Lotka-Volterа Impulsive Model Control or How not to fish

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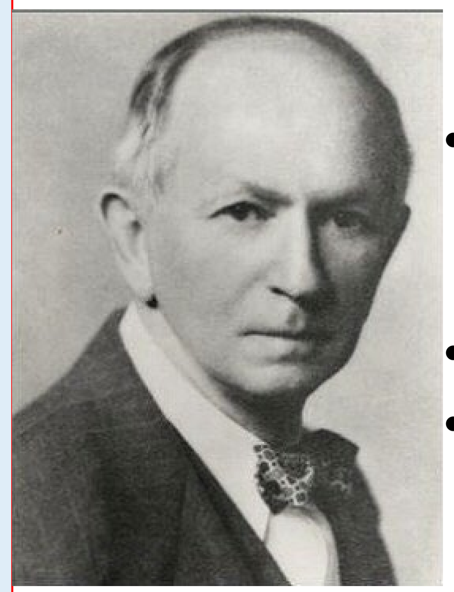
Abstract

Human intervention in nature often destroys the equilibrium that has been established in it. It is very important to find controls that would not destroy the stability of the natural equilibrium in the ecosystem. The impulsive systems theory allows us to investigate such effects on a biological system with a simpler example that do not destroy its equilibrium.

Lotka–Volterra equations

$$\frac{dx}{dt} = Ax - Bxy,$$
$$\frac{dy}{dt} = -Cy + Dxy;$$

- Discovered independently by Alfred Lotka and Vito Volterra.
- Describe interaction between two species (prey and predator) in simplified biological system.
- x and y are the numbers of prey and predator respectively; A, B, C, D are the constants that represent the interaction.



Alfred James Lotka

(March 2, 1880 – December 5, 1949)

- a US mathematician, founder of mathematical demography;
- born in Lviv, Ukraine;
- statistician at the Metropolitan Life Insurance Co..

Vito Volterra

(May 3, 1860 – October 11, 1940)

- Italian mathematician, founder of functional analysis;
- born in Ancona, Italy;
- refuse to sign an oath of loyalty to the Fascist regime;
- professor at the University of Turin and the University of Rome La Sapienza.

Lotka–Volterra impulsive model

$$\begin{cases} \dfrac{dN_1}{dt} = \alpha N_1 - \beta N_1 N_2, & t \neq k\theta, \\ N_1(t+0) = \rho_{11}N_1(t) + \rho_{12}N_2(t) + \rho_{10}, & t = k\theta, \\ \dfrac{dN_2}{dt} = -\gamma N_2 + \delta N_1 N_2, & t \neq k\theta, \\ N_2(t+0) = \rho_{21}N_1(t) + \rho_{22}N_2(t) + \rho_{20}, & t = k\theta; \end{cases}$$

$N_1^* = \frac{\gamma}{\delta}, N_2^* = \frac{\alpha}{\beta}$ — equilibrium state.

$$x = N_1 - N_1^*, y = N_2 - N_2^*;$$

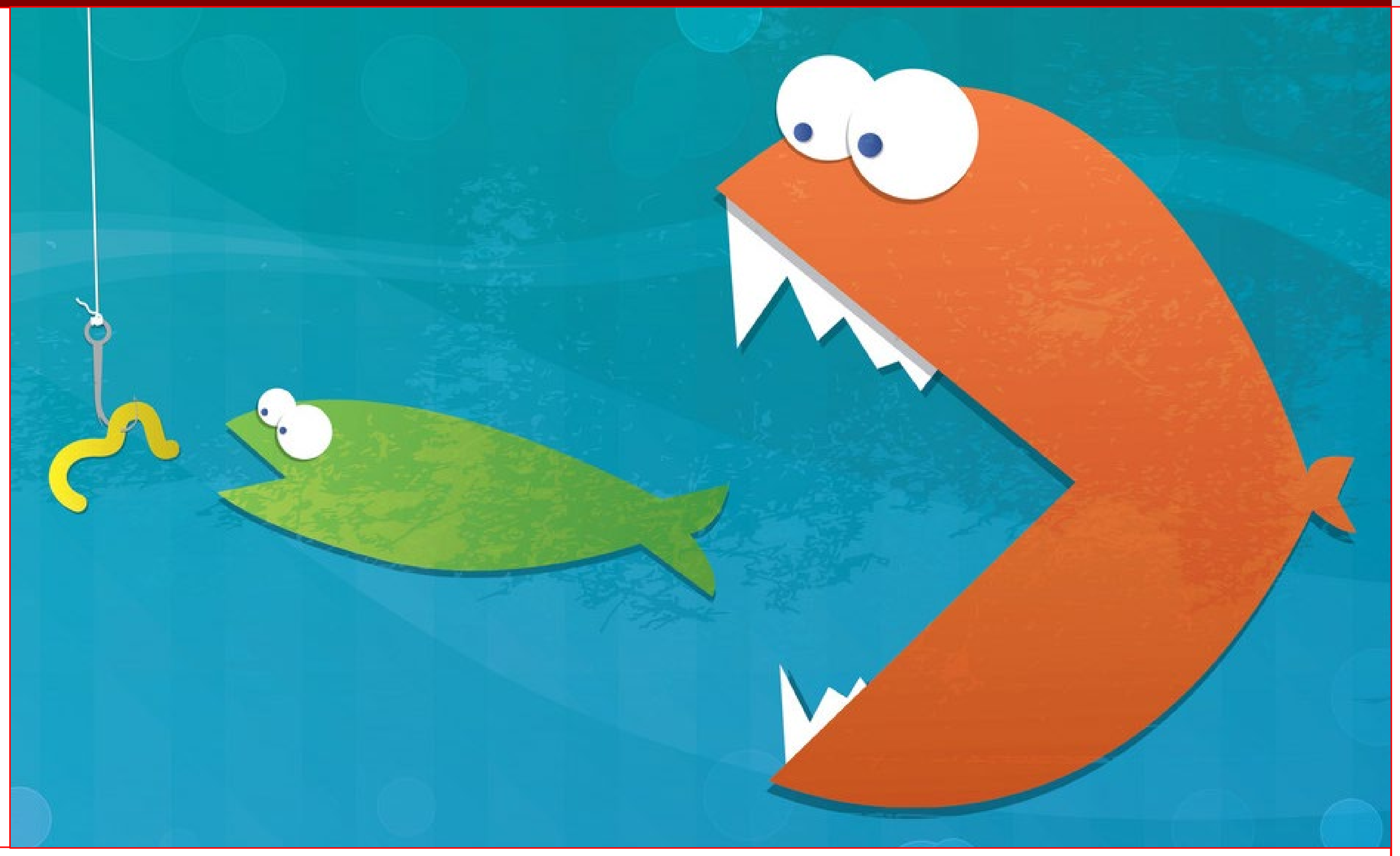
Since impulses do not ruin the equilibrium, the following condition has to be satisfied

$$\begin{cases} \rho_{10} + (\rho_{11} - 1)N_1^* + \rho_{12}N_2^* = 0, \\ \rho_{20} + (\rho_{22} - 1)N_2^* + \rho_{21}N_1^* = 0. \end{cases}$$

$$\begin{cases} \dot{x} = -\dfrac{\beta\gamma}{\delta} y, & t \neq k\theta, \\ \dot{y} = \dfrac{\alpha\delta}{\beta} x, & t \neq k\theta, \\ x(t+0) = \rho_{11}x(t) + \rho_{12}y(t), & t = k\theta, \\ y(t+0) = \rho_{21}x(t) + \rho_{22}y(t), & t = k\theta; \end{cases}$$

$$\Omega = \sqrt{\alpha\gamma};$$

$$\Phi = \begin{bmatrix} \rho_{11} & \dfrac{\delta}{\beta}\sqrt{\dfrac{\alpha}{\gamma}}\rho_{12} \\ \dfrac{\beta}{\delta}\sqrt{\dfrac{\gamma}{\alpha}}\rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} \cos\Omega\theta & \sin\Omega\theta \\ -\sin\Omega\theta & \cos\Omega\theta \end{bmatrix}.$$



Conditions for asymptotic stability of equilibrium state are

$$-1 + |\mathrm{tr}\,\Phi| < \det\Phi < 1,$$
$$\det\Phi = \det\rho = \rho_{11}\rho_{22} - \rho_{12}\rho_{21},$$
$$\mathrm{tr}\,\Phi = (\rho_{11} + \rho_{22})\cos\Omega\theta + \left(\frac{\rho_{12}\beta}{\delta}\sqrt{\frac{\gamma}{\alpha}} - \frac{\rho_{21}\delta}{\beta}\sqrt{\frac{\alpha}{\gamma}}\right)\sin\Omega\theta,$$

Conditions for stability (nonasymtotic) of equilibrium state

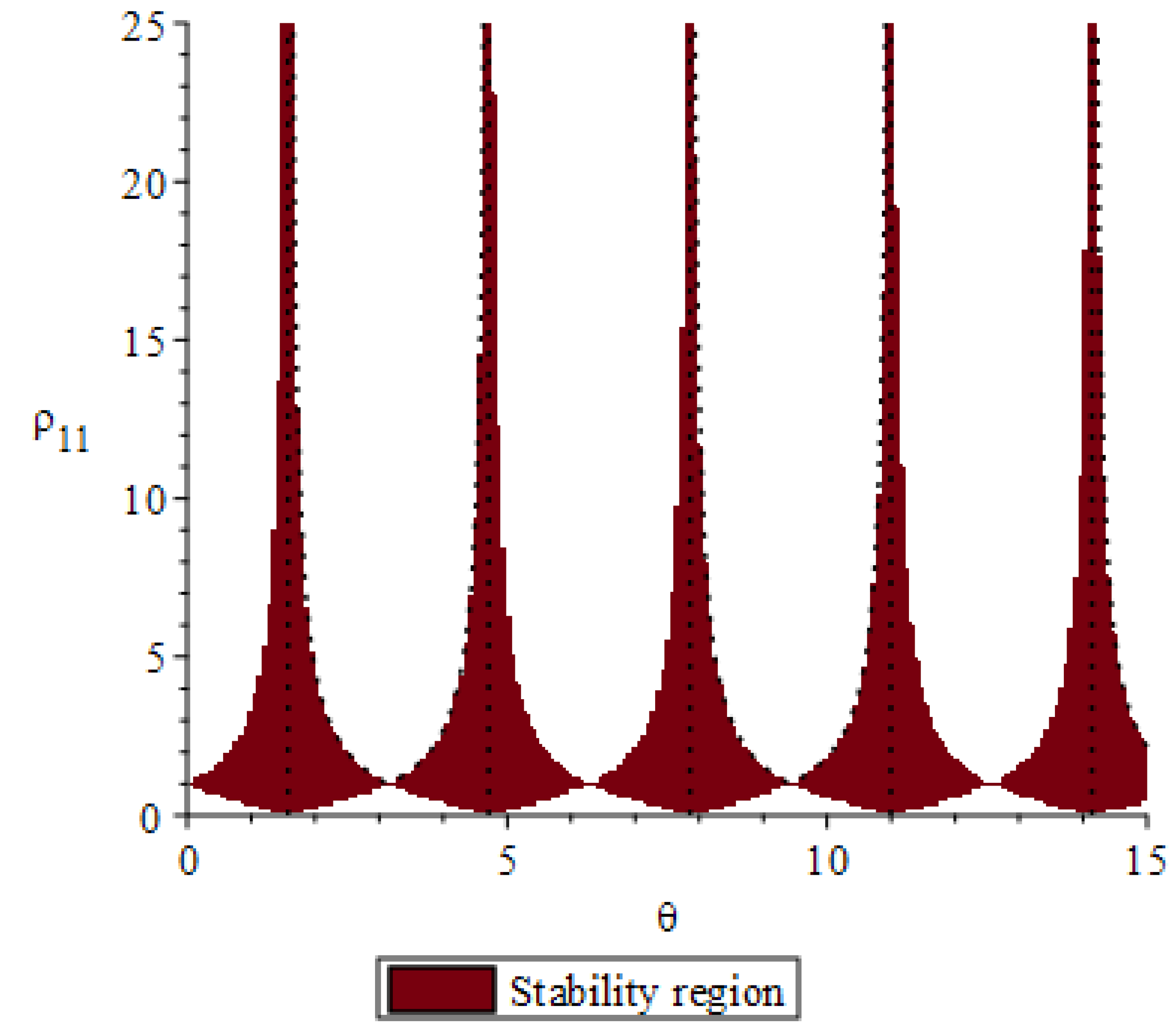
$$\det\Phi = 1, |\mathrm{tr}\,\Phi| < 2.$$

Example

If $\rho_{12} = \rho_{21} = 0, \rho_{11}\rho_{22} = 1, \rho_{11} = \frac{1}{\rho_{22}}$, then the stability condition is

$$\left(\rho_{11} + \frac{1}{\rho_{11}}\right)|\cos\Omega\theta| < 2.$$

Example. Regions for parameters ρ_{11} and θ that satisfy stability conditions.



Conclusion

This research reveals stability and asymptotic stability conditions of an impulsive Lotka–Volterra model. The simple example shows that mathematical modeling of such a human intervention is necessary before the actual intrusion into the ecosystem. Modern methods of applied mathematics and mathematical modeling can be useful in the study of artificial ecological systems and the influence of anthropogenic factors on their evolution.

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