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Regge Calculus as a Fourth Order Method in Numerical Relativity

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Abstract

The convergence properties of numerical Regge calculus as an approximation to continuum vacuum General Relativity is studied, both analytically and numerically. The Regge equations are evaluated on continuum spacetimes by assigning squared geodesic distances in the continuum manifold to the squared edge lengths in the simplicial manifold. It is found analytically that, individually, the Regge equations converge to zero as the second power of the lattice spacing, but that an average over local Regge equations converges to zero as (at the very least) the third power of the lattice spacing. Numerical studies using analytic solutions to the Einstein equations show that these averages actually converge to zero as the fourth power of the lattice spacing.

1 Introduction

In 1961, Regge introduced a discrete form of General Relativity (which has since been named Regge calculus) [1]. Like General Relativity, Regge calculus is a geometrical theory of gravity. However, instead of using infinitesimal distances (the metric) as the dynamical variables of the theory, Regge calculus employs *finite* distances as dynamical variables. Since its introduction, Regge calculus has been used for a variety of purposes (see [2]), but was initially envisioned as a tool for numerical relativity [1]. Although work has been done in numerical relativity using Regge calculus, most applications share at least one of the two following undesirable (from the standpoint of practical numerical relativity) features:

1. The method used is based on non-simplicial (non-rigid) blocks (i.e. prisms or hypercubes).

2. The method uses blocks with Euclidean signature.

The first feature is undesirable for two reasons. First, not only must one specify extra "rigidifying" conditions on each block, but one must also make sure that the geometry on a face of a block matches up with the geometry of the face of an adjacent block (these "rigidifying" conditions have non-local constraints). Simplices, on the other hand, are rigid: fixing the values of the squared edge lengths completely determines the geometry inside the simplex, so there is no problem in trying to match up the geometry of two adjacent simplices. Second, it has recently been realized that Sorkin triangulations (triangulations in which the Regge equations decouple into single vertex evolutions) can be constructed from virtually any initial 3-dimensional simplicial manifold, thus forming a triangulation of the 4-dimensional spacetime [3, 4, 5].

The second feature (Euclidean spacetime) is undesirable since the goal of numerical relativity is to model physically realistic processes. The fact is that once one has a Euclidean implementation of numerical Regge calculus, it is not an entirely trivial task to go over to Minkowskian signature.[1] When taking derivatives of square roots of negative functions (as one does in Regge calculus with a $-+++$ signature), one gets sign flips due to branch cuts in the square root function. One has to be very careful to avoid errors that are typified by the following "equation":

$$i = \sqrt{-1} = \sqrt{\frac{1}{-1}} = \frac{1}{\sqrt{-1}} = \frac{1}{i} = -i$$

With a goal of using Regge calculus in numerical relativity, I have developed a code that calculates the Regge equations (and their derivatives with respect to the squared length of any link) on a simplicial manifold of arbitrary topology, with each simplex having a flat, Minkowskian metric. As a test for the code, I pose the following question: What is the order of the Regge calculus numerical method? That is, how fast do the Regge equations, when "evaluated" on a generic solution to the vacuum Einstein equations, approach zero as the lattice spacing approaches zero? (How one goes about "evaluating" the discrete Regge equations on a continuous solution of the Einstein equations is described in the next section.)

In contrast to Regge calculus, the order of a given finite differencing scheme is clear from the beginning: it is the order of the error introduced by truncating the Taylor expansion when approximating functions and their derivatives on a discrete lattice. For example, the derivative of a function $f(x)$ defined on grid points x_i (evenly spaced; $x_{i+1} - x_i = h$) can be approximated at x_0 with a Taylor expansion about $x = x_0$:

$$f(x_1) = f(x_0 + h) = f(x_0) + f'(x_0)h + \mathcal{O}(h^2)$$

[1] This is a technical point. Minkowskian signature introduces a bit of added complexity that must be dealt with, and it may not be as simple as adding a C++ tag on timelike legs to an existing Euclidean code, as suggested in [6].

namely,

$$f'(x_0) = \frac{f(x_1) - f(x_0)}{h} + \mathcal{O}(h)$$

This is a *first order* approximation to the derivative of $f(x)$ at x_0. In a similar manner, any differential operator, $\mathcal{L}$, acting on a set of functions, $\vec{f}(\vec{x})$, can be approximated by an n^{th} order finite difference operator, $\hat{\mathcal{L}}_n$, acting on functions $\vec{f}$ defined on a lattice $\vec{x}_i$ characterized by uniform lattice spacing h. If $\vec{f}_0(\vec{x})$ is a solution to the differential equation

$$\mathcal{L}\vec{f} = 0$$

then, by construction, the finite difference operator $\hat{\mathcal{L}}_n$ acting on $\vec{f}_0(\vec{x}_i)$ (the continuum solution $\vec{f}_0(\vec{x})$ evaluated at lattice sites $\vec{x}_i$) will converge to zero as the n^{th} power of h:

$$\hat{\mathcal{L}}_n \vec{f}_0(\vec{x}_i) = 0 + p_0 h^n + \mathcal{O}(h^{n+1})$$

where p_0 is some constant.

In the case of Regge calculus, it is not known, a priori, what order (in lattice spacing h) of error is introduced by discretizing the Einstein equations. This question is the subject of this paper. I will show analytically that, individually, the Regge equations, when evaluated on a continuum spacetime, converge to zero as the second power of the lattice spacing (there is a non-zero second order term).[2] This result will be seen to be independent of whether or not the continuum metric satisfies the Einstein equations. However, it will be shown analytically that averages (with weightings given by variations of each component of the continuum metric) over local Regge equations have a vanishing second order term only when the continuum metric satisfies the Einstein equations. Studying the behavior of the third order term must be done numerically, using analytic solutions to the Einstein equations. These results show that, for the specific solutions studied, the third order term of the average of the Regge equations also vanishes.

2 Regge Calculus

Regge calculus is based on the idea of approximating a curved continuum spacetime with a collection of flat simplices. For instance, a curved 2-dimensional surface can be approximated by a collection of 2-simplices (triangles). A general property of simplicial manifolds is that the curvature always resides on subspaces of codimension two. In the 2-dimensional case, this means that the curvature is concentrated on the vertices. In fact, the integral of the scalar curvature over a region of the simplicial manifold is simply a sum of the deficit angles associated with each vertex.

[2]This fact has been independently observed by Leo Brewin[7]

In four dimensions, the curvature is concentrated on 2-dimensional simplices (triangles). The Regge action is simply the Hilbert action evaluated on the simplicial manifold:

$$S(l^2) = \int \sqrt{-g}R = \sum_i A_i \epsilon_i \tag{1}$$

where the sum on the right hand side is over all triangles (labeled by i), A_i is the area of triangle i, and ϵ_i is the deficit angle associated with the i^{th} triangle (boundary terms in the action are treated in [8]). Here, we have set $16\pi G = 1$. Demanding that the action be stationary with respect to small variations in the squared edge lengths yields the Regge equations. Each dynamical variable (squared edge length) has an associated Regge equation. The Regge equation associated with leg (i, j) is

$$\mathcal{R}_{ij}(l^2) = \sum_k \epsilon_{ijk} \cot\theta_{ijk} = 0 \tag{2}$$

where the sum is over all triangles (i, j, k) that have (i, j) as one edge, θ_{ijk} is the angle $\angle ikj$ of triangle (i, j, k), and ϵ_{ijk} is the deficit angle associated with triangle (i, j, k).

One property of the Regge equations that will be important in the following sections is the fact that the equations are dimensionless. Only angles appear in the Regge equations. Specifically, look at the expression for $\cot\theta_{ijk}$ in terms of the squared edge lengths l_{ij}^2, l_{ik}^2, and l_{jk}^2:

$$\cot\theta_{ijk} = \frac{l_{ik}^2 + l_{jk}^2 - l_{ij}^2}{\sqrt{2l_{ij}^2 l_{ik}^2 + 2l_{ij}^2 l_{jk}^2 + 2l_{ik}^2 l_{jk}^2 - l_{ij}^4 - l_{ik}^4 - l_{jk}^4}} \tag{3}$$

We see explicitly that both the numerator and denominator have units of L^2. In section 4, I will want to substitute a power series (in smallness parameter δ) for each l^2 which has the form

$$l_{ij}^2 = (l_2^2)_{ij}\delta^2 + (l_3^2)_{ij}\delta^3 + (l_4^2)_{ij}\delta^4 + \cdots \tag{4}$$

Observe that $\cot\theta_{ijk}$ (and, therefore, the Regge equation) would be indeterminate (0/0) when $\delta = 0$. In doing the actual calculations, equation (4) is scaled by a factor of $1/\delta^2$. The first non-zero term in a typical series expansion for l^2 will now be of order unity, and the Regge equations will be perfectly well defined at $\delta = 0$. Due to the complexity of the Regge equations, MATHEMATICA was used to perform the analytic calculations in section (4).

3 A Formalism for Studying the Convergence Properties of the Regge Equations

In the past, various methods have been used to study the relationship between Einstein's General Relativity in the continuum and the discrete Regge calculus. Both Sorkin [3] and

Friedberg and Lee [9] derive the Regge action from the Hilbert action by considering the limit of a sequence of continuum surfaces that approximate a piecewise flat surface. Barrett [10] showed the equivalence between the Regge equations and the vanishing of the flow of energy-momentum across a certain hypersurface. Brewin [11] has shown the equivalence of the Regge action and the Hilbert action on almost flat simplicial spacetimes by treating the Riemann tensor as a distribution.

The basic idea I use for comparing the continuum Einstein equations to the Regge equations is the same as used in a paper by Cheeger *et al* [12] (this idea can also be found in [3]): assign the (signed) square of the geodesic length between two points in the continuum manifold whose metric satisfies the vacuum Einstein equations to the squared edge length of the corresponding link in the simplicial manifold. In [12], it was shown (for Euclidean metrics) that the Regge action converges to the Hilbert action, in the sense of measures, as long as certain fatness conditions are satisfied by the simplices. Here, I ask a different question: how *fast* do the Regge equations converge to zero as the lattice spacing goes to zero?

3.1 Evaluating the Regge Equations on a Continuum Spacetime

For simplicity, this paper will use a hypercubic simplicial topology (hypercubes split into simplices). Consider a 4-manifold, $\mathcal{M}$, equipped with a metric ($g_{\mu\nu}$) whose signature is everywhere $(-+++)$. Furthermore, let $g_{\mu\nu}$ be a solution to the vacuum Einstein equations $R_{\mu\nu} = 0$. Let $\{x^\mu\}$ be coordinates on a region of $\mathcal{M}$ containing some point $\mathcal{O}$ with coordinates $x^\mu_{\mathcal{O}} = (0,0,0,0)$. Now, introduce the hypercubic simplicial triangulation in the following way. First, construct a rectangular lattice in the coordinates $\{x^\mu\}$. The lattice site $x_{\vec{n}}$ (where $\vec{n} = (n^0, n^1, n^2, n^3)$) is located at coordinates $x^\mu_{\vec{n}} = (n^0\ \Delta x^0, n^1\ \Delta x^1, n^2\ \Delta x^2, n^3\ \Delta x^3)$, where Δx^μ are 4 arbitrary lattice spacing constants of order unity. To construct a triangulation from this rectangular lattice, we construct 15 links at each lattice site $x_{\vec{n}}$:

`diagonal links`	`rectangular links`
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_0+\hat{n}_1})$	
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_0+\hat{n}_2})$	
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_0+\hat{n}_3})$	
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_1+\hat{n}_2})$	$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_0})$
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_1+\hat{n}_3})$	$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_1})$
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_2+\hat{n}_3})$	$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_2})$
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_0+\hat{n}_1+\hat{n}_2})$	$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_3})$
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_0+\hat{n}_1+\hat{n}_3})$	
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_0+\hat{n}_2+\hat{n}_3})$	
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_1+\hat{n}_2+\hat{n}_3})$	
$(x_{\vec{n}}, x_{\vec{n}+\hat{n}_0+\hat{n}_1+\hat{n}_2+\hat{n}_3})$	

In this way, each hypercube is split into 24 4-simplices. We now have a simplicial manifold with each vertex $x_{\vec{n}}$ associated with a point on $\mathcal{M}$ (the point with coordinates $(n^0\ \ \Delta x^0, n^1\ \ \Delta x^1, n^2\ \ \Delta x^2, n^3\ \ \Delta x^3)$). In order to study the limit of small lattice spacings, we multiply the coordinates corresponding to each lattice site with a "smallness parameter", δ. The coordinates associated with lattice site $x_{\vec{n}}$ are now $(\delta\ n^0\ \Delta x^0, \delta\ n^1\ \Delta x^1, \delta\ n^2\ \Delta x^2, \delta\ n^3\ \Delta x^3)$.

We now assign the (signed) square of the geodesic length of separation between points on the continuum manifold to the squared lengths of the links in the simplicial manifold (see Figure 1). For instance, to find the squared length of the link $(x_{(0,0,0,0)}, x_{(1,0,0,0)})$ in the simplicial manifold,

1. find the solution to the geodesic equation,

$$\frac{d^2x^\mu(\lambda)}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}(x(\lambda))\frac{dx^\alpha(\lambda)}{d\lambda}\frac{dx^\beta(\lambda)}{d\lambda} = 0$$

   with boundary conditions $x^\mu(\lambda = 0) = (0, 0, 0, 0)$ and $x^\mu(\lambda = 1) = (\delta\ \Delta x^0, 0, 0, 0)$, where λ is an affine parameter ranging for 0 to 1.

2. integrate the length of the geodesic :

$$l = \int_0^1 d\lambda\ \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

3. set the squared length of the link equal to l^2 (this quantity will be positive for spacelike geodesics, and negative for timelike geodesics).

4 Analytic Results

The geodesic equation cannot be solved analytically for an arbitrary metric. However, since all we want to do is find the first non-vanishing order of the Regge equations, we will be content with a power expansion (in δ) for the squared geodesic distance of curves in the continuum manifold.

4.1 Series Expansion for the Squared Geodesic Distances

In general, we will need to find the power series expansion in δ for the squared geodesic distances between two points $\mathcal{P}_i$ and $\mathcal{P}_f$ with respective coordinates $\{\delta\ x_i^\mu\}$ and $\{\delta\ x_f^\mu\}$ in terms of the metric at $\mathcal{O}$, the Christoffel symbols at $\mathcal{O}$, the Riemann tensor at $\mathcal{O}$, the first derivatives of the Riemann tensor at $\mathcal{O}$, etc. Let $x^\mu(\lambda)$ be a parametrization of the geodesic with affine parameter $\lambda \in [0, \delta]$ (so that the affine parameter is of the same order as the length of the geodesic) that satisfies the boundary conditions

$$\begin{aligned} x^{\mu}(\lambda = 0) &= \delta \; x_i^{\mu} \\ x^{\mu}(\lambda = \delta) &= \delta \; x_f^{\mu} \end{aligned} \tag{5}$$

Now, expand $x^{\mu}(\lambda)$ about the point $\lambda = 0$:

$$x^{\mu}(\lambda) = x^{\mu}(0) + \frac{dx^{\mu}}{d\lambda}(0)\,\lambda + \frac{1}{2!}\frac{d^2x^{\mu}}{d\lambda^2}(0)\,\lambda^2 + \frac{1}{3!}\frac{d^3x^{\mu}}{d\lambda^3}(0)\,\lambda^3 + \cdots \tag{6}$$

Taking this expansion, along with an expansion of the Christoffel symbols about $\mathcal{O}$ and substituting into the geodesic equation allows one to solve for the coefficients of eq. (6) in terms of the boundary values (the coordinates of the endpoints of the geodesic) and the Christoffel symbols and their derivatives at point $\mathcal{O}$. One then expands the metric as a power series about $\mathcal{O}$, and substitutes it into the final expression,

$$l_{i,f}^2 = \left(\int_0^{\delta} d\lambda \sqrt{g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}} \right)^2 = \left(g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} \right)_{\lambda=0} \delta^2 \tag{7}$$

along with the power series of $\frac{dx^{\mu}}{d\lambda}$ (obtained by rearranging eq. (6)) to get the final power series expansion for the square of the geodesic distance between points $\mathcal{P}_i$ and $\mathcal{P}_f$.

4.2 Expanding the Regge Equations as a Power Series

With a power series expansion for the squared geodesic distances in hand, we now evaluate the Regge equations on the simplicial manifold, where each squared length in the simplicial manifold is assigned the squared geodesic distance between the corresponding two points in the continuum manifold. A typical Regge equation can then be expressed as a power series in δ,

$$\mathcal{R}_{ij}(\delta) = \mathcal{R}_{ij}(0) + \left(\frac{d\mathcal{R}_{ij}}{d\delta}(0) \right) \delta + \frac{1}{2!} \left(\frac{d^2\mathcal{R}_{ij}}{d\delta^2}(0) \right) \delta^2 + \cdots \tag{8}$$

where the coefficients of the series are functions of the metric at point $\mathcal{O}$, the Christoffel symbols at point $\mathcal{O}$, the first derivatives of the Christoffel symbols at point $\mathcal{O}$, etc. Recall that, due to the dimensionlessness of the Regge equations, in order to calculate the n^{th} order coefficient in the expansion of a particular Regge equation, all of the expressions for the squared geodesic distances must be accurate up to and including the $(n+2)^{th}$ order in δ.

4.2.1 Zeroth Order Results

To calculate the zeroth order term in the expansion of a typical Regge equation, eq. (8), we require expressions for squared geodesic distances that are accurate up to and including the second order in δ. The analysis in section 4.1 gives us the answer:

$$l_{if}^2 = (g_{\mu\nu})_{\mathcal{O}}(x_f^\mu - x_i^\mu)(x_f^\nu - x_i^\nu)\delta^2 + \mathcal{O}(\delta^3) \quad (9)$$

Substituting these expressions into a typical Regge equation, and setting $\delta = 0$ is equivalent to evaluating the Regge equations on a continuum manifold with constant metric $(g_{\mu\nu})_{\mathcal{O}}$. Therefore, all of the deficit angles vanish, and the zeroth order term in the expansion of a typical Regge equation, eq. (8), is zero.

4.2.2 First Order Results

To calculate the first order term in the expansion of a typical Regge equation, eq. (8), we require expressions for squared geodesic distances that are accurate up to and including the third order in δ. Again, the analysis in section 4.1 gives us the answer:

$$l_{if}^2 = (g_{\mu\nu})_{\mathcal{O}}(x_f^\mu - x_i^\mu)(x_f^\nu - x_i^\nu)\delta^2 + (\Gamma_{\alpha\mu\nu})_{\mathcal{O}}(x_i^\mu + x_f^\mu)(x_f^\nu - x_i^\nu)(x_f^\alpha - x_i^\alpha)\delta^3 + \mathcal{O}(\delta^4) \quad (10)$$

The first order term in the expansion of a typical Regge equation, $(\frac{d\mathcal{R}_{ij}}{d\delta})_{\delta=0}$, has been computed on a number of different lattices (hypercubic lattice, randomly generated lattice, quantity production lattice [13]), all with the same result:

$$(\frac{d\mathcal{R}_{ij}}{d\delta})_{\delta=0} = 0 \quad (11)$$

This result can be interpreted geometrically as a statement of the fact that there is no first order change in a vector that is parallel transported around a small closed loop. From the standpoint of numerical relativity, this result means that, at the very least, Regge calculus is a second order numerical method (Of course, the result is independent of whether or not the metric satisfies the Einstein equations, since eq. (10) depends only on the metric and its first derivatives (Christoffels) at point $\mathcal{O}$).

4.2.3 Second Order Results

Unfortunately, the problem of expressing the squared geodesic distances (in an arbitrary coordinate system), accurate up to and including the fourth order in δ (needed to study the second order behavior of the Regge equations), in terms of the Riemann tensor (as well as the metric and Christoffel symbols) at point $\mathcal{O}$ seems to be intractable. To remedy this, I will use Riemann normal coordinates [14] about point $\mathcal{O}$. One property of Riemann normal coordinates is that the Christoffel symbols vanish at point $\mathcal{O}$. From eq. (10), it can be seen that this causes the third order term in the expression for the squared geodesic distance to vanish. Using other properties of Riemann normal coordinates, namely,

$$(g_{\mu\nu})_{\mathcal{O}} = \eta_{\mu\nu} \tag{12}$$

$$(\partial_\alpha \Gamma^\nu{}_{\beta\mu})_{\mathcal{O}} = -\frac{1}{3}(R_{\alpha\mu\beta}{}^\nu + R_{\alpha\beta\mu}{}^\nu)_{\mathcal{O}} \tag{13}$$

$$(\partial_\alpha \partial_\beta g_{\mu\nu})_{\mathcal{O}} = -\frac{1}{3}(R_{\alpha\mu\beta\nu} + R_{\beta\mu\alpha\nu})_{\mathcal{O}} \tag{14}$$

one can obtain the expression for the squared geodesic distance that is accurate up to and including the fourth order in δ:

$$l_{if}^2 = (\eta_{\mu\nu})(x_f^\mu - x_i^\mu)(x_f^\nu - x_i^\nu)\delta^2 - \frac{1}{3}(R_{\alpha\mu\beta\nu})_{\mathcal{O}} x_i^\alpha x_f^\beta (x_f^\mu - x_i^\mu)(x_f^\nu - x_i^\nu)\delta^4 + \mathcal{O}(\delta^5) \tag{15}$$

One would hope that when substituting the above expression for squared geodesic distances into the Regge equation expansion (8), one would observe the vanishing of the second order term when the Einstein equations are imposed ($(R_{\mu\nu})_{\mathcal{O}} = 0$). Although there are cases (particularly when there is high lattice symmetry) where the second order term does vanish when the Einstein equations are imposed, it does not vanish in general.

4.3 Averaging Over a Region of Spacetime

From the above analysis, we see that only the zeroth and first order terms in the expansion of a typical Regge equation vanish. However, this result holds even if the continuum metric does *not* satisfy the Einstein equations!! Therefore, how confident can one be that Regge calculus is a good approximation to General Relativity? To motivate an answer to this, consider the analogous problem of approximating a continuous curve with a series of straight line segments. If we look closely at any one particular line segment, it does not look as if we are approximating the continuum curve very well. However, if we "step back" and look at many line segments, the approximation is better (see Figure 2).

Therefore, we might expect a suitable average of the Regge equations contained in some region of spacetime to have a vanishing second order term when the Einstein equations are imposed on the continuum metric [15]. How should one go about constructing this average? What should one use as a reasonable weighting? The continuum action principle suggests a partial answer[15]. If we view the squared leg lengths in the Regge action as functions of the continuum metric,

$$S = S(l_{ij}^2(g_{\mu\nu})) \tag{16}$$

then the variation of the action with respect to variations in the metric ($\delta g_{\mu\nu}$) is simply

$$\delta S = \sum_{ij} \left(\frac{\partial S}{\partial l_{ij}^2} \right) \left(\frac{\partial l_{ij}^2}{\partial g_{\mu\nu}} \right) \delta g_{\mu\nu} \tag{17}$$

The first factor in the above sum, $\partial S / \partial l_{ij}^2$, is the Regge equation that corresponds to leg (i, j). The second factor can be used as the weighting of that particular Regge equation. There will be 10 independent weightings, corresponding to the 10 independent components of the metric.

With the lattice constructed in section 3, one could average over all Regge equations contained in a unit hypercube (it turns out that there are 65 of them in this particular case). I define $\mathcal{R}_{avg}$ as

$$\mathcal{R}_{avg}(\delta) = \sum_{ij} \left(\frac{m_{ij}}{\delta^2} \frac{\partial l_{ij}^2}{\partial g_{\mu\nu}} \right) \mathcal{R}_{ij}(\delta) \tag{18}$$

where the sum is over all legs (ij) contained in a unit hypercube, and m_{ij} is the combinatorial factor

$$m_{ij} = \frac{\text{number of 4-simplices inside unit hypercube that have (ij) as a leg}}{\text{total number of 4-simplices in the simplicial manifold that have (ij) as a leg}} \tag{19}$$

The $1/\delta^2$ in the weighting is to keep the weighting factor of order unity. There are ten independent weightings, each one corresponding to an independent component of the metric. Plugging in the fifth order accurate expressions for squared geodesic distances (15) into the Regge equations and expanding $\mathcal{R}_{avg}(\delta)$ as a power series about $\delta = 0$ reveals that the second order term of $\mathcal{R}_{avg}(\delta)$ vanishes when the Einstein equations are imposed at $\mathcal{O}$.

$$\left(\frac{d^2 \mathcal{R}_{avg}}{d\delta^2} \right)_{\delta=0} = 0 \tag{20}$$

This result holds for each of the ten independent weightings that correspond to variations of the ten independent metric components.

One might wonder whether or not this result is an artifact of the usage of Riemann normal coordinates. To check, one can make an arbitrary coordinate transformation on equation (15) before substituting the l_{ij}^2's into the Regge equations. It turns out that the result, eq. (20), is independent of the coordinate transformation, thus, the result does not depend on the usage of Riemann normal coordinates.

5 Numerical Results

In this section, I study the convergence properties of the Regge equations numerically. Instead of finding a power expansion for squared geodesic distances (as in the previous section), I numerically solve the geodesic equation for different analytic solutions of the vacuum Einstein equations (using the relaxation methods found in [16]). I then evaluate the Regge equations on simplicial manifolds using these values. The first non-zero order term in a power expansion of $\mathcal{R}_{ij}(\delta)$ or $\mathcal{R}_{avg}(\delta)$ can be found as the slope of the line of a $ln(\mathcal{R}(\delta))$ vs. $ln(1/\delta)$ plot. In all of the plots, the error bars are actually smaller than the data points appearing on the graph. This fact is a result of the accuracy of the relaxation methods used to solve the geodesic equations. I have verified this by increasing the resolution of the grid points by more than one order of magnitude, and observing no change (to one part in 10^9) in the values for the computed geodesic distances.

Instead of picking on one Regge equation to evaluate and plot, I actually plot the sum of the absolute values of the Regge equations associated with the links in a unit hypercube (summing over the same links as in equation (18)). I define $\mathcal{R}_{abs}$ to be

$$\mathcal{R}_{abs} = \sum_{ij} |\mathcal{R}_{ij}| \tag{21}$$

I also plot the weighted average of the Regge equations $\mathcal{R}_{avg}$ defined by equation (18).

The results from section 4 predict that $\mathcal{R}_{abs}$ should converge like δ^2. Also, in section 4 it was shown that the zeroth, first, and second order terms of $\mathcal{R}_{avg}$ vanish. This leaves on open question: What is the first non-vanishing term of $\mathcal{R}_{avg}(\delta)$?

5.1 Schwarzschild Metric

The metric of the Schwarzschild solution has the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \tag{22}$$

I set $M = 1$, and put the origin of the lattice just outside the event horizon: $x^\mu_{\mathcal{O}} = (t_{\mathcal{O}}, r_{\mathcal{O}}, \theta_{\mathcal{O}}, \phi_{\mathcal{O}}) = (0, 3, \pi/2, \pi)$. The coordinate lattice spacing is $\Delta x^\mu = (\Delta t, \Delta r, \Delta\theta, \Delta\phi) = (\delta, \delta/2, \delta/4, \delta/4)$. The plot of $\mathcal{R}_{abs}$ is shown in Figure 3. The plot of $\mathcal{R}_{avg}$, with a weighting induced by the variation of the time-time component of the continuum metric (the other weightings give the same results), is shown in Figure 4.

The figures show us a surprise. First, we would expect $\mathcal{R}_{abs}$ to converge as the second power in δ, but Figure 3 shows a fourth order convergence. Also, note the fourth order convergence of $\mathcal{R}_{avg}$. To check these results, I did the same calculation, using Kruskal coordinates, and moving $\mathcal{O}$ inside the event horizon. In this case, the convergence properties are shown to be exactly the same as in Figures 3 and 4.

Something to notice about both figures is the fact that, even though the lengths of the geodesics are of the order of the length scale defined by the curvature of the problem, the points corresponding to $\delta = 1$ are already on the converging line. This suggests that

Regge calculus may be superior to finite differencing in tracking the geometry of black hole spacetimes.

5.2 Kerr Metric

The Kerr metric, which is the vacuum solution corresponding to a rotating black hole, has the form

$$\begin{aligned} ds^2 &= -\left(\frac{r^2 + A^2\cos^2\theta - 2Mr}{r^2 + A^2\cos^2\theta}\right) dt^2 - \left(\frac{4MAr\sin^2\theta}{r^2 + A^2\cos^2\theta}\right) dt\, d\phi + \left(\frac{r^2 + A^2\cos^2\theta}{r^2 + A^2 - 2Mr}\right) dr^2 \\ &\quad + (r^2 + A^2\cos^2\theta) d\theta^2 + \left(\frac{(r^2 + A^2)^2 - (r^2 + A^2 - 2Mr)A^2\sin^2\theta}{r^2 + A^2\cos^2\theta}\right) \sin^2\theta\, d\phi^2 \end{aligned} \quad (23)$$

where M is the mass of the black hole and A is the angular momentum of the black hole divided by the mass: $A = J/M$. I set $M = 1$, $A = 1/2$ (so that the spacetime is asymptotically predictable), and put the origin of the lattice at $x^\mu_{\mathcal{O}} = (t_{\mathcal{O}}, r_{\mathcal{O}}, \theta_{\mathcal{O}}, \phi_{\mathcal{O}}) = (0, 5, \pi/2, \pi)$. The coordinate lattice spacing is $\Delta x^\mu = (\Delta t, \Delta r, \Delta\theta, \Delta\phi) = (1.5\delta, \delta, 0.2\delta, 0.3\delta)$. The plot of $\mathcal{R}_{abs}$ is shown in Figure 5, while the plot of $\mathcal{R}_{avg}$, with a weighting induced by the variation of the time-time component of the continuum metric (the other weightings give the same results), is shown in Figure 6.

As can be seen, this example shows $\mathcal{R}_{abs}$ converging as the second power of δ, while $\mathcal{R}_{avg}$ still converges as the fourth power of δ.

5.3 Kasner Metric

The Kasner metric, which corresponds to a spatially homogeneous anisotropic universe, has the form

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 \quad (24)$$

where the constants p_i satisfy $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$. I set $p_1 = p_2 = 2/3$, and $p_3 = -1/3$, and put the origin of the lattice at $x^\mu_{\mathcal{O}} = (t_{\mathcal{O}}, x_{\mathcal{O}}, y_{\mathcal{O}}, z_{\mathcal{O}}) = (2, 0, 0, 0)$. The coordinate lattice spacing is $\Delta x^\mu = (\Delta t, \Delta x, \Delta y, \Delta z) = (\delta, \delta, \delta, \delta)$. The plot of $\mathcal{R}_{abs}$ is shown in Figure 7, while the plot of $\mathcal{R}_{avg}$, with a weighting induced by the variation of the time-time component of the continuum metric (the other weightings give similar results), is shown in Figure 8.

We see that $\mathcal{R}_{abs}$ converges like the second power of δ and $\mathcal{R}_{avg}$ converges like the fourth power of δ. These are the same convergence properties found with the Kerr metric.

6 Conclusions

Both analytically and numerically, we see that the Regge equations individually converge as the second power of the lattice spacing, although this result is independent of whether or not the continuum metric satisfies the Einstein equations. We see that the second order term of an average over local Regge equations vanishes only when the continuum metric satisfies the Einstein equations. Numerical studies (Figures 4, 6, and 8) show that, for certain analytic solutions of the Einstein equations, the third order term of $\mathcal{R}_{avg}$ also vanishes. Thus, Regge calculus can be used as a fourth order method in numerical relativity.

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I thank the members of the Syracuse Relativity Group for useful discussions and suggestions. I particularly want to thank Rafael Sorkin, without whose suggestions and ideas, the results of this paper would not be nearly as strong. I also want to thank the Northeast Parallel Architecture Center (NPAC) for the use of their computer facilities. This work was partially supported by a Fellowship from Syracuse University and NSF ASC 93 18152 / PHY 93 18152 (ARPA supplemented).

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Figure Captions

1. Method of "evaluating" a simplicial spacetime on a continuum spacetime.
2. Approximating a continuous curve with line segments. The approximation is better if one looks at many line segments, rather than focusing in on any one particular segment.
3. A plot of $\ln(\mathcal{R}_{abs})$ vs. $\ln(1/\delta)$ for the Schwarzschild metric.
4. A plot of $\ln(\mathcal{R}_{avg})$ vs. $\ln(1/\delta)$ for the Schwarzschild metric. The particular weighting in $\mathcal{R}_{avg}$ (eq. 18) is the time-time variation of the metric. The graphs for the other 9 independent weightings are identical.
5. A plot of $\ln(\mathcal{R}_{abs})$ vs. $\ln(1/\delta)$ for the Kerr metric.
6. A plot of $\ln(\mathcal{R}_{avg})$ vs. $\ln(1/\delta)$ for the Kerr metric. The particular weighting in $\mathcal{R}_{avg}$ (eq. 18) is the time-time variation of the metric. The graphs for the other 9 independent weightings are identical.
7. A plot of $\ln(\mathcal{R}_{abs})$ vs. $\ln(1/\delta)$ for the Kasner metric.
8. A plot of $\ln(\mathcal{R}_{avg})$ vs. $\ln(1/\delta)$ for the Kasner metric. The particular weighting in $\mathcal{R}_{avg}$ (eq. 18) is the time-time variation of the metric. The graphs for the other 9 independent weightings have the same convergence rates.

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Figure 1

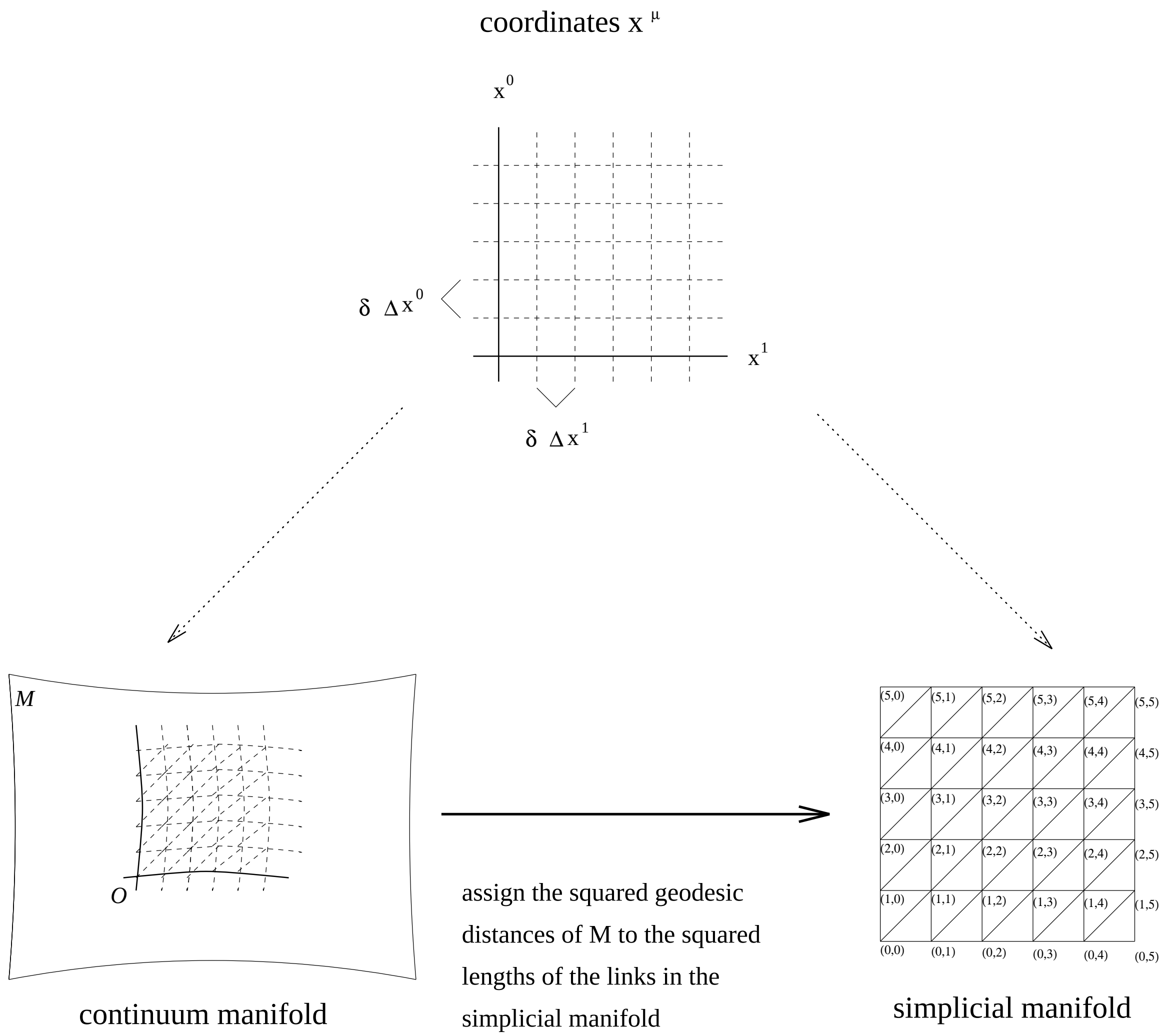

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Figure 2

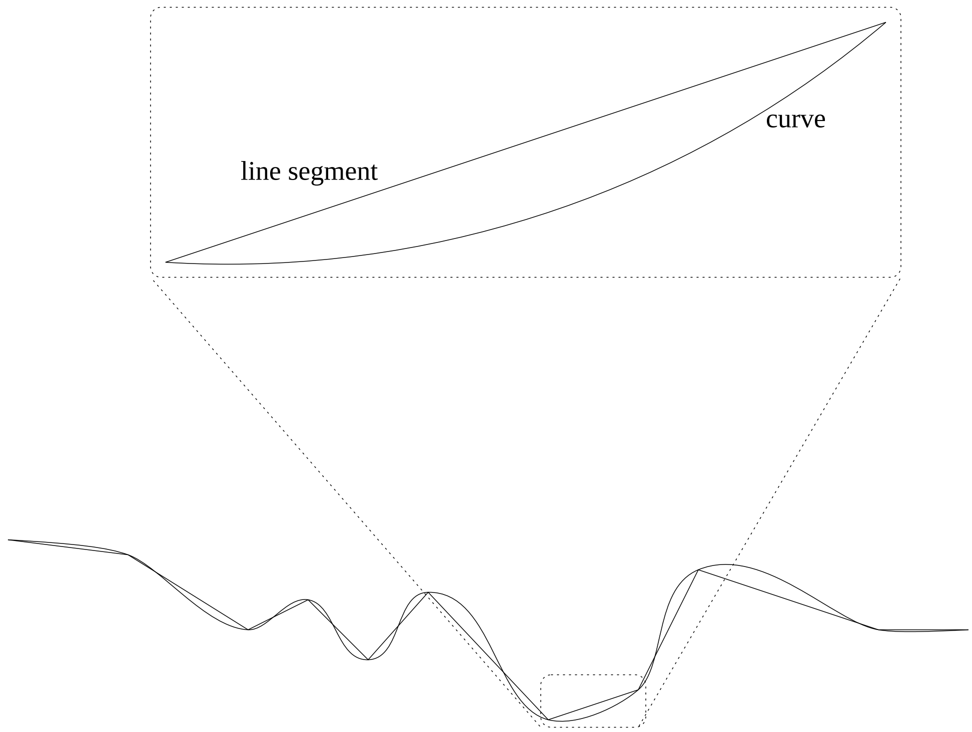

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Figure 3

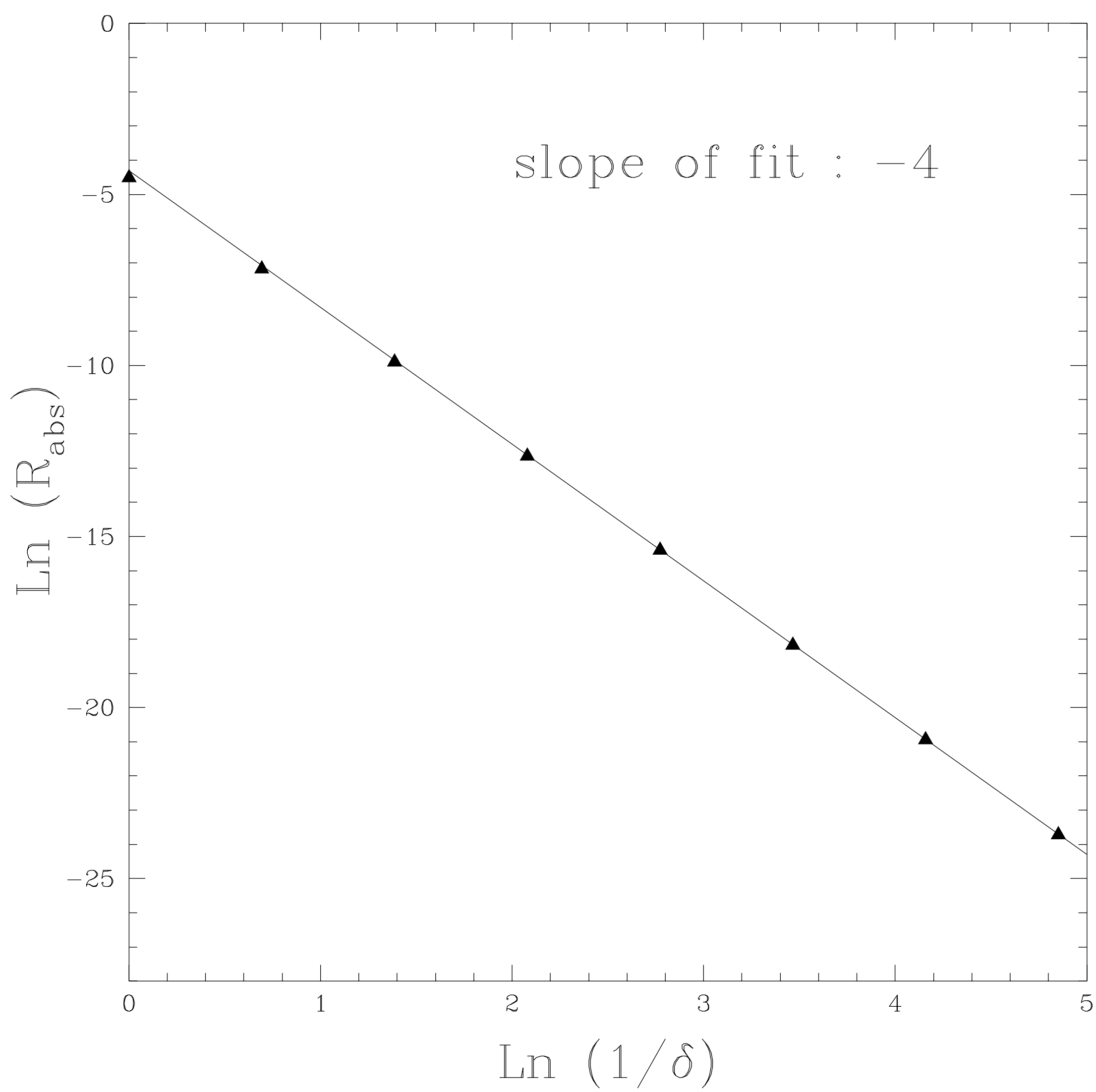

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Figure 4

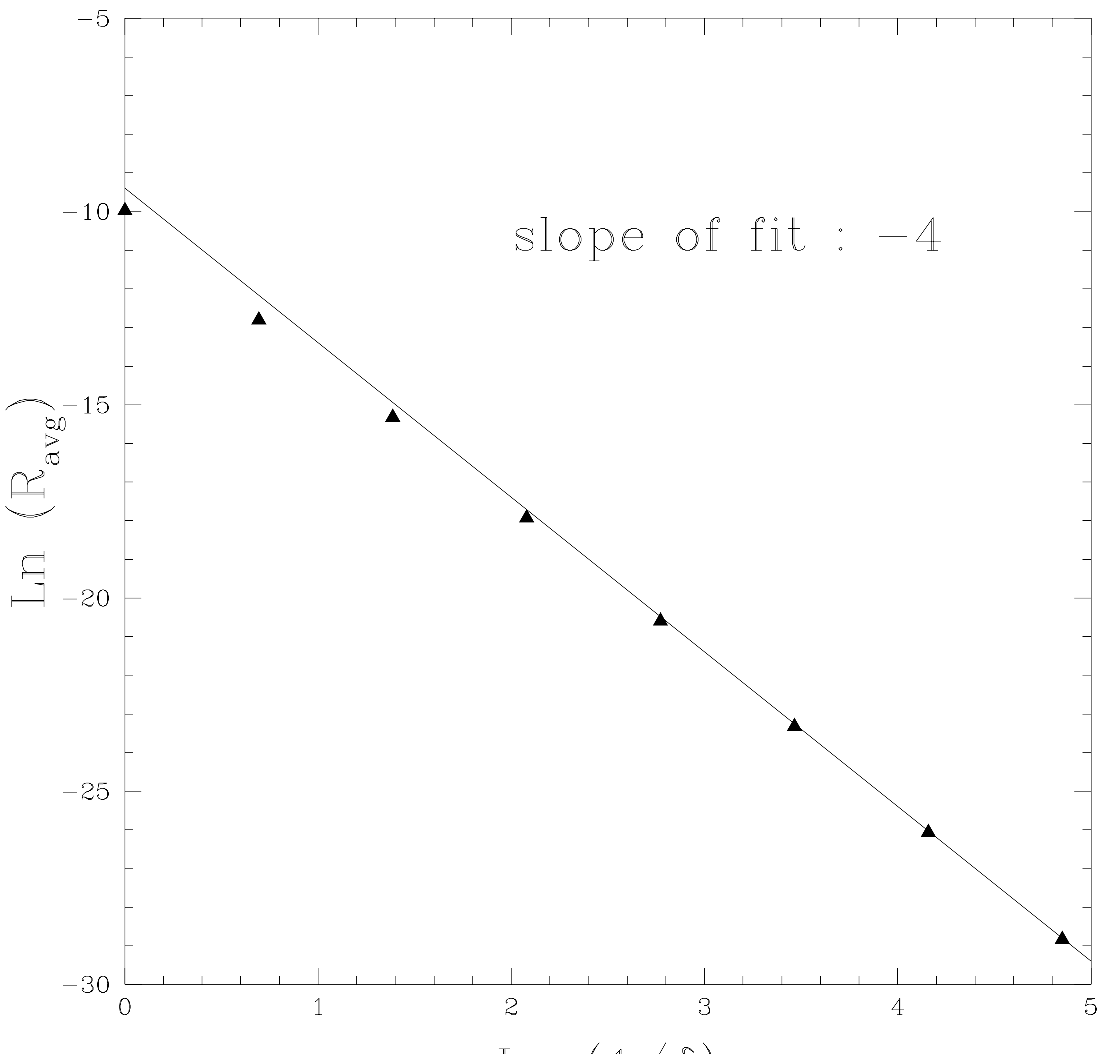

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Figure 5

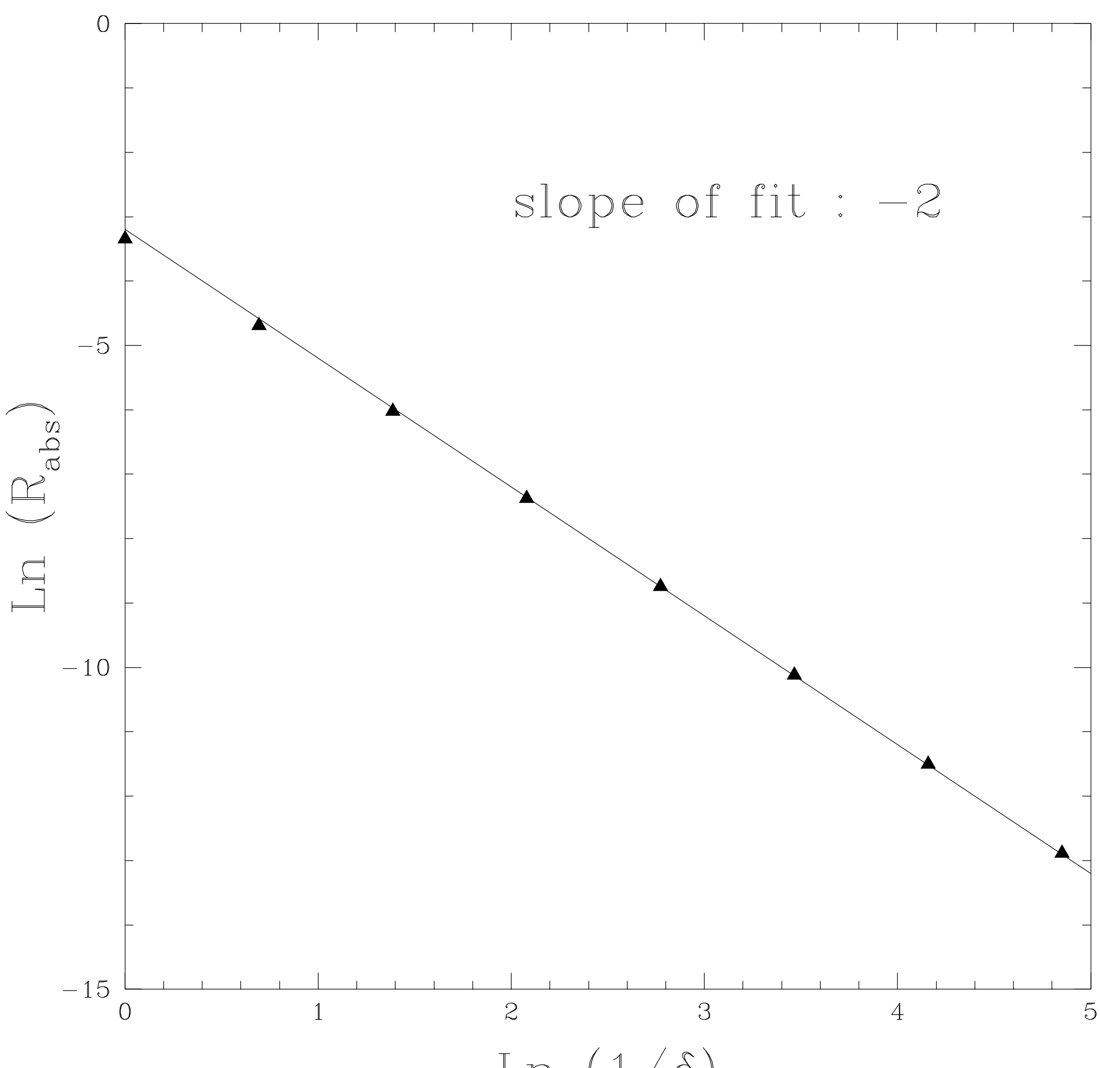

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Figure 6

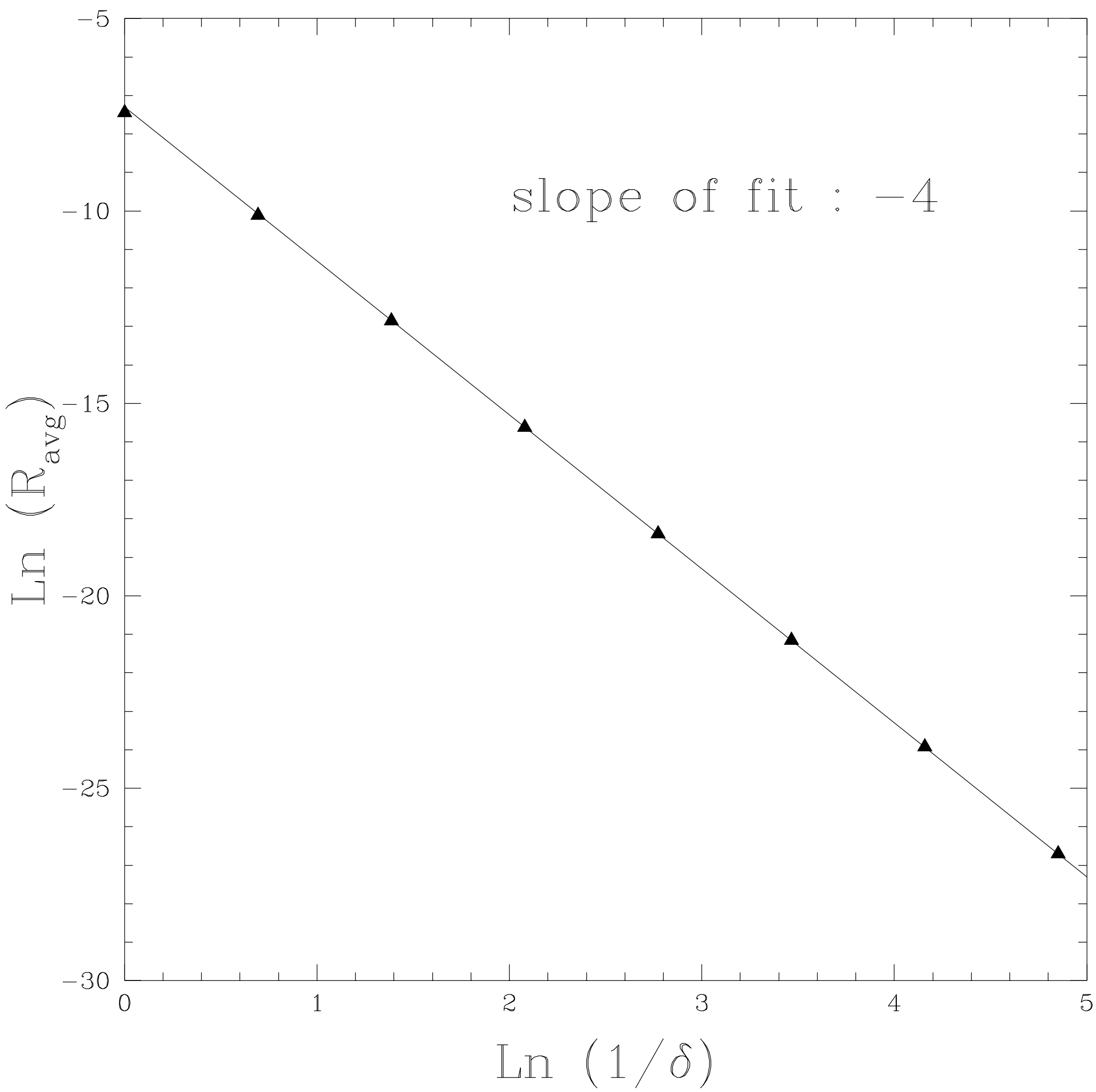

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Figure 7

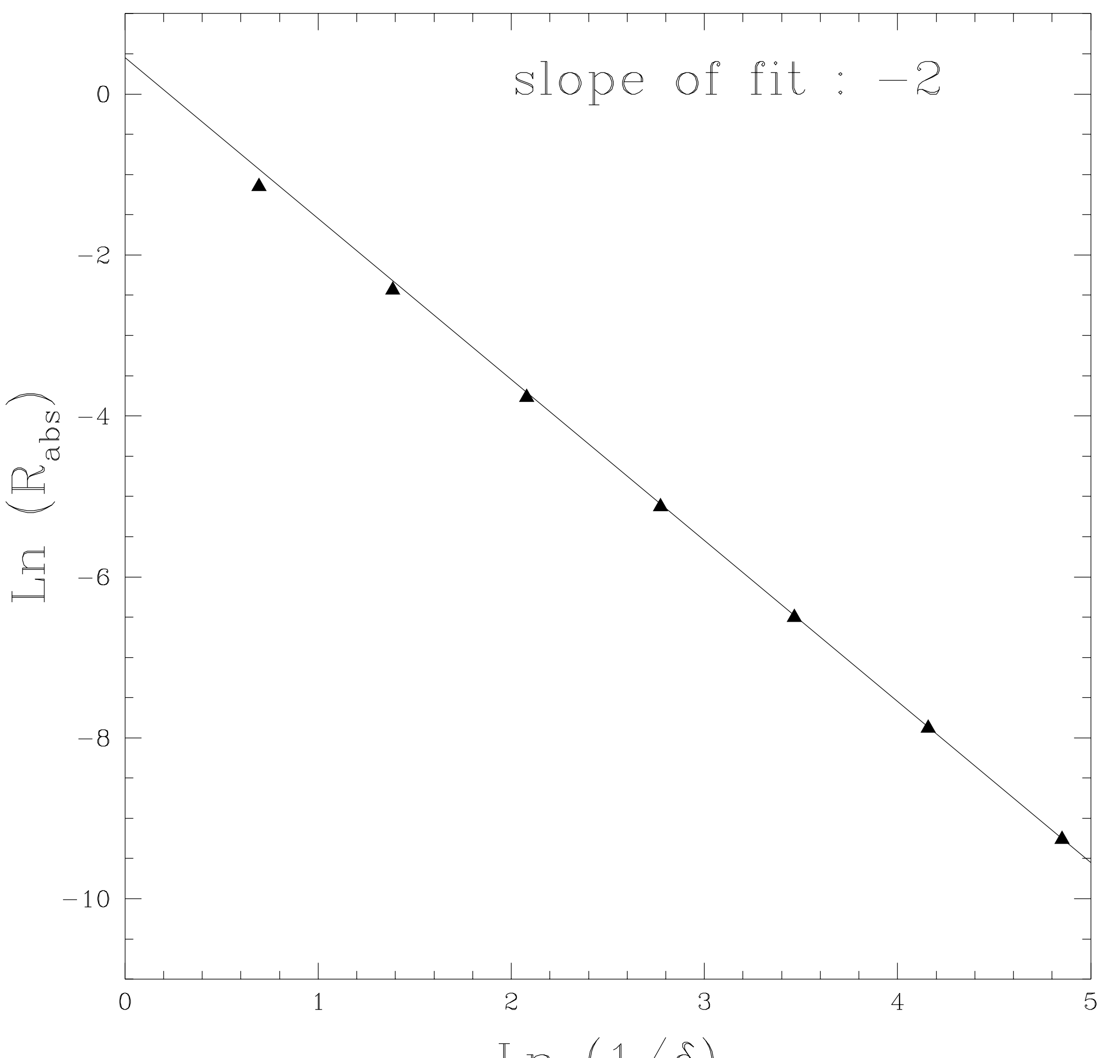

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Figure 8