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Dan Coman  
*Syracuse University*

Vincent Guedj
*Université Aix-Marseille I*

Ahmed Zeriahi
*Université Paul Sabatier,*

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EXTENSION OF PLURISUBHARMONIC FUNCTIONS WITH GROWTH CONTROL

DAN COMAN, VINCENT GUEDJ AND AHMED ZERIAHI

Abstract. Suppose that $X$ is an analytic subvariety of a Stein manifold $M$ and that $\varphi$ is a plurisubharmonic (psh) function on $X$ which is dominated by a continuous psh exhaustion function $u$ of $M$. Given any number $c > 1$, we show that $\varphi$ admits a psh extension to $M$ which is dominated by $cu$ on $M$.

We use this result to prove that any $\omega$-psh function on a subvariety of the complex projective space is the restriction of a global $\omega$-psh function, where $\omega$ is the Fubini-Study Kähler form.

Introduction

Let $X \subset \mathbb{C}^n$ be a (closed) analytic subvariety. In the case when $X$ is smooth it is well known that a plurisubharmonic (psh) function on $X$ extends to a psh function on $\mathbb{C}^n$ [Sa] (see also [BL, Theorem 3.2]). Using different methods, Coltoiu generalized this result to the case when $X$ is singular [Co, Proposition 2].

In this article we follow Coltoiu’s approach and show that it is possible to obtain extensions with global growth control:

**Theorem A.** Let $X$ be an analytic subvariety of a Stein manifold $M$ and let $\varphi$ be a psh function on $X$. Assume that $u$ is a continuous psh exhaustion function on $M$ so that $\varphi(z) < u(z)$ for all $z \in X$. Then for every $c > 1$ there exists a psh function $\psi = \psi_c$ on $M$ so that $\psi|_X = \varphi$ and $\psi(z) < c \max\{u(z), 0\}$ for all $z \in M$.

We recall that a function $\varphi : X \to [\infty, +\infty)$ is called psh if $\varphi \not\equiv -\infty$ on $X$ and if every point $z \in X$ has a neighborhood $U$ in $\mathbb{C}^n$ so that $\varphi = u|_U$ for some psh function $u$ on $U$. We refer to [FN] and [D2, section 1] for a detailed discussion of this notion. We note here that if $\varphi$ is not identically $-\infty$ on an irreducible component $Y$ of $X$ then $\varphi$ is locally integrable on $Y$ with respect to the area measure of $Y$. Let us stress that the more general notion of weakly psh function is not appropriate for the extension problem (see section 3).

We then look at a similar problem on a compact Kähler manifold $V$. Here psh functions have to be replaced by quasip plurisubharmonic (qpsh) ones. Given a Kähler form $\omega$, we let

$$PSH(V, \omega) = \{ \varphi \in L^1(V, [\infty, +\infty)) : \varphi \text{ upper semicontinuous, } dd^c \varphi \geq -\omega \}$$

denote the set of $\omega$-plurisubharmonic ($\omega$-psh) functions. If $X \subset V$ is an analytic subvariety, we define similarly the class $PSH(X, \omega|_X)$ of $\omega$-psh functions on $X$ (see section 2 for precise definitions).

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By restriction, ω-psh functions on V yield ω|_X-psh functions on X. Assuming that ω is a Hodge form, i.e. a Kähler form with integer cohomology class, our second result is that every ω|_X-psh function on X arises in this way.

**Theorem B.** Let X be a subvariety of a projective manifold V equipped with a Hodge form ω. Then any ω|_X-psh function on X is the restriction of an ω-psh function on V.

Note that in the assumptions of Theorem B there exists a positive holomorphic line bundle L on V whose first Chern class c_1(L) is represented by ω. In this case the ω-psh functions are in one-to-one correspondence with the set of (singular) positive metrics of L (see [GZ]). Thus an alternate formulation of Theorem B is the following:

**Theorem B’.** Let X be a subvariety of a projective manifold V and L be an ample line bundle on V. Then any (singular) positive metric of L|_X is the restriction of a (singular) positive metric of L on V.

Recall that it is possible to regularize qpsh functions on \( \mathbb{P}^n \), since it is a homogeneous manifold. Hence Theorem B has the following immediate corollary:

**Corollary C.** Let X be a subvariety of a projective manifold V equipped with a Hodge form ω. If \( \varphi \in PSH(X, \omega|_X) \) then there exists a sequence of smooth functions \( \varphi_j \in PSH(V, \omega) \) which decrease pointwise on V so that \( \lim \varphi_j = \varphi \) on X.

When X is smooth this regularization result is well known to hold even when the cohomology class of ω is not integral (see [D3], [BK]).

Corollary C allows to show that the singular Kähler-Einstein currents constructed in [EGZ1] have continuous potentials, a result that has been obtained recently in [EGZ2] by completely different methods (see also [DZ] for partial results in this direction).

We prove Theorem A in section 1. The compact setting is considered in section 2, where Theorem B is derived from Theorem A. In section 3 we discuss the special situation when X is an algebraic subvariety of \( \mathbb{C}^n \). As an application of Theorem B, we give a characterization of those psh functions in the Lelong class \( \mathcal{L}(X) \) which admit an extension in the Lelong class \( \mathcal{L}(\mathbb{C}^n) \) (see section 3 for the necessary definitions). In particular, we give simple examples of algebraic curves \( X \subset \mathbb{C}^2 \) and of functions \( \eta \in \mathcal{L}(X) \) which do not have extensions in \( \mathcal{L}(\mathbb{C}^2) \).

1. **Proof of Theorem A**

The following proposition will allow us to reduce the proof of Theorem A to the case \( M = \mathbb{C}^n \). We include its short proof for the convenience of the reader.

**Proposition 1.1.** Let V be a complex submanifold of \( \mathbb{C}^N \) and u be a continuous psh exhaustion function on V. Then there exists a continuous psh exhaustion function \( \tilde{u} \) on \( \mathbb{C}^N \) so that \( \tilde{u}|_V = u \).

**Proof.** The argument is very similar to the one of Sadullaev ([Sa], [BL, Theorem 3.2]). By [Si], there exists an open neighborhood \( W \) of V in \( \mathbb{C}^N \) and a holomorphic retraction \( r : W \to V \). We can find an open neighborhood \( U \) of V so that \( U \subset W \) and \( \|r(z) - z\| < 2 \) for every \( z \in U \). Indeed, if \( B(p,r) \) denotes the open ball in \( \mathbb{C}^N \) centered at \( p \) and of radius \( r \), then \( U_p = r^{-1}(B(p,1)) \cap B(p,1) \) is an open...
neighborhood of \( p \in V \), and we let \( U = \bigcup_{p \in V} U_p \). Since \( u \) is a continuous psh exhaustion function on \( V \), it follows that the function \( u(r(z)) \) is continuous psh on \( U \) and \( \lim_{z \in U, \|z\| \to +\infty} u(r(z)) = +\infty \).

It is well known that there exist entire functions \( f_0, \ldots, f_N \), so that \( V = \{ z \in \mathbb{C}^N : f_k(z) = 0, 0 \leq k \leq N \} \) (see [Ch, p.63]). The function \( \rho = \log(\sum |f_k|^2) \) is psh on \( \mathbb{C}^N \) and \( V = \{ \rho = -\infty \} \).

Let \( D \) be an open set so that \( V \subset D \subset \overline{D} \subset U \). Since \( \rho \) is continuous on \( \mathbb{C}^N \setminus V \), we can find a convex increasing function \( \chi \) on \([0, +\infty)\) which verifies for every \( R \geq 0 \) the following two properties:

(i) \( \chi(R) > R - \rho(z) \) for all \( z \in \mathbb{C}^N \setminus D \) with \( \|z\| = R \).

(ii) \( \chi(R) > u(r(z)) - \rho(z) \) for all \( z \in \partial D \) with \( \|z\| = R \).

Then
\[
\tilde{u}(z) = \begin{cases} 
\max\{u(r(z)), \chi(\|z\|) + \rho(z)\}, & \text{if } z \in D, \\
\chi(\|z\|) + \rho(z), & \text{if } z \in \mathbb{C}^N \setminus D,
\end{cases}
\]
is a continuous psh exhaustion function on \( \mathbb{C}^N \) and \( \tilde{u} = u \) on \( V \).

Employing the methods of Coltoiu [Co] we now construct psh extensions with growth control over bounded sets in \( \mathbb{C}^n \).

**Proposition 1.2.** Let \( \chi \) be a psh function on a subvariety \( X \subset \mathbb{C}^n \) and let \( v \) be a continuous psh function on \( \mathbb{C}^n \) with \( \chi < v \) on \( X \). If \( R > 0 \), there exists a psh function \( \tilde{\chi} = \tilde{\chi}_R \) on \( \mathbb{C}^n \) so that \( \tilde{\chi}|_X = \chi \) and \( \tilde{\chi}(z) < v(z) \) for all \( z \in \mathbb{C}^n \) with \( \|z\| \leq R \).

**Proof.** We use a similar argument to the one in the proof of Proposition 2 in [Co]. Consider the subvariety \( A = (X \times \mathbb{C}) \cup (\mathbb{C}^n \times \{0\}) \subset \mathbb{C}^{n+1} \), and let
\[
D = \{(z, w) \in X \times \mathbb{C} : \log |w| + \chi(z) < 0\} \cup (\mathbb{C}^n \times \{0\}) \subset A.
\]

Since \( D \cap (X \times \mathbb{C}) \) is Runge in \( X \times \mathbb{C} \), it follows that \( D \) is Runge in \( A \). Let
\[
K = \{(z, w) \in \mathbb{C}^{n+1} : \rho(z, w) = \max\{\log^+ (\|z\|/R), \log |w| + v(z)\} \leq 0\}.
\]
Since \( v \) is continuous, \( \rho \) is a continuous psh exhaustion function on \( \mathbb{C}^{n+1} \), so \( K \) is a polynomially convex compact set. As \( \chi < v \) on \( X \), we have \( K \cap A \subset D \). By [Co, Theorem 3] there exists a Runge domain \( \tilde{D} \subset \mathbb{C}^{n+1} \), with \( D \cap A = \tilde{D} \cap A = \tilde{D} \). Let \( \delta(z, w) \) denote the distance from \((z, w) \in \tilde{D}\) to \( \partial \tilde{D} \) in the \( w \)-direction. Since \( \tilde{D} \) is pseudoconvex, \(-\log \delta \) is psh on \( \tilde{D} \) (see e.g. [FS, Proposition 9.2]). Hence \( \tilde{\chi}(z) = -\log \delta(z, 0) \) is psh on \( \mathbb{C}^n \), as \( \mathbb{C}^n \times \{0\} \subset \tilde{D} \). Since \( \tilde{D} \cap A = D \), it follows that \( \tilde{\chi}|_X = \chi \). Moreover, \( K \subset \tilde{D} \) implies that \( \tilde{\chi}(z) < v(z) \) for all \( z \in \mathbb{C}^n \) with \( \|z\| \leq R \).

The proof of Theorem A proceeds like this. Given a partition
\[
\mathbb{C}^n = \bigcup \{m_{j-1} < u \leq m_j\},
\]
where \( m_j \nearrow +\infty \), we apply Proposition 1.2 inductively to construct an extension dominated in each “annulus” \( \{m_{j-1} < u \leq m_j\} \) by \( \gamma_j u \), where \( \gamma_j > 1 \) is an increasing sequence defined in terms of the \( m_j \)'s. Theorem A will follow by showing that it is possible to choose \( \{m_j\} \) rapidly increasing so that \( \lim \gamma_j \) is arbitrarily close to 1.

We fix next an increasing sequence \( \{m_j\}_{j \geq -1} \) so that
\[
m_{-1} = m_0 = 0 < m_1 < m_2 < \ldots, \{u < m_1\} \neq \emptyset, \ m_j \nearrow +\infty.
\]
Define inductively a sequence \( \{\gamma_j\}_{j \geq 0} \), as follows:

\[
\gamma_0 = 1, \quad \gamma_j(m_j - m_{j-1}) = \gamma_{j-1}(m_j - m_{j-2}) + 1 \quad \text{for } j \geq 1.
\]

Clearly, \( \gamma_j > \gamma_{j-1} > 1 \) for all \( j > 1 \).

**Proposition 1.3.** Let \( X, \varphi, u \) be as in Theorem A with \( M = \mathbb{C}^n \), and let \( \{m_j\} \), \( \{\gamma_j\} \) be as above. There exists a psh function \( \psi \) on \( \mathbb{C}^n \) so that \( \psi |_X = \varphi \) and for all \( z \in \mathbb{C}^n \) we have

\[
\psi(z) < \begin{cases} 
\gamma_j u(z), & \text{if } m_{j-1} < u(z) \leq m_j, \ j \geq 2, \\
\gamma_1 \max\{u(z), 0\}, & \text{if } u(z) \leq m_1.
\end{cases}
\]

**Proof.** We introduce the sets

\[
D_j = \{z \in \mathbb{C}^n : u(z) < m_j\}, \quad K_j = \{z \in \mathbb{C}^n : u(z) \leq m_j\}.
\]

Since \( u \) is a continuous psh exhaustion function, \( K_j \) is a compact set. Let

\[\rho_j = \gamma_j \max\{u - m_{j-1}, 0\} - j, \ j \geq 0.\]

Then \( \rho_j \) is psh on \( \mathbb{C}^n \) and (1) implies that

\[
(2) \quad \rho_j(z) = \rho_{j-1}(z) \text{ if } u(z) = m_j, \ j \geq 1.
\]

We claim that

\[
(3) \quad \rho_j(z) \geq u(z) \text{ if } z \in \mathbb{C}^n \setminus D_j, \ j \geq 0.
\]

Indeed, since \( \gamma_j \geq 1 \) and using (1) we obtain

\[
\rho_j(z) - u(z) = (\gamma_j - 1)u(z) - \gamma_j m_{j-1} - j \geq (\gamma_j - 1)m_j - \gamma_j m_{j-1} - j
\]

\[
= (\gamma_{j-1} - 1)m_j - \gamma_{j-1} m_{j-2} - j + 1
\]

\[
\geq (\gamma_{j-1} - 1)m_{j-1} - \gamma_{j-1} m_{j-2} - (j - 1).
\]

So \( x_j := (\gamma_j - 1)m_j - \gamma_j m_{j-1} - j \geq x_0 = 0 \), and (3) is proved.

Let \( \varphi_j = \max\{\varphi, -j\} \). We construct by induction on \( j \geq 1 \) a sequence of continuous psh functions \( \psi_j \) on \( \mathbb{C}^n \) with the following properties:

\[
(4) \quad \psi_j(z) > \varphi_j(z) \quad \text{for } z \in X, \quad \int_{X \setminus K_{j-1}} (\psi_j - \varphi_j) < 2^{-j}.
\]

\[
(5) \quad \psi_j(z) \geq \rho_j(z) \quad \text{for } z \in D_j, \quad \psi_j(z) = \rho_j(z) \quad \text{for } z \in \mathbb{C}^n \setminus D_j.
\]

\[
(6) \quad \psi_j(z) < \psi_{j-1}(z) \quad \text{for } z \in K_{j-1}, \ \text{where } \psi_0 = \rho_0 = \max\{u, 0\}.
\]

Here the integral in (4) is with respect to the area measure on each irreducible component, i.e.

\[
\int_{X \setminus K} f := \sum \int_{Y \setminus K} f \beta^\dim Y,
\]

where the sum is over all irreducible components \( Y \) of \( X \) which intersect \( K \) and \( \beta \) is the standard Kähler form on \( \mathbb{C}^n \). (Note that this is a finite sum.)

Assume that the function \( \psi_{j-1} \) is constructed with the desired properties. We construct \( \psi_j \) by applying Proposition 1.2 with \( \chi = \varphi_j \) and \( v = \psi_{j-1} \). (If \( j = 1 \), \( \psi_1 \) is constructed in the same way by applying Proposition 1.2 with \( \chi = \varphi_1 \) and \( v = \psi_0 \).) By (4), \( \varphi_j \leq \varphi_{j-1} < \psi_{j-1} \) on \( X \) (and for \( j = 1 \), clearly \( \varphi_1 \leq \psi_0 \) on \( X \)). Therefore Proposition 1.2 yields a psh function \( \varphi_j \) on \( \mathbb{C}^n \) so that \( \varphi_j |_X = \varphi_j \) and \( \varphi_j < \psi_{j-1} \) on \( K_j \). Using the standard regularization of \( \varphi_j \) and the dominated
convergence theorem (as $\varphi_j \geq -j$) we obtain a continuous psh function $\tilde{\psi}_j$ on $\mathbb{C}^n$ which verifies

$$
\tilde{\psi}_j(z) > \varphi_j(z) \text{ for } z \in X, \quad \int_{X \cap K_j} (\tilde{\psi}_j - \varphi_j) < 2^{-j}.
$$

Moreover, since $\psi_{j-1}$ is continuous, we can ensure by the Hartogs lemma that we also have $\tilde{\psi}_j(z) < \psi_{j-1}(z)$ for $z \in K_j$.

We now define

$$
\psi_j(z) = \begin{cases} 
\max\{\tilde{\psi}_j(z), \rho_j(z)\}, & \text{if } z \in D_j, \\
\rho_j(z), & \text{if } z \in \mathbb{C}^n \setminus D_j.
\end{cases}
$$

By (5) and (2) we have $\tilde{\psi}_j < \psi_{j-1} = \rho_{j-1} = \rho_j$ on $\partial D_j$ (for $j = 1$, recall that $\psi_0 = \rho_0$ by definition). So $\psi_j$ is a continuous psh function on $\mathbb{C}^n$ which verifies (5).

On $X \setminus D_j$ we have by (3) that $\psi_j = \rho_j \geq u > \varphi_j$, while on $X \cap D_j$, $\psi_j \geq \tilde{\psi}_j > \varphi_j$.

Since $\rho_j = -j \leq \varphi_j < \tilde{\psi}_j$ on $X \cap K_{j-1}$, we see that $\psi_j = \tilde{\psi}_j$ on $X \cap K_{j-1}$ so

$$
\int_{X \cap K_{j-1}} (\psi_j - \varphi_j) \leq \int_{X \cap K_{j-1}} (\tilde{\psi}_j - \varphi_j) < 2^{-j}.
$$

Hence $\psi_j$ verifies (4). Finally, we have by (5), $\rho_j = -j < \rho_{j-1} \leq \psi_{j-1}$ on $K_{j-1}$ (and for $j = 1$, $\rho_1 = -1 < \psi_0 = 0$ on $K_0$). Since $\psi_j < \psi_{j-1}$ on $K_j$ we conclude that $\psi_j < \psi_{j-1}$ on $K_{j-1}$, so (6) is verified.

So we have constructed a sequence of continuous psh functions $\psi_j$ on $\mathbb{C}^n$ verifying properties (4)-(6). Since $\bigcup_{j \geq 1} D_j = \mathbb{C}^n$, we have by (6) that the function

$$
\psi(z) = \lim_{j \to \infty} \psi_j(z)
$$

is well defined and psh on $\mathbb{C}^n$. As $\ldots < \psi_{j+2} < \psi_{j+1} < \psi_j$ on $K_j$, it follows that $\psi < \psi_j$ on $K_j$.

Suppose now that $z \in K_j \setminus D_{j-1}$, for some $j \geq 2$, so $m_{j-1} \leq u(z) \leq m_j$. By the above construction and property (5), we have

$$
\tilde{\psi}_j(z) < \psi_{j-1}(z) = \rho_{j-1}(z) \implies \psi(z) < \psi_j(z) \leq \max\{\rho_{j-1}(z), \rho_j(z)\} \leq \gamma_j u(z).
$$

Similarly, for $z \in K_1$ we have

$$
\psi(z) < \psi_1(z) \leq \max\{\rho_0(z), \rho_1(z)\} \leq \gamma_1 \max\{u(z), 0\}.
$$

Hence $\psi$ satisfies the desired global upper estimates on $\mathbb{C}^n$.

Property (4) implies that $\psi(z) \geq \varphi(z)$ for every $z \in X$. Let $K$ be a compact in $\mathbb{C}^n$ and $Y$ be an irreducible component of $X$ so that $\varphi|_Y \equiv -\infty$. By (4) we have that for all $j$ sufficiently large

$$
0 \leq \int_{Y \cap K} (\psi_j - \varphi) = \int_{Y \cap K} (\tilde{\psi}_j - \varphi_j) + \int_{Y \cap K} (\varphi_j - \varphi) \leq 2^{-j} + \int_{Y \cap K} (\varphi_j - \varphi).
$$

Hence by dominated convergence, $\int_{Y \cap K} (\psi - \varphi) = 0$, which shows that $\psi = \varphi$ on $Y$.

Assume now that $Y$ is an irreducible component of $X$ so that $\varphi|_Y \equiv -\infty$. Then using (4) and the monotone convergence theorem we conclude that

$$
\int_{Y \cap K} \psi = \lim_{j \to \infty} \int_{Y \cap K} \psi_j = \lim_{j \to \infty} \left( \int_{Y \cap K} (\tilde{\psi}_j - \varphi_j) + \int_{Y \cap K} \varphi_j \right) = -\infty,
$$

so $\psi|_Y \equiv -\infty$. Therefore $\psi = \varphi$ on $X$, and the proof is finished. \qed
Proof of Theorem A. We consider first the case $M = \mathbb{C}^n$. Fix $c > 1$. We define inductively a sequence $\{m_j\}$ with the following properties: $m_{-1} = m_0 = 0 < m_1$, $\{u < m_1\} \neq \emptyset$, and for $j \geq 1$, $m_j > m_{j-1}$ is chosen large enough so that

$$a_j = \frac{m_{j-1} - m_{j-2} + 1}{m_j - m_{j-1}} \leq \frac{\log c}{2^j}.$$ 

Since $\gamma_j \geq \gamma_0 = 1$ we have by (1),

$$\gamma_j(m_j - m_{j-1}) \leq \gamma_j(1 + a_j) \implies \gamma_j \leq \gamma_j(1 + a_j).$$

Thus

$$\gamma_j < \gamma = \prod_{j=1}^{\infty} (1 + a_j), \quad \log \gamma \leq \sum_{j=1}^{\infty} a_j \leq \log c.$$ 

Let $\psi = \psi_\varepsilon$ be the psh extension of $\varphi$ provided by Proposition 1.3 for this sequence $\{m_j\}$. Then for every $z \in \mathbb{C}^n$ we have

$$\psi(z) < \gamma \max\{u(z), 0\} \leq c \max\{u(z), 0\}.$$ 

Assume now that $M$ is a Stein manifold of dimension $n$. Then $M$ can be properly embedded in $\mathbb{C}^{2n+1}$, hence we may assume that $M$ is a complex submanifold of $\mathbb{C}^{2n+1}$ (see e.g. [Ho, Theorem 5.3.9]). Proposition 1.1 implies the existence of a continuous psh exhaustion function $\tilde{u}$ on $\mathbb{C}^{2n+1}$ so that $\tilde{u} = u$ on $M$. By what we already proved, given $c > 0$ there exists a psh function $\tilde{\psi}$ on $\mathbb{C}^{2n+1}$ which extends $\varphi$ and such that $\tilde{\psi} < c \max\{\tilde{u}, 0\}$ on $\mathbb{C}^{2n+1}$. We let $\psi = \tilde{\psi} |_M$. \(\square\)

We end this section by noting that some hypothesis on the growth of $u$ is necessary in Theorem A. Indeed, suppose that $X$ is a submanifold of $\mathbb{C}^n$ for which there exists a non-constant negative psh function $\varphi$ on $X$. Then any psh extension of $\varphi$ to $\mathbb{C}^n$ cannot be bounded above. However, by Theorem A, given any $\varepsilon > 0$ there exists a psh function $\psi = \psi_\varepsilon$ so that $\psi |_X = \varphi$ and $\psi(z) < \varepsilon \log^+ \|z\|$ on $\mathbb{C}^n$.

2. EXTENSION OF QPSh FUNCTIONS

Let $V$ be a compact Kähler manifold equipped with a Kähler form $\omega$. We let $\text{PSH}(V, \omega)$ denote the set of $\omega$-psh functions on $V$. These are upper semicontinuous functions $\varphi \in L^1(V, [-\infty, +\infty])$ such that $\omega + dd^c \varphi \geq 0$, where $d = \partial + \overline{\partial}$ and $dd^c = \frac{1}{2\pi i} (\partial - \overline{\partial})$. We refer the reader to [GZ] for basic properties of $\omega$-psh functions.

Let $X$ be an analytic subvariety of $V$. Recall that an upper semicontinuous function $\varphi : X \to [-\infty, +\infty)$ is called $\omega |_X$-psh if $\varphi \neq -\infty$ on $X$ and if there exist an open cover $\{U_i\}_{i \in I}$ of $X$ and psh functions $\varphi_i, \rho_i$ defined on $U_i$, where $\rho_i$ is smooth and $dd^c \rho_i = \omega$, so that $\rho_i + \varphi = \varphi_i$ holds on $X \cap U_i$, for every $i \in I$. Moreover, $\varphi$ is called strictly $\omega |_X$-psh if it is $(1 - \varepsilon)\omega |_X$-psh for some small $\varepsilon > 0$. The current $\omega |_X + dd^c \varphi$ is then called a Kähler current on $X$ (see [EGZ1, section 5.2]). We denote by $\text{PSH}(X, \omega |_X)$, resp. $\text{PSH}^+(X, \omega |_X)$, the class of $\omega |_X$-psh, resp. strictly $\omega |_X$-psh functions on $X$.

Every $\omega$-psh function $\varphi$ on $V$ yields, by restriction, an $\omega |_X$-psh function $\varphi |_X$ on $X$, as soon as $\varphi |_X \neq -\infty$. The question we address here is whether this restriction operator is surjective. In other words, is there equality

$$\text{PSH}(X, \omega |_X) \ni \text{PSH}(V, \omega) |_X.$$
2.1. The smooth case. We start with the elementary observation that smooth strictly $\omega$-psh functions can easily be extended.

Proposition 2.1. Let $V$ be a compact Kähler manifold equipped with a Kähler form $\omega$, and let $X$ be a complex submanifold of $V$. Then

$$PSH^+(X, \omega|_X) \cap C^\infty(X, \mathbb{R}) = (PSH^+(V, \omega) \cap C^\infty(V, \mathbb{R}))|_X.$$ 

We include a proof for the convenience of the reader, although this is probably part of the “folklore” (see e.g. [Sch] for the case where $\omega$ is a Hodge form).

Proof. Let $\varphi \in C^\infty(X, \mathbb{R})$ be such that $(1 - \varepsilon)\omega|_X + dd^c \varphi \geq 0$ on $X$, for some $\varepsilon > 0$. We first choose $\tilde{\varphi}$ to be any smooth extension of $\varphi$ to $V$. Consider

$$\psi := \tilde{\varphi} + A\chi \text{dist}(\cdot, X)^2,$$

where $\chi$ is a test function supported in a small neighborhood of $X$ and such that $\chi \equiv 1$ near $X$. Here $\text{dist}$ is any Riemannian distance on $V$, for instance the distance associated to the Kähler metric $\omega$. Then $\psi$ is yet another smooth extension of $\varphi$ to $V$, which now satisfies $(1 - \varepsilon/2)\omega + dd^c \psi \geq 0$ near $X$, if $A$ is chosen large enough.

The function $\log(\text{dist}(\cdot, X)^2)$ is well defined and qpsh in a neighborhood of $X$. Let $\chi$ be a test function supported in this neighborhood so that $\chi \equiv 1$ near $X$. The function $u = \chi \log(\text{dist}(\cdot, X)^2)$ is $N\omega$-psh on $V$ for a large integer $N$. Moreover, $\exp(u)$ is smooth and $X = \{u = -\infty\}$. Replacing $\omega$ by $N\omega$, $\varphi$ by $N\varphi$, and $\psi$ by $N\psi$, we may assume that $N = 1$. Set now

$$\psi_C := \frac{1}{2} \log [e^{2\psi} + e^u + C].$$

This again is a smooth extension of $\varphi$, and a straightforward computation yields

$$dd^c \psi_C \geq \frac{2e^{2\psi}dd^c \psi + e^u + C dd^c u}{2(e^{2\psi} + e^u + C)}.$$

Hence

$$\left(1 - \frac{\varepsilon}{2}\right)\omega + dd^c \psi_C \geq \frac{2e^{2\psi} \left(\left(1 - \frac{\varepsilon}{2}\right)\omega + dd^c \psi\right) + (1 - \varepsilon)e^u + C \omega}{2(e^{2\psi} + e^u + C)} \geq 0,$$

if $C$ is chosen large enough. \hfill \Box

This proof breaks down when $\varphi$ is singular and hence a different approach is needed. We consider in the next section the particular case when $\omega$ is a Hodge form.

2.2. Proof of Theorem B. We assume here that $\omega$ is a Hodge form, i.e. that the cohomology class $\{\omega\}$ belongs to $H^2(V, \mathbb{Z})$ (more precisely to the image of $H^2(V, \mathbb{Z})$ in $H^2(V, \mathbb{R})$ under the mapping induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$). We prove the following more precise version of Theorem B.

Theorem 2.2. Let $X$ be a subvariety of a projective manifold $V$ equipped with a Hodge form $\omega$. If $\varphi \in PSH(X, \omega|_X)$ then given any constant $a > 0$ there exists $\psi \in PSH(V, \omega)$ so that $\psi|_X = \varphi$ and $\max_V \psi < \max_X \varphi + a$. 

In the assumptions of Theorem 2.2 there exists a positive holomorphic line bundle $L$ on $V$ whose first Chern class $c_1(L)$ is represented by $\omega$. By Kodaira’s embedding theorem $L$ is ample, hence for large $k$ there exists an embedding $\pi : V \hookrightarrow \mathbb{P}^n$ such that $L^k = \mathcal{O}(1)$.

Replacing $\omega$ by $k\omega$, $\varphi$ by $k\varphi$, we can assume that $L = \mathcal{O}(1)$, $V$ is an algebraic submanifold of the complex projective space $\mathbb{P}^n$, and $\omega = \omega_{FS}|_V$ is the Fubini-Study Kähler form. Hence $X$ is an algebraic subvariety of $\mathbb{P}^n$, and Theorem 2.2 follows if we show that $\omega_{FS}$-psh functions on $X$ extend to $\omega_{FS}$-psh functions on $\mathbb{P}^n$.

Therefore we assume in the sequel that $X \subset V = \mathbb{P}^n$ and $\omega$ is the Fubini-Study Kähler form on $\mathbb{P}^n$. Let $[z_0 : \ldots : z_n]$ denote the homogeneous coordinates.

Without loss of generality, we may assume that they are chosen so that no coordinate hyperplane $\{z_j = 0\}$ contains any irreducible component of $X$.

Let
\[ \theta(z) = \log \frac{\max\{|z_0|, \ldots, |z_n|\}}{\sqrt{|z_0|^2 + \ldots + |z_n|^2}}, \quad z = [z_0 : \ldots : z_n] \in \mathbb{P}^n. \]
This is an $\omega$-psh function and for all $z \in \mathbb{P}^n$,
\[-m \leq \theta(z) \leq 0, \quad \text{where} \quad m = \log \sqrt{n+1}. \]

We start by noting that Theorem A yields special subextensions of $\omega$-psh functions on $X$.

**Lemma 2.3.** Let $\varepsilon \geq 0$ and $u$ be a continuous $(1 + \varepsilon)\omega$-psh function on $\mathbb{P}^n$ so that $u(z) \leq 0$ for all $z \in \mathbb{P}^n$. If $c > 1$ and $\varphi$ is an $\omega$-psh function on $X$ so that $\varphi < u$, then there exists a $c\omega$-psh function $\psi$ on $\mathbb{P}^n$ so that
\[ \frac{1}{c} \psi(z) \leq \frac{1}{1 + \varepsilon} u(z), \quad \forall z \in \mathbb{P}^n, \]
and
\[ \psi(z) = \varphi(z) + (c - 1)\theta(z) + (c - 1) \min_{\zeta \in \mathbb{P}^n} u(\zeta), \quad \forall z \in X. \]

**Proof.** Let
\[ M = - \min_{\zeta \in \mathbb{P}^n} u(\zeta) \geq 0. \]
We work first in an affine chart $\{z_j = 1\} \equiv \mathbb{C}^n$. Let $X_j = X \cap \{z_j = 1\}$ and let $\rho_j \geq 0$ be the potential of $\omega$ in this chart with $\rho_j(0) = 0$. Then $\varphi + \rho_j$ is psh on $X_j$ and since $u \leq 0$,
\[ \varphi + \rho_j + M < u + \rho_j + M \leq \frac{1}{1 + \varepsilon} u + \rho_j + M \quad \text{on} \quad X_j. \]
Note that $(1 + \varepsilon)^{-1}u + \rho_j + M \geq 0$ is a continuous psh exhaustion function on $\mathbb{C}^n$.

Theorem A yields a psh function $\tilde{\psi}$ on $\mathbb{C}^n$ so that
\[ \tilde{\psi} < \frac{c}{1 + \varepsilon} u + c\rho_j + cM \quad \text{on} \quad \mathbb{C}^n, \quad \tilde{\psi} = \varphi + \rho_j + M \quad \text{on} \quad X_j. \]

The function $\psi_j = \tilde{\psi} - c\rho_j - cM$ extends uniquely to a $c\omega$-psh function on $\mathbb{P}^n$ which verifies
\[ \psi_j \leq \frac{c}{1 + \varepsilon} u \quad \text{on} \quad \mathbb{P}^n. \]
Moreover on $X \cap \{z_j = 1\}$ we have
\[ \psi_j = \varphi - (c - 1)\rho_j - (c - 1)M = \varphi + (c - 1)\theta_j - (c - 1)M, \]
where
\[ \theta_j(z) = \log \frac{|z_j|}{\sqrt{|z_0|^2 + \ldots + |z_n|^2}}. \]

Hence \( \psi_j = -\infty \) on \( X \cap \{ z_j = 0 \} \).

We finally let \( \psi = \max \{ \psi_0, \ldots, \psi_n \} \). This is a \( cw \)-psh function on \( \mathbb{P}^n \) which verifies the desired conclusions, since \( \theta = \max \{ \theta_0, \ldots, \theta_n \} \). \( \square \)

**Proof of Theorem 2.2.** Fix \( a > 0 \). Replacing \( \varphi \) by \( \varphi - \max_X \varphi - a \) we may assume that \( \max_X \varphi = -a \). We will show that there exists a sequence of smooth \( \omega \)-psh functions \( \varphi_j \) on \( \mathbb{P}^n \) which decrease pointwise on \( \mathbb{P}^n \) to a negative \( \omega \)-psh function \( \psi \) so that \( \psi = \varphi \) on \( X \).

Let \( X' \) be the union of the irreducible components \( W \) of \( X \) so that \( \varphi|_W \not\equiv -\infty \). We first construct by induction on \( j \geq 1 \) a sequence of numbers \( \varepsilon_j \wedge 0 \) and a sequence of negative smooth \( (1 + \varepsilon_j)\omega \)-psh functions \( \psi_j \) on \( \mathbb{P}^n \) so that for all \( j \geq 2 \)

\[ \frac{\psi_j}{1 + \varepsilon_j} < \frac{\psi_{j-1}}{1 + \varepsilon_{j-1}} \text{ on } \mathbb{P}^n, \quad \psi_{j-1} > \varphi \text{ on } X, \quad \int_{X'} (\psi_j - \varphi) < \frac{1}{j}, \quad \int_W \psi_j < -j, \]

for every irreducible component \( W \) of \( X \) where \( \varphi|_W \equiv -\infty \). Here the integrals are with respect to the area measure on each irreducible component \( X_j \) of \( X \), i.e.

\[ \int_X f := \sum_{X_j} \int_{X_j} f \omega^{\dim X_j}. \]

Let \( \varepsilon_1 = 1, \psi_1 = 0 \), and assume that \( \varepsilon_{j-1}, \psi_{j-1} \), where \( j \geq 2 \), are constructed with the above properties. Since \( \varphi < \psi_{j-1}|_X \) and the latter is continuous on the compact set \( X \), we can find \( \delta > 0 \) so that \( \varphi < \psi_{j-1} - \delta \) on \( X \).

Let \( c > 1 \). By Lemma 2.3, there exists a \( cw \)-psh function \( \psi_c \) so that

\[ \frac{\psi_c}{c} \leq \frac{\psi_{j-1} - \delta}{1 + \varepsilon_{j-1}} \text{ on } \mathbb{P}^n, \quad \psi_c = \varphi + (c - 1)\theta - (c - 1)M_{j-1} \text{ on } X, \]

where

\[ M_{j-1} = \delta - \min_{\zeta \in \mathbb{P}^n} \psi_{j-1}(\zeta) \geq 0. \]

We can regularize \( \psi_c \) on \( \mathbb{P}^n \); there exists a sequence of smooth \( cw \)-psh functions decreasing to \( \psi_c \) on \( \mathbb{P}^n \). Therefore we can find a smooth \( cw \)-psh function \( \psi'_c \) on \( \mathbb{P}^n \) so that

\[ \frac{\psi'_c}{c} < \frac{\psi_{j-1} - \delta}{1 + \varepsilon_{j-1}} \text{ on } \mathbb{P}^n, \quad \psi'_c > \varphi + (c - 1)\theta - (c - 1)M_{j-1} \geq \varphi - (c - 1)(m + M_{j-1}) \text{ on } X. \]

By dominated, resp. monotone convergence, we can in addition ensure that

\[ \int_{X'} (\psi'_c - \varphi) \leq \int_{X'} (\psi'_c - \varphi - (c - 1)\theta + (c - 1)M_{j-1}) < c - 1, \]

\[ \int_W \psi'_c < -j - (c - 1)(m + M_{j-1})|W|, \]

for every irreducible component \( W \) of \( X \) where \( \varphi|_W \equiv -\infty \). Here \( |W| \) denotes the (projective) area of \( W \).
Now let $\psi_c'' = \psi'_c + (c-1)(m+M_{j-1})$. Then on $\mathbb{P}^n$ we have

$$\frac{\psi_c''}{c} < \frac{\psi_{j-1} - \frac{c}{2} + (c-1)(m+M_{j-1})}{1 + \varepsilon_{j-1}} < \frac{\psi_{j-1} - \frac{c}{4} + (c-1)(m+M_{j-1})}{1 + \varepsilon_{j-1}}.$$

Moreover, $\psi_c'' > \varphi$ on $X$ and

$$\int_{X'} (\psi_c'' - \varphi) = \int_{X'} (\psi'_c - \varphi) + (c-1)(m+M_{j-1})|X'| < \int_{X'} (c-1)(1+m|X'| + M_{j-1}|X'|),$$

$$\int_W \psi_c'' = \int_W \psi'_c + (c-1)(m+M_{j-1})|W| < -j,$$

for every irreducible component $W$ of $X$ where $\varphi|_W \equiv -\infty$.

We take $c = 1 + \varepsilon_j$ and $\psi_j = \psi_c''$, where $\varepsilon_j > 0$ is so that

$$\varepsilon_j < \varepsilon_{j-1}/2, \quad \varepsilon_j(m+M_{j-1}) < \frac{\delta}{4}, \quad \varepsilon_j(1+m|X'| + M_{j-1}|X'|) < \frac{1}{j}.$$

Then $\varepsilon_j$, $\psi_j$ have the desired properties.

We conclude that $\varphi_j = (1 + \varepsilon_j)^{-1} \psi_j$ is a decreasing sequence of smooth negative $\omega$-psh function on $\mathbb{P}^n$, so that $\varphi_j > (1 + \varepsilon_j)^{-1} \varphi > \varphi$ on $X$. Hence $\psi = \lim_{j \to \infty} \varphi_j$ is a negative $\omega$-psh function on $\mathbb{P}^n$ and $\psi \geq \varphi$ on $X$. Note that

$$\int_{X'} (\varphi_j - \varphi) = \frac{1}{1 + \varepsilon_j} \int_{X'} (\psi_j - \varphi) - \frac{\varepsilon_j}{1 + \varepsilon_j} \int_{X'} \varphi < \frac{1}{j} - \frac{\varepsilon_j}{1 + \varepsilon_j} \int_{X'} \varphi,$$

$$\int_W \varphi_j = \frac{1}{1 + \varepsilon_j} \int_W \psi_j < \frac{j}{2},$$

for every irreducible component $W$ of $X$ where $\varphi|_W \equiv -\infty$. It follows that $\psi = \varphi$ on $X$ and the proof of Theorem 2.2 is finished. $\Box$

3. Algebraic subvarieties of $\mathbb{C}^n$

If $X$ is an analytic subvariety of $\mathbb{C}^n$ and $\gamma$ is a positive number, we denote by $\mathcal{L}_\gamma(X)$ the *Lelong class* of psh functions $\varphi$ on $X$ which verify $\varphi(z) \leq \gamma \log^+ \|z\| + C$ for all $z \in X$, where $C$ is a constant that depends on $\varphi$. We let $\mathcal{L}(X) = \mathcal{L}_{1}(X)$. By Theorem A, functions $\varphi \in \mathcal{L}(X)$ admit a psh extension in each class $\mathcal{L}_{\gamma}(\mathbb{C}^n)$, for every $\gamma > 1$.  

We assume in the sequel that $X$ is an algebraic subvariety of $\mathbb{C}^n$ and address the question whether it is necessary to allow the arbitrarily small additional growth. More precisely, is it true that

$$\mathcal{L}(X) \equiv \mathcal{L}(\mathbb{C}^n)|_X,$$

i.e. is every psh function with logarithmic growth on $X$ the restriction of a globally defined psh function with logarithmic growth? We will give a criterion for this to hold, but show that in general this is not the case.

1If $X$ is algebraic this result is claimed in [BL, Proposition 3.3], but there is a gap in their proof.
3.1. Extension preserving the Lelong class. Consider the standard embedding
\[ z \in \mathbb{C}^n \mapsto [1 : z] \in \mathbb{P}^n, \]
where \([t : z]\) denote the homogeneous coordinates on \(\mathbb{P}^n\). Let \(\omega\) be the Fubini-Study Kähler form and let
\[ \rho(t, z) = \log \sqrt{|t|^2 + \|z\|^2} \]
be its logarithmically homogeneous potential on \(\mathbb{C}^{n+1}\).

We denote by \(\overline{X}\) the closure of \(X\) in \(\mathbb{P}^n\), so \(\overline{X}\) is an algebraic subvariety of \(\mathbb{P}^n\).

It is well known that the class \(PSH(\mathbb{P}^n, \omega)\) is in one-to-one correspondence with the Lelong class \(\mathcal{L}(\mathbb{C}^n)\) (see [GZ]). Let us look at the connection between \(\omega\)-psh functions on \(\overline{X}\) and the class \(\mathcal{L}(X)\).

The mapping
\[ F_X : PSH(\overline{X}, \omega|_{\overline{X}}) \rightarrow \mathcal{L}(X), \quad (F_X \varphi)(z) = \rho(1, z) + \varphi([1 : z]), \]
is well defined and injective. However, it is in general not surjective, as shown by Examples 3.2 and 3.3 that follow.

Conversely, a function \(\eta \in \mathcal{L}(X)\) induces an upper semicontinuous function \(\tilde{\eta}\) on \(\overline{X}\) defined in the obvious way:
\[ \tilde{\eta}([t : z]) = \begin{cases} \eta(z) - \rho(1, z), & \text{if } t = 1, \ z \in X, \\ \limsup_{[1:\zeta] \to [0:z]} (\eta(\zeta) - \rho(1, \zeta)), & \text{if } t = 0, \ [0 : z] \in \overline{X} \setminus X. \end{cases} \]
The function \(\tilde{\eta}\) is in general only weakly \(\omega\)-psh on \(\overline{X}\), i.e. it is bounded above on \(\overline{X}\) and it is \(\omega|_{\overline{X}_r}\)-psh on the set \(\overline{X}_r\) of regular points of \(\overline{X}\). This notion is in direct analogy to that of weakly psh function on an analytic variety (see [D2, section 1]). We do not pursue it any further here.

Note that \(\eta \in F_X \left(PSH(\overline{X}, \omega|_{\overline{X}})\right)\) if and only if \(\tilde{\eta} \in PSH(\overline{X}, \omega|_{\overline{X}})\). The following simple characterization is a consequence of Theorem B.

**Proposition 3.1.** Let \(\eta \in \mathcal{L}(X)\). The following are equivalent:

(i) There exists \(\psi \in \mathcal{L}(\mathbb{C}^n)\) so that \(\psi = \eta\) on \(X\).

(ii) \(\tilde{\eta} \in PSH(\overline{X}, \omega|_{\overline{X}})\).

(iii) For every point \(a \in \overline{X} \setminus X\) the following holds: if \((X_j, a)\) are the irreducible components of the germ \((\overline{X}, a)\) then the value
\[ \limsup_{X_j \ni [1:1] \to a} (\eta(\zeta) - \rho(1, \zeta)) \]
is independent of \(j\).

In particular, if the germs \((\overline{X}, a)\) are irreducible for all points \(a \in \overline{X} \setminus X\) then \(\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)|_X\).

**Proof.** Assume that (i) holds. It follows that \(\tilde{\eta} = \varphi|_{\overline{X}}\), where
\[ \varphi([t : z]) = \begin{cases} \psi(z) - \rho(1, z), & \text{if } t = 1, \\ \limsup_{[0:z] \to [1:1]} (\psi(\zeta) - \rho(1, \zeta)), & \text{if } t = 0, \end{cases} \]
is an \(\omega\)-psh function on \(\mathbb{P}^n\). Hence \(\tilde{\eta} \in PSH(\overline{X}, \omega|_{\overline{X}})\).
 Conversely, if (ii) holds then by Theorem B there exists an \( \omega \)-psh function \( \phi \) on \( \mathbb{P}^n \) which extends \( \tilde{\eta} \). Hence \( \psi(z) = \rho(1, z) + \varphi((1 : z)) \) is an extension of \( \eta \) and \( \psi \in \mathcal{L}(\mathbb{C}^n) \).

The equivalence of (ii) and (iii) follows easily from [D2, Theorem 1.10]. \( \square \)

### 3.2. Explicit examples. In view of section 3.1, it is easy to construct examples of algebraic curves \( X \subset \mathbb{C}^2 \) and functions in \( \mathcal{L}(X) \) which do not admit an extension in \( \mathcal{L}(\mathbb{C}^2) \). We write \( z = (x, y) \in \mathbb{C}^2 \).

**Example 3.2.** Let \( X = \{y = 0\} \cup \{y = 1\} \subset \mathbb{C}^2 \) and \( \eta \in \mathcal{L}(X) \), where

\[
\eta(z) = \begin{cases} 
\rho(1, z), & \text{if } z = (x, 0), \\
\rho(1, z) + 1, & \text{if } z = (x, 1).
\end{cases}
\]

The function \( \tilde{\eta} \) is not \( \omega \)-psh on \( \overline{X} = \{y = 0\} \cup \{y = t\} \), hence \( \eta \) does not have an extension in \( \mathcal{L}(\mathbb{C}^2) \). Indeed, the maximum principle is violated along \( \{y = 0\} \) near the point \( a = [0 : 1 : 0] \), since \( \tilde{\eta}([t : 1 : 0]) = 0 \) for \( t \neq 0 \), while \( \tilde{\eta}([t : 1 : t]) = 1 \).

With a little more effort we can give an example as above where \( X \) is an irreducible curve. Let \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \).

**Example 3.3.** Let \( X \subset \mathbb{C}^2 \) be the irreducible cubic with equation \( xy = x^3 + 1 \). Then

\[
\overline{X} = \{[t : x : y] \in \mathbb{P}^2 : xy = x^3 + t^3\}, \quad X = \{a\}, \quad a = [0 : 1 : 1].
\]

The germ \( (\overline{X}, a) \) has two irreducible components \( X_1, X_2 \), both are smooth at \( a \), \( X_1 \) being tangent to the line \( \{x = 0\} \), and \( X_2 \) to the line \( \{t = 0\} \).

Note that in fact \( X \subset \mathbb{C}^* \times \mathbb{C} \) is the graph of the rational function \( y = x^2 + x^{-1}, x \in \mathbb{C}^* \). If \( (x, y) \in X \) and \( x \to 0 \) then \( (x, y) \to a \) along \( X_1 \), while as \( x \to \infty \) then \( (x, y) \to a \) along \( X_2 \). The function

\[
u(x, y) = \max\{-\log |x|, 2 \log |x| + 1\}
\]

is psh in \( \mathbb{C}^* \times \mathbb{C} \). It is easy to check that \( \eta := \nu \big|_X \in \mathcal{L}(X) \) and

\[
\limsup_{X_1 \ni \zeta \to a} (\eta(\zeta) - \rho(1, \zeta)) = 0, \quad \limsup_{X_2 \ni \zeta \to a} (\eta(\zeta) - \rho(1, \zeta)) = 1.
\]

Hence \( \eta \) does not admit an extension in \( \mathcal{L}(\mathbb{C}^2) \).

We conclude this section with an example of a cubic \( X \subset \mathbb{C}^2 \) and a psh function on \( X \) of the form \( \eta = \log |P| \), where \( P \) is a polynomial, so that \( \eta \) admits a “transcendental” extension with exactly the same growth, but small additional growth is necessary if we look for an “algebraic” extension.

**Proposition 3.4.** Let \( X = \{x = y^3\} \) and \( \eta(x, y) = \log |1 + y| \), so \( \eta \big|_X \in \mathcal{L}_{1/3}(X) \).

Given \( k \geq 1 \), there is a polynomial \( Q_k(x, y) \) of degree \( k + 1 \) so that \( Q_k(y^3, y) = (y + 1)^{3k} \). In particular, \( \psi_k = \frac{1}{3k} \log |Q_k| \in \mathcal{L}_{(k+1)/3k}(\mathbb{C}^2) \) is an extension of \( \eta \big|_X \).

There exists no polynomial \( Q(x, y) \) of degree \( k \) so that \( Q(y^3, y) = (y + 1)^{3k} \).

However, \( \eta \big|_X \) has an extension in \( \mathcal{L}_{1/3}(\mathbb{C}^2) \).

**Proof.** We construct \( Q_k \) by replacing \( y^3 \) by \( x \) in the polynomial

\[
(y + 1)^{3k} = \sum_{j=0}^{3k} \binom{3k}{j} y^j.
\]
Since $j = 3\lfloor j/3 \rfloor + r_j$, $r_j \in \{0, 1, 2\}$, it follows that

$$Q_k(x, y) = \sum_{j=0}^{3k} \binom{3k}{j} x^{\lfloor j/3 \rfloor} y^{r_j} = 3kx^{k-1}y^2 + \text{l.d.t.}.$$ 

We now check that there is no polynomial $Q(x, y)$ of degree $k$ so that $Q(y^3, y) = (y + 1)^{3k}$. Indeed, if $Q(x, y) = \sum_{j+l \leq k} c_{jl}x^jy^l$ then

$$Q(y^3, y) = c_{k0}y^{3k} + c_{k-1,1}y^{3k-2} + \text{l.d.t.}$$ 

does not contain the monomial $y^{3k-1}$. 

Note that $\mathbb{X} = \{xt^2 = y^3\} = X \cup \{a\}$, where $a = [0 : 1 : 0]$, so the germ $(\mathbb{X}, a)$ is irreducible. Proposition 3.1 implies that $\eta|_X$ has an extension in $\mathcal{L}_{1/3}(\mathbb{C}^2)$. □

We conclude with some remarks regarding our last example. If $X$ is an algebraic subvariety of $\mathbb{C}^n$ and $f$ is a holomorphic function on $X$, $f$ is said to have polynomial growth if there is an integer $N(f)$ and a constant $A$ so that

$$|f(z)| \leq A(1 + ||z||)^{N(f)}, \quad \forall z \in X.$$ 

Then it is well known that there exists a polynomial $P$ of degree at most $N(f) + \varepsilon(X)$ so that $P|_X = f$, where $\varepsilon(X) > 0$ is a constant depending only on $X$ (see e.g. [Bj] and references therein). However, if $X \subset \mathbb{P}^n$ is irreducible at each of its points at infinity then by Proposition 3.1 the psh function $\eta = N(f)^{-1}\log|f| \in \mathcal{L}(X)$ has a psh extension in the Lelong class $\mathcal{L}(\mathbb{C}^n)$.

On the other hand, Demailly [D1] has shown that in the case of the transcendental curve $X = \{e^x + e^y = 1\}$ any holomorphic function $f$ on $X$, of polynomial growth, has a polynomial extension of the same degree to $\mathbb{C}^n$. Hence it is natural to ask if for this curve one has that $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)|_X$.

References


D. Coman: dcoman@syr.edu, Department of Mathematics, Syracuse University, Syracuse, NY 13244-1150, USA

V. Guedj: guedj@cmi.univ-mrs.fr, Université Aix-Marseille 1, LATP, 13453 Marseille Cedex 13, FRANCE

A. Zeriahi: zeriahi@picard.ups-tlse.fr, Laboratoire Emile Picard, UMR 5580, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 04, FRANCE