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QUASIPLURISUBHARMONIC GREEN FUNCTIONS

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Abstract. Given a compact Kähler manifold X, a quasiplurisubharmonic function is called a Green function with pole at $p \in X$ if its Monge-Ampère measure is supported at p. We study in this paper the existence and properties of such functions, in connection to their singularity at p. A full characterization is obtained in concrete cases, such as (multi)projective spaces.

Introduction

Let X be a compact Kähler manifold of complex dimension n. We pursue the study started in [Y], [Ko1], [Ko2], [GZ2], [EGZ], [BGZ] of the range of the complex Monge-Ampère operator. Given a Kähler class $\alpha \in H^{1,1}(X,\mathbb{R})$ and a positive Radon measure μ, the problem is to solve the equation $T^n = \mu$, where T is a positive closed (1,1)-current in α. When μ does not charge pluripolar sets, a complete answer was given in [GZ2]. The main purpose of this article is to start and study the case when μ charges pluripolar sets by looking at measures μ which are sums of Dirac masses. The equation now reads

$$T^n = \sum_{j=1}^{k} c_j \delta_{p_j}. \tag{1}$$

We seek solution(s) $T \in \alpha$ whose potentials are locally bounded away from the poles $p_1, \dots, p_k$. An obvious necessary condition in order to solve (1) is that the volume of α,

$$V_\alpha := \mathrm{Vol}\,(\alpha) = \alpha^n,$$

is equal to the total mass of μ, $\mu(X) = \sum c_j = \mathrm{Vol}\,(\alpha)$.

Fix θ a Kähler form representing α and let $PSH(X,\theta)$ denote the set of θ-plurisubharmonic (θ-psh) functions: these are functions $\varphi \in L^1(X,\mathbb{R})$ which are upper semicontinuous and such that $T = \theta + dd^c\varphi$ is a positive current. Here $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$. Solving (1) is therefore equivalent to finding a *"quasiplurisubharmonic Green function"*:

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Definition. *A function $\varphi \in PSH(X,\theta)$ is called a θ-psh Green function with (isolated) poles at $p_1,\dots,p_k \in X$ if it is locally bounded in $X \setminus \{p_1,\dots,p_k\}$ and*

$$(\theta + dd^c\varphi)^n = V_\alpha \sum_{j=1}^{k} m_j \delta_{p_j},\ \textit{where } m_j > 0,\ \sum_{j=1}^{k} m_j = 1.$$

In [CGZ], the domain $DMA(X,\theta)$ of the Monge-Ampère operator was defined as the largest set of θ-psh functions on which the operator is continuous along decreasing sequences of bounded θ-psh functions. Hence one can consider a more general notion of θ-psh Green function, by only requiring in the above definition that $\varphi \in DMA(X,\theta)$, instead of φ being locally bounded away from the poles. We will not pursue this here.

Similar objects were considered by several authors in a local context ([Lm], [Kl], [D1], [Le], [CP], [Co1], [CN]), and have found important applications (see e.g. [BP], [He], [DH]). In our global context their existence depends on the geometry of X and on the local positivity properties of α at the poles.

We therefore study in *section 1* several indicators of the local positivity properties of α, following Demailly [D2]. Recall that the Lelong number $\nu(\varphi,x)$ of a θ-psh function φ at x is the largest constant ν for which $\varphi(p) \le \nu \log dist(p,x) + O(1)$ holds for p near x. If $\varphi(p) = \nu \log dist(p,x) + O(1)$ for p near x and $\nu > 0$, we say that φ has an *isotropic pole* at x with Lelong number ν.

We let $\nu(\alpha,x)$ (resp. $\varepsilon(\alpha,x)$) denote the maximal (resp. maximal isotropic) logarithmic singularity that a positive closed current $T \in \alpha$ can have at the point x. The indicator $\varepsilon(\alpha,x)$, introduced by Demailly [D2], is called the Seshadri constant of α at x and was intensively studied in algebraic geometry. We note in *section 1* that for all $x \in X$,

$$\nu(\alpha,x) \ge \mathrm{Vol}\,(\alpha)^{1/n} \ge \varepsilon(\alpha,x).$$

Thus a necessary condition for the existence of a α-Green function with one isotropic pole at x is that $\mathrm{Vol}\,(\alpha)^{1/n} = \varepsilon(\alpha,x)$. This is far from being true in general: we observe for instance in Proposition 3.1 that this is never the case when X is a multiprojective space. Even if this condition is satisfied, it is not clear whether it is sufficient, nor is it clear that the supremum in the definition of ε is attained. We observe in section 4.3.2 that the following properties are equivalent:

- existence of a Green function with 9 isotropic poles in general position in $\mathbb{P}^2$;
- existence of a Green function with one isotropic pole in generic position on a degree 1 Del Pezzo surface;
- existence of a positive metric with bounded potentials for $c_1(Y)$, where $Y \to \mathbb{P}^2$ denotes the blow up of $\mathbb{P}^2$ at 9 points in general position,

the last one being a famous open problem [DPS]. We therefore introduce in *section 1* weaker notions of Green functions. We show in Theorems 1.4, 1.5 and Proposition 1.6 how to construct these by a balayage procedure. It is a delicate and interesting problem to determine whether θ-psh Green functions always exist. As already observed, we have to consider arbitrary singularities. The balayage procedure depends on the choice of local data $(u_1,\dots,u_k)$ encoding the singularities at the poles $(p_1,\dots,p_k)$. In particular, the problem of constructing θ-psh Green functions is reduced to finding local data for which the functions g constructed in Theorems 1.4 and 1.5 have isolated singularities at p_j.

In *section 2* we give a complete description of all these notions on the complex projective space $\mathbb{P}^n$. In particular, we characterize in Theorem 2.4 Green functions arising naturally from rational maps $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ with finite indeterminacy set. We end section 2 by constructing interesting dynamical Green functions.

In *section 3* we compute similar quantities for multiprojective spaces, focusing on $\mathbb{P}^1 \times \mathbb{P}^1$. We show in Proposition 3.4 that Green functions with one pole correspond to a certain class of Green functions with three poles on $\mathbb{P}^2$. A large class of examples of these can be constructed using Theorem 2.4 (see Example 3.5). However, there is no Green function with one isotropic pole on $\mathbb{P}^1 \times \mathbb{P}^1$ (Corollary 3.2).

In *section 4* we turn our attention to the case of smooth Del Pezzo surfaces, focusing on those of degree 1, i.e. blow ups X of $\mathbb{P}^2$ at 8 points in general position. Let α be the first Chern class of X. We prove in Proposition 4.1 that $\nu(\alpha, x) = 1$ if $x \in X \setminus S$, and $\nu(\alpha, x) = 2$ if $x \in S$. Here S is the set of singular points on the singular cubics passing through the 8 blown up points, and $1 \leq |S| \leq 12$. The results of Proposition 4.1 allow us to compute, using currents, the exact value of Tian's "α-invariant", and to deduce that X has a Kähler-Einstein metric (section 4.2). We conclude the paper with the discussion in section 4.3 of ω-psh Green functions with one pole $x \in X$, where $\omega \in \alpha$ is a Kähler form. Such functions are easy to construct when $x \in S$. For generic points $x \notin S$ the existence of Green functions with an isotropic pole at x of maximal Lelong number $1 = \varepsilon(\alpha, x)$ is equivalent to a famous open problem in algebraic geometry (see section 4.3.2).

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1. Local positivity of (1,1) classes and Green functions

Let $\mathcal{P}(X)$ be the set of all positive closed currents of bidegree (1,1) on X. For $\alpha \in H^{1,1}(X, \mathbb{R})$ we let

$$\mathcal{P}(\alpha) = \{T \in \mathcal{P}(X) : T \in \alpha\}$$

be the set of positive closed currents whose cohomology class is α. By definition, a class α is *pseudoeffective* if $\mathcal{P}(\alpha) \neq \emptyset$. Let $H^{1,1}_{psef}(X, \mathbb{R})$ denote the closed convex cone of all pseudoeffective (1,1) classes.

There are two other interesting cones in $H^{1,1}_{psef}(X, \mathbb{R})$ which correspond to stronger notions of positivity. We let $H^{1,1}_{Kaehler}(X, \mathbb{R})$ denote the cone of Kähler classes and $H^{1,1}_{nef}(X, \mathbb{R})$ denote its closure. Then $H^{1,1}_{Kaehler}(X, \mathbb{R})$ is the interior of $H^{1,1}_{nef}(X, \mathbb{R})$.

Following Demailly [D2], we would like to measure the local positivity of a class α. There are two main indicators, in connection to the various types of positivity. In the sequel we denote by $\nu(T, x)$ the Lelong number of $T \in \mathcal{P}(X)$ at a point x.

Definition 1.1. *Let $\pi : \widetilde{X} \to X$ denote the blow up of X at a point x, and let $E = \pi^{-1}(x)$ denote the exceptional divisor.*

1) For $\alpha \in H^{1,1}_{psef}(X, \mathbb{R})$ we set

$$\nu(\alpha, x) := \sup\{\nu \geq 0 : \pi^\star\alpha - \nu E \in H^{1,1}_{psef}(\widetilde{X}, \mathbb{R})\}.$$

2) For $\alpha \in H^{1,1}_{nef}(X,\mathbb{R})$ we set

$$\varepsilon(\alpha,x) := \sup\{\varepsilon \geq 0 : \pi^\star\alpha - \varepsilon E \in H^{1,1}_{nef}(\widetilde{X},\mathbb{R})\}.$$

The indicator $\nu(\alpha,x)$ is the maximal Lelong number that a current $T \in \mathcal{P}(\alpha)$ can have at x. In this case the supremum is attained, because $\mathcal{P}(\alpha)$ is a compact set (in the weak topology of currents).

The indicator $\varepsilon(\alpha,x)$ is called the Seshadri constant of α at x. It has been intensively studied since it was introduced by Demailly. We refer the reader to [La, Chapter 5] for a detailed account of this notion.

By definition we have $0 \leq \varepsilon(\alpha,x) \leq \nu(\alpha,x)$. It follows from the characterization of the Kähler cone obtained in [DP] that if $\alpha \in H^{1,1}_{nef}(X,\mathbb{R})$ and $x \in X$ then

$$\varepsilon(\alpha,x) = \min_V \left(\frac{(\alpha^{\dim V}\cdot V)}{\operatorname{mult}_x V}\right)^{\frac{1}{\dim V}},$$

where the minimum is taken over all irreducible subvarieties $V \subseteq X$ with $\dim V \geq 1$ and $x \in V$ (see e.g. Proposition 5.1.9 and Remark 1.5.32 in [La]). With $V = X$, this yields the estimate (recall that $V_\alpha = \operatorname{Vol}(\alpha)$):

$$\varepsilon(\alpha,x) \leq V_\alpha^{1/n},\ \forall\, x \in X. \tag{2}$$

On the other hand, it follows easily from Theorem 1.4 below that if $\alpha \in H^{1,1}_{Kaehler}(X,\mathbb{R})$

$$\nu(\alpha,x) \geq V_\alpha^{1/n},\ \forall\, x \in X.$$

Both bounds are sharp in the case of $\mathbb{P}^n$.

Remark 1.2. *If $\alpha \in H^2(X,\mathbb{Z})$ is an integral class, then $\nu(\alpha,x) \geq V_\alpha^{1/n} \geq 1$ for all $x \in X$. Note also that if α is very ample then $\varepsilon(\alpha,x) \geq 1$.*

An alternate description of the Seshadri constant $\varepsilon(\alpha,x)$ can be given in terms the maximal Lelong number of currents in $\mathcal{P}(\alpha)$ whose potentials have an isolated singularity at x [D2]. Let $\alpha \in H^{1,1}_{Kaehler}(X,\mathbb{R})$ and θ be a Kähler form representing α. It follows as in [D2, Theorem 6.4] that for every $x \in X$,

$$\begin{aligned} \varepsilon(\alpha,x) &= \sup\{\gamma : \exists\, \varphi \in PSH(X,\theta),\ \|\varphi - \gamma \log dist(\cdot,x)\|_{L^\infty(X)} < +\infty\} \\ &= \sup\{\gamma : \exists\, \varphi \in PSH(X,\theta),\ \nu(\varphi,x) = \gamma,\ \varphi \in L^\infty_{loc}(U \setminus \{x\})\}, \end{aligned} \tag{3}$$

where U is a neighborhood of x depending on φ. Recall that $PSH(X,\theta)$ is the set of θ-psh functions. The set of normalized θ-psh functions, for example by the condition $\max_X \varphi = 0$, is isomorphic to $\mathcal{P}(\alpha)$ via $\varphi \to \theta + dd^c\varphi \in \mathcal{P}(\alpha)$. The fact that the two supremums are equal is straightforward. Moreover, in this case we have $\varepsilon(\alpha,x) > 0$ for all $x \in X$.

We now list a few elementary properties of these numerical indicators.

Proposition 1.3. *1) The functions $\alpha \to \nu(\alpha,x), \varepsilon(\alpha,x)$ are homogeneous and superadditive (i.e. $\nu(\alpha+\beta,x) \geq \nu(\alpha,x) + \nu(\beta,x)$).*

2) The function $x \to \nu(\alpha,x)$ is upper semicontinuous.

3) If α is Kähler the function $x \to \varepsilon(\alpha,x)$ is lower semicontinuous.

Proof. The upper semicontinuity property of $x \to \nu(\alpha, x)$ follows since $\mathcal{P}(\alpha)$ is compact and from the well known fact that $\limsup \nu(T_j, x_j) \le \nu(T, x)$ as positive closed (1,1)-currents $T_j \to T$ and $x_j \to x$.

To prove 3), let $\theta \in \alpha$ be a Kähler form, $x \in X$, $0 < \epsilon < 1$, and $0 < \nu < \varepsilon(\alpha, x)$. We construct for all y near x a θ-psh function φ_y with $\varphi_y = (1-\epsilon)\nu \log dist(\cdot, y) + O(1)$. Using (3), this shows that $\liminf_{y \to x} \varepsilon(\alpha, y) \ge \varepsilon(\alpha, x)$.

By (3) there exists $\varphi \in PSH(X, \theta)$ such that $\varphi = \nu \log dist(\cdot, x) + O(1)$. Let $B_2 \subset \mathbb{C}^n$ be the ball of radius 2 centered at 0. We can find a coordinate chart $f : B_2 \longrightarrow U \subset X$, $f(0) = x$, and a function $\rho \in C^\infty(U)$ so that $dd^c \rho = \theta$ and

$$\nu \log \|z\| - C \le v(z) := (\rho + \varphi) \circ f(z) \le \nu \log \|z\| + C, \ z \in B_2,$$

for some constant $C > 0$. Fix $r > 0$ small enough so that

$$(1-\epsilon)\left(\nu \log \frac{r}{2} - 2C\right) \ge \nu \log r + 2C.$$

Next, let T_w be an automorphism of the unit ball $B_1 \subset \mathbb{C}^n$ with $T_w(w) = 0$. There exists $\delta(r) < r$ such that $\|T_w(z)\| \ge r/2$, if $\|z\| = r$ and $\|w\| < \delta(r)$. For such w we define the function v_w on B_2 by

$$v_w(z) = \begin{cases} v(z) + C, \ 1 \le \|z\| < 2, \\ \max\{v(z) + C, (1-\epsilon)(v \circ T_w(z) - C)\}, \ r < \|z\| < 1, \\ (1-\epsilon)(v \circ T_w(z) - C), \ \|z\| \le r. \end{cases}$$

Note that if $\|z\| = 1$ then $v(z) + C \ge 0 \ge (1-\epsilon)(v \circ T_w(z) - C)$, while if $\|z\| = r$,

$$(1-\epsilon)(v \circ T_w(z) - C) \ge (1-\epsilon)\left(\nu \log \frac{r}{2} - 2C\right) \ge \nu \log r + 2C \ge v(z) + C.$$

Hence v_w is psh on B_2 and $v(z) = (1-\epsilon)\nu \log \|z - w\| + O(1)$ for z near w.

For $y = f(w)$, where $\|w\| < \delta(r)$, we finally let

$$\varphi_y = \begin{cases} \varphi + C, \ \text{on } X \setminus f(B_1), \\ v_w \circ f^{-1} - \rho, \ \text{on } f(B_1). \end{cases}$$

Then φ_y is θ-psh and $\varphi_y = (1-\epsilon)\nu \log dist(\cdot, y) + O(1)$ near y. □

In general, the functions $\nu(\alpha, \cdot), \varepsilon(\alpha, \cdot)$ are not continuous (see e.g. Proposition 4.1 and section 4.3). Note that in the special case when X is projective and α is an integral class, it follows from [La, Example 5.1.11] that $\varepsilon(\alpha, \cdot)$ is constant outside a countable union of proper subvarieties of X.

If $\theta \in \alpha$ is a Kähler form, we have by (2) and (3) that a necessary condition for the existence of a θ-psh Green function with an *isotropic* pole at p is

$$\varepsilon(\alpha, p) = V_\alpha^{1/n}.$$

Since this fails to hold in general (see Proposition 3.1), one has to consider other singularities. Following ideas of Demailly [D5], we will show that local fundamental solutions of the Monge-Ampère operator have θ-psh subextensions to X.

We will consider the slightly more general situation when the class α is represented by a smooth closed (1,1) form $\theta \ge 0$ and $V_\alpha > 0$. Recall that the *unbounded locus* $M(\varphi)$ of $\varphi \in PSH(X, \theta)$ is defined as the set of all points $p \in X$ such that φ is unbounded in every neighborhood of p. We denote by $PSH^-(X, \theta)$ the set of θ-psh functions $\varphi \le 0$ on X. For $p \in X$, let $\mathcal{G}_p(V_\alpha)$ be the set of germs of functions u at p with the following properties: there exists an open set $U \subset X$ containing p such that u is psh on U and locally bounded on $U \setminus \{p\}$, $u(p) = -\infty$, and $(dd^c u)^n = V_\alpha \delta_p$ as measures on U.

Theorem 1.4. *Let $p \in X$ and $u \in \mathcal{G}_p(V_\alpha)$. There exists a unique function $g = g_{u,p} \in PSH^-(X,\theta)$ such that*

(i) $g \le u + C$ holds near p, for some constant C.

(ii) If $\varphi \in PSH^-(X,\theta)$ and $\liminf_{q\to p} \varphi(q)/u(q) \ge 1$ then $\varphi \le g$ on X.

In addition, g has the following properties:

(a) $(\theta + dd^c g)^n = 0$ on the open set $X \setminus (M(g) \cup \{g = 0\})$.

(b) If p is an isolated point of $M(g)$ then $M(g) = \{p\}$ and g is a θ-psh Green function on X with pole at p.

(c) The open set $D_{u,p} = \{g < 0\}$ is connected.

It should be noted that the existence of a global θ-psh function φ subextending u (i.e. such that $\varphi \le u$ near p) is a nontrivial matter. We use Yau's solution in the spirit of [D5], [DP]. Producing the "best subextension" g proceeds using a classical balayage procedure (see [R] for recent similar local extremal problems).

Proof. The uniqueness of a function with properties (i), (ii) is clear. Fix $U \subset X$ an open coordinate ball around p, so that u is psh on U, locally bounded on $U \setminus \{p\}$ and $(dd^c u)^n = V_\alpha \delta_p$ as measures on U. We divide the proof in three steps.

Step 1. Using a mass concentration technique of Demailly [D5], we construct a function $\varphi \in PSH(X,\theta)$ so that $\varphi \le u$ near p. Let ω_0 be a Kähler form on X.

Let $W \subset\subset W' \subset\subset U$ be open and connected, with $p \in W$, and let χ be a smooth function on X with compact support in W', such that $0 \le \chi \le 1$ and $\chi = 1$ on W. We may assume that $u \ge 0$ on ∂W. Let ρ, ρ_0 be negative smooth functions on W' with $dd^c\rho = \theta$, $dd^c\rho_0 = \omega_0$.

Let $u_j \searrow u$ be a sequence of smooth psh functions on W' and let $\omega_j = \theta + j^{-1}\omega_0$. We define measures

$$\mu_j = C_j \chi\, (dd^c\, u_j)^n,$$

where the constants $C_j > 0$ are chosen so that $\mu_j(X) = \int_X \omega_j^n$. Note that μ_j has support in W', and $(dd^c\, u_j)^n \to V_\alpha \delta_p$ in the weak sense of measures on W'. Hence

$$\lim_{j\to\infty} \int \chi\, (dd^c\, u_j)^n = V_\alpha \chi(p) = V_\alpha, \text{ so } \lim_{j\to\infty} C_j = 1.$$

Yau's theorem (see [Y], also [Ko1]) implies that there exist continuous functions $\varphi_j \in PSH(X,\omega_j)$ such that

$$(\omega_j + dd^c\varphi_j)^n = \mu_j,\ \max_X \varphi_j = 0.$$

By [GZ1, Proposition 1.7] we may assume after passing to a subsequence that $\{\varphi_j\}$ converges in $L^1(X)$ to a function $\varphi \in PSH(X,\theta)$. Moreover, by [Ho, Theorem 4.1.8] we have $\varphi = (\limsup_{j\to\infty} \varphi_j)^\star$ on X.

Choose a sequence $a_j \ge 1$ so that $a_j^n C_j > 1$ and $a_j \to 1$. We have

$$a_j(\varphi_j + \rho + j^{-1}\rho_0) \le 0 \le u_j \text{ on } \partial W.$$

On the other hand

$$a_j^n (dd^c(\varphi_j + \rho + j^{-1}\rho_0))^n = a_j^n C_j \chi\, (dd^c\, u_j)^n \ge (dd^c\, u_j)^n$$

holds on W, as $\chi = 1$ on W. The minimum principle of Bedford and Taylor [BT1, Theorem A] implies that $a_j(\varphi_j + \rho + j^{-1}\rho_0) \leq u_j$ on W. Letting $j \to \infty$ we obtain that $\varphi + \rho \leq u$ holds on W. This concludes Step 1.

Step 2. We construct the function g using an upper envelope method. Consider the family

$$\mathcal{F} = \left\{ \varphi \in PSH^-(X, \theta) : \liminf_{q \to p} \frac{\varphi(q)}{u(q)} \geq 1 \right\}.$$

In the terminology of Rashkovskii, this is the family of negative θ-psh functions whose relative type with respect to u is at least 1 (see [R]).

By Step 1, $\mathcal{F} \neq \emptyset$. If $g = \sup\{\varphi : \varphi \in \mathcal{F}\}$, then the upper semicontinuous regularization $g^\star \in PSH^-(X, \theta)$. We will show that $g^\star \leq u + C$ holds near p for some constant C. This implies that $g = g^\star \in \mathcal{F}$, so g verifies properties (i), (ii).

We can find $M > 0$ such that the connected component D of $\{u < -M\}$ which contains p is relatively compact in U. Let $\rho < 0$ be a smooth function on U so that $dd^c\rho = \theta$. Fix $\varphi \in \mathcal{F}$. There exists a sequence of relatively compact domains $D_j \subset D$, $j > 0$, with the following properties:

$$D_{j+1} \subset D_j, \ \bigcap_{j>0} D_j = \{p\}, \ \varphi(q) \leq (1 - j^{-1})u(q) \text{ for } q \in \overline{D}_j.$$

We have $\rho + \varphi \leq 0 \leq (1 - j^{-1})(u + M)$ on ∂D, and clearly $\rho + \varphi \leq (1 - j^{-1})(u + M)$ on ∂D_j. Since the psh function u is maximal on $U \setminus \{p\}$, it follows that the last inequality holds on $D \setminus D_j$. As $j \to \infty$ we see that $\rho + \varphi \leq u + M$ on D. Since $\varphi \in \mathcal{F}$ was arbitrary, this implies that $g^\star \leq u + C$ on D, where $C = M - \min_D \rho$.

Step 3. We prove the remaining properties of g.

(a) Note that $M(g)$ is closed and since $g \leq 0$ is upper semicontinuous the set $\{g = 0\}$ is closed. Let $q \in X \setminus (M(g) \cup \{g = 0\})$ and let ρ be a smooth function in a neighborhood of q such that $dd^c\rho = \theta$ and $\rho(q) = 0$. We can find $\varepsilon > 0$ and a small neighborhood G of q such that $G \subset X \setminus (M(g) \cup \{g = 0\})$ and $g < -\varepsilon$, $|\rho| < \varepsilon/2$ on G. Let W be a relatively compact open subset of G and v be psh on W so that $v^\star \leq \rho + g$ on ∂W. The function

$$\varphi = g \ \text{ on } X \setminus W, \quad \varphi = \max\{\rho + g, v\} - \rho \ \text{ on } W,$$

is θ-psh and $\varphi \leq 0$ on X. Since $\varphi = g$ in a neighborhood of p, we conclude that $\varphi \in \mathcal{F}$, hence $v \leq \rho + g$ on W. This shows that the psh function $\rho + g$ is maximal on G. By [BT2], $(\theta + dd^c g)^n = 0$ in G, and hence on $X \setminus (M(g) \cup \{g = 0\})$.

(b) If $p \in M(g)$ is isolated, there exists a closed ball K centered at p so that $K \cap M(g) = \{p\}$. Hence g is bounded below on ∂K. It follows that if $C > 0$ is large enough the function φ defined by $\varphi = g$ on K, $\varphi = \max\{g, -C\}$ on $X \setminus K$, is θ-psh and $\varphi \in \mathcal{F}$. Thus $\varphi \leq g$, so $M(g) = \{p\}$. By (i) and [D4], $(\theta + dd^c g)^n(\{p\}) \geq (dd^c u)^n(\{p\}) = V_\alpha$. Mass considerations imply that g is a θ-psh Green function.

(c) Suppose that there exists a connected component W of $D_{u,p}$ not containing p. The function φ defined by $\varphi = g$ on $X \setminus W$ and $\varphi = 0$ on W, verifies $\varphi \in \mathcal{F}$, so $\varphi \leq g$. This contradicts our assumption that $g < 0$ on W, so $D_{u,p}$ is connected. $\square$

The following theorem produces Green functions with several poles. Its proof is a straightforward adaptation of the proof of Theorem 1.4.

Theorem 1.5. *For $1 \leq j \leq k$, let $p_j \in X$, $u_j \in \mathcal{G}_{p_j}(V_\alpha)$, and $m_j > 0$ with $\sum_{j=1}^k m_j = 1$. There exists a unique function $g \in PSH^-(X, \theta)$ such that*

(i) $g \leq m_j^{1/n} u_j + C$ holds near each p_j, for some constant C.

(ii) If $\varphi \in PSH^-(X, \theta)$ and for each j, $\liminf_{q \to p_j} \varphi(q)/u_j(q) \geq m_j^{1/n}$, then $\varphi \leq g$ on X.

Moreover, we have $(\theta + dd^c g)^n = 0$ on $X \setminus (M(g) \cup \{g = 0\})$. If all p_j are isolated points of $M(g)$ then g is a θ-psh Green function with poles at $p_1, \ldots, p_k$.

It is an intricate problem to decide whether there always exist local models u at $p \in X$ such that $g_{u,p}$ is a Green function. As an alternate approach, we introduce a partial Green function associated to an isotropic singularity.

Proposition 1.6. *Let $\theta \in \alpha$ be a Kähler form, let $p \in X$ and $0 < \gamma < \varepsilon(\alpha, p)$. There exists a unique function $\psi_{\gamma,p} \in PSH^-(X, \theta)$ so that $\nu(\psi_{\gamma,p}, p) = \gamma$ and with the property that if $\varphi \in PSH^-(X, \theta)$ and $\nu(\varphi, p) \geq \gamma$ then $\varphi \leq \psi_{\gamma,p}$. Moreover,*

$$\|\psi_{\gamma,p} - \gamma \log dist(\cdot, p)\|_{L^\infty(X)} < +\infty, \ (\theta + dd^c \psi_{\gamma,p})^n = \gamma^n \delta_p + \mu_{\gamma,p},$$

where $\mu_{\gamma,p}$ is a positive measure supported on the compact $\{\psi_{\gamma,p} = 0\}$.

Proof. The uniqueness of $\psi_{\gamma,p}$ is clear. Let us fix a biholomorphic map $f : B \to U$ from the unit ball $B \subset \mathbb{C}^n$ onto a neighborhood U of p, with $f(0) = p$. Let $\rho < 0$ be a smooth function on U with $dd^c \rho = \theta$.

By (3) there exists $\psi \in PSH^-(X, \theta)$ so that $\psi = \gamma \log dist(\cdot, p) + O(1)$. Let

$$\psi_{\gamma,p}(q) = \sup\{\varphi(q) : \varphi \in PSH^-(X, \theta), \ \nu(\varphi, p) \geq \gamma\}.$$

For such φ, we have $(\rho + \varphi)(f(z)) \leq \gamma \log \|z\|$ on B. This implies $\psi^\star_{\gamma,p} \in PSH^-(X, \theta)$ and $\nu(\psi^\star_{\gamma,p}, p) \geq \gamma$. Thus $\psi_{\gamma,p} = \psi^\star_{\gamma,p}$. Since $\psi \leq \psi_{\gamma,p}$, it follows that $\nu(\psi_{\gamma,p}, p) = \gamma$ and the function $\psi_{\gamma,p} - \gamma \log dist(\cdot, p)$ is bounded on X.

Arguing as in the proof of Theorem 1.4 (a) we show that $(\theta + dd^c \psi_{\gamma,p})^n = 0$ in $\{\psi_{\gamma,p} < 0\} \setminus \{p\}$. By [D4], $(\theta + dd^c \psi_{\gamma,p})^n(\{p\}) = \gamma^n$, and the proof is complete. □

We refer to [R] for similar extremal problems on domains in $\mathbb{C}^n$. In the following sections, we are going to compute the functions ν, ε and $g_{u,p}$, $\psi_{\nu,p}$ in a number of interesting cases.

2. Green functions on $\mathbb{P}^n$

Let $[z_0 : \ldots : z_n]$ be homogeneous coordinates on $\mathbb{P}^n$ and $\pi_n : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the standard projection. Let $\alpha_n = \{\omega_n\}$, where ω_n is the Fubini-Study form, so $\pi_n^\star \omega_n = dd^c \log \|z\|$ and $\mathrm{Vol}\,(\alpha_n) = 1$.

2.1. Maximal Lelong number.

Proposition 2.1. *We have $\nu(\alpha_n, x) = \varepsilon(\alpha_n, x) = 1$ for all $x \in \mathbb{P}^n$. If $T \in \mathcal{P}(\alpha_n)$ and $\nu(T, x) = 1$ then $T = \wp_x^\star S$, where $\wp_x : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ is the projection with center x onto a hyperplane $\mathbb{P}^{n-1} \not\ni x$ and $S \in \mathcal{P}(\alpha_{n-1})$. Moreover, the following are equivalent:*

(i) the potentials of T have isotropic pole at x with Lelong number 1.

(ii) T has locally bounded potentials on $\mathbb{P}^n \setminus \{x\}$.

(iii) S has bounded potentials.

Proof. Let $\pi : X \to \mathbb{P}^n$ denote the blow up of $\mathbb{P}^n$ at x, and let E be the exceptional divisor. The map $\Phi = \wp_x \circ \pi : X \to \mathbb{P}^{n-1}$ is a holomorphic fibration, whose fibers are the projective lines through x. Moreover, $\pi^\star\alpha_n - E = \Phi^\star\alpha_{n-1}$.

If $\nu(T,x) = 1$ then $\widetilde{T} = \pi^\star T - [E]$ is a positive closed (1,1)-current on X in the cohomology class $\Phi^\star\alpha_{n-1}$. It follows that $\widetilde{T} = \Phi^\star S$ for some $S \in \mathcal{P}(\alpha_{n-1})$, hence $T = \wp_x^\star S$. The potentials of T have isotropic pole at x with Lelong number 1 if and only if $\widetilde{T}$ has bounded potentials, hence if and only if S has bounded potentials.

It is well known that currents in $\mathcal{P}(\alpha_n)$ have Lelong number at most 1 at each point x. The above construction shows that $\nu(\alpha_n, x) = \varepsilon(\alpha_n, x) = 1$. □

We now explore further the geometry of sublevel sets of high Lelong numbers, in the spirit of [Co2]. For $c > 0$ and $T \in \mathcal{P}(\alpha_n)$ a theorem of Siu [Si] states that

$$E_c(T) := \{x \in \mathbb{P}^n : \nu(T,x) \geq c\}$$

is an algebraic subset of dimension at most $n-1$. We also consider the set

$$E_c^+(T) := \{x \in \mathbb{P}^n : \nu(T,x) > c\}.$$

Proposition 2.2. *The set $E^+_{n/(n+1)}(T)$ is contained in a hyperplane of $\mathbb{P}^n$.*

Proof. Let $T = \omega_n + dd^c\varphi$ and set $E_c(\varphi) = E_c(T)$ and $E_c^+(\varphi) = E_c^+(T)$. The proof is by induction on n. If $n = 1$, T is a probability measure, $\nu(T,p) = T(\{p\})$, so $E^+_{1/2}(T)$ contains at most one point.

Let $c_n = n/(n+1)$. If $n \geq 2$ we assume for a contradiction that $E^+_{c_n}(\varphi)$ contains the points $q, p_1, \ldots, p_n$ in general position. Let H be the hyperplane determined by $p_1, \ldots, p_n$, so $q \notin H$. By a theorem of Siu [Si], $T = c[H] + R$, where $0 \leq c \leq 1$ and $R \in \mathcal{P}((1-c)\alpha_n)$ has generic Lelong number 0 along H. Thus

$$c_n < \nu(\varphi, q) = \nu(R,q) \leq 1 - c,\ \nu(R,p_j) = \nu(\varphi,p_j) - c > c_n - c,\ 1 \leq j \leq n.$$

Consider the current $S = R/(1-c) = \omega_n + dd^c\psi \in \mathcal{P}(\alpha_n)$. Since $c < 1 - c_n$,

$$\nu(\psi, p_j) > \frac{c_n - c}{1-c} > \frac{2c_n - 1}{c_n} = c_{n-1},\ 1 \leq j \leq n.$$

By [D3, Proposition 3.7], there exist $\epsilon_k \searrow 0$ and currents $S_k = (1+\epsilon_k)\omega_n + dd^c\psi_k \geq 0$, where ψ_k have analytic singularities, such that $S_k \to S$ and $0 \leq \nu(\psi,p) - \nu(\psi_k,p) \leq \epsilon_k$ for all $p \in \mathbb{P}^n$. Since S does not charge H, it follows that $\psi_k \not\equiv -\infty$ on $H \equiv \mathbb{P}^{n-1}$. Hence $\psi_k\,|_H \in PSH(\mathbb{P}^{n-1}, \omega_{n-1})$ and

$$\nu(\psi_k\,|_H, p_j) \geq \nu(\psi_k, p_j) > c_{n-1},\ 1 \leq j \leq n,$$

for k sufficiently large. This yields a contradiction, since by our induction hypothesis the set $E^+_{(n-1)/n}(\psi_k\,|_H)$ is contained in a hyperplane of $\mathbb{P}^{n-1}$. □

The value $n/(n+1)$ in the previous theorem is sharp. Indeed, let S be a set of $n+1$ points $p_j \in \mathbb{P}^n$ in general position, and let $[H_j]$ be the current of integration along the hyperplane H_j determined by $S \setminus \{p_j\}$. If $T = ([H_1] + \ldots + [H_{n+1}])/(n+1)$ then the set $E_{n/(n+1)}(T) = S$ is not contained in any hyperplane.

We are now in position to make the result of Proposition 2.1 more precise, by giving a characterization of the currents T for which $E_1(T) \neq \emptyset$.

Proposition 2.3. *If $T \in \mathcal{P}(\alpha_n)$ and $E_1(T) \neq \emptyset$ then $E_1(T)$ is a k-dimensional linear subspace of $\mathbb{P}^n$ for some integer $0 \leq k \leq n-1$. Let $\wp$ denote the projection with center $E_1(T)$ onto a linear subspace $L \equiv \mathbb{P}^{n-k-1}$ such that $L \cap E_1(T) = \emptyset$. Then $T = \wp^\star S$ for a unique current $S \in \mathcal{P}(\alpha_{n-k-1})$, and $E_1(S) = \emptyset$.*

Proof. Let $T = \omega_n + dd^c\varphi$ and $k \geq 0$ be the largest integer for which there exist $k+1$ points $p_0, \dots, p_k \in E_1(T)$ in general position (i.e. not contained in a $(k-1)$-dimensional subspace). Proposition 2.2 implies $k \leq n-1$. Using an automorphism of $\mathbb{P}^n$, we may assume $p_0 = [1 : 0 : \dots : 0]$, $p_1 = [0 : 1 : \dots : 0]$, and so on. Consider the projection f_0 of $\mathbb{P}^n$ with center p_0 onto the hyperplane $\mathbb{P}^{n-1} \equiv \{z_0 = 0\}$. Proposition 2.1 shows that $\varphi = u + h_0 \circ f_0$, where $h_0 \in PSH(\mathbb{P}^{n-1}, \omega_{n-1})$ and

$$u([z_0 : \dots : z_n]) = \frac{1}{2} \log \frac{|z_1|^2 + \dots + |z_n|^2}{|z_0|^2 + \dots + |z_n|^2} .$$

It follows that $f_0(p_j) \in E_1(h_0)$, $j = 1, \dots, k$, and Proposition 2.1 can be applied to h_0 and the point $f_0(p_1)$. Continuing like this we get

$$\varphi([z_0 : \dots : z_n]) = \frac{1}{2} \log \frac{|z_{k+1}|^2 + \dots + |z_n|^2}{|z_0|^2 + \dots + |z_n|^2} + h([z_{k+1} : \dots : z_n]),$$

with $h \in PSH(\mathbb{P}^{n-k-1}, \omega_{n-k-1})$. The definition of k implies $E_1(h) = \emptyset$, so $E_1(\varphi) = \{z_{k+1} = \dots = z_n = 0\}$. $\square$

2.2. **Green functions.**

2.2.1. *Green functions with one pole.* It follows from Proposition 2.1 that if $T = \wp_x^\star S$, where $S \in \mathcal{P}(\alpha_{n-1})$ has bounded potentials and $\wp_x : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ is the projection from x, then $T = \omega_n + dd^c g$ with $g = g_{S,x} \in PSH(\mathbb{P}^n, \omega_n) \cap L^\infty_{loc}(\mathbb{P}^n \setminus \{x\})$, g has an isotropic pole at x with Lelong number 1 and

$$(\omega_n + dd^c g)^n = \delta_x.$$

Conversely, any ω_n-psh Green function g with pole at x and maximal Lelong number $\nu(g, x) = 1$ is of this form, and in particular it must have an isotropic pole at x. Observe that the set of such functions is large.

2.2.2. *Multipole Green functions.* We push further the result of Proposition 2.1 and study multipole Green functions which arise naturally from rational maps.

Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$, $f = [P_1 : \dots : P_n]$, be a rational map with finite indeterminacy set I_f, where P_j are homogeneous polynomials of degree d on $\mathbb{C}^{n+1}$. Then f determines an ω_n-psh Green function,

$$g_f(\pi_n(z)) = d^{-1} \log \|F(z)\| - \log \|z\|,\ z \in \mathbb{C}^{n+1} \setminus \{0\}, \tag{4}$$

where $F : \mathbb{C}^{n+1} \to \mathbb{C}^n$, $F(z) = (P_1(z), \dots, P_n(z))$. The function g_f is continuous, $I_f = \{g_f = -\infty\}$, and g_f has an isolated pole at each point of I_f. Moreover, g_f verifies the Monge-Ampère equation

$$(\omega_n + dd^c g_f)^n = \sum_{p \in I_f} m_p \delta_p, \text{ where } m_p > 0,\ m_p \in \mathbb{Q},\ \sum_{p \in I_f} m_p = 1.$$

Our next result shows that this function has an extremal property (see [Co1] for a similar characterization of classes of pluricomplex Green functions on $\mathbb{C}^n$):

Theorem 2.4. *If $\varphi \in PSH(\mathbb{P}^n, \omega_n)$ and $\varphi \le g_f$, then there exists a unique function $h \in PSH(\mathbb{P}^{n-1}, \omega_{n-1})$ such that $\varphi = g_f + d^{-1} h \circ f$. Conversely, any such function φ is ω_n-psh. We have that φ is locally bounded on $\mathbb{P}^n \setminus I_f$ if and only if h is bounded. In this case, φ satisfies*

$$(\omega_n + dd^c\varphi)^n = \sum_{p \in I_f} m_p \delta_p.$$

Proof. Since the indeterminacy set I_f is finite, we can find a hyperplane H which does not intersect I_f. Let L be a linear polynomial defining H, and let $P_0 = L^d$. The map $\hat{f} = [P_0 : P_1 : \ldots : P_n] : \mathbb{P}^n \to \mathbb{P}^n$ is holomorphic and $f = \wp \circ \hat{f}$, where

$$\wp : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}, \ \wp([z_0 : z_1 : \ldots : z_n]) = [z_1 : \ldots : z_n],$$

is the projection with center $[1 : 0 : \ldots : 0]$.

For every $p \in \mathbb{P}^{n-1}$ the fiber $X_p := f^{-1}(p) = \hat{f}^{-1}(\wp^{-1}(p))$ is one-dimensional and is connected by [FH, Proposition 1], since $\wp^{-1}(p)$ is a line in $\mathbb{P}^n$. This implies in particular the uniqueness of h.

Fix now an arbitrary $p \in \mathbb{P}^{n-1}$, and let us assume $p = [a_1 : \ldots : a_{n-1} : 1]$. Then X_p is defined by the equations $P_j = a_j P_n$. Let $q = [b_0 : \ldots : b_n]$ be a point in $X_p \setminus I_f$. We assume that $b_0 = 1$. Then q has a neighborhood where $P_n(1, z_1, \ldots, z_n) \neq 0$. So, for some constant c, we have $\log \|F\| = \log |P_n| + c$ in this neighborhood. It follows that $\varphi - g_f$ is psh in some open set which contains $X_p \setminus I_f$. Since $\varphi - g_f \le 0$ and I_f is a finite set, $\varphi - g_f$ extends to a subharmonic function on X_p. But X_p is compact and connected, so $\varphi - g_f$ is constant on X_p. We conclude that $\varphi = g_f + (h \circ f)/d$, for some function h on $\mathbb{P}^{n-1}$. Since $\varphi \le g_f$ and g_f is continuous, it follows easily that h is upper semicontinuous.

We now show that $h \in PSH(\mathbb{P}^{n-1}, \omega_{n-1})$. By using an automorphisms of $\mathbb{P}^n$ we may assume that the hyperplane $H = \{z_0 = 0\}$ does not intersect I_f. We claim that the map $F' : \mathbb{C}^n \to \mathbb{C}^n$, $F'(z') = F(1, z')$, is proper. Indeed, if $P_j^d(z')$ is the homogeneous part of degree d of $P_j(1, z')$, then $P_j^d(z')$, $j = 1, \ldots, n$, have no common zeros except at 0. The homogeneity of P_j^d yields

$$\sum_{j=1}^{n} |P_j^d(z')|^2 \ge M \|z'\|^{2d},$$

for some constant $M > 0$, which implies that F' is proper. The function

$$u(z') = \varphi([1 : z']) + \log \sqrt{1 + \|z'\|^2} = \frac{1}{d} \log \|F'(z')\| + \frac{1}{d} h \circ \pi_{n-1}(F'(z'))$$

is psh on $\mathbb{C}^n$. Since F' is proper, the function

$$v(w) = d \max\{u(z') : F'(z') = w\} = \log \|w\| + h \circ \pi_{n-1}(w)$$

is psh on $\mathbb{C}^n$. This proves that $h \in PSH(\mathbb{P}^{n-1}, \omega_{n-1})$.

For the converse, note that

$$\omega_n + dd^c(g_f + (h \circ f)/d) = d^{-1} f^\star(\omega_{n-1} + dd^c h) \ge 0,$$

so $g_f + (h \circ f)/d$ is ω_n-psh.

Finally, it is clear that $\varphi \in L^\infty_{loc}(\mathbb{P}^n \setminus I_f)$ if and only if h is bounded. Then we infer by [D4] that $m_p = (\omega_n + dd^c g_f)^n(\{p\}) = (\omega_n + dd^c\varphi)^n(\{p\})$. The conclusion follows since $\sum_{p \in I_f} m_p = 1$. $\square$

Note that Proposition 2.1 follows from Theorem 2.4 applied to rational maps of degree $d = 1$. We will see in section 3.2 that Green functions determined by certain rational maps $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with three points of indeterminacy provide rich classes of examples of Green functions with one pole on $\mathbb{P}^1 \times \mathbb{P}^1$ (see Example 3.5).

Example 2.5. *An important particular case of Theorem 2.4 is the one of rational functions $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $f = [P_1 : P_2]$, where P_j are homogeneous polynomials of degree d whose common zero set I_f consists of d^2 distinct points of $\mathbb{P}^2$. Then g_f is a ω_2-psh Green function with d^2 isotropic poles and Lelong number $1/d$ at each pole. If $d = 2$ we observe that any set of four points in general position is the complete intersection of two conics, hence it can be realized as the indeterminacy set I_f for a rational map f of degree $d = 2$ as described above. It follows that the ω_2-psh Green functions with four isotropic poles are described by Theorem 2.4. However, if $d \geq 3$ a set of d^2 points of $\mathbb{P}^2$ in general position is not the complete intersection of two curves of degree d (in fact when $d \geq 4$, there is no curve of degree d passing through d^2 points in general position). So the Green functions g_f with d^2 isotropic poles, $d \geq 3$, only exist for very special sets of poles.*

2.2.3. *Partial Green functions.* We compute here in the case of $(\mathbb{P}^n, \omega_n)$ the functions $\psi_{\nu,p}$ constructed in Proposition 1.6. Assume without loss of generality that $p = 0 \in \mathbb{C}^n$. For $\nu < 1$ define R_ν, C_ν by

$$R_\nu = [\nu/(1-\nu)]^{1/2},\ \nu \log R_\nu + C_\nu = \log \sqrt{1 + R_\nu^2}.$$

For $z \in \mathbb{C}^n$ let

$$V(z) = \begin{cases} \nu \log \|z\| + C_\nu,\ \|z\| \leq R_\nu, \\ \log \sqrt{1 + \|z\|^2}\ ,\ \|z\| \geq R_\nu. \end{cases}$$

Proposition 2.6. *For $\nu < 1$ and $z \in \mathbb{C}^n$ we have $\psi_{\nu,p}(z) = V(z) - \log \sqrt{1 + \|z\|^2}$.*

Proof. Note that $\psi_{\nu,p}(z) = W(z) - \log \sqrt{1 + \|z\|^2}$, where

$$W(z) = \sup \left\{ v(z) :\ v \in PSH(\mathbb{C}^n),\ v \leq \log \sqrt{1 + \|\cdot\|^2},\ \nu(v, 0) \geq \nu \right\}.$$

Since $\max_{\|z\|=r} v(z)$ is a convex increasing function of $\log r$, and since $x = \log R_\nu$ is the solution of the equation $\frac{d}{dx} \log \sqrt{1 + e^{2x}} = \nu$, it follows that $W = V$. $\square$

Letting $\nu \nearrow 1$ it follows that $\psi_{1,p}(z) = \log(\|z\|/\sqrt{1 + \|z\|^2})$, $z \in \mathbb{C}^n$, is the Green function constructed in Theorem 1.4 for $u(z) = \log \|z\|$.

2.2.4. *Dynamical Green functions.* We now consider the problem of constructing Green functions on $\mathbb{P}^2$ with one pole at p and Lelong number at p less than 1. Let $\omega = \omega_2$, let $[t : x : y]$ denote the homogeneous coordinates on $\mathbb{P}^2$, and identify $z = (x, y) \in \mathbb{C}^2$ to $[1 : x : y]$. Simple examples can be obtained by considering a smooth curve with a flex at p, i.e. the tangent line at p does not intersect the curve at any other points. More generally, for integers $1 \leq k < n$, the function

$$g([t : x : y]) = \frac{1}{2n} \log(|y^k t^{n-k} - x^n|^2 + |y^n|^2) - \frac{1}{2} \log(|t|^2 + |x|^2 + |y|^2)$$

is ω-psh and smooth away from $p = 0 \in \mathbb{C}^2$, $\nu(g, p) = k/n$ and $(\omega + dd^c g)^2 = \delta_p$.

We describe next more elaborate constructions using complex dynamics. Let $h : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping of algebraic degree $\lambda > 1$. Then h extends to a rational self-map of $\mathbb{P}^2$, denoted again by h, with finite indeterminacy set $I \subset \{t = 0\}$. We call h *weakly regular* if h maps $\{t = 0\} \setminus I$ to a point $Z \notin I$ (see [GS]). Such h is algebraically stable ($\deg h^n = \lambda^n$). It was shown in [S] that the currents $\lambda^{-n}(h^n)^\star\omega$ converge weakly to an invariant positive closed current $T = T_h$ on $\mathbb{P}^2$, $T = \omega + dd^c g$. We call T the dynamical Green current and g a dynamical Green function of h. By [GS, Theorem 2.2], g is continuous on $\mathbb{P}^2 \setminus I$, $T \wedge T$ is supported on I, so g is a ω-psh Green function with poles in I.

If $|I| = 1$ then $T \wedge T = \delta_I$. Our goal is to compute the Lelong number $\nu(T, I)$.

Proposition 2.7. *Let h be a weakly regular polynomial endomorphism of $\mathbb{C}^2$ of degree $\lambda > 1$, with $|I| = 1$, and such that*

$$dist(h(p), I) \geq C\, dist(p, I)^\delta,\ p \in \mathbb{P}^2 \setminus \{I\}, \tag{5}$$

for constants $0 < C < 1$, $1 < \delta < \lambda$. Then $\nu(\lambda^{-n}(h^n)^\star\omega, I) \nearrow \nu(T, I)$ as $n \nearrow \infty$.

Proof. If $\lambda^{-1}h^\star\omega = \omega + dd^c\psi$, where $\psi \leq 0$ is ω-psh, then by [G, Theorem 2.1]

$$T_n := \lambda^{-n}(h^n)^\star\omega = \omega + dd^c g_n\ ,\ g_n = \sum_{j=0}^{n-1} \lambda^{-j}\psi \circ h^j \ \searrow\ g = \sum_{j=0}^{\infty} \lambda^{-j}\psi \circ h^j,$$

and $T = \omega + dd^c g$. Hence $\{\nu(T_n, I)\}$ is increasing and $\nu(T_n, I) \leq \nu(T, I)$.

It follows from (5) that there is $C' > 0$ so that for every n and $p \in \mathbb{P}^2 \setminus \{I\}$

$$dist(h^n(p), I) \geq (C'\, dist(p, I))^{\delta^n}.$$

Note that the function ψ is smooth except at I, and $\psi \geq \gamma \log dist(\cdot, I) - M$ holds on $\mathbb{P}^2$ for some constants $\gamma, M > 0$. Writing $g = g_n + \rho_n$, we deduce that

$$\rho_n(p) \geq \sum_{j=n}^{\infty} \lambda^{-j}\left(\gamma \log dist(h^j(p), I) - M\right) \geq \gamma'(\delta/\lambda)^n \log dist(p, I) - \epsilon_n,$$

with some $\gamma' > 0$ and $\epsilon_n \to 0$. Thus $\nu(T_n, I) \leq \nu(T, I) \leq \nu(T_n, I) + \gamma'(\delta/\lambda)^n$. $\square$

Note that (5) holds for Hénon maps $h(x, y) = (P(x) + ay, x)$, $\deg P = \lambda$, with $\delta = 1$, since $I = [0 : 0 : 1]$ is an attracting fixed point for h^{-1}. However, the map $h(x, y) = (x^\lambda - y^{\lambda-1}, y^{\lambda-1})$ shows that (5) does not hold for $\delta < \lambda$.

Proposition 2.8. *Let $h(x, y) = (x^\lambda + y^\mu, x)$, where $\lambda > \mu \geq 1$ are integers, so $I = [0 : 0 : 1]$. The Green current T of h verifies $T \wedge T = \delta_I$, $\nu(T, I) = (\lambda - \mu)/\lambda$.*

Proof. We show first that (5) holds with $\delta = \lambda - 1$. Note that h is weakly regular and in local coordinates (t, x) near I we have

$$h(t, x) = \left(\frac{t}{x}\ , \frac{x^\lambda + t^{\lambda-\mu}}{xt^{\lambda-1}}\right).$$

It is enough to prove (5) for $p = (t, x)$ with $0 < |x|, |t| < 1$. If $|t| \geq |x|$, or if $|x^\lambda + t^{\lambda-\mu}| \geq |xt^{\lambda-1}|$, then $\|h(t, x)\| \geq 1$ and the estimate follows. Otherwise, we have $|t| < |x| < 1$ and $|x^\lambda + t^{\lambda-\mu}| < |xt^{\lambda-1}|$, so $|x|^\lambda < 2|t|^{\lambda-\mu}$. Therefore

$$\|h(t, x)\| \geq \frac{|t|}{|x|} \geq C|x|^{\mu/(\lambda-\mu)} \geq C|x|^{\lambda-1} \geq C'\, dist(p, I)^{\lambda-1}.$$

Next we compute $\nu_n := \nu(\lambda^{-n}(h^n)^\star\omega, I)$. Let $h^n([t:x:y] = [t^{\lambda^n} : p_n(t,x,y) : q_n(t,x,y)]$, where p_n, q_n are homogeneous polynomials of degree λ^n, and

$$v_n(t,x) = \log(|t|^{2\lambda^n} + |p_n(t,x,1)|^2 + |q_n(t,x,1)|^2)^{1/2}\,.$$

It follows by induction that $\nu(v_n, 0) = \lambda^n - \max\{\deg_y p_n, \deg_y q_n\} = \lambda^n - \mu\lambda^{n-1}$, where $\deg_y p_n$ denotes the degree in y of p_n. Hence $\nu_n = (\lambda-\mu)/\lambda = \nu(T,I)$. $\square$

If h is Hénon map of degree λ a similar argument shows $\nu(T_h, I) = 1 - \lambda^{-1}$.

3. Green functions on $\mathbb{P}^1 \times \mathbb{P}^1$

It is possible to describe the functions ν, ε, g, ψ on a multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. For simplicity, we only consider the case $X = \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}^1_z \times \mathbb{P}^1_w$. Let $\pi_z : X \to \mathbb{P}^1_z$, $\pi_w : X \to \mathbb{P}^1_w$, denote the canonical projections and set

$$\alpha_{a,b} := a\alpha_z + b\alpha_w,\ \omega_{a,b} := a\omega_z + b\omega_w,\ a,b \geq 0,$$

where $\alpha_z = \pi_z^\star\alpha_1$, $\alpha_w = \pi_w^\star\alpha_1$, $\omega_z = \pi_z^\star\omega_1$, $\omega_w = \pi_w^\star\omega_1$, and $\omega_1 \in \alpha_1$ is the Fubini-Study form on $\mathbb{P}^1$. Note that $\alpha_{a,b}$ is a Kähler class if and only if $a, b > 0$.

For concrete computations, it will be convenient to use coordinates on X. Let

$$\pi : (\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\}) \to X\ ,\ \pi(z_0, z_1, w_0, w_1) = ([z_0 : z_1], [w_0 : w_1]),$$

and identify $(z_1, w_1) \in \mathbb{C}^2$ to $\pi(1, z_1, 1, w_1) \in X$. The currents $T \in \mathcal{P}(\alpha_{a,b})$ can be described using the class $P_{a,b}$ of bihomogeneous psh functions $\widetilde{u}$ on $\mathbb{C}^4$ (see [G]):

$$\widetilde{u}(\lambda z_0, \lambda z_1, \mu w_0, \mu w_1) = a\log|\lambda| + b\log|\mu| + \widetilde{u}(z_0, z_1, w_0, w_1),\ \lambda, \mu \in \mathbb{C}.$$

Then $\pi^\star T = dd^c\widetilde{u}$, for some $\widetilde{u} \in P_{a,b}$ which is unique up to additive constants.

For a point $p = (x,y) \in X$ we denote by

$$V_x = \pi_z^{-1}(x) = \{z = x\}\ ,\ H_y = \pi_w^{-1}(y) = \{w = y\},$$

the vertical, and respectively horizontal, line through p.

3.1. Maximal Lelong numbers.

Proposition 3.1. *For all $p = (x,y) \in X$, we have*

$$\nu(\alpha_{a,b}, p) = a + b,\ \varepsilon(\alpha_{a,b}, p) = \min\{a, b\}.$$

If $T \in \mathcal{P}(\alpha_{a,b})$ and $\nu(T,p) = a + b$ then $T = a[V_x] + b[H_y]$. Moreover, if T does not charge V_x and H_y then $\nu(T,p) \leq \min\{a,b\}$.

Proof. Let $T \in \mathcal{P}(\alpha_{a,b})$. We can assume that $p = (0,0)$ and let $m = \min\{a,b\}$. The current $R_{a,b} \in \mathcal{P}(\alpha_{a,b})$ defined by $\pi^\star R_{a,b} = dd^c\widetilde{u}_{a,b}$, where $\widetilde{u}_{a,b} \in P_{a,b}$,

$$\widetilde{u}_{a,b}(z_0, z_1, w_0, w_1) := m\log\sqrt{|z_1w_0|^2 + |w_1z_0|^2} + (a-m)\log|z_0| + (b-m)\log|w_0|,$$

shows that $\varepsilon(\alpha_{a,b}, p) \geq m$. Moreover, the measure $T \wedge R_{1,1}$ is well defined and

$$\nu(T,p) = T \wedge R_{1,1}(\{p\}) \leq \int_X T \wedge R_{1,1} = \int_X \omega_{a,b} \wedge \omega_{1,1} = a + b.$$

Assume now that T does not charge the subvarieties V_x and H_y. By [D3], there exist $\epsilon_j \searrow 0$ and currents $T_j \in \mathcal{P}(\alpha_{a,b} + \epsilon_j\alpha_{1,1})$ with analytic singularities, so that

$0 \leq \nu(T,q) - \nu(T_j,q) \leq \epsilon_j$ for every $q \in X$. Since T does not charge V_x, the measure $T_j \wedge [V_x]$ is well defined. If v_j is a psh potential of T_j near p then

$$\nu(T_j,p) \leq \nu(v_j|_{V_x},p) = T_j \wedge [V_x](\{p\}) \leq \int_X T_j \wedge [V_x] = b + \epsilon_j.$$

We replace V_x by H_y in this argument and let $j \to +\infty$ to get $\nu(T,p) \leq m$. By (3) it follows that $\varepsilon(\alpha_{a,b},p) \leq m$.

Assume finally that $\nu(T,p) = a + b$. By [Si], we can write

$$T = a'[V_x] + b'[H_y] + T',\ T' \in \mathcal{P}(\alpha_{a-a',b-b'}),$$

where T' does not charge V_x and H_y. By what we have already shown,

$$a + b = \nu(T,p) \leq a' + b' + \min\{a - a', b - b'\}.$$

This implies that $a' = a$, $b' = b$, and $T' = 0$. □

Observe that the functions ν, ε are constant here, as well as in the case of $\mathbb{P}^n$, because α is invariant under a compact group of automorphisms that acts transitively on X.

Note that $\mathrm{Vol}\,(\alpha_{a,b})^{1/2} = \sqrt{2ab} > \min\{a,b\}$, hence the upper bound given in (2) is not sharp in this case. Another obvious consequence of the previous proposition is the following:

Corollary 3.2. *There is no Green function with one isotropic pole on $\mathbb{P}^1 \times \mathbb{P}^1$.*

We can however compute the partial Green functions with isotropic singularity $\psi_{\nu,p}$ constructed in Proposition 1.6. Assume that $p = (0,0) \in \mathbb{C}^2 \subset X$, and let $a = b = 1$, $\nu = \varepsilon(\alpha_{1,1},p) = 1$. A psh potential of $\omega_{1,1}$ on $\mathbb{C}^2$ is given by

$$\rho(z_1,w_1) = \log\sqrt{1+|z_1|^2} + \log\sqrt{1+|w_1|^2}.$$

Proposition 3.3. *We have $\psi_{1,p}(z_1,w_1) = \log(|z_1|+|w_1|) - \rho(z_1,w_1)$ if $|z_1w_1| \leq 1$, and $\psi_{1,p}(z_1,w_1) = 0$ if $|z_1w_1| \geq 1$.*

Proof. We have to obtain upper estimates for psh functions v on $\mathbb{C}^2$ which verify $v \leq \rho$ and $\nu(v,0) \geq 1$. We do this first along a complex line $z_1 = s\zeta$, $w_1 = t\zeta$. Using the same convexity argument as in the proof of Proposition 2.6, we obtain

$$v(s\zeta,t\zeta) \leq \begin{cases} \log|\zeta| + C, & |\zeta| \leq R, \\ \rho(s\zeta,t\zeta), & |\zeta| \geq R. \end{cases}$$

Here $R = |st|^{-1/2}$, $x = \log R$ is the solution of the equation

$$\frac{d}{dx}\left(\log\sqrt{1+|s|^2e^{2x}} + \log\sqrt{1+|t|^2e^{2x}}\right) = 1,$$

and $C = \log(|s|+|t|)$ verifies $\log R + C = \rho(sR,tR)$. If $s = 1$, $t = w_1/z_1$, we get

$$v(z_1,w_1) \leq V(z_1,w_1) = \begin{cases} \log(|z_1|+|w_1|), & |z_1w_1| \leq 1, \\ \rho(z_1,w_1), & |z_1w_1| \geq 1. \end{cases}$$

Since $\log(|z_1|+|w_1|) \leq \rho(z_1,w_1)$ on $\mathbb{C}^2$, with equality when $|z_1w_1| = 1$, the function V is psh. It follows that $\psi_{1,p} = V - \rho$. □

Note that the (unbounded) hyperconvex domain

$$D_{1,p} = \{\psi_{1,p} < 0\} = \{(z_1, w_1) \in \mathbb{C}^2 : |z_1 w_1| < 1\}$$

does not have a pluricomplex Green function: if $v < 0$ is psh on $D_{1,p}$ and $v(0,0) = -\infty$ then $v = -\infty$ along the lines $\{z_1 = 0\}, \{w_1 = 0\}$.

3.2. **Green functions with one pole.** It is clear from Proposition 3.1 and Corollary 3.2 that the characterization of Green functions in $PSH(X, \omega_{a,b})$ with one pole at $p \in X$ is more involved. Using a birational map, we will show that they correspond to a certain class of Green functions with three poles on $\mathbb{P}^2$. A rich class of examples of the latter can be constructed using (4) (see also Theorem 2.4). This will show that the Green functions of X with pole at p have many different types of singularities, even if one asks that the Lelong number at p is maximal.

We may assume that $p = (0,0) \in \mathbb{C}^2 \subset X$ and $a = 1 \leq b$. Let $\omega = \omega_{FS}$ on $\mathbb{P}^2$ and consider the rational map $\Phi : \mathbb{P}^2 \dashrightarrow X$ defined by

$$\Phi([t_0 : t_1 : t_2]) = ([t_0 : t_1], [t_0 : t_2]).$$

It is a birational map, with rational inverse

$$\Phi^{-1}([z_0 : z_1], [w_0 : w_1]) = [z_0 w_0 : z_1 w_0 : w_1 z_0].$$

Note that Φ is the identity on $\mathbb{C}^2 \equiv \{[1 : t_1 : t_2] \in \mathbb{P}^2\} \equiv \{([1 : z_1], [1 : w_1]) \in X\}$, Φ blows up the points $A = [0 : 1 : 0]$, $B = [0 : 0 : 1]$, to the lines $\{z_0 = 0\}$, respectively $\{w_0 = 0\}$, and Φ contracts the line $\{t_0 = 0\}$ to the point $q = (\infty, \infty)$.

We denote by $\mathcal{S}_b$ the set of the currents $S \in \mathcal{P}(\alpha_{1,b})$ with locally bounded potentials on $X \setminus \{p\}$ and such that $S \wedge S = 2b\delta_p$. A potential of S is then a $\omega_{1,b}$-psh Green function on X with pole at p.

Let $\mathcal{R}_b$ be the set of currents $R \in \mathcal{P}((1+b)\omega)$ on $\mathbb{P}^2$ whose potentials are locally bounded on $\mathbb{P}^2 \setminus \{p, A, B\}$, have isotropic poles at A, B with Lelong numbers $\nu(R, A) = b$, $\nu(R, B) = 1$, and such that $R \wedge R = 0$ on $\mathbb{P}^2 \setminus \{p, A, B\}$. It follows that a potential v of R is a $(1+b)\omega$-psh Green function on $\mathbb{P}^2$ with poles at p, A, B:

$$R \wedge R = ((1+b)\omega + dd^c v)^2 = b^2 \delta_A + \delta_B + 2b\,\delta_p.$$

Proposition 3.4. *The mapping $\Phi^\star : \mathcal{S}_b \to \mathcal{R}_b$ is well defined and bijective. Its inverse is the mapping*

$$G : R \in \mathcal{R}_b \mapsto (\Phi^{-1})^\star R - b[z_0 = 0] - [w_0 = 0] \in \mathcal{S}_b.$$

Proof. Let $S \in \mathcal{S}_b$ and $\widetilde{u} \in P_{1,b}$ be a potential of S. Then

$$\widetilde{v}(t_0, t_1, t_2) := \widetilde{u}(t_0, t_1, t_0, t_2),\ \widetilde{v}(\lambda t_0, \lambda t_1, \lambda t_2) = \widetilde{v}(t_0, t_1, t_2) + (1+b) \log |\lambda|,$$

is a logarithmically homogeneous potential for $R = \Phi^\star S$, so $R \in \mathcal{P}((1+b)\omega)$. In particular, it follows that R has locally bounded potentials on $\mathbb{P}^2 \setminus \{p, A, B\}$. Near the point A, assuming wlog that $|t_0| \leq |t_2|$ we have

$$\widetilde{v}(t_0, 1, t_2) = \widetilde{u}(t_0, 1, t_0/t_2, 1) + b \log |t_2| = b \log \sqrt{|t_0|^2 + |t_2|^2} + O(1).$$

So R has potentials with an isotropic pole at A and $\nu(R, A) = b$. One proves in the same way that R has potentials with an isotropic pole at B and $\nu(R, B) = 1$. We have $R \wedge R = S \wedge S = 0$ on $\mathbb{C}^2 \setminus \{0\}$. Since R has locally bounded potentials near each point of $\{t = 0\} \setminus \{A, B\}$ we have $R \wedge R(\{t = 0\} \setminus \{A, B\}) = 0$, so $R \in \mathcal{R}_b$.

Conversely, let $R \in \mathcal{R}_b$ with logarithmically homogeneous potential $\widetilde{v}$. Then

$$\widetilde{u}(z_0, z_1, w_0, w_1) := \widetilde{v}(z_0 w_0, z_1 w_0, w_1 z_0) - b \log |z_0| - \log |w_0| \in P_{1,b}$$

is a bihomogeneous potential of $G(R)$. We show that $G(R)$ has locally bounded potentials in a neighborhood of any point at infinity $\zeta \neq q$. Suppose wlog $\zeta \in \{z_0 = 0\}$. Then for $|z_0|$ small enough we have that $[z_0 : 1 : z_0 w_1]$ is near A, so

$$\widetilde{u}(z_0, 1, 1, w_1) = \widetilde{v}(z_0, 1, w_1 z_0) - b \log |z_0| = b \log \sqrt{1 + |w_1|^2} + O(1) = O(1).$$

Next we study the potentials of $G(R)$ in a neighborhood of q. We have

$$\widetilde{u}(z_0, 1, w_0, 1) = \widetilde{v}(z_0 w_0, w_0, z_0) - b \log |z_0| - \log |w_0|,$$

where $|z_0|, |w_0|$ are small. If $|w_0/z_0|$ is small, then $[w_0 : w_0/z_0 : 1]$ is near B so

$$\widetilde{u}(z_0, 1, w_0, 1) = \widetilde{v}(w_0, w_0/z_0, 1) + \log |z_0| - \log |w_0| = \log \sqrt{|z_0|^2 + 1} + O(1).$$

Similarly, $\widetilde{u}(z_0, 1, w_0, 1) = O(1)$ if $|z_0/w_0|$ is small. If $\epsilon \leq |w_0/z_0| \leq M$ then

$$\widetilde{u}(z_0, 1, w_0, 1) = \widetilde{v}(w_0, w_0/z_0, 1) + \log(|z_0|/|w_0|) = O(1).$$

It follows that $G(R)$ has locally bounded potentials in $X \setminus \{p\}$, hence $G(R) \in \mathcal{S}_b$.

Since Φ is the identity on $\mathbb{C}^2$ and the currents in $\mathcal{R}_b$, resp. $\mathcal{S}_b$, do not charge the line(s) at infinity, we conclude by the support theorem that $\Phi^\star$ is bijective and G is its inverse. □

Example 3.5. *Let $1 \leq b = m/n \in \mathbb{Q}$ and $f = [P_1 : P_2] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, where*

$$P_1(t_0, t_1, t_2) = t_1^{nk} t_2^{mk}, \; P_2(t_0, t_1, t_2) = t_1^{nk} t_0^{mk} + t_2^{mk} t_0^{nk} + t_1 t_2 Q(t_0, t_1, t_2),$$

$k \geq 1$ is an integer, and Q is a homogenous polynomial of degree $(m+n)k - 2$ with $\deg_{t_1} Q \leq nk - 1$ and $\deg_{t_2} Q \leq mk - 1$. Note that the indeterminacy set $I_f = \{p, A, B\}$ and the current

$$R_f := (1 + b)(\omega + dd^c g_f) \in \mathcal{R}_b,$$

where g_f is the Green function associated to f defined in (4). Then $S_f = G(R_f)$ has bihomogeneous potential $\widetilde{u}_f \in P_{1,b}$ given by

$$\widetilde{u}_f(1, z_1, 1, w_1) = \frac{1}{2nk} \log \left(|z_1^{nk} w_1^{mk}|^2 + |z_1^{nk} + w_1^{mk} + z_1 w_1 Q(1, z_1, w_1)|^2 \right),$$

where $Q(1, z_1, w_1) = \sum_{i_1=0}^{nk-1} \sum_{i_2=0}^{mk-1} c_{i_1 i_2} z_1^{i_1} w_1^{i_2}$. Depending on the vanishing order of $Q(1, \cdot)$ at the origin, one sees that the Lelong number $\nu(S_f, p)$ can take any value of the form $\frac{j}{nk}$, $2 \leq j \leq nk$. It follows that for any rational number $r \in (0, 1]$ there exist $\omega_{1,b}$-psh Green functions on X with one pole at p and Lelong number equal to r there, but with different types of singularities at p.

We finally give an alternate way to construct $\omega_{1,1}$-psh Green functions on X with pole at $q = (\infty, \infty)$, using currents on $\mathbb{P}^2$ arising from psh functions in the Lelong class $\mathcal{L}^\star(\mathbb{C}^2)$. This is the class of psh functions v on $\mathbb{C}^2$ so that

$$\limsup_{\|s\| \to \infty} v(s) / \log \|s\| = 1.$$

If R is the trivial extension of $dd^c v$ to $\mathbb{P}^2$ then $R \in \mathcal{P}(\omega)$.

Proposition 3.6. *Let $R \in \mathcal{P}(\omega)$ be a current with locally bounded potentials in $\mathbb{P}^2 \setminus \{t_0 = 0\}$ and near the points A, B. Then the current $S = (\Phi^{-1})^\star R \in \mathcal{P}(\alpha_{1,1})$, $\nu(S, q) = 1$, and S has locally bounded potentials on $X \setminus \{q\}$. Moreover, we have*

$$S \wedge S = 2\delta_q \Longleftrightarrow R \wedge R = 0 \text{ on } \mathbb{P}^2 \setminus \{t_0 = 0\}.$$

Proof. By considering (bi)homogeneous potentials as in the proof of Proposition 3.4, it follows that $S \in \mathcal{P}(\alpha_{1,1})$ and S has locally bounded potentials on $X \setminus \{q\}$. So $S \wedge S(\{z_0 = 0\} \cup \{w_0 = 0\} \setminus \{q\}) = 0$, and $S \wedge S = 0$ on $\mathbb{C}^2$ implies $S \wedge S = 2\delta_q$.

Let $\nu := \nu(S, q)$. Since Φ contracts the line $\{t_0 = 0\}$ to q, we have that $\Phi^\star S = \nu[t_0 = 0] + T$, where $T \in \mathcal{P}((2 - \nu)\omega)$ does not charge the line $\{t_0 = 0\}$. Note that $R = T$ on $\mathbb{C}^2$. By the support theorem we conclude that $R = T$, so $\nu = 1$. □

Proposition 3.6 shows how Green functions can be constructed on X by using currents R on $\mathbb{P}^2$ possessing the right properties at any two points A, B and outside the line joining them. Indeed, we pull back R by an automorphism of $\mathbb{P}^2$ which maps the points $[0 : 1 : 0]$, $[0 : 0 : 1]$ to A, B, and then apply Proposition 3.6.

Example 3.7. *The Green currents T^+, T^- of a Hénon map h on $\mathbb{C}^2$ yield by the preceding considerations Green functions on X with pole at q. More generally, let h be a weakly regular polynomial endomorphism of $\mathbb{C}^2$ with indeterminacy set I (see section 2.2.4). Then its Green current T has continuous local potentials on $\mathbb{P}^2 \setminus I$ and $T \wedge T = \sum_{s \in I} m_s \delta_s$. So T yields a Green function on X with pole at q.*

4. Del Pezzo Surfaces

We evaluate here the functions ν, ε, g when X is a (smooth) Del Pezzo surface, i.e. $\dim_{\mathbb{C}} X = 2$ and $c_1(X) > 0$. It is well known (see e.g. [De]) that such X is biholomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, or $\mathbb{P}^2$ blown up at r points in general position, $1 \le r \le 8$. Here general position means the following:

– no three points are collinear;
– no six points lie on a conic;
– when $r = 8$, the points do not lie on a cubic that is singular at one of them.

The cases $X = \mathbb{P}^2$, $X = \mathbb{P}^1 \times \mathbb{P}^1$, have already been considered in Sections 2 and 3. We focus here on the case when X is the blow up of $\mathbb{P}^2$ at 8 points in general position, which we consider to be the most interesting one. The other cases could be handled similarly. Note that the Seshadri constants ε are computed in [Br].

4.1. **Maximal Lelong numbers.** Let $\pi : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at 8 points $p_1, \ldots, p_8$ in general position, and let $E_j = \pi^{-1}(p_j)$ denote the exceptional divisors. We let

$$\alpha := c_1(X) = K_X^{-1} = \pi^\star \mathcal{O}(3) - \sum_{j=1}^{8} E_j$$

denote the (ample) anticanonical class of X. It is well known [De] that 2α is very ample. It follows from Remark 1.2 that

$$\nu(\alpha, x) \ge 1,\ \varepsilon(\alpha, x) \ge 1/2,\ \forall\, x \in X. \tag{6}$$

We can actually be much more precise. Let $\mathcal{V}$ be the pencil of cubics in $\mathbb{P}^2$ passing through $p_1, \ldots, p_8$. It contains at most 12 singular cubics [De]. We let $S \subset X$ denote the set of the corresponding singular points, $|S| \leq 12$. These points do not belong to the exceptional divisors, by the general position assumption.

Proposition 4.1. *We have*

$$\nu(\alpha, x) = \begin{cases} 1, & \text{if } x \in X \setminus S, \\ 2, & \text{if } x \in S. \end{cases}$$

Moreover, if $x \in S$ and $T \in \mathcal{P}(\alpha)$ does not charge the strict transform of the singular cubic in $\mathcal{V}$ passing through x then $\nu(T, x) \leq 1/2$.

Proof. For $x \in X$ there exists a unique cubic $\mathcal{C}_x \in \mathcal{V}$ whose strict transform $\mathcal{C}'_x$ contains x. (If $x \in E_j$ this is the cubic whose strict transform intersects E_j at x.) Note that $\mathcal{C}'_x$ is irreducible.

Let $T \in \mathcal{P}(\alpha)$. We assume at first that T does not charge $\mathcal{C}'_x$ and let ω be a fixed Kähler form on X. By [D3] there exist $\epsilon_j \searrow 0$ and currents $T_j \in \mathcal{P}(\alpha + \epsilon_j \omega)$ with analytic singularities, such that $T_j \to T$ and $0 \leq \nu(T, z) - \nu(T_j, z) \leq \epsilon_j$ for all $z \in X$. Since T does not charge $\mathcal{C}'_x$, the measure $T_j \wedge [\mathcal{C}'_x]$ is well defined. As $\mathrm{Vol}\,(\alpha) = 1$ it follows that

$$1 + O(\epsilon_j) = \int_X T_j \wedge [\mathcal{C}'_x] \geq T_j \wedge [\mathcal{C}'_x](\{x\}) \geq \nu(T_j, x) m(\mathcal{C}'_x, x),$$

where $m(\mathcal{C}'_x, x)$ denotes the multiplicity of $\mathcal{C}'_x$ at x. The last inequality can be seen by using a local normalization at x for each irreducible component of $\mathcal{C}'_x$ and since local psh potentials of T_j are subharmonic along $\mathcal{C}'_x$.

Letting $j \to +\infty$, we have shown that $\nu(T, x) \leq 1/m(\mathcal{C}'_x, x) \leq 1$, if $T \in \mathcal{P}(\alpha)$ does not charge $\mathcal{C}'_x$. In particular, if $x \in S$ then $\nu(T, x) \leq 1/2$ since $m(\mathcal{C}'_x, x) = 2$.

In the general case, we can write by [Si]

$$T = a[\mathcal{C}'_x] + (1 - a)R, \ 0 \leq a \leq 1,$$

where $R \in \mathcal{P}(\alpha)$ does not charge $\mathcal{C}'_x$. Then

$$\nu(T, x) = am(\mathcal{C}'_x, x) + (1 - a)\nu(R, x) \leq a(m(\mathcal{C}'_x, x) - 1) + 1 \leq m(\mathcal{C}'_x, x),$$

which concludes the proof. □

4.2. **Uniform integrability exponent.** We fix $\omega \in \alpha = c_1(X)$ a Kähler form and we denote by $PSH_0(X, \omega)$ the set of ω-psh functions φ normalized by $\max_X \varphi = 0$. This is a compact subset of $L^1(X)$. Set

$$\sigma(X) = \sup\{c \geq 0 : \, e^{-2c\varphi} \in L^1(X), \ \forall \, \varphi \in PSH_0(X, \omega)\}.$$

This number clearly depends only on $\alpha = c_1(X)$, rather than on the particular choice of ω. By the compactness of $PSH_0(X, \omega)$ and the semicontinuity of the "complex singularity exponent" [DK], $\sigma(X)$ coincides with the exponent introduced by Tian in [T] (the so-called "α-invariant of Tian").

We assume here again that X is the blow up of $\mathbb{P}^2$ at 8 points in general position. Since $\nu(\alpha, x) \leq 2$ for all $x \in X$, it follows from Skoda's integrability theorem [Sk] that $\sigma(X) \geq 1/2$. One can however obtain sharp estimates, thanks to the full characterization given in Proposition 4.1:

Proposition 4.2. *If there is a singular cubic in $\mathcal{V}$ with a cusp then $\sigma(X) = 5/6$. Otherwise, $\sigma(X) = 1$.*

Recall that there is no cuspidal cubic in $\mathcal{V}$ when the points $p_1, \dots, p_8$ are in very general position [De].

Proof of Proposition 4.2. Let $s = |S| \leq 12$ and $\mathcal{C}'_j$, $1 \leq j \leq s$, denote the strict transforms of the singular cubics in $\mathcal{V}$. We write $[\mathcal{C}'_j] = \omega + dd^c\varphi_j$, where $\varphi_j \in PSH_0(X, \omega)$.

Fix now $\varphi \in PSH_0(X, \omega)$ and let $T = \omega + dd^c\varphi \in \mathcal{P}(\alpha)$. By [Si],

$$T = a_0 T_0 + \sum_{j=1}^{s} a_j [\mathcal{C}'_j], \text{ where } a_j \geq 0, \ \sum_{j=0}^{s} a_j = 1,$$

and $T_0 = \omega + dd^c\varphi_0 \in \mathcal{P}(\alpha)$ does not charge any curve $\mathcal{C}'_j$. Hölder's inequality shows that $e^{-2c\varphi} \in L^1(X)$ if $e^{-2c\varphi_j} \in L^1(X)$ for all $j = 0, \dots, s$.

For $j \geq 1$, a direct computation in local coordinates shows that $e^{-2c\varphi_j} \in L^1(X)$ for every $c < 1$ if $\mathcal{C}'_j$ is non-singular or has a simple node, while $e^{-2c\varphi_j} \in L^1(X)$ for every $c < 5/6$ if $\mathcal{C}'_j$ has a cusp. In the latter case, $e^{-2c\varphi_j} \notin L^1(X)$ if $c = 5/6$.

Since T_0 does not charge any curve $\mathcal{C}'_j$, it follows from Proposition 4.1 that $\nu(T_0, x) \leq 1$ for all $x \in X$. By [Sk] we see that $e^{-2c\varphi_0} \in L^1(X)$ for every $c < 1$. This completes the proof of the proposition. □

Note that $\sigma(X)$ is also called the (global) "log-canonical threshold" of X. It has been the subject of intensive studies in the last decade. The above result has been recently obtained by Cheltsov [Ch] by more algebraic methods.

The importance of this notion is seen in its connection with the existence of Kähler-Einstein metrics: it was shown by Tian [T] that a Fano surface admits a Kähler-Einstein metric if $\sigma(X) > 2/3$. The exponent $\sigma(X)$ was previously estimated by Tian and Yau in [TY].

4.3. **Green functions.** In this section X denotes again the blow up of $\mathbb{P}^2$ at 8 points in general position.

4.3.1. *Special points.* For $x \in S$, let $\mathcal{C}_x$ be the cubic in $\mathcal{V}$ which is singular at x, and let $\mathcal{C}'_x$ be its strict transform.

Counting dimension we see that there exists an irreducible sextic $Z \subset \mathbb{P}^2$ passing through x and with multiplicity 2 at each point p_j. By Bezout we see that Z and $\mathcal{C}_x$ intersect only at x and at the points p_j and the intersection numbers $(Z \cdot \mathcal{C}_x)_{p_j} = (Z \cdot \mathcal{C}_x)_x = 2$. This implies that the strict transform $Z' \subset X$ of Z intersects $\mathcal{C}'_x$ only at x with $(Z' \cdot \mathcal{C}'_x)_x = 2$.

We write $(1/2)[Z'] = \omega + dd^c u$, $[\mathcal{C}'_x] = \omega + dd^c v$, and set

$$g_x := (1/2) \log(e^{2u} + e^{2v}) \in PSH(X, \omega) \cap C^\infty(X \setminus \{x\}).$$

Proposition 4.3. *If $x \in S$ we have $(\omega + dd^c g_x)^2 = \delta_x$, and the function g_x is a ω-psh Green function with Lelong number $\nu(g_x, x) = 1/2$.*

Proof. Since Z' is smooth at x we have $\nu(g_x, x) = 1/2$. Moreover, $(Z' \cdot \mathcal{C}'_x)_x = 2$ implies that $(\omega + dd^c g_x)^2(\{x\}) = 1$. We conclude by mass considerations. □

Observe that the singularity of g_x at x is not isotropic, since an isotropic pole with Lelong number $1/2$ would produce a Dirac mass at x with coefficient $1/4$. However, the existence of a Green function which is locally bounded away from x has interesting consequences:

Corollary 4.4. *If $x \in S$ then $\varepsilon(\alpha, x) = 1/2$. Moreover, the supremum is attained in the formula (3) of $\varepsilon(\alpha, x)$, i.e.*

$$\exists\, \varphi \in PSH(X,\omega) \cap L^\infty_{loc}(X \setminus \{x\}),\ \|\varphi - (1/2)\log dist(\cdot, x)\|_{L^\infty(X)} < +\infty.$$

Proof. It follows from (6) and Proposition 4.1 that $\varepsilon(\alpha, x) = 1/2$. Let g_x be the function constructed in Proposition 4.3. Fix $\chi \in C^\infty(X)$ a test function with $\chi \equiv 1$ on $\overline{U}$, where U is a small open neighborhood of x. We define

$$\varphi := \max\{g_x, (1/2)\chi \log dist(\cdot, x) - C\},$$

where C is large so that $\varphi = g_x$ on $X \setminus U$. Since $\chi \log dist(\cdot, x)$ is psh on U we see that $\varphi \in PSH(X, \omega)$. Now $\nu(g_x, x) = 1/2$, therefore $\varphi - (1/2)\log dist(\cdot, x)$ is bounded on X. □

4.3.2. *Generic points.* Assume now that $x \in X \setminus S$. The bound (6) is not sharp: by [Br] we have $\varepsilon(\alpha, x) = 1$.

It is easy to see that the supremum in formula (3) is attained if x is the ninth base point of the pencil of cubics $\mathcal{V}$. In this case we write $[\mathcal{C}_1'] = \omega + dd^c u$, $[\mathcal{C}_2'] = \omega + dd^c v$, where $\mathcal{C}_j'$ are the strict transforms of two cubics generating $\mathcal{V}$, and we set

$$g_x := (1/2)\log(e^{2u} + e^{2v}) \in PSH(X, \omega) \cap C^\infty(X \setminus \{x\}).$$

We have that $(\omega + dd^c g_x)^2 = \delta_x$ and g_x is a ω-psh Green function with an isotropic pole at x with $\nu(g_x, x) = 1$.

However, it is unclear whether this holds at arbitrary points $x \in X \setminus S$. If this was the case, it would imply that K_Y^{-1} admits a positive metric with bounded potentials, where $Y \to \mathbb{P}^2$ is the blow up of $\mathbb{P}^2$ at 9 points in general position, which is a famous open problem (see [DPS]). Observe that the existence of such a metric is equivalent to constructing a ω_{FS}-psh Green function with isotropic poles of Lelong number 1/3 at 9 points in general position in $\mathbb{P}^2$.

More generally, finding a ω_{FS}-psh Green function with isotropic poles of Lelong number $1/\sqrt{s}$ at s points in general position in $\mathbb{P}^2$ is equivalent to the celebrated (strong version of) Nagata's conjecture (see [La, Remark 5.1.14]).

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