* $\nu = \nu(T, 0)$. *If* R *has psh potentials in* $W^{1,2}_{loc}(Y)$, *then* T *has psh potentials in* $W^{1,2}_{loc}(B)$.

Recall that any positive closed $(1, 1)$-current in the blow up of B at the origin writes $R = \pi^*T - \nu[E] + \lambda[E]$, where T is a positive closed current in B, $\nu = \nu(T, 0)$ is the Lelong number of T at the origin, and $\lambda \geq 0$. Clearly R cannot have potentials in $W^{1,2}_{loc}$ if $\lambda > 0$.

Proof. Let u be a psh potential of T on B. We only have to check that the gradient of u is L^2-integrable in a neighborhood of the origin. Using the notation from the proof of Lemma 4.3, a psh potential for R on Δ is $v(s, t) = u(s, st) - \nu \log |s|$. Hence on K we have $u(x, y) = v(x, y/x) + \nu \log |x|$. Therefore for almost all $(x, y) \in K$

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial s}(x, y/x) - \frac{y}{x^2} \frac{\partial v}{\partial t}(x, y/x) + \frac{\nu}{2x}\ ,\ \frac{\partial u}{\partial y}(s, t) = \frac{1}{x} \frac{\partial v}{\partial t}(x, y/x).$$

Now

$$\begin{aligned}\int_K \frac{|y|^2}{|x|^4}\left|\frac{\partial v}{\partial t}(x,y/x)\right|^2 &= \int_\Delta |t|^2\left|\frac{\partial v}{\partial t}(s,t)\right|^2,\\ \int_K \frac{1}{|x|^2}\left|\frac{\partial v}{\partial t}(x,y/x)\right|^2 &= \int_\Delta \left|\frac{\partial v}{\partial t}(s,t)\right|^2.\end{aligned}$$

Since the function $(x,y)\to 1/x$ is in $L^2(K)$, we conclude that the partial derivatives of u belong to $L^2(K)$. A similar argument shows that the partial derivatives of u are in $L^2(K')$, where $K'=\{(x,y):\,|y|<a,\ |x|<M|y|\}$. $\square$

4.3. **Compact singularities.** We give here an important class of functions in $DMA_{loc}(X,\omega)$. Let D be a divisor on X and set

$$L^\infty_D(X,\omega)=\{\varphi\in PSH(X,\omega)\,/\,\varphi \text{ is bounded near } D\}.$$

Thus the singularities of $\varphi\in L^\infty_D(X,\omega)$ are constrained to a compact subset of $X\setminus D$. When $D=H$ is a hyperplane of the complex projective space $X=\mathbb{P}^n$, the set $L^\infty_D(X,\omega)$ is in one-to-one correspondence with the Lelong class $\mathcal{L}^+(\mathbb{C}^n)$ of psh functions u in $\mathbb{C}^n$ such that $u(z)-\log||z||$ is bounded near infinity. So these are the ω-psh analogues of the psh functions with compact singularities introduced by Sibony in [Sib] (see also [D2]).

Proposition 4.6. *If D is an ample divisor then $L^\infty_D(X,\omega)\subset DMA_{loc}(X,\omega)$.*

Proof. Fix $\varphi\in L^\infty_D(X,\omega)$ and let V be a small neighborhood of D, so that φ is bounded in V. We can assume that $\varphi\le 0$ and $\int_X\omega^n=1$.

Let ω' be a smooth semi-positive closed $(1,1)$ form in the cohomology class of D, such that $\omega'\equiv 0$ in $X\setminus V$. Since D is ample, ω' is cohomologous to a Kähler form ω_0. For simplicity, we assume $\omega_0=\omega$ (otherwise we bound $\omega\le C\omega_0$ in all arguments below). Hence $\omega=\omega'+dd^c\chi$, where χ is a smooth function on X, chosen to be either negative or positive, as we like.

We assume here $\chi\ge 0$, and we first observe that $\varphi\in L^1(\omega_\varphi\wedge\omega^{n-1})$:

$$\begin{aligned}\int_X(-\varphi)\,\omega_\varphi\wedge\omega^{n-1} &= \int_X(-\varphi)\omega_\varphi\wedge\omega'\wedge\omega^{n-2}+\int_X(-\varphi)\omega_\varphi\wedge dd^c\chi\wedge\omega^{n-2}\\ &\le\ ||\varphi||_{L^\infty(V)}\int_X\omega_\varphi\wedge\omega'\wedge\omega^{n-2}+\int_X\chi\omega_\varphi\wedge(-dd^c\varphi)\wedge\omega^{n-2}\\ &\le\ ||\varphi||_{L^\infty(V)}+||\chi||_{L^\infty(X)}<+\infty,\end{aligned}$$

since $-dd^c\varphi\le\omega$, $\chi\omega_\varphi\ge 0$ and $\int_X\omega_\varphi\wedge\omega'\wedge\omega^{n-2}=\int_X\omega^n=1$.

It follows that the positive current $\omega_\varphi^2:=\omega_\varphi\wedge\omega+dd^c(\varphi\omega_\varphi)$ is well defined. We can thus show by a similar argument that $\varphi\in L^1(\omega_\varphi^2\wedge\omega^{n-2})$, so that ω_φ^3 is also well defined, and so on. At last, we show that $\varphi\in L^1(\omega_\varphi^{n-1}\wedge\omega)$.

We now prove that $\varphi^2\in L^1(\omega_\varphi^{n-2}\wedge\omega^2)$. We assume here that $\chi\le 0$. Observe that $-dd^c\varphi^2=-2d\varphi\wedge d^c\varphi-2\varphi dd^c\varphi\le 2(-\varphi)\omega_\varphi$, therefore

$$\int_X\varphi^2\omega_\varphi^{n-2}\wedge\omega^2\le\int_X\varphi^2\omega_\varphi^{n-2}\wedge\omega'\wedge\omega+2\|\chi\|_{L^\infty(X)}\int_X(-\varphi)\omega_\varphi^{n-1}\wedge\omega.$$

The integrals are finite because ω' has support in V, where φ is bounded, and because $\varphi\in L^1(\omega_\varphi^{n-1}\wedge\omega)$. Thus $\varphi^2\in L^1(\omega_\varphi^{n-2}\wedge\omega^2)$. Similar integration by parts allows us to show that $\varphi^3\in L^1(\omega_\varphi^{n-3}\wedge\omega^3)$, by using $\varphi^2\in L^1(\omega_\varphi^{n-2}\wedge\omega^2)$. Continuing like this, we see that $\varphi\in DMA_{loc}(X,\omega)$. $\square$

Proposition 4.6 shows that $PSH(X,\omega)\cap L^{\infty}_{loc}(X\setminus\{p\})\subset DMA_{loc}(X,\omega)$. Indeed, if X is *projective*, one can find for each $p\in X$ a divisor $D\not\ni p$. In the general case, it follows from the proof that one only needs to construct a smooth semi-positive form ω' cohomologous to a Kähler form, and such that $\omega'\equiv 0$ near p. This can be achieved on any *Kähler* manifold.

5. Concluding remarks

In this section we restrict our attention to the two-dimensional case. In the local setting of an open subset $U\subset\mathbb{C}^2$, a psh function u on U belongs to the domain of definition $\mathcal{D}(U)$ if and only if the gradient of u is locally square integrable ($W^{1,2}$) on U [Bl1]. Such characterization allows to prove important properties of $\mathcal{D}(U)$, such as convexity and stability under taking the maximum of elements of $\mathcal{D}(U)$ with arbitrary psh functions (see [Bl1]).

Let X be a compact Kähler surface and ω be a Kähler form on X. In order to prove further properties of $DMA(X,\omega)$ it would be useful to obtain equivalent characterizations for this domain. We present here the connection in certain cases between $DMA(X,\omega)$ and certain energy classes.

5.1. **Direct sums.** Let $X=\mathbb{P}^1\times\mathbb{P}^1$ and $\omega=\omega_1+\omega_2$, where $\omega_i=\pi_i^*\omega'$ is the pull-back of the Fubini-Study form ω' of $\mathbb{P}^1$ by the projection onto the i^{th} factor, $i=1,2$.

Proposition 5.1. *Let $\varphi\in PSH(X,\omega)$ be of the form*

$$\varphi(x,y)=u(x)+v(y),\ \textit{where } u,v\in PSH(\mathbb{P}^1,\omega').$$

Then
(i) $\varphi\in\mathcal{E}^1(\omega,\omega)\Longleftrightarrow u,v\in\mathcal{E}^1(\mathbb{P}^1,\omega')\Longleftrightarrow\varphi\in\mathcal{E}^1(X,\omega)$;
(ii) $\varphi\in\mathcal{E}(\omega,\omega)\Longleftrightarrow u,v\in\mathcal{E}(\mathbb{P}^1,\omega')\Longleftrightarrow\varphi\in\mathcal{E}(X,\omega)$;
(iii) $\varphi\in DMA(X,\omega)\Longleftrightarrow\varphi\in\mathcal{E}(\omega,\omega)$.

Proof. (i) Note that

$$\begin{aligned}\int\varphi\,\omega_\varphi\wedge\omega &= \int u\,\omega_{1,u}\wedge\omega_2+\int v\,\omega_1\wedge\omega_{2,v}+\\ &\quad\int u\,\omega_1\wedge\omega_{2,v}+\int v\,\omega_{1,u}\wedge\omega_2,\end{aligned}$$

where the integrals $\int u\,\omega_1\wedge\omega_{2,v}$ and $\int v\,\omega_{1,u}\wedge\omega_2$ are always finite by Fubini's theorem. We use here the obvious notations $\omega_{1,u}:=(\omega_1+dd^cu)(x)$ and $\omega_{2,v}:=(\omega_2+dd^cv)(y)$. This shows that $\varphi\in\mathcal{E}^1(\omega,\omega)$ if and only if $u,v\in\mathcal{E}^1(\mathbb{P}^1,\omega')$.

If $u,v\in\mathcal{E}^1(\mathbb{P}^1,\omega')$ then $\varphi\in W^{1,2}$, hence $\varphi\in\widehat{DMA}(X,\omega)$ and $\omega_\varphi^2=2\omega_{1,u}\wedge\omega_{2,v}$. By Fubini's theorem

$$\int\varphi\,\omega_\varphi^2=2\int_{\mathbb{P}^1}u\,\omega_{1,u}+2\int_{\mathbb{P}^1}v\,\omega_{2,v},$$

so $\omega_\varphi^2(\{\varphi=-\infty\})=0$. We conclude that $\varphi\in\mathcal{E}(X,\omega)$, hence $\varphi\in\mathcal{E}^1(X,\omega)$. This formula also shows that $\varphi\in\mathcal{E}^1(X,\omega)$ implies that $u,v\in\mathcal{E}^1(\mathbb{P}^1,\omega')$.

(ii) and (iii). The equivalence $\varphi \in \mathcal{E}(\omega,\omega) \iff u, v \in \mathcal{E}(\mathbb{P}^1,\omega')$ is a direct consequence of the following equality:

$$\int_{\{\varphi=-\infty\}} \omega_\varphi \wedge \omega = \int_{\{u=-\infty\}} \omega_{1,u} + \int_{\{v=-\infty\}} \omega_{2,v}.$$

We show next that $u, v \in \mathcal{E}(\mathbb{P}^1,\omega') \Longrightarrow \varphi \in \mathcal{E}(X,\omega)$. Let φ_j, u_j, v_j be the canonical approximants and set $E_j = \{u > -j\} \cap \{v > -j\}$. Since the bounded ω-psh functions φ_{2j} and $u_j + v_j$ coincide on the plurifine open set E_j, we have by [BT4] that

$$\mathbf{1}_{E_j}\omega^2_{\varphi_{2j}} = 2\cdot \mathbf{1}_{E_j}\omega_{u_j} \wedge \omega_{v_j} = 2\cdot \mathbf{1}_{\{u>-j\}}\omega_{u_j} \wedge \mathbf{1}_{\{v>-j\}}\omega_{v_j}.$$

Note that the product measure $\omega_u \wedge \omega_v$ puts full mass 2 on the set $\{u > -\infty\} \cap \{v > -\infty\}$, and that $\mathbf{1}_{\{\varphi>-2j\}}\omega^2_{\varphi_{2j}} \geq \mathbf{1}_{E_j}\omega^2_{\varphi_{2j}}$. This shows that the sequence of measures $\mathbf{1}_{\{\varphi>-2j\}}\omega^2_{\varphi_{2j}}$ increases to a measure with total mass 2, hence $\varphi \in \mathcal{E}(X,\omega)$.

We conclude the proof by showing that $\varphi \in DMA(X,\omega)$ implies that $u, v \in \mathcal{E}(\mathbb{P}^1,\omega')$. Assume for a contradiction that $v \notin \mathcal{E}(\mathbb{P}^1,\omega')$. We may assume that there exists a compact $K \subset \{v = -\infty\} \cap \{[1 : w] : w \in \mathbb{C}\}$ so that

$$\int_K dd^c V = a > 0\ ,\quad V(w) := \log\sqrt{1+|w|^2} + v([1 : w]).$$

We use here (bi)homogeneous coordinates $[z_0 : z_1], [w_0 : w_1]$ on X. Let u_j be smooth ω'-psh functions decreasing to u on $\mathbb{P}^1$, and set

$$\begin{aligned} U_j(z) &= \log\sqrt{1+|z|^2} + u_j([1 : z]), \\ \Phi_j(z,w) &= \max(U_j(z) + V(w), \log|z-\zeta| - j), \end{aligned}$$

where $\zeta \in \mathbb{C}$. The functions Φ_j yield functions $\varphi_j \in DMA(X,\omega)$ decreasing to φ, hence $(dd^c\Phi_j)^2 \to \omega^2_\varphi$ on $\mathbb{C}^2 \subset X$. We will show that

$$\int_{\{z=\zeta\}\times K} \omega^2_\varphi \geq a.$$

Since ζ is arbitrary, we get a contradiction.

For $r > 0$ let $\chi_1 \geq 0$ be a smooth function such that $\chi_1(z) = 1$ in the closed disc E_r of radius r centered at ζ and χ_1 is supported in the disc D_r of radius $2r$ centered at ζ. For fixed j let N be an open neighborhood of K so that $V(w) < \log r - j - \max_{D_r} U_j$ for $w \in N$, and let $\chi_2 \geq 0$ be a smooth function supported in N such that $\chi_2(w) = 1$ on K. Let $\chi(z,w) = \chi_1(z)\chi_2(w)$. Since $dd^c\chi \wedge dd^c\Phi_j$ is supported on the open set $\{U_j(z) + V(w) < \log|z-\zeta| - j\}$ it follows that

$$\begin{aligned} \int \chi\,(dd^c\Phi_j)^2 &= \int (U_j + V)\, dd^c\chi \wedge dd^c\Phi_j \\ &\geq \int \chi\, dd^c V \wedge dd^c\Phi_j = \int \Phi_j\chi_2\, dd^c\chi_1(z) \wedge dd^c V(w). \end{aligned}$$

Note that

$$U_j(z) + V(w) < \log r - j < \log|z-\zeta| - j$$

on the support of $\chi_2 dd^c\chi_1$, thus

$$\begin{aligned}\int \chi\,(dd^c\Phi_j)^2 &\geq \int (\log|z-\zeta|-j)\chi_2\, dd^c\chi_1 \wedge dd^c V \\ &= \int \chi\, dd^c \log|z-\zeta| \wedge dd^c V \geq \int_K dd^c V = a.\end{aligned}$$

We conclude that

$$\int_{E_r \times K} \omega_\varphi^2 \geq \limsup_{j\to\infty} \int_{E_r\times K} (dd^c\Phi_j)^2 \geq a,$$

and as $r \to 0$, that $\int_{\{z=\zeta\}\times K} \omega_\varphi^2 \geq a$. □

5.2. **The case $X = \mathbb{P}^2$.** We produce now similar examples in the case of $X = \mathbb{P}^2$ with ω the Fubini-Study form. Let $[t : z : w]$ denote the homogeneous coordinates and φ be a ω-psh function with Lelong number 1 at point $p = [1 : 0 : 0]$. It is easy to see that φ can be written as

$$\varphi[t : z : w] = \frac{1}{2}\log\frac{|z|^2+|w|^2}{|t|^2+|z|^2+|w|^2} + u[z : w] \tag{11}$$

where u is a ω'-psh function on $\{t = 0\} \simeq \mathbb{P}^1$.

Proposition 5.2. *If $u \notin \mathcal{E}(\mathbb{P}^1, \omega)$ then $\varphi \notin DMA(\mathbb{P}^2, \omega)$.*

Proof. Suppose $\varphi \in DMA(\mathbb{P}^2, \omega)$ and let φ_j be functions defined by (11) with u replaced by u_j, where u_j are bounded ω-psh on $\mathbb{P}^1$ decreasing to u. Then φ_j decreases to φ and $\omega_{\varphi_j}^2 = \delta_p$ is the Dirac mass at p, hence $\omega_\varphi^2 = \delta_p$.

On the other other hand, we are going to construct another sequence of functions $\psi_j \in DMA(\mathbb{P}^2, \omega)$ decreasing to φ such that $\omega_{\psi_j}^2$ does not converge to δ_p. Let $K \subset \{t = 0\}$ be a compact so that $u = -\infty$ on K and $\omega_u(K) = a > 0$, and let

$$\psi_j([t : z : w]) = \max\left(\varphi([t : z : w]), \log|t| - \frac{1}{2}\log(|t|^2+|z|^2+|w|^2) - j\right).$$

Then $\psi_j \in DMA_{loc}(\mathbb{P}^2, \omega)$ by Proposition 4.6. One can show, as in the proof of Proposition 5.1, that $\omega_{\psi_j}^2(K) \geq a$, for all j. This contradicts that $\omega_\varphi^2 = \delta_p$. □

For functions φ as in (11), it is easy to show that $\varphi \in DMA_{loc}(\mathbb{P}^2, \omega)$ if and only if $u \in W^{1,2}(\mathbb{P}^1)$. It is an interesting question whether $\varphi \in DMA(\mathbb{P}^2, \omega)$ if $u \in \mathcal{E}(\mathbb{P}^1, \omega')$. A concrete example is

$$\varphi_\alpha[t : z : w] := \frac{1}{2}\log\frac{|z|^2+|w|^2}{|t|^2+|z|^2+|w|^2} - \left[1 - \frac{1}{2}\log\frac{|z|^2}{|z|^2+|w|^2}\right]^\alpha,$$

where $0 < \alpha < 1$. Then $\varphi_\alpha \notin \mathcal{E}(\mathbb{P}^2, \omega)$ since it has positive Lelong number at p, and $\varphi_\alpha \notin DMA_{loc}(\mathbb{P}^2, \omega)$ if $\alpha \geq 1/2$. It would be of interest to know if $\varphi_\alpha \in DMA(\mathbb{P}^2, \omega)$ for some $\alpha \in [1/2, 1]$.

5.3. **A candidate?** Previous examples indicate that in dimension $n = 2$, the class

$$\mathcal{E}(\omega,\omega) := \{\varphi \in PSH(X,\omega) \,/\, (\omega + dd^c\varphi)(\{\varphi = -\infty\}) = 0\}$$

plays a central role. Note that it enjoys several interesting properties:

- $\mathcal{E}(\omega,\omega)$ is convex and stable under maximum;
- $\mathcal{E}(\omega,\omega) = \mathcal{E}(\omega',\omega') \cap PSH(X,\omega)$ whenever $\omega \leq \omega'$;
- $DMA_{loc}(X,\omega) \subset \mathcal{E}^1(\omega,\omega) \subset \mathcal{E}(\omega,\omega)$;
- $\mathcal{E}(X,\omega) \subset \mathcal{E}(\omega,\omega)$.

Together with the special examples analyzed in sections 5.1 and 5.2, this motivates the following:

Question 5.3. *Assume $n = \dim_{\mathbb{C}} X = 2$.*
Do we have $DMA(X,\omega) \subset \mathcal{E}(\omega,\omega)$ and/or $\mathcal{E}(\omega,\omega) \subset DMA(X,\omega)$?

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